EXERCISE n°1 : Fréchet differentiability

Definition : $f : \mathbb{R}^n \to \mathbb{R}^m$ is Fréchet differentiable at point x if there exists a linear form $df_x : \mathbb{R}^n \to \mathbb{R}^m$ such that $f(x+h) = f(x) + df_x(h) + \|h\|\varepsilon(h)$, $\forall h \in \mathbb{R}^n$, where $\varepsilon(h) \to 0$ when $h \to 0$ (instead of $\|h\|\varepsilon(h)$, one can write $o(h)$).
The gradient of $f : \mathbb{R}^n \to \mathbb{R}$ is the vector $\nabla f = \left(\frac{\partial f}{\partial x_i}\right)$. The Hessian of f is the matrix $\nabla^2 f = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$. The Jacobian of $F : \mathbb{R}^n \to \mathbb{R}^m$ is the matrix $JF = \left(\frac{\partial F_i}{\partial x_j}\right)$.

Exercise 1 : Computation of Fréchet differentials

1.1. Let $f : \mathbb{R}^n \to \mathbb{R}$,
$x \to f(x) :=< c, x > +b$, where $c \in \mathbb{R}^n$, $b \in \mathbb{R}$.
Compute the differential df_x of f at point x, then the gradient $\nabla f(x)$. Compute the second order differential $d^2 f_x$, then the hessian $\nabla^2 f(x)$.

1.2. Let $F : \mathbb{R}^n \to \mathbb{R}^m$,
$x \to F(x) := Ax + b$, where $A \in \mathcal{M}_{m,n}(\mathbb{R})$, $b \in \mathbb{R}^m$.
Compute the differential dF_x, then the Jacobian $JF(x)$.

1.3. Let $f : \mathbb{R}^n \to \mathbb{R}$,
$x \to f(x) := \frac{1}{2} < Ax, x > + < b, x > +c$, where $A \in \mathcal{M}_{n,n}(\mathbb{R})$, $b \in \mathbb{R}^n$, $c \in \mathbb{R}$.
Compute the differential df_x, then the gradient $\nabla f(x)$. Compute the second order differential $d^2 f_x$, then the hessian $\nabla^2 f(x)$.

1.4. Let $g : \mathbb{R}^n \to \mathbb{R}$,
$$x \to g(x) := \sum_{i=1}^{m} (r_i(x))^2,$$ where the $r_i : \mathbb{R}^n \to \mathbb{R}$ are twice differentiable.
Compute the differential dg_x, then the gradient $\nabla g(x)$. Compute the second order differential $d^2 g_x$, then the hessian $\nabla^2 g(x)$.

Exercise 2 : Differentiability

2.1. Let $g : \mathbb{R}^n \to \mathbb{R}$,
$$x \to g(x) := \sum_{i=1}^{m} \left(f_i^+(x)\right)^2,$$
where the $f_i : \mathbb{R}^n \to \mathbb{R}$ are twice differentiable and $a^+ = \max(0, a)$. Is the function g differentiable, twice differentiable ?

2.2. Let
$$\begin{array}{ccc} \mathbb{R}^m & \xrightarrow{A} & \mathbb{R}^n \\ & \searrow^{g:=foA} & \downarrow f \\ & & \mathbb{R} \end{array}$$
, where $A : u \to A_0 u + b$ with $A_0 \in \mathcal{M}_{n,m}(\mathbb{R})$, $b \in \mathbb{R}^n$ and f twice differentiable.
Determine ∇g and $\nabla^2 g$ in terms of ∇f and $\nabla^2 f$.

Application : let $f : \mathbb{R}^n \to \mathbb{R}$, twice differentiable, $x_0 \in \mathbb{R}^n$ and $d \in \mathbb{R}^n$. One sets $\varphi(t) := f(x_0 + td)$.
Compute $\varphi'(t)$ (resp. $\varphi''(t)$) in terms of $\nabla f(x_0 + td)$ (resp. $\nabla^2 f(x_0 + td)$).