Exercise n°5 : Minimization under (equality and) inequality constraints

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Exercise 1 :

One considers the following minimization problem in $\mathbb{R}^2$:

$$\inf f_a(x_1, x_2) = x_1^2 + ax_2^2 + x_1x_2 + x_1 \text{ under the constraint } x_1 + x_2 - 1 \leq 0. \tag{1}$$

1.1. For which values of $a \in \mathbb{R}$, the function f_a is convex ?

1.2. For these values of a, can we apply the Kuhn-Tucker theorem ? If yes, write the Kuhn-Tucker relations.

1.3. Solve then problem (1) according to the values of a.

Exercise 2 :

Let C be the set of $\mathbb{R}^2$ defined by $C = \{(x, y)/x + y \leq 1,\ x \geq 0,\ y \geq 0\}$, and f the function of $\mathbb{R}^2$ to $\mathbb{R}$ defined by $f(x, y) = -x - 2y - xy + x^2 + y^2$.

2.1. Is f convex ? concave ?

2.2. Is C open ? closed ? convex ? is it an affine sub-space ?

2.3. One wants to minimize f on C. Prove that each minimum (including the local minima) is located on the boundary of C.

2.4. Determine the minima of f on C (with the help of Kuhn-Tucker, or directly by using question 2.3).

Exercise 3 :

Let the following problem :

$$\min \frac{1}{2}(x_1 - 1)^2 + \frac{1}{2}(x_2 - 2)^2 \text{ under the constraints } x_1 - x_2 = 1,\ x_1 + x_2 \leq 2,\ x_1 \geq 0 \text{ and } x_2 \geq 0. \tag{2}$$

3.1. Prove that problem (2) has a unique solution.

3.2. Solve problem (2) with the help of Lagrange multipliers.