# Nudging-based observers for geophysical data assimilation and joint state-parameters estimation

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**Abstract** Oceans and atmosphere are governed by the general equations of fluid dynamics. Data assimilation consists in estimating the state of a system by combining via numerical methods two different sources of information: models and observations.

The Back and Forth Nudging (BFN) algorithm is a prototype of a new class of data assimilation methods. The nudging technique consists in adding a feedback term in the model equations, measuring the difference between the observations and the corresponding space states. The BFN algorithm is an iterative sequence of forward and backward resolutions, all of them being performed with an additional nudging feedback term in the model equations.

These nudging-based algorithms can be extended with the aim of correcting non-observed variables. This concerns in particular model parameters identification, with the potential of improving the quality and the confidence in model state for future data assimilation processes.

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# 1 Introduction

It is well established that the quality of weather and ocean circulation forecasts is highly dependent on the quality of the initial conditions. Geophysical fluids (air, atmospheric, oceanic, surface or underground water) are governed by the general equations of fluid dynamics. Geophysical processes are hence nonlinear because of their fluid component. Such nonlinearities impose a huge sensitivity to the initial conditions, and then an ultimate limit to deterministic prediction (estimated to be about two weeks for weather prediction for example). This limit is still far from being reached, and substantial gain can still be obtained in the quality of forecasts.

Data assimilation (DA) is precisely the domain at the interface between observations and models which makes it possible to identify the global structure of a system from a set of discrete space-time data. DA covers all the mathematical and numerical techniques in which the observed information is accumulated into the model state by taking advantage of consistency constraints with laws of time evolution and physical properties, and which allow us to blend as optimally as possible all the sources of information coming from theory, models and other types of data.

There are two main categories of data assimilation techniques [1], variational methods based on the optimal control theory [2] and statistical methods based on the theory of optimal statistical estimation (see, for example, [3, 4, 5] for an overview of inverse methods, both for oceanography and meteorology).

We study here the Back and Forth Nudging (BFN) algorithm, which is the prototype of a new class of data assimilation methods, although the standard nudging algorithm is known for a couple of decades. The nudging technique consists in adding a feedback term in the model equations, measuring the difference between the observations and the corresponding space states. The idea is to apply the standard nudging algorithm to the backward (in time) nonlinear model in order to stabilize it. The BFN algorithm is an iterative sequence of forward and backward resolutions, all of them being performed with an additional nudging feedback term in the model equations. We also present the Diffusive Back and Forth Nudging (DBFN) algorithm, which is a natural extension of the BFN to some particular diffusive models, and the P-BFN for parameter estimation (possibly jointly with state estimation).

# 2 Back and Forth Nudging

## *2.1 The nudging algorithm*

The standard nudging algorithm consists in adding to the state equations a feedback term, which is proportional to the difference between the observation and its equivalent quantity computed by the resolution of the state equations. The model appears then as a weak constraint, and the nudging term forces the state variables to fit as well as possible to the observations.

Let us consider a very generic model

$$\begin{cases} \dfrac{dX}{dt} = F(X,U), \quad 0 < t < T, \\ X(0) = V. \end{cases} \tag{1}$$

We assume that we have an observation $X_{obs}(t)$ of the state variable $X(t)$. The nudging algorithm simply gives

$$\begin{cases} \dfrac{dX}{dt} = F(X,U) + K(X_{obs} - HX), \quad 0 < t < T, \\ X(0) = V, \end{cases} \tag{2}$$

where $H$ is the observation operator, allowing us to compare the observation $X_{obs}$ with the corresponding quantity of the model solution $X$, and $K$ is the nudging matrix. It is quite easy to understand that if $K$ is large enough, then the state vector transposed into the observation space (through the observation operator) $HX(t)$ will tend towards the observation vector $X_{obs}(t)$. In the linear case (where $F$ and $H$ are linear operators), the forward nudging method is nothing else than the Luenberger observer [6], a deterministic and time continuous alternative method to statistical Kalman filtering method first introduced in 1966. The operator $K$ can be chosen so that the error goes to zero when time goes to infinity, hence its name of asymptotic observer.

