1 Introduction to optimization

1.1 Notations and definitions

1. a **criterion**, or **cost function**, or **objective function**: a function J defined over V with values in $\mathbb{R}$, where V is the space (normed vector space) in which the problem lies, also called the space of *command variables*.

2. some **constraints**: for example,

   (a) $v \in K$, where K is a subset of V

   (b) **equality constraints**: $F(v) = 0$, where $F : V \to \mathbb{R}^m$ (m *real* constraints: $F_i(v) = 0$)

   (c) **inequality constraints**: $G(v) \leq 0$, where $G : V \to \mathbb{R}^p$ (p constraints $G_i(v) \leq 0$).
   If $G_i(v) = 0$, the constraint is **active**, or **saturated**.
   If $G_i(v) < 0$, the constraint is **inactive**.

   (d) Functional equation: the constraint is given by an ODE, or a PDE, to be satisfied $\rightsquigarrow$ optimal command of an evolutive problem.

The different types of constraints can be mixed: $v \in K$ and $F(v) = 0$ and $G(v) \leq 0$.

We denote by U the set of **admissible** elements of the problem:

$$U = \{v \in V; v \text{ satisfies all the constraints}\}.$$

Minimization problem:

$$(\mathcal{P}) \qquad \text{Find } u \in U \text{ such that } \quad J(u) \leq J(v), \quad \forall v \in U$$

Maximization problem: idem.

u is the **optimal** solution, or the solution of the optimization problem. $J(u)$ is the optimal value of the criterion.

Local optimum: $\bar{u}$ is a local optimum if there exists a neighborhood $\mathcal{V}(\bar{u})$ such that $J(\bar{u}) \leq J(v)$, $\forall v \in \mathcal{V}(\bar{u})$.

Note that a global optimum is a local optimum, but the converse proposition is not true (except in a convex case).

1.2 Examples

- Finite dimension: $J : \mathbb{R}^n \to \mathbb{R}$
- Infinite dimension: $J : V \to \mathbb{R}$

Finite dimension:

1.2.1 Example 1: linear problem

Food rationing (e.g. during wars):
n types of food
m food components (proteins, vitamins, ...)
c_j: unitary price of food j

v_j: quantity of food j
a_{ij} quantity of component i per unit of food j
b_i: minimal (vital) quantity of component i

Food ration of minimal cost: minimize

$$J(v) = \sum_{j=1}^{n} c_j v_j,$$

under the constraints $v_j \geq 0$, $\displaystyle\sum_{j=1}^{n} a_{ij} v_j \geq b_i$, $i = 1, \ldots, m$.

1.2.2 Example 2: least squares problem

$Av = b$, where A is a $m \times n$ matrix, of rank $n < m$, and $v \in \mathbb{R}^n$.

$$J(v) = \|Av - b\|^2_{\mathbb{R}^m}$$

Infinite dimension:

1.2.3 Calculus of variations

V functional space,

$$J(v) = \int_\Omega L(x, v(x), Pv(x), \ldots)\, dx,$$

where P is a differential operator.
Problem: $\inf_{v \in V} J(v)$.

1.2.4 Example 3: optimal trajectory

From point (a, y_0), reach (b, y_1) as soon as possible.
The speed at point $(x, y(x))$ is $c(x, y(x))$.
Boundary conditions: $y(a) = y_0$, $y(b) = y_1$.

$$c(x, y(x)) = \frac{\sqrt{dx^2 + y'^2 dx^2}}{dt} = \frac{dx}{dt}\sqrt{1 + y'^2}$$

$$dt = \frac{\sqrt{1 + y'^2}\, dx}{c(x, y(x))}$$

and then

$$J(y) = \int_a^b \frac{\sqrt{1 + y'^2}}{c(x, y(x))}\, dx$$

to be minimized under the constraint $y(a) = y_0$, $y(b) = y_1$, and $y \in V = C^1([a, b])$.

1.2.5 Example 4: geodesic

A geodesic is the shortest path between points on a given space (e.g. on the Earth's surface). $x = x(u,v),\ y = y(u,v),\ z = z(u,v) \Rightarrow ds^2 = dx^2 + dy^2 + dz^2 = (x_u du + x_v dv)^2 + (y_u du + y_v dv)^2 + (z_u du + z_v dv)^2 = e\, du^2 + 2f\, dudv + g\, dv^2$, where $e = x_u^2 + y_u^2 + z_u^2$, $f = x_u x_v + y_u y_v + z_u z_v$, $g = x_v^2 + y_v^2 + z_v^2$.
The shortest path between points (u_0, v_0) and (u_1, v_1) is given by a function $v : [u_0, u_1] \to \mathbb{R}$ solution of

$$\inf J(v) = \int_{u_0}^{u_1} \sqrt{e + 2fv' + gv'^2}\, du$$

under the constraints $v(u_0) = v_0$, $v(u_1) = v_1$, and $v \in C^1([u_0, u_1])$.

1.2.6 Other examples

• Energy principle (or variational principle) for a PDE:

$$\inf J(v) = \frac{1}{2} \int_\Omega |\nabla v|^2\, dx - \int_\Omega fv\, dx$$

• Inverse problems: identification and estimation of parameters:

$$-div(K(x)\nabla u) = f$$

where K is unknown:

$$J = \sum_i [u(x_i) - u_i]^2$$