IMAGE PROCESSING BY TOPOLOGICAL ASYMPTOTIC ANALYSIS

Didier Auroux, Mohamed Masmoudi

Institut de Mathématiques de Toulouse
Université Paul Sabatier Toulouse 3
31062 Toulouse cedex 9
France
auroux@mip.ups-tlse.fr

ABSTRACT

We present in this paper a new way for modeling and solving image processing problems (restoration, classification,...), the topological gradient method. This method is considered in the frame of variational approaches and the minimization of potential energy with respect to conductivity. The numerical experiments show the efficiency of the topological gradient approach. The image is most of the time processed at the first iteration of the optimization process. Moreover, the computational cost of this iteration is reduced drastically using spectral methods.

Index Terms— Topological gradient, image edge analysis, image classification, image restoration, image reconstruction.

1. INTRODUCTION

The goal of topological optimization and most image processing problems is to create a partition of a given domain (or set). In topological optimization, we look for the optimal design and its complementary; in image processing problems like edge detection, classification, segmentation, or inpainting, the goal is to split the image in several parts. For this reason, topological shape optimization and image processing problems have common mathematical methods like level set approaches, material properties optimization, variational methods,...

In this paper, we consider the topological gradient approach that has been introduced for topological optimization purpose [1, 2, 3, 4, 5, 6, 7]. The basