VARIATIONAL FORMULATIONS IN MECHANICS:
THEORY AND APPLICATIONS
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IMAGE RESTORATION AND CLASSIFICATION BY TOPOLOGICAL ASYMPTOTIC EXPANSION

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Abstract. We present in this paper a new way for modeling and solving image restoration and classification problems, the topological gradient method. This method is considered in the frame of variational approaches and the minimization of potential energy with respect to conductivity. The numerical experiments show the efficiency of the topological gradient approach. The image is most of the time restored or classified at the first iteration of the optimization process. Moreover, the computational cost of this iteration is reduced drastically using spectral methods. We also propose an algorithm which provides the optimal classes (number and values) for the unsupervised regularized classification problem.

Key words: topological gradient, topological asymptotic expansion, edge detection, image restoration, unsupervised classification

1 INTRODUCTION

The goal of topological optimization and most image processing problems is to create a partition of a given domain (or set) Ω:

- in topological optimization, we look for the optimal design $\omega \subset \Omega$ and its complementary;
- in image processing problems like edge detection, classification, and segmentation, the goal is to split the image in several parts.

For this reason, topological shape optimization and image processing problems have common mathematical methods like level set approaches, material properties optimization, variational methods,...

Level set approaches have been applied to image processing[8,30,26,10] and it gave very promizing results in topological shape optimization[33,2,36]. Diffusive methods in image restoration are based on the optimization of conductive material properties[8,39]. Like in topological optimization[1,11], isotropic and anisotropic material properties have been considered.

In this paper, we consider the topological gradient approach that has been introduced for topological optimization purpose[34,24,20,5,6,22,21]. The basic idea is to adapt the topological gradient approach used for crack detection[5]: an image can be viewed as a piecewise smooth function and edges can be considered as a set of singularities. It has been applied to diffusive grey image restoration giving very promizing results[23]. An optimal material distribution is obtained at the first iteration. Our objective is to apply topological gradient approach to color images, and also to the image classification problem. We show that it is possible to solve these image processing problems using topological optimization tools for the detection of edges. Then the restoration or classification operations become straightforward.

More precisely, let Ω be an open bounded domain of $I\!R^2$ and $j(\Omega) = J(u_\Omega)$ be a cost function to be minimized, where u_Ω is the solution to a given Partial Differential Equations (PDE) problem defined in Ω. For a small $\rho \geq 0$, let $\Omega_\rho = \Omega \backslash \overline{\omega_\rho}$ be the perturbed domain by the insertion of a small hole $\omega_\rho = x_0 + \rho\omega$, where $x_0 \in \Omega$ and ω is a fixed bounded domain of $I\!R^2$ containing the origin. The topological sensitivity theory provides an asymptotic expansion of j when ρ tends to zero. It takes the general form

$$j(\Omega_\rho) - j(\Omega) = f(\rho)G(x_0) + \circ(f(\rho)), \tag{1}$$

where $f(\rho)$ is an explicit positive function going to zero with ρ and $G(x_0)$ is called the topological gradient at point x_0. Then to minimize the criterion j, we have to insert small holes at points where g is negative. Using this gradient type information, it is possible to build fast algorithms. In most applications, a satisfying approximation of the optimal solution is reached at the first iteration of the optimization process. A topological sensitivity framework allowing to obtain such an expansion for general cost functions has been proposed in the work of Masmoudi[24,5].

We recall that a classical way to restore an image u from its noisy version v defined in a domain $\Omega \subset I\!R^2$ is to solve the following PDE problem

$$\begin{cases} -div\,(c\nabla u) + u = v & \text{in} \quad \Omega, \\ \partial_n u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2}$$

where c is a small positive constant, ∂_n denotes the normal derivative and n is the outward unit normal to $\partial\Omega$. This method is well known to give poor results: it blurs important structures like edges. In order to improve this method, nonlinear isotropic and anisotropic methods were introduced, we can cite here the work of Perona and Malik[28], Catté *et al.*[15] and more recently Weickert[38,37] and Aubert[8].

In topological gradient approach, c takes only two values: c_0 in the smooth part of the image and a small value ϵ on edges. For this reason, classical nonlinear diffusive approaches, where c takes all the values of the interval $[\epsilon, c_0]$, could be seen as a relaxation of our method. By enlarging the set of admissible solutions, relaxation increases the instability of the restoration process and this could explain why our method is so efficient.

Then, this paper is concerned with the problem of classifying an image using n predefined (supervised classification) classes $C_i, 1 \leq i \leq n$, by choosing the grey or color level intensity as a classifier. Let us first recall the general mathematical formulation of image classification problem, which consists to find a regular and homogeneous partition of Ω. A partitionning of Ω consists in searching for a family of open sets $\{\Omega_i\}_{i=1,..,n}$, such that $\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$, and $\Omega = \cup_{i=1}^{n} \Omega_i \cup \Gamma$. Γ is the union of all interfaces between two different subsets: $\Gamma = \cup_{i\neq j} \Gamma_{ij}$ where Γ_{ij} represents the interface between Ω_i and Ω_j. A regular partition means that Γ is of minimal length and an homogeneous partition implies that each set Ω_i is homogeneous with respect to the grey level intensity criterion.

Many classification models have been studied and tested on synthetic and real images in image processing literature, and results are more or less comparative taking account of the complexity of algorithms suggested and/or the cost of operations defined. We can cite here some models enough used like the structural approach by regions growth[27], the stochastic approaches[13,14] and the variational approaches which are based on various strategies like level set formulations, the Mumford-Shah functionnal, active contours and geodesic active contours methods or wavelet transforms[8,25,30,31,26,10,7,39].

In section 2, we review the classical approaches for image restoration. The nonlinear diffusion method according to Aubert *et al.*[8,9] is in particular presented. The topological gradient method[24] and its application to image restoration is developed in section 3 and then compared with the nonlinear diffusion approach in section 4, in which several numerical experiments show the efficiency of our method. In section 5, we remind the variational classification formulation when the grey or color levels are given, and we present an application of the topological gradient to this problem. We present then in section 6 a restoration-based preprocessing algorithm for the classification problem. Several numerical experiments are given. We finally present in section 7 a way to solve the unsupervised classification problem (i.e. when the levels are not given) by determining in an optimal way the number of levels and their values. A conclusion ends this paper, recalling the main results of this work and presenting developments under progress.

2 CLASSICAL APPROACHES FOR IMAGE RESTORATION

2.1 Linear approach

Let K be the canonical embedding operator defined by

$$\begin{aligned} K : H^1(\Omega) &\longrightarrow L^2(\Omega), \\ u &\longmapsto Ku = u \end{aligned} \tag{3}$$

For a given $v \in L^2(\Omega)$, we consider the problem

$$Ku = v, \tag{4}$$

which can be formulated as a minimization problem

$$\inf_{u \in H^1(\Omega)} \int_\Omega |v - Ku|^2 \, dx. \tag{5}$$

A necessary optimality condition of (5) is given by

$$K^* K u = K^* v, \tag{6}$$

where K^* is the adjoint of K. Solving (6) is in general an ill-posed problem. The classical idea is to apply the Tikhonov regularization[18,35]

$$K^*Ku + cu = K^*v, \tag{7}$$

where c is a small constant called the regularization coefficient. Problems (2) and (7) are equivalent.

The variational formulation associated to problem (7) is given by

$$(u, w)_{L^2(\Omega)} + c \int_{\Omega} \nabla u \nabla w dx = (v, w)_{L^2(\Omega)} . \tag{8}$$

2.2 Nonlinear diffusion method

In order to avoid the blurring drawback of linear diffusion approach, non linear approaches has been considered. The basic idea is to reduce the diffusion coefficient around the edges of the image. In other words, the diffusion coefficient is a decreasing function of $|\nabla u|$. The first non linear technique due to Perona and Malik[28] is not suitable for very noisy images, the problem is that noise cannot be removed along edges. In order to achieve this, other models using isotropic and anisotropic techniques are proposed. In this section, we present a model proposed by Aubert and Vese[9,8]. To study the influence of the smoothing term, the authors consider the following energy

$$E(u) = \frac{1}{2} \int_{\Omega} |v - u|^2 \ dx + \lambda \int_{\Omega} \psi(|\nabla u|) \ dx. \tag{9}$$

The first term in $E(u)$ measures the misfit to data and the second is a smoothing term. The parameter λ is a positive constant. Note that if we choose $\psi(|\nabla u|) = |\nabla u|^2$, we obtain the linear approach.

If $E(u)$ has a minimizer u, then it satisfies the following Euler-Lagrange equation

$$-\lambda \ div(\psi'(|\nabla u|) \frac{\nabla u}{|\nabla u|}) + u = v. \tag{10}$$

We summarize the assumptions imposed on the function ψ, as follows

- $\psi'(0) = 0$ and $\lim\limits_{t \to 0^+} \dfrac{\psi'(t)}{t} = \psi''(0) > 0$: isotropic smoothing condition at locations where the variations of the intensity are weak (low gradients).

- $\lim\limits_{t \to +\infty} \dfrac{\psi'(t)}{t} = \lim\limits_{t \to +\infty} \psi''(t) = 0$ and $\lim\limits_{t \to +\infty} \dfrac{\psi''(t)}{\psi'(t)/t} = 0$: anisotropic smoothing condition at locations such as edges (high gradients).

- $\lim\limits_{t \to +\infty} \psi(t) = +\infty$: condition to prove that the model is well posed mathematically.

We refer the reader to Aubert and Vese[8,9] for more details about these conditions imposed to the function ψ and for the existence and uniqueness of the solution for the minimization problem (9). However, we note that many functions ψ satisfying

Figure 1: Top left: initial Lena image (512×512 pixels), top right: noisy image (SNR=17), down left: restored image using the linear diffusion method (SNR=23), down right: restored image using the nonlinear diffusion method (SNR=27).

the preceding conditions can be found in literature. For our numerical experiences, we considered the function

$$\psi(t) = 2\sqrt{1+t^2} - 2 \tag{11}$$

and $\lambda = 10$.

We present in Figure 2.2 numerical tests for both the linear diffusion and the non linear diffusion methods, such that a gaussian noise (corresponding to a signal to noise ratio SNR=17) is added to the original image. One may remark that for the non linear approach, convergence is achieved after 53 iterations.

3 A TOPOLOGICAL GRADIENT APPROACH FOR IMAGE RESTORATION

In this section, we use the topological gradient as a tool for detecting edges for image restoration. First, we recall the principle of the topological asymptotic expansion adapted to our case, according to Masmoudi *et al*[24,5].

Let Ω be an open bounded domain of $I\!R^2$. For v a given function in $L^2(\Omega)$, the initial problem is defined on the safe domain and reads as follows: find $u \in H^1(\Omega)$ such that

$$\begin{cases} -div\,(c\nabla u) + u = v & \text{in} \quad \Omega, \\ \partial_n u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{12}$$

where n denotes the outward unit normal to $\partial\Omega$ and c is a constant function.
For a given $x_0 \in \Omega$ and a small $\rho \geq 0$, let us now consider $\Omega_\rho = \Omega \backslash \overline{\sigma_\rho}$ the perturbed domain by the insertion of a crack $\sigma_\rho = x_0 + \rho\sigma(n)$, where $x_0 \in \Omega$, $\sigma(n)$ is a straight crack, and n a unit vector normal to the crack. Then, the new solution $u_\rho \in H^1(\Omega_\rho)$

satisfies

$$\begin{cases} -div\,(c\nabla u_\rho) + u_\rho = v & \text{in} \quad \Omega_\rho, \\ \partial_n u_\rho = 0 & \text{on} \quad \partial\Omega_\rho, \end{cases} \tag{13}$$