This algorithm was first used in meteorology [7], and then has been used with success in oceanography [8] and applied to a mesoscale model of the atmosphere [9]. Many results have also been carried out on the optimal determination of the nudging coefficients $K$ [10, 11, 12].

The backward nudging algorithm consists in solving the state equations of the model backwards in time, starting from the observation of the state of the system at the final instant. A nudging term, with the opposite sign compared to the standard nudging algorithm, is added to the state equations, and the final obtained state is in fact the initial state of the system [13, 14].

## 2.2 The BFN algorithm

The Back and Forth Nudging (BFN) algorithm consists in solving first the forward (standard) nudging equation, and then the backward nudging equation. After resolution of this backward equation, one obtains an estimate of the initial state of the system. We repeat these forward and backward resolutions with the feedback terms until convergence of the algorithm [14].

The BFN algorithm is then the following:

$$\begin{cases} \dfrac{dX_k}{dt} = F(X_k,U) + K(X_{obs} - HX_k), \quad 0 < t < T, \\ X_k(0) = \tilde{X}_{k-1}(0), \end{cases}$$
$$\begin{cases} \dfrac{d\tilde{X}_k}{dt} = F(\tilde{X}_k,U) - K(X_{obs} - H\tilde{X}_k), \quad T < t < 0, \\ \tilde{X}_k(T) = X_k(T), \end{cases} \tag{3}$$

with $X_0(0) = V$ as initial condition. Starting from $V$, a resolution of the direct model gives $X_0(T)$ and hence $\tilde{X}_0(T)$. Then a resolution of the backward model provides $\tilde{X}_0(0)$, which is equal to $X_1(0)$, and so on.

This algorithm can be compared to the variational algorithm (4D-VAR, based on optimal control theory), which also consists in a sequence of forward and backward resolutions. In the BFN algorithm, even for nonlinear problems, it is useless to linearize the system and the computation of the backward system is as easy as the direct system, unlike adjoint equation which determination can be a playful task. In the case of ill-posed backward resolution, the extra feedback term in backward equation has the additional property to stabilize the numerical resolution.

The BFN algorithm has been tested successfully for the system of Lorenz equations, Burgers equation and a quasi-geostrophic ocean model in [15], for a shallow-water model in [16] and compared with a variational approach for all these models. It has been used to assimilate the wind data in a mesoscale model [17] and for the reconstruction of quantum states in [18].

### 2.3 DBFN: Diffusive Back and Forth Nudging algorithm

In the framework of oceanographic and meteorological problems, there is usually no diffusion in the model equations. However, the numerical equations that are solved contain some diffusion terms in order to both stabilize the numerical integration (or the numerical scheme is set to be slightly diffusive) and model some subscale turbulence processes. We can then separate the diffusion term from the rest of the model terms, and assume that the partial differential equations read:

$$\frac{dX}{dt} = F(X) + \nu \Delta X, \quad 0 < t < T, \tag{4}$$

where $F$ has no diffusive terms, $\nu$ is the diffusion coefficient, and we assume that the diffusion is a standard second-order Laplacian (note that it could be a fourth or sixth order derivative in some oceanographic models, but for clarity, we assume here that it is a Laplacian operator).

We introduce the D-BFN algorithm in this framework, for $k \geq 1$:

$$\begin{cases} \dfrac{dX_k}{dt} = F(X_k) + \nu \Delta X_k + K(X_{obs} - H(X_k)), \quad 0 < t < T, \\ X_k(0) = \tilde{X}_{k-1}(0), \end{cases}$$
$$\begin{cases} \dfrac{d\tilde{X}_k}{dt} = F(\tilde{X}_k) - \nu \Delta \tilde{X}_k - K'(X_{obs} - H(\tilde{X}_k)), \quad T > t > 0, \\ \tilde{X}_k(T) = X_k(T). \end{cases} \tag{5}$$

It is straightforward to see that the backward equation can be rewritten, using $t' = T - t$:

$$\begin{cases} \dfrac{d\tilde{X}_k}{dt'} = -F(\tilde{X}_k) + \nu \Delta \tilde{X}_k + K'(X_{obs} - H(\tilde{X}_k)), \quad 0 < t' < T, \\ \tilde{X}_k(t' = 0) = X_k(T). \end{cases} \tag{6}$$

where $\tilde{X}_k$ is evaluated at time $t'$, the backward equation is well-posed, with an initial condition and the same diffusion operator as in the forward equation. The diffusion term both takes into account the subscale processes and stabilizes the numerical backward integrations, and the feedback term still controls the trajectory with the observations.

