# Data assimilation: variational methods and back and forth nudging algorithm; Application to thermoacoustic tomography

**D. Auroux**[1,*], **J. Blum**[1], **S. Marinesque**[2]
[1] Laboratoire J. A. Dieudonné, Université de Nice Sophia Antipolis, Nice, France.
[2] Institut de Mathématiques de Toulouse, Université Paul Sabatier, Toulouse, France.
*Email: auroux@unice.fr

**Abstract**

We consider several data assimilation techniques for thermoacoustic tomography (TAT), which is a non invasive medical imaging technique. The inverse problem can be formulated as an initial condition reconstruction. Variational data assimilation schemes are compared with the back and forth nudging algorithm.

**Introduction**

ThermoAcoustic Tomography (TAT) is a hybrid imaging technique that uses ultrasound waves produced by a body submitted to a radiofrequency pulse, uniformly deposited throughout the body. The absorption of this initial energy causes a non-uniform thermal expansion, leading to the propagation of a pressure wave outside the body to investigate. This wave is then measured all around the body.

The physiological properties of the tissue are highly related to the absorption of the initial pulse. Considering that the initial illumination is a Dirac distribution in time, the problem of recovering the absorptivity of the investigated body from the thermoacoustic signal is equivalent to recovering the initial condition of a Cauchy problem involving the wave equation from the knowledge of the solution on a surface surrounding the imaging object [11].

Data assimilation consists in estimating the state of a system by combining via numerical methods two different sources of information: models and observations. Data assimilation makes it possible to answer a wide range of questions such as: optimal identification of the initial state of a system, perform reliable numerical forecasts, identify or extrapolate non observed variables by using a numerical model . . . [6]. Most data assimilation methods are either variational methods such as 4D-VAR (based on optimal control theory) or sequential methods (filtering theory: Kalman filters). In linear situations, these two approaches are usually equivalent.

Variational data assimilation methods consider the equations governing the system as constraints and the problem is closed by using a variational principle. The well-known 4D-VAR, four dimensional variational data assimilation algorithm, is based on the minimization of a global cost function, which measures the discrepancy between the observations and the corresponding system states. Based on optimal control theory, the adjoint method allows one to compute the gradient of the cost function in a single numerical integration of the adjoint equation (see e.g. [8]). One iteration of the minimization process consists then in one forward integration of the model (in order to compute the cost function) and one backward integration of the adjoint model (in order to compute its gradient).

Nudging can be seen as a degenerate Kalman filter. Also known as the Luenberger or asymptotic observer [9], it consists in applying a Newtonian recall of the state value towards its direct observation. A main disadvantage of such sequential data assimilation methods is that it only takes into account past observations at a given time, and not future ones. Auroux and Blum proposed in [1] an original approach of backward and forward nudging (or *back and forth* nudging, BFN), which consists in initially solving the forward equations with a nudging term, and then, using the final state as an initial condition, in solving the same equations in a backward direction with a feedback term (with the opposite sign compared to the feedback term of forward nudging). This process is then repeated iteratively until convergence. The implementation of the BFN algorithm has been shown to be very easy, compared to other data assimilation methods [2].

This algorithm has been successfully applied to various problems: ODEs, PDEs, linear and nonlinear equations, . . . , including viscous irreversible equations [2], [3], [4]. Note that for linear reversible systems, there has been a recent theoretical study of a similar algorithm [10].

From a practical point of view, these methods can be successfully used to manage the usual issues of the TAT inverse problem as incomplete data, external source and variable sound speed (when given, however). So far, the theoretical convergence result for the nudging technique is based on a classical result about stabilization of the wave equation, which requires somehow a geometric optics condition. Numerical comparisons between variational and nudging algorithms, and also time reversal, have been performed.

Consider the following problem:

$$\begin{cases} \partial_{tt} u - \Delta u = 0, \quad (x,t) \in R^3 \times R_+, \\ u(x,0) = u_0(x), \quad x \in R^3, \\ \partial_t u(x,0) = 0, \quad x \in R^3, \end{cases} \tag{1}$$

where $u_0$ is the object to reconstruct. We assume that the support of $u_0$ is compact and included in the unit ball. The problem is the following: from the knowledge of $u$ (possibly with noise) on a surface surrounding the unit ball, can we reconstruct $u_0$? Let $u_{data}$ be the observed data.