The variation formulation of problem 13 is given by

$$\begin{cases} \text{Find } u_\rho \in H^1(\Omega_\rho) \text{ such that} \\ a_\rho(u_\rho, w) = l_\rho(w) \quad \forall w \in H^1(\Omega_\rho), \end{cases} \tag{14}$$

where a_ρ is the following bilinear form, defined on $H^1(\Omega_\rho)^2$ by

$$a_\rho(u, w) = \int_{\Omega_\rho} (c\nabla u \nabla w + uw)\ dx, \tag{15}$$

and l_ρ is the linear form defined on $L^2(\Omega_\rho)$ by

$$l_\rho(w) = \int_{\Omega_\rho} vw\ dx. \tag{16}$$

Edge detection is equivalent to look for a subdomain of Ω where the energy is small. So our goal is to minimize the energy norm outside edges

$$j(\rho) = J_\rho(u_\rho) = \int_{\Omega_\rho} \|\nabla u_\rho\|^2. \tag{17}$$

To study the asymptotic behaviour when ρ tends to zero of the criterion $j(\rho) = J_\rho(u_\rho)$, and in order to apply the topological asymptotic theory, we suppose that there exist a function $f : I\!R^+ \to I\!R^+$ going to zero with ρ, a linear form L_ρ, and four real numbers δJ_1, δJ_2, δa and δl such that the following assumptions are satisfied:

1. $J_\rho\,(u_\rho) - J_\rho(u_0) = L_\rho(u_\rho - u_0) + f(\rho)\delta J_1 + \circ(f(\rho))$,
2. $J_\rho\,(u_0) - J_0(u_0) = f(\rho)\delta J_2 + \circ(f(\rho))$.
3. $(a_\rho - a_0)\,(u_0, v_\rho) = f(\rho)\delta a + \circ(f(\rho))$,
4. $(l_\rho - l_0)\,(v_\rho) = f(\rho)\delta l + \circ(f(\rho))$,

where v_ρ is the solution to the adjoint problem

$$a_\rho(w, v_\rho) = -L_\rho(w) \quad \forall w \in H^1(\Omega_\rho). \tag{18}$$

It is supposed that for all $\rho \geq 0$, problem (18) has a unique solution. This expression is only used for the theoretical analysis part, but numerically we just consider the function v_0 the adjoint state, for the case $\rho = 0$. Then, if the previous hypothesis are satisfied, the asymptotic expansion of $j(\rho)$ is given by

$$j(\rho) = j(0) + f(\rho)(\delta a - \delta l + \delta J_1 + \delta J_2) + \circ(f(\rho)). \tag{19}$$

In our case, the cost function j has the following asymptotic expansion

$$j(\rho) - j(0) = \rho^2 G(x_0, n) + \circ(\rho^2), \tag{20}$$

with

$$G(x_0, n) = -\pi(\nabla u_0(x_0).n)(\nabla v_0(x_0).n) - \pi|\nabla u_0(x_0).n|^2. \tag{21}$$

and where v_0 is the solution to the adjoint problem

$$\begin{cases} -div(c\nabla v_0) + v_0 = -\partial_u J(u) & \text{in} \quad \Omega, \\ \partial_n v_0 = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{22}$$

The topological gradient could be written as

$$G(x, n) =< M(x)n, n >, \tag{23}$$

where $M(x)$ is the 2×2 symmetric matrix defined by

$$M(x) = -\pi \frac{\nabla u_0(x)\nabla v_0(x)^T + \nabla v_0(x)\nabla u_0(x)^T}{2} - \pi\nabla u_0(x)\nabla u_0(x)^T. \tag{24}$$

For a given x, $G(x, n)$ takes its minimal value when n is the eigenvector associated to the lowest eigenvalue λ_{min} of M. This value will be considered as the topological gradient associated to the optimal orientation of the crack $\sigma_\rho(n)$.

4 NUMERICAL APPLICATIONS

4.1 Grey level images

The goal of this section is to prove that the topological gradient method is able to denoise an image and preserve features such as edges. In order to avoid blurring edges, non linear isotropic and anisotropic methods are proposed[28,15,16,8,38]. Hence, due to this large number of approaches, it clearly appears that it is important to compare our experimental results with techniques already proposed in literature. Particularly, we compare our method with both the classical linear diffusion and the nonlinear diffusion methods tested previously.

Our algorithm consists in inserting small heterogeneities in regions where the topological gradient is smaller than a given threshold $\alpha < 0$. These regions are the edges ω_ρ of the image. Our method can be interpreted as a linear isotropic diffusion scheme. The algorithm is as follows

- Initialization : $c = c_0$.
- Calculation of u_0 and v_0 : solutions of the direct (13) and adjoint (22) problems.
- Computation of the 2×2 matrix M and its lowest eigenvalue λ_{min} at each point of the domain.
- Set

$$c_1 = \begin{cases} \varepsilon \quad if \quad x \in \Omega \quad such \quad that \quad \lambda_{min} < \alpha < 0, \quad \varepsilon > 0 \\ c_0 \quad elsewhere. \end{cases} \tag{25}$$

- Calculation of u_1 solution to problem (13) using c_1.

Figure 2: Top left: initial Lena image (512×512 pixels), top right: noisy image (SNR=17), down left: restored image using an homogeneous diffusion method (SNR=27), down right: restored image using topological gradient method (SNR=29).

