ERASMUS MUNDUS - Optimization Exercise
Year 2010-2011

Exercise n°3 : Least Square Minimization

Let A be a $m \times n$ matrix. One tries to solve "at best" the linear system

$$Ax = b,\ x \in \mathbb{R}^n,\ b \in \mathbb{R}^m, \tag{1}$$

when $n \neq m$ or when $n = m$, but with A non inversible.

One notes

$$||y|| = \left(\sum_{i=1}^{n} y_i^2 \right)^{1/2}$$

the euclidian norm of y on $\mathbb{R}^n$.

Part 1 : One notes $J(x) = ||Ax - b||^2$ for all $x \in \mathbb{R}^n$.

1.1. Prove that the minimization problem :

$$\text{Find } x \in \mathbb{R}^n \text{ such that } J(x) = \min_{y \in \mathbb{R}^n} J(y) \tag{2}$$

has at least one solution. One notes X_b the set of solutions of (2).

1.2. Prove that (2) is equivalent to :

$$\text{Find } x \in \mathbb{R}^n \text{ such that } {}^tAAx = {}^tAb. \tag{3}$$

1.3. Discuss the existence and uniqueness of a solution of (3) according to the rank of A.

1.4. Prove that the minimization problem :

$$\text{Find } x \in X_b \text{ such that } ||x||^2 = \min_{y \in X_b} ||y||^2 \tag{4}$$

has a unique solution $\overline{x}$ that one calls a pseudo-solution of (1).

1.5. Prove that $\overline{x}$ is characterized by $\overline{x} \in X_b \cap (\text{Ker}\,{}^tAA)^{\perp}$.

Part 2 : Let $\varepsilon > 0$, one notes $J_\varepsilon(x) = ||Ax - b||^2 + \varepsilon ||x||^2$ for all $x \in \mathbb{R}^n$.

One considers then the problem :

$$\text{Find } x \in \mathbb{R}^n \text{ such that } J_\varepsilon(x) = \min_{y \in \mathbb{R}^n} J_\varepsilon(y). \tag{5}$$

2.1. Prove that (5) has a unique solution x_ε.

2.2. Prove that for all $y \in X_b$, $||x_\varepsilon|| \leq ||y||$.

2.3. Express x_ε in terms of A, b and ε.

2.4. Deduce that x_ε converges to $\overline{x}$ solution of (4) when $\varepsilon \to 0$.

Part 3 : Let $\varepsilon > 0$, one notes $\overline{J_\varepsilon}(x) = \dfrac{1}{\varepsilon}||^t AAx -^t Ab||^2 + ||x||^2$ for all $x \in \mathbb{R}^n$.

One considers then the problem :

$$\text{Find } x \in \mathbb{R}^n \text{ such that } \overline{J_\varepsilon}(x) = \min_{y \in \mathbb{R}^n} \overline{J_\varepsilon}(y). \tag{6}$$

3.1. Prove that (6) has a unique solution $\overline{x_\varepsilon}$.

3.2. Express $\overline{x_\varepsilon}$ in terms of A, b and ε.

3.3. Deduce that $\overline{x_\varepsilon}$ converges to $\overline{x}$ solution of (4) when $\varepsilon \to 0$.