ERASMUS MUNDUS - Optimization Exercice **Year 2010-2011**

Exercise n°2 : Gâteaux-differentiability – Convexity

Exercice 1 : Convexity

Are the following functions from $\mathbb{R}$ to $\mathbb{R}$ convex ? strictly convex ? α-convex ?

1.1. $x \to x^+ = \max(x, 0)$.

1.2. $x \to |x|$.

1.3. $x \to x^2$.

1.4. $x \to x^3$.

Exercice 2 : Convexity

Is the function $f(x) = ||x||^2$ defined on $\mathbb{R}^n$ convex ? Is it strictly convex ? Is it α-convex ?

Exercice 3 : Computation of Gâteaux-derivatives and convexity

One considers $x \in \mathbb{R}^n \to f(x) = ||Ax - b||^2$, with $A \in \mathcal{M}_{n,n}(\mathbb{R})$ and $b \in \mathbb{R}^n$.

3.1. Compute the Gâteaux-derivative of f and deduce the gradient $\nabla f(x)$.

3.2. Compute the second-order Gâteaux-derivative of f and deduce the hessian $\nabla^2 f(x)$.

3.3. For which type of matrices $A \in \mathcal{M}_{n,n}(\mathbb{R})$, is the function f convex ? strictly convex ? α-convex ?

Exercice 4 : Characterization of strict convexity

4.1. Prove that if f is Gâteaux-differentiable in U, f is strictly convex in U iff for all $u, v \in U$, $u \neq v$, $f(v) > f(u) + (f'(u), v - u)$.

4.2. Prove that if f is Gâteaux-differentiable in U, f is strictly convex in U iff for all $u, v \in U$, $u \neq v$, $(f'(v) - f'(u), v - u) > 0$.

4.3. Prove that if f is twice Gâteaux-differentiable in U and that for all $u, w \in U$, $w \neq 0$, $(f''(u); w, w) > 0$, then f is strictly convex on U.