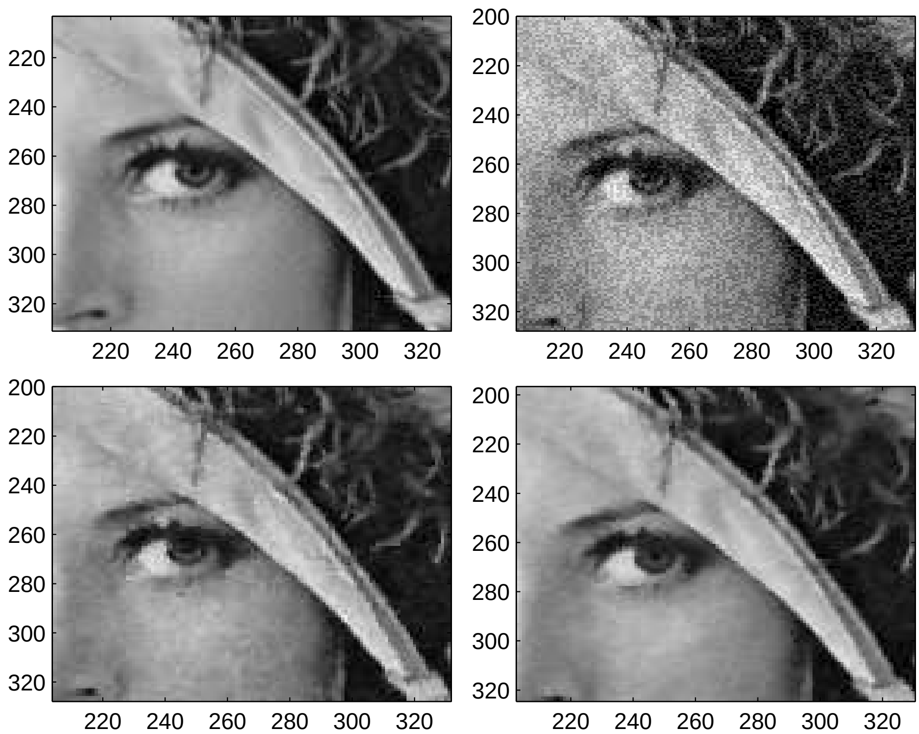


Figure 3: Top: zoom of the original (left) and noisy (right) images, down: zoom of the restored images using the non linear diffusion method (left) and the topological gradient method (right).

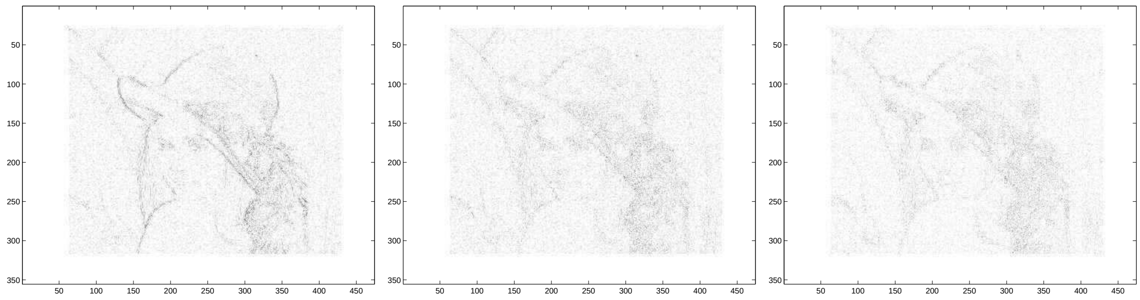


Figure 4: Left: error on restored Lena image by linear diffusion method, middle: error on restored Lena image by non linear diffusion method, right: error on restored Lena image by topological gradient method

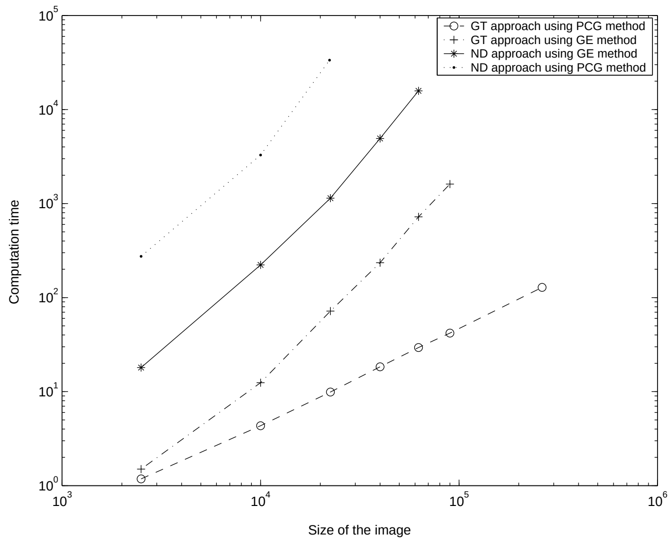


Figure 5: Variation of the function Computation time according to the size of the image using logarithmic scale, for both topological gradient and non linear diffusion approaches ($c(t) = \frac{1}{\sqrt{1+t^2}}$) by GE and PCG methods.

From the numerical point of view, it is more convenient to simulate the cracks by a small value of c.

Figure 2 shows the restored Lena image using topological gradient approach: the original image (512×512 pixels), the perturbed image which is still obtained with an additive gaussian noise (with a SNR equal to 17), the restored image with an homogeneous diffusion method, and the restored image using the topological gradient method.

Figure 3 shows a zoom of the previous images, i.e. Lena image restored by both the nonlinear diffusion method and the topological gradient method.

In order to distinguish the differences obtained between the restored images using these different approaches (linear diffusion, non linear diffusion and topological gradient), we highlight the comparison by showing the error between the original image and the restored image for the three approaches. These numerical results are given in Figure 4.

To allow a better comparison from numerical point of view between the topologi-

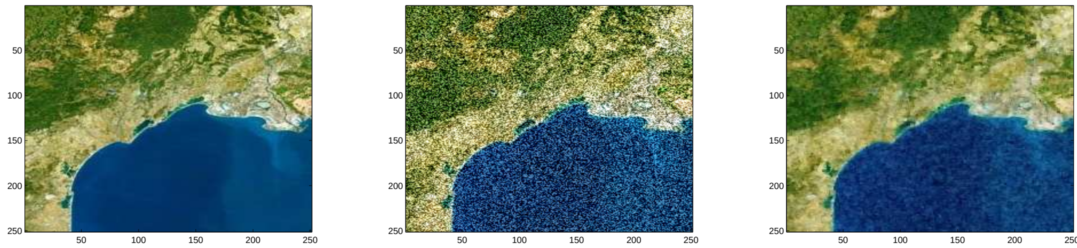


Figure 6: Original image (left), noised image (SNR=10) (center), and restored image by topological gradient (SNR=23) (right).

