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A new method for data assimilation: the back and forth nudging algorithm

Didier Auroux[1], **Jacques Blum**[1] **and Maëlle Nodet**[2]

[1]Laboratoire J.-A. Dieudonné, UMR 7351
Université de Nice Sophia-Antipolis
Parc Valrose, 06108 Nice cedex 2, France
{auroux,jblum}@unice.fr

[2] Université Grenoble 1 / INRIA
Laboratoire Jean Kuntzmann, BP 53
38041 Grenoble cedex 9, France
maelle.nodet@inria.fr

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Abstract. *In this paper, we propose an improvement to the Back and Forth Nudging algorithm for handling diffusion in the context of geophysical data assimilation. We detail the Diffusive Back and Forth Nudging algorithm, in which the sign of the diffusion term is changed in the backward integrations. We study the convergence of this algorithm, in particular for linear transport equations.*

1 Introduction

Data assimilation consists in estimating the state of a system by combining via numerical methods two different sources of information: models and observations. Data assimilation makes it possible to answer a wide range of questions such as: optimal identification of the initial state of a system, perform reliable numerical forecasts, identify or extrapolate non observed variables by using a numerical model . . . [4].

Nudging can be seen as a degenerate Kalman filter. Also known as the Luenberger or asymptotic observer [6], it consists in applying a Newtonian recall of the state value towards its direct observation [5, 7, 8]. A main disadvantage of such sequential data assimilation methods is that it only takes into account past observations at a given time, and not future ones.

Auroux and Blum proposed in [1] an original approach of backward and forward nudging (or *back and forth* nudging, BFN), which consists in initially solving the forward equations with a nudging term, and then, using the final state as an initial condition, in solving the same equations in a backward direction with a feedback term (with the opposite sign compared to the feedback term of forward nudging). This process is then repeated iteratively until convergence. The implementation of the BFN algorithm has been shown to be very easy, compared to other data assimilation methods [2].

However, several theoretical and numerical studies showed that it was difficult to deal with diffusion processes during backward integrations, leading to instabilities or explosion of the numerical solutions [3]. We present here an improved Back and Forth Nudging algorithm for diffusive equations in the context of meteorology and oceanography. In these applications, the theoretical equations are usually diffusive free (e.g. Euler's equation for meteorological processes). But then, in a numerical framework, a diffusive term is often added to the equations (or a diffusive scheme is used), in order to both stabilize the numerical integration of the equations, and take into consideration some subscale phenomena. In such situations, it is physically coherent to change the sign of the diffusion term in the backward integrations, in order to keep unchanged both roles of the diffusion term.

2 Diffusive Back and Forth Nudging algorithm for quasi-inviscid models

2.1 Standard Back and Forth Nudging algorithm

We first briefly recall the Back and Forth Nudging (BFN) algorithm, introduced in [1]. We assume that the time continuous model satisfies dynamical equations of the form:

$$\partial_t X = F(X), \quad 0 < t < T, \tag{1}$$

with an initial condition $X(0) = x_0$, and where F is the model operator (including spatial derivative operators). We will denote by H the observation operator, allowing one to compare the observations $X_{obs}(t)$ with the corresponding $H(X(t))$, deduced from the state vector $X(t)$.

The back and forth nudging algorithm consists in first solving the forward nudging equation and then the backward nudging equation. The *initial* condition of the backward integration is the final state obtained after integration of the forward nudging equation. At the end of this process, one obtains an estimate of the initial state of the system. These forward and backward integrations (with the feedback terms) are repeated until convergence of the algorithm: for $k \geq 1$,

$$\begin{aligned} &\begin{cases} \partial_t X_k = F(X_k) + K(X_{obs} - H(X_k)), \\ X_k(0) = \tilde{X}_{k-1}(0), \quad 0 < t < T, \end{cases} \\ &\begin{cases} \partial_t \tilde{X}_k = F(\tilde{X}_k) - K'(X_{obs} - H(\tilde{X}_k)), \\ \tilde{X}_k(T) = X_k(T), \quad T > t > 0, \end{cases} \end{aligned} \tag{2}$$

with the notation $\tilde{X}_0(0) = x_0$ and where K and K' are gain matrices. We refer to [1, 2, 3] for theoretical and numerical results about this algorithm. As explained in the introduction, when the model F contains some diffusion processes, the backward integration may be ill posed, and the numerical experiments may require quite large nudging coefficients and enough observations in order to stabilize the backward integrations.

2.2 Diffusive Back and Forth Nudging (D-BFN) algorithm

In the framework of oceanographic and meteorologic problems, there is usually no diffusion in the model equations. However, the numerical equations that are solved contain some diffusion terms in order to both stabilize the numerical integration (or the numerical scheme is set to be slightly diffusive) and model some subscale processes. We can then separate the diffusion term from the rest of the model terms, and assume that the model equations read:

$$\partial_t X = F(X) + \nu \Delta X, \quad 0 < t < T, \tag{3}$$

where F has no diffusive terms, ν is the diffusion coefficient, and we assume that the diffusion is a standard second-order Laplacian (note that it could be a fourth or sixth order derivative in some oceanographic models, but for clarity, we assume here that it is a Laplacian operator).