The main interest of this new algorithm is that for many geophysical problems, the non diffusive part of the model is reversible, and the backward model is then stable. Moreover, the forward and backward equations are now consistent in the sense that they will be both diffusive in the same way (as if the numerical schemes were the same in forward and backward integrations), and only the non-diffusive part of the physical model is solved backwards. Note that in this case, it is reasonable to set $K' = K$.

The DBFN algorithm has been tested successfully for a linear transport equation in [19] and for non-linear Burgers equation in [20].

## 2.4 Theoretical considerations

Data Assimilation is the ensemble of techniques combining the mathematical information provided by the equations of the model and the physical information given by the observations in order to retrieve the state of a flow. In order to show that both BFN and DBFN algorithms achieve this double objective, let us give a formal explanation of the way these algorithms proceed.

If $K' = K$ and the forward and backward limit trajectory are equal, i.e $\tilde{X}_\infty = X_\infty$, then taking the sum of the two equations in (3) shows that the limit trajectory $X_\infty$ satisfies the model equation (1) (including possible model viscosity). Moreover, the difference between the two equations in (3) shows that the limit trajectory is solution of the following equation:

$$K(X_{obs} - H(X_\infty)) = 0. \tag{7}$$

Equation (7) shows that the limit trajectory perfectly fits the observations (through the observation operator, and the gain matrix). In a similar way, for the DBFN algorithm, taking the sum of the two equations in (5) shows that the limit trajectory $X_\infty$ satifies the model equations without diffusion:

$$\frac{dX_\infty}{dt} = F(X_\infty) \tag{8}$$

while taking the difference between the two same equations shows that $X_\infty$ satisfies the Poisson equation:

$$\Delta X_\infty = -\frac{K}{\nu}(X_{obs} - H(X_\infty)) \tag{9}$$

which represents a smoothing process on the observations for which the degree of smoothness is given by the ratio $\frac{\nu}{K}$ [19]. Equation (9) corresponds, in the case where $H$ is a matrix and $K = kH^T R^{-1}$, to the Euler equation of the minimization of the following cost function

$$J(X) = k\langle R^{-1}(X_{obs} - HX), (X_{obs} - HX)\rangle + \nu \int_\Omega \|\nabla X\|^2 \tag{10}$$

where the first term represents the quadratic difference to the observations and the second one is a first order Tikhonov regularisation term over the domain of resolution $\Omega$. The vector $X_\infty$, solution of (9), is the point where the minimum of this cost function is reached. This is a nice increment to the BFN algorithm, in which the limit trajectory fits the observations, while in the DBFN algorithm, the limit trajectory is the result of a smoothing process on the observations (which are often very noisy).

## *2.5 P-BFN for parameter estimation*

In many physical or biological dynamical systems the state equations contain parameters which are not well known and need to be estimated. The usual approach to achieve this identification is to include the unknown parameters together with the initial conditions into the set of control variables and to minimize a cost function measuring the discrepancy between the model outputs and the observations. The drawback of this approach is that it necessitates the computation of derivatives with respect to the parameters.

On the other hand the relative simplicity of the BFN framework is attractive to perform parameter identification. The state equations can be augmented with equations expressing the stationarity of the parameters, the initial conditions being the parameters values:

$$\begin{cases} \dfrac{dX}{dt} = F(X,P), & 0 < t < T, \quad X(0) = V, \\ \dfrac{dP}{dt} = 0, & 0 < t < T, \quad P(0) = U. \end{cases} \tag{11}$$

A Lyapunov functional is then formulated and enables to obtain the expression of the nudging term to be added to these parameter equations [21]. Note that using observations on the state only, it is possible to add feedback terms (to the observations) on both parameter and state equations.