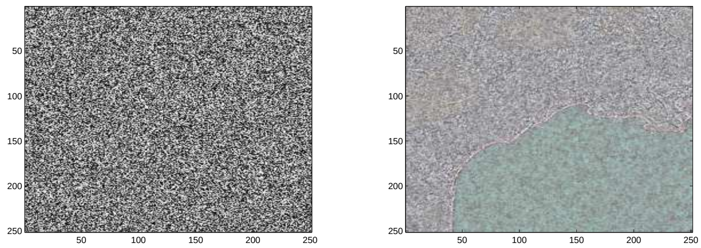


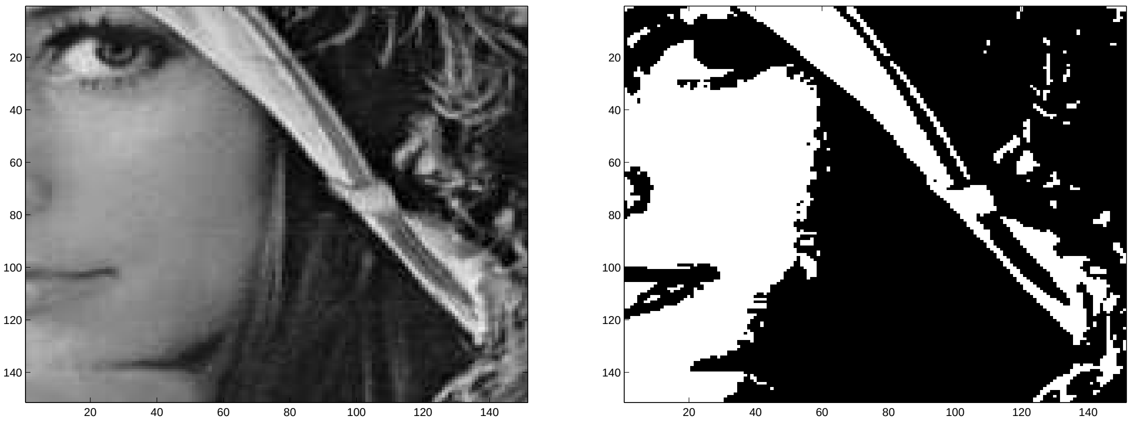
Figure 7: Difference between the original and noised images (left) ; Error on the restored image by topological gradient (right).

cal gradient approach and the non linear diffusion approach, the graphs of computation times according to the size of the image for both approaches using GE (Gauss Elimination) and PCG (Preconditioned Conjugate Gradient) methods, are given in Figure 5. The DCT (Discrete Cosine Transform) is used as a preconditioner for the conjugate gradient. These computation times are represented using a logarithmic scale for both X and Y axis. Figure 5 illustrates the efficiency of the topological gradient approach using the PCG method, in fact according to our algorithm, c is a constant in step 2 and c is equal to a constant except on the edges, as given in step 4. This makes the preconditioning by the DCT very efficient in our case and not suitable for the non linear diffusion approach.

4.2 Color images

The restoration algorithm for color images is almost the same. We first decompose the color image v in the HSV (Hue-Saturation-Value) color space, which provides 3 new images v_1, v_2 and v_3 corresponding to the three constituent components of the original image in this space. Each of these new images can be seen as a grey level image as it represents only one scalar component of the image. We then simply apply the previous restoration algorithm to each of these three *grey level* images, and we simply obtain the restored color image by reassembling the three restored components.

Figure 6 presents this restoration algorithm for a color image. One can notice that the performance is almost the same as for the grey level images. Figure 7 shows the reduction of the noise before and after the restoration process (the scale is the same for the two images).



(a) (b)

Figure 8: Original image (a) and 2-classes ($C = \{0; 255\}$) classified image obtained using the *closest class* algorithm (b).

5 VARIATIONAL SUPERVISED CLASSIFICATION FORMULATION AND TOPOLOGICAL GRADIENT APPROACH

5.1 Without regularization

Let u_0 be the original image defined on an open set Ω of $\mathbb{R}^2$. We want to classify the image u_0 using n predefined classes $C_i, 1 \leq i \leq n$, and we choose the grey level intensity as a classifier. The goal of image classification is then to find a partition of Ω in subsets $\{\Omega_i\}_{i=1,..,n}$, such that u_0 is close to C_i in Ω_i. The classified image u will then be defined by

$$u(x) = C_i \quad \forall x \in \Omega_i, \tag{26}$$

where $\{\Omega_i\}_{i=1,..,n}$ are defined by

$$\Omega_i = \left\{ x \in \Omega; x \textit{ belongs to the } i^{th} \textit{ class} \right\}. \tag{27}$$

The variational approach consists in defining a cost function measuring the root mean square difference between the original image and the classified image

$$J\left((\Omega_i)_{i=1,..,n}\right) = \sum_{i=1}^{n} \int_{\Omega_i} (u_0(x) - C_i)^2 \, dx. \tag{28}$$

The minimization of J is easy, because for each point $x \in \Omega$, we only have to find $i_x = \arg\min\{|u_0(x) - C_i|; i = 1, .., n\}$ and add x to subset Ω_{i_x}. This can be called the *closest class* algorithm because each pixel of the original image is assigned in the classified image to its closest class.

Figure 8-a shows the original 151×151 image u_0, using 256 grey levels: the grey level of a pixel is an integer $u_0(x) \in \{0; 255\}$. We have chosen $C_1 = 0$ (black) and $C_2 = 255$ (white). Figure 8-b shows the computed image using the *closest class* algorithm.

5.2 With regularization

In order to obtain a classified image with smoother contours, we may add a regularization term to the cost function

$$J\left((\Omega_i)_{i=1,..,n}\right) = \sum_{i=1}^{n} \int_{\Omega_i} (u_0(x) - C_i)^2 \, dx + \sum_{i \neq j} |\Gamma_{ij}|, \tag{29}$$

where $|\Gamma_{ij}|, i \neq j$ represents the one-dimensional Hausdorff measure of Γ_{ij}[8].