We introduce the D-BFN algorithm in this framework, for $k \geq 1$:

$$\begin{cases} \partial_t X_k = F(X_k) + \nu\Delta X_k + K(X_{obs} - H(X_k)), \\ X_k(0) = \tilde{X}_{k-1}(0), \quad 0 < t < T, \end{cases}$$
$$\begin{cases} \partial_t \tilde{X}_k = F(\tilde{X}_k) - \nu\Delta \tilde{X}_k - K'(X_{obs} - H(\tilde{X}_k)), \\ \tilde{X}_k(T) = X_k(T), \quad T > t > 0. \end{cases} \tag{4}$$

It is straightforward to see that the backward equation can be rewritten, using $t' = T - t$:

$$\begin{aligned} \partial_{t'} \tilde{X}_k &= -F(\tilde{X}_k) + \nu\Delta\tilde{X}_k + K'(X_{obs} - H(\tilde{X}_k)), \\ \tilde{X}_k(t'=0) &= X_k(T), \end{aligned} \tag{5}$$

where $\tilde{X}$ is evaluated at time t'. Then the backward equation can easily be solved, with an initial condition, and the same diffusion operator as in the forward equation. The diffusion term both takes into account the subscale processes and stabilizes the numerical backward integrations, and the feedback term still controls the trajectory with the observations.

The main interest of this new algorithm is that for many geophysical applications, the non diffusive part of the model is reversible, and the backward model is then stable. Moreover, the forward and backward equations are now consistent in the sense that they will be both diffusive in the same way (as if the numerical schemes were the same in forward and backward integrations), and only the non-diffusive physical model is solved backwards. Note that in this case, it is reasonable to set $K' = K$.

3 Convergence of the algorithm

3.1 General result

In this section only, we suppose that the model F and the observation operator H are linear, and that the Cauchy problem for equation (3) is well posed with an initial data $X(0) = X_0 \in E$, where E is a suitable Hilbert space. We now consider only one iteration of the D-BFN algorithm (e.g. for $k = 1$). Let us define the following operator that corresponds to one forward and one backward integrations:

$$\psi \quad : \quad E \times E \rightarrow E, \qquad (X_1(0), X_{obs}(0)) \mapsto \tilde{X}_1(0), \tag{6}$$

where X_k and $\tilde{X}_k$ satisfy equations (4), and X_{obs} is solution of equation (1). This operator is linear in the initial conditions, so that there exist C and D

linear operators on E such that

$$\begin{aligned} X_2(0) &= \psi(X_1(0), X_{obs}(0)) = \psi(X_1(0), 0) + \psi(0, X_{obs}(0)) \\ &= CX_1(0) + DX_{obs}(0). \end{aligned} \tag{7}$$

Then, if the spectrum of C is included in $\{X \in E, \|X\|_E \le \rho\}$, with $\rho < 1$, then we can prove that $X_k(0)$ converges as k goes to infinity to X_∞ solution of

$$X_\infty = (I - C)^{-1} D X_{obs}(0) \tag{8}$$

3.2 Application to a linear transport equation

Let us consider a simple situation, in which the physical model is a linear transport equation on a periodic domain $\Omega = \mathbb{R}/\mathbb{Z}$:

$$\partial_t u + a(x)\,\partial_x u = 0, \quad t \in [0, T],\ x \in \Omega, \quad u(t = 0) = u_0 \in L^2(\Omega) \tag{9}$$

with periodic boundary conditions, and we assume that $a \in W^{1,\infty}(\Omega)$. Numerically, for both stability and subscale modelling, some diffusion is added to the model equations. Let us assume that the observations satisfy the physical model (without diffusion):

$$\partial_t u_{obs} + a(x)\,\partial_x u_{obs} = 0, \quad t \in [0, T], x \in \Omega, \quad u_{obs}(t = 0) = u^0_{obs} \in L^2(\Omega). \tag{10}$$

We assume in this idealized situation that the system is fully observed (and H is then the identity operator).

Then the D-BFN algorithm applied to this problem gives, for $k \ge 1$:

$$\begin{cases} \partial_t u_k + a(x)\,\partial_x u_k = \nu \partial_{xx} u_k + K(u_{obs,k} - u_k), \\ u_k(0, x) = \tilde{u}_{k-1}(0, x) \end{cases}$$
$$\begin{cases} \partial_t \tilde{u}_k + a(x)\,\partial_x \tilde{u}_k = -\nu \partial_{xx} \tilde{u}_k - K(\tilde{u}_{obs,k} - \tilde{u}_k), \\ \tilde{u}_k(T, x) = u_k(T, x). \end{cases} \tag{11}$$

In this case, we can prove that $\|C\| < 1$ and convergence is assured.

In the special case where $a(x) = a \in \mathbb{R}$, we can change variables to straighten characteristics: $v_k(t, y) = u_k(t, y + at)$ and $\tilde{v}_k(t, z) = \tilde{u}_k(t, z - at)$, and at the limit $k \to \infty$, v_k and $\tilde{v}_k$ tend to $v_\infty(x)$ solution of

$$\nu \partial_{xx} v_\infty + K(u^0_{obs}(x) - v_\infty) = 0, \quad x \in \mathbb{R} \tag{12}$$

Note that equation (12) is well known in signal or image processing, as being the standard linear diffusion restoration equation. In some sense, v_∞ is the result of a smoothing process on the observations u_{obs}, where the degree of smoothness is given by the ratio $\frac{\nu}{K}$.

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