Finally BFN-like iterations provide after convergence an estimate of the the initial conditions for the state equations and the parameters supplementary equations thus giving an estimate of the unknown parameters.

## 3 Numerical results

Let assume now that the parameter $a(x)$ of the transport equation is unknown. We want to estimate both the model state $u$ and parameter $a$. We add an ad hoc equation for the time independent parameter:

$$\begin{cases} \partial_t u(t,x) + a(x)\partial_x u(t,x) = 0, & u(0,x) = u_0(x), \\ \partial_t a(t,x) = 0, & a(0,x) = a(x). \end{cases}$$

Then we apply the BFN algorithm to this coupled system, and we add feedback terms to both equations, using only observations on the state $u$:

$$\begin{cases} \partial_t \hat{u}(t,x) + \hat{a}(t,x)\partial_x \hat{u}(t,x) = K_u(u_{obs}(t,x) - \hat{u}(t,x)), \\ \partial_t \hat{a}(t,x) = K_a \mathscr{F}(u_{obs}(t,x) - \hat{u}(t,x)), \end{cases}$$

where $\mathscr{F}$ is a feedback function involving spatial differential operators, such that there exists a Lyapunov function which decreases in time.

Then, we can prove that both $u$ and $a$ can be reconstructed, as it can be seen on figure 1.

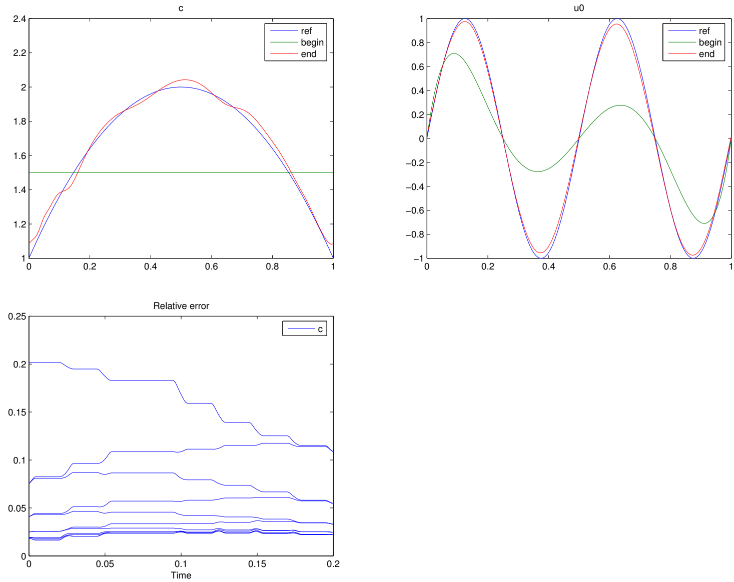


**Fig. 1** Comparisons between true augmented state (state and parameter) and its estimation by P-BFN algorithm: reconstruction of parameter (left) and state (right). Relative error of parameter reconstruction (bottom) with regards to P-BFN iterations in a time window of $[0,T]$ equal to $[0,0.2]$.

## 4 Conclusion

For state estimation, the BFN algorithm is a valuable technique for many reasons: its ease of implementation (simple combination of model and observation functions), its robustness and fast convergence, without requiring any linearization or optimization processes. When it comes to dealing with large scale problems, as is already the case in meteorology and oceanography, the use of high computational cost methods is hardly possible and BFN can be a suitable solution. In term of efficiency, the estimation provided by the BFN is comparable to other data assimilation methods in all carried out tests.

Its range of application can be extended to parameter estimation. By taking advantage of model parameter information contained in the state solution, the P-BFN identifies the model parameter exclusively from state observations.

The numerical tests on the identification of the velocity parameter in a transport equation confirm that both state and parameter can be estimated without increasing the computational cost.

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