In order to solve this problem, variational models were proposed[30,31]. The minimization of J is no more easy to compute, the main difficulty comes from the fact that the unknowns are sets and not variables. It is then possible to use the topological gradient theory to solve the regularized classification problem.

5.3 Topological gradient for the image classification

The classification model that we propose is based on the topological gradient method[29]. We prove that we can solve the regularised classification problem using topological optimization tools. In fact, to assign each pixel of the original image to one of the classes $C_i, 1 \leq i \leq n$, it sufficies to suppose first that all pixels are assigned to the same class, and then to find subsets of pixels that should be reassigned to the other classes.

The initial guess will be

$$u = C_n \quad \text{in } \Omega, \tag{30}$$

and then, perturbing the domain with $n-1$ small cracks, we will have to solve

$$u = \begin{cases} C_1 & \text{in } \omega_{\rho_1}, \\ \vdots & \\ C_{n-1} & \text{in } \omega_{\rho_{n-1}}, \\ C_n & \text{in } \Omega \backslash (\bigcup_{i=1}^{n-1} \overline{\omega_{\rho_i}}), \end{cases} \tag{31}$$

where ω_{ρ_i} is the subset of pixels that should be reassigned to the class C_i.

We first work with the unregularized cost function, measuring the root mean square difference between the solution u_ρ of (30) and the original image u_0

$$J_\rho(u) = \int_\Omega |u - u_0|^2 \, dx, \tag{32}$$

and we can then define a cost function depending only on each ρ_i

$$j(\rho) = J_\rho(u_\rho) = \int_\Omega |u_\rho - u_0|^2 \, dx, \tag{33}$$

where u_ρ is the solution of (31).

5.4 Variation of the cost function

In the present case, the topological expansion analysis gives no real improvement compared to a classical minimization approach because it is possible to compute exactly the variation of the cost function. It is indeed possible to study the variations of J_ρ with respect to each ρ_i, and use these variations to define a topological gradient for each ρ_i, $1 \leq i \leq n-1$. For each i, we have

$$g_i(x) = (C_n - C_i)^2 - 2(C_n - C_i)(C_n - u_0(x)), \tag{34}$$

if x has never been reassigned (otherwise, $g_i(x)$ is set equal to 0).

The implementation of this method is quite easy, because we only have to compute each g_i, which is an affine function of the original image u_0, and then find the pixels x where $g_i(x) < 0$ (or $g_i(x) < -\varepsilon$), in order to minimize the cost function J, and reassign them to the corresponding class.

The algorithm is then the following

- for each $1 \leq i \leq n-1$, compute $g_i(x)$ for each pixel x,
- for each pixel x, find i_0 so that $g_{i_0}(x) \leq g_i(x)\ \forall i$,
- if $g_{i_0}(x) < 0$ (or $< -\varepsilon$), reassign x to the class C_{i_0}.

It is easy to prove that, at the end of the algorithm, each pixel x will be assigned to its closest class, i.e. to the class C_{i_0} with $i_0 = \arg\min\{|C_i - u(x)|^2\}$.

If we add a regularization term to the cost function, the variation of the cost function upon reassigning the pixel x to the class C_i is

$$\begin{aligned} \delta\tilde{J}_i(x) = & \int_{\omega_{\rho_i}} (C_n - C_i)(2u_0(x) - C_i - C_n)\, dx \\ & + \alpha(|\partial(\Omega_i \cup \omega_{\rho_i})| - |\partial\Omega_i|), \end{aligned} \tag{35}$$

and the pixel x is still reassigned to the class which minimizes mostly the cost function.

Let us remark that (35) is not an asymptotic expansion, because the first part of (35) is proportional to ρ^2 whereas the second one is proportional to ρ. Hence, if $\rho \to 0$, only the second term subsists. In our problem, ρ will be set so that the hole $\rho\mathcal{B}$ is one pixel, and then (35) is valid for this given value of ρ.

Because of the regularization term, it is important to run again the algorithm with the classified image as initial guess, because some pixels which had positive topological gradients may have negative ones at next iteration. If for example all neighbours of x have been reassigned to class i but not x, which is still assigned to the class C_n, at the next iteration, the regularization term in $\delta\tilde{J}_i(x)$ may be strongly negative, and then x may be reassigned to the class C_i. So, we have to iterate the algorithm until all functional variations $\delta\tilde{J}_i$ are everywhere non negative.

5.5 Numerical results

Figure 9-a shows the computed image using the topological gradient algorithm with the unregularized cost function and $n = 2$ classes ($C = \{0; 255\}$). This obviously gives the same result as the closest class algorithm because the asymptotic

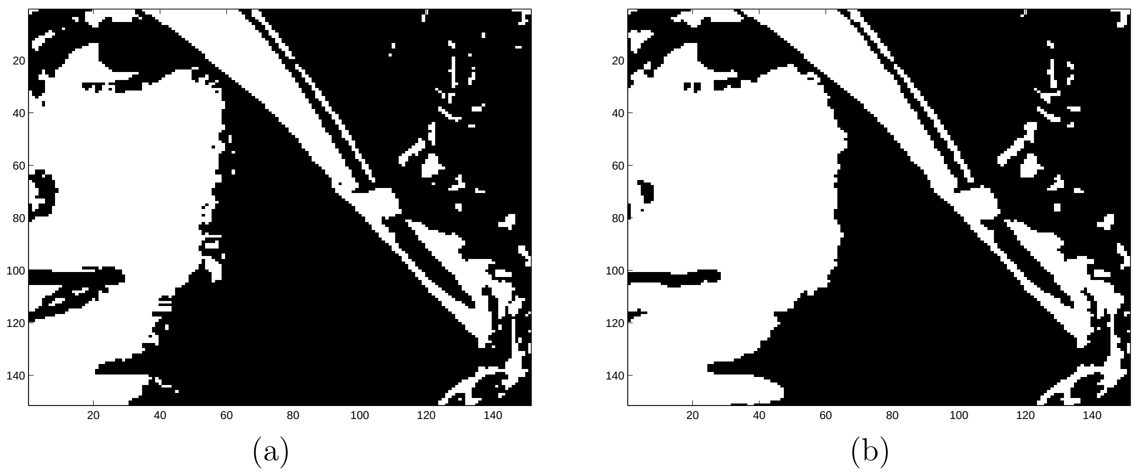


Figure 9: Two-classes classified images obtained using the topological gradient algorithm: unregularized (a) and regularized (b).

expansion of the unregularized cost function is indeed an exact variation. Figure 9-b shows the result of the topological gradient algorithm with the regularized cost function. We can clearly see that the resulting image has smoother contours and fewer isolated points.

Figure 10 shows the same results as in figure 9 for 3 and 5 classes. The conclusions are obviously the same.

6 A RESTORATION-BASED PREPROCESSING ALGORITHM FOR IMAGE CLASSIFICATION

6.1 Presentation of the algorithm

Inspired by the work of Aubert *et al.*[31,8] in which the authors propose a classification model coupled with a restoration process, we propose in this section to use the topological gradient approach applied to image restoration problem[23] for the regularized classificaton problem. It consists firstly in an iteration of the topological asymptotic analysis for the image smoothering and secondly in the closest class algorithm for its classification.

We consider the following equation

$$\begin{cases} -div(c\nabla u) + u = u_0 & \text{in } \Omega, \\ \partial_n u = 0 & \text{in } \Gamma = \partial\Omega, \end{cases} \tag{36}$$

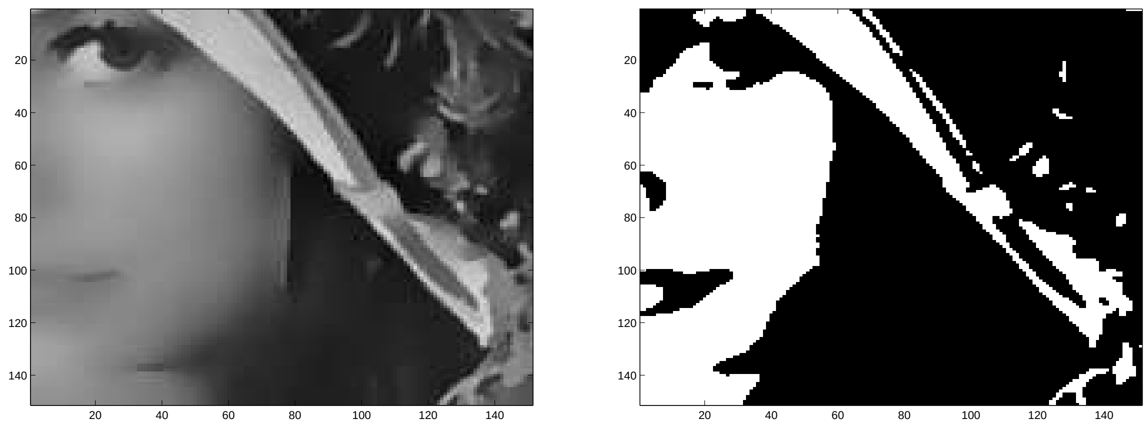
but with $c = \dfrac{1}{\varepsilon}$ in Ω_1 and $c = \varepsilon$ in Ω_ρ. Ω_ρ still represents the contours of the image. As ε is supposed to be a small positive real number, if we are on a contour, $c = \varepsilon$ and then u and u_0 are almost the same. But otherwise, $c = \dfrac{1}{\varepsilon}$ and then the p.d.e. is nearly equivalent to $\Delta u = 0$, which will provide a really smooth image.

6.2 Numerical results for grey level images

Figure 11-a shows the computed image using the topological gradient algorithm applied to our new equation. Figure 11-b shows the result of the closest class algorithm applied to figure 11-a with $n = 2$ classes ($C = \{0; 255\}$). Figure 11-b

(a) (b)

Figure 10: n-classes classified images obtained using the topological gradient algorithm: $n = 3$ and $C = \{34; 112; 165\}$ (top) and $n = 5$ and $C = \{29; 71; 117; 146; 184\}$ (bottom) ; unregularized (a) and regularized (b).



(a) (b)

Figure 11: Two-classes classified image obtained using the topological gradient algorithm applied to the new equation for smoothering (a) and then the closest class algorithm for classification (b).

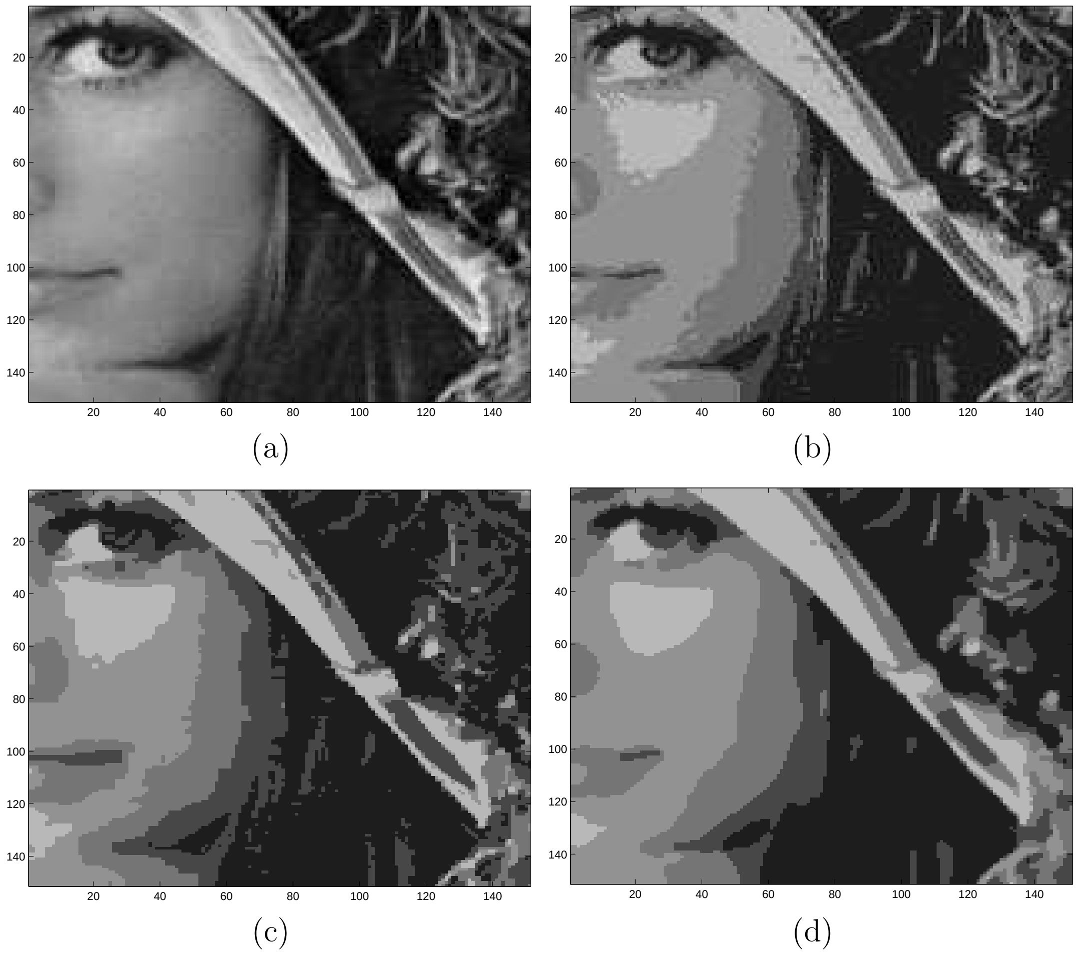


Figure 12: Original image (a); 5-classes ($C = \{29; 71; 117; 146; 184\}$) classified images obtained using: the closest class algorithm (no regularization) (b), the topological gradient method for the computation of the exact variation of the cost function (with regularization) (c), and the topological method applied to the improved restoration method (d).

should be compared to figure 9-b. We can see that the smoothering of contours is much more efficient in this case, and the computational cost of our latest algorithm is exactly the same as the previous one (only one iteration of topological gradient computation).

Figure 12 allows one to compare the 5-classes (with $C = \{29; 71; 117; 146; 184\}$) classified images computed with these different algorithms. This figure clearly shows that the last algorithm produces much smoother contours than the previous one.

6.3 Color images

As in the restoration process, the last classification algorithm can be easily extended to color images. We simply decompose the image in the HSV space, and deal separately with the three component images. When these three images (which can be seen as grey level images) are classified, we obtain the classified color image by recomposing the three classified component images.

Figure 13 shows the classification of a color image, using 40 color levels (whereas the original image has 256^3 different colors), without regularization and with regularization.

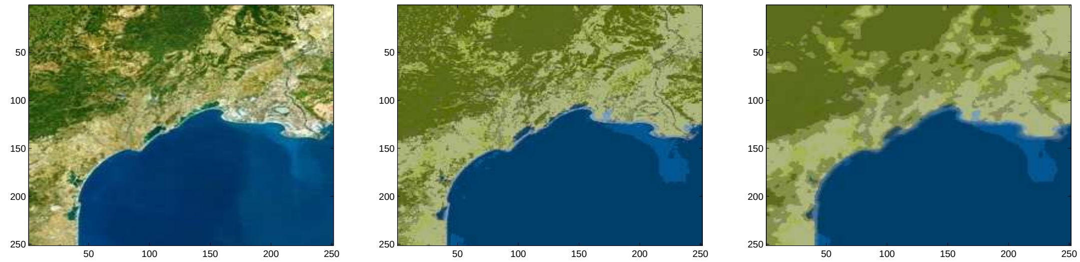


Figure 13: Original image (left), unregularized classified image (center), and regularized and classified image (right).

7 AN EXTENSION TO UNSUPERVISED CLASSIFICATION

The unsupervised classification corresponds to a classification problem in which the classes are not given. In this case, it is possible to determine them in an optimal way, still by using the topological gradient method. The idea is to study the impact of changing the value of a class $C_i := C_i + 1$ or $C_i - 1$ on the cost function. As in section 5, this variation can be exactly computed, and then, for each class, if the variation is negative, we add (or substract, depending on which variation provides the most negative variation) one to the value of the class. This algorithm has been applied in the previous sections in order to determine the optimal value of the classes (for example, in figure 10, the values of the 3 and 5 classes have been determined with this algorithm).

One may notice that this algorithm needs at least that the number of classes is known. But in unsupervised classification, it is not the case. The idea is then to add another term in the cost function defined in equation (28), measuring the number of classes. This can be seen as a regularization term, because we usually don't want to get too many classes. Without regularization, we have seen that the subsets Ω_i are uniquely defined, and then we can rewrite this cost function as a function of the (C_i):

$$J\left(n,(C_i)_{i=1\dots n}\right) = \sum_{i=1}^{n} \int_{\Omega_i} (u_0 - C_i)^2 \, dx \; + \; \alpha \, n, \tag{37}$$

where the Ω_i are defined as in subsection 5.1, and α is a positive regularization coefficient. It is then possible to bound n as the two terms of the cost function are positive. We then propose an iterative algorithm:

- we first assume that $n = 1$ and we determine the optimal value of the class (see previous paragraph);

- while n is smaller than its upper bound, $n := n + 1$ and we determine the optimal values of the new n classes;

- if the value of the cost function at the optimum (with respect to (C_i)) is larger than the previous optimal value (with $n - 1$ classes), stop.

This algorithm has been applied in figure 13 in order to find the optimal number of classes, and their optimal values.

8 CONCLUSION

A new method for image restoration with edge detection has been presented in this work. To make this method relevant with real life applications, we have used the Discrete Cosine Transform as a preconditioner for the conjugate gradient method. The results obtained are very promising especially with computation time and number of iterations. The extension of this method to other problems in image processing such as classification have also been presented, and the results are still obtained very quickly. We also presented a way to use the topological gradient for the optimal determining of the classes in unsupervised classification.

One of our main goals is now to extend these methods to videos, which can be considered as three-dimensional images.

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