The iterative BFN algorithm for TAT is the following [7]:

- Forward evolution:

$$\begin{cases} \partial_{tt} u_i - \Delta u_i = k\partial_t(u_{data} - u_i), \quad (x,t) \in R^3 \times [0;T], \\ u_i(x,0) = \tilde{u}_{i-1}(x,0), \quad x \in R^3, \\ \partial_t u_i(x,0) = \partial_t \tilde{u}_{i-1}(x,0), \quad x \in R^3, \end{cases} \tag{2}$$

where $T$ is such that the solution vanishes on the unit ball.

- Backward evolution:

$$\begin{cases} \partial_{tt} \tilde{u}_i - \Delta \tilde{u}_i = -\tilde{k}\partial_t(u_{data} - \tilde{u}_i), \quad (x,t) \in R^3 \times [0;T], \\ \tilde{u}_i(x,T) = u_i(x,T), \quad x \in R^3, \\ \partial_t \tilde{u}_i(x,0) = \partial_t u_i(x,T), \quad x \in R^3. \end{cases} \tag{3}$$

After each iteration, $\tilde{u}_i(x,0)$ is a new estimate of the object to reconstruct.

The nudging terms are added only on the observed domain, and parameters $k$ and $\tilde{k}$ can be chosen equal as the considered equation (wave equation) is reversible. If we add a numerical or physical attenuation in the equation, then the backward nudging parameter might be increased, or one should refer to the recent improvement of the BFN algorithm in diffusive or attenuated situations [5].

In [7], the authors show that this algorithm converges (under standard hypotheses) with a geometric decay rate of the $H_0^1$ norm of the error.

## References


[1] D. Auroux and J. Blum, *Back and forth nudging algorithm for data assimilation problems*, C. R. Acad. Sci. Sér. I, **340** (2005), pp. 873–878.

[2] D. Auroux and J. Blum, *A nudging-based data assimilation method: the Back and Forth Nudging (BFN) algorithm*, Nonlin. Proc. Geophys., **15** (2008), pp. 305–319.

[3] D. Auroux, P. Bansart, and J. Blum, *An evolution of the Back and Forth Nudging for geophysical data assimilation: application to Burgers equation and comparisons*, Inv. Prob. Sci. Eng., **21**(3) (2013), pp.399–419.

[4] D. Auroux and M. Nodet, *The back and forth nudging algorithm for data assimilation problems: theoretical results on transport equations*, ESAIM Control Optim. Calc. Var., **18**(2) (2012), pp.318–342.

[5] D. Auroux, J. Blum, and M. Nodet, *Diffusive Back and Forth Nudging algorithm for data assimilation*, C. R. Acad. Sci. Paris, Ser. I, **349**(15-16) (2011), pp.849-854.

[6] A. F. Bennett, *Inverse Modeling of the Ocean and Atmosphere*, Cambridge University Press, Cambridge, 2002.

[7] X. Bonnefond and S. Marinesque, *Application of a nudging technique to thermoacoustic tomography*, Inverse Problems Imaging, (2013), accepted for publication.

[8] F.-X. Le Dimet and O. Talagrand, *Variational algorithms for analysis and assimilation of meteorological. observations: theoretical aspects*, Tellus, **38A** (1986), pp. 97–110.

[9] D. Luenberger, *Observers for mutivariable systems*, IEEE Trans. Autom. Contr., **11** (1966), pp. 190–197.

[10] K. Ramdani, M. Tucsnak, and G. Weiss, *Recovering the initial state of an infinite-dimensional system using observers*, Automatica, **46** (2010), pp. 1616–1625.

[11] O. Scherzer, M. Grasmair, H. Grossauer, M. Haltmeier, and F. Lenzen, *Variational Methods in Imaging*, Appl. Math. Sci., vol. 167, Springer, New York, 2009.