3 Existence of a minimum

3.1 Existence of minima in finite dimension

We now study the **existence of minima** for optimization problems in finite dimension.

Definition 3.1 *A* **minimizing sequence** *of the criterion J on the set K is a sequence $(u^n)_{n\in\mathbb{N}}$ such that $u^n \in K$ and*

$$\lim_{n\to+\infty} J(u^n) = \inf_{v\in K} J(v).$$

We now study the particular case $V = \mathbb{R}^n$, provided with the Euclidean norm.

Theorem 3.1 *Let K be a nonempty closed set of $\mathbb{R}^n$, and J a continuous function over K with values in $\mathbb{R}$, such that $J(v) \to +\infty$ if $\|v\| \to +\infty$. Then there exists at least one minimum of J over K. Moreover, from every minimizing sequence of J over K, one can extract a subsequence converging to a minimum of J.*

Proof
Let (u^n) be a minimizing sequence of J over K. As $J(u^n)$ is bounded, (u^n) is bounded. Then there exists a subsequence (u^{n_k}) which converges to $u \in \mathbb{R}^n$. As K is closed, $u \in K$. As J is continuous, $J(u^{n_k}) \to J(u)$. Then $J(u) = \inf_{v\in K} J(v)$. □

Remark: If K is bounded, the assumption "J infinite at infinity" can be relaxed.

3.2 Infinite dimension

3.2.1 Convex analysis: existence of a minimum in a Hilbert space

Theorem 3.2 *Let K be a nonempty closed convex set of a Hilbert space V, and $J : V \to \mathbb{R}$ be a convex Gâteaux-differentiable function. If K is bounded, or if J is infinite at infinity, then there exists at least one minimum of J over K*

$$J(u) = \inf_{v\in K} J(v).$$

3.2.2 Application to quadratic functions:

$$J(v) = \frac{1}{2}a(v,v) - L(v)$$

where a is a continuous symmetric bilinear coercive form over V, and L is a continuous linear form over V. Let K be a nonempty closed convex subset of V.

Proposition 3.1 *There exists a unique u such that*

$$J(u) = \inf_{v\in K} J(v).$$

It is given by

$$a(u, v-u) \geq L(v-u),\ \forall v \in K,$$

and if $K = V$, by

$$a(u,v) = L(v),\ \forall v \in K.$$

All the following is introduced for the proof of theorem 3.2.

3.2.3 Weak convergence

Definition 3.2 *Let x_n be a sequence of V, $x_n \rightharpoonup x$ in V (x_n converges* **weakly** *to x) if*

$$\langle f, x_n \rangle \to \langle f, x \rangle, \ \forall f \in V.$$

Remark: The definition is the same in a Banach space E, with $f \in E'$.

Proposition 3.2 *Strong convergence implies weak convergence.*

Corollary 3.1 *A weakly closed set is strongly closed.*

Proposition 3.3 *Let C be a convex subset of V. Then C is weakly closed iff C is strongly closed.*

Proposition 3.4 *In a Hilbert space of finite dimension, strong and weak convergence are equivalent.*

Theorem 3.3 *Weak compacity of the closed unit ball: Let V be a Hilbert space, and (x_n) a bounded sequence of V. Then there exists a subsequence (x_{n_k}) that converges weakly in V.*

3.2.4 Lower semicontinuous functions and epigraphs

Definition 3.3 *A function $J : V \to]-\infty; +\infty[$ is lower semicontinuous if $\forall x \in V$, $\forall \varepsilon > 0$, $\exists$ a neighborhood $\mathcal{V}(x)$ such that*

$$J(y) \geq J(x) - \varepsilon, \ \forall y \in \mathcal{V}(x).$$

Example: $J(x) = 0$ if $x \leq 0$, and $J(x) = 1$ if $x > 0$.

Proposition 3.5 *J is lower semicontinuous iff*

$$J(x) \leq \liminf_{x_n \to x} J(x_n).$$

Definition 3.4 *The* **epigraph** *of J is the set*

$$Epi(J) = \{(\lambda, v) \in \mathbb{R} \times V, \quad \lambda \geq J(v)\}.$$

Example: the previous function J.

Proposition 3.6 *J is convex iff $Epi(J)$ is convex.*

Proposition 3.7 *J is lower semicontinuous iff $Epi(J)$ is closed.*

Proposition 3.8 *If J is convex lower semicontinuous over V, then J is also lower semicontinuous for the weak topology.*

Proof
$Epi(J)$ is convex, and strongly closed. Then it is convex and weakly closed. □

3.2.5 Proof of the existence of a minimum in a Hilbert space

Proof
Let (v^n) be a minimizing sequence of J over K: $J(v^n) \to \inf_{v\in K} J(v)$ (note that the infimum can be $-\infty$).
As (v^n) is bounded, there exists a subsequence v^{n_k} such that $v^{n_k} \rightharpoonup u \in V$ (weak convergence).
We now prove that $u \in K$. Let $\bar{u} = Proj_K(u)$ the projection of u on K. As K is convex, $\langle u - \bar{u}, w - \bar{u}\rangle \leq 0$, $\forall w \in K$. Considering $w = v^{n_k}$, $\langle \bar{u} - u, v^{n_k}\rangle \to \langle \bar{u} - u, u\rangle$ and $\langle \bar{u} - u, v^{n_k} - \bar{u}\rangle \leq 0$. Then $0 \leq \langle u - \bar{u}, u - \bar{u}\rangle \leq 0$. Then $\bar{u} = u$ and $u \in K$.
We now prove that $J(u) \leq \liminf_{k\to+\infty} J(v^{n_k})$. As J is convex and Gâteaux-differentiable, $J(v^{n_k}) \geq J(u) + (J'(u), v^{n_k} - u) \to J(u)$. Then $J(u) \leq \liminf J(v^{n_k})$. Then J is weakly lower semicontinuous.
Finally, $J(v^{n_k}) \to \inf J(v)$.

$$J(u) \leq \inf_{k\to+\infty} J(v^{n_k}) = \inf_{v\in K} J(v)$$

and then u is the optimum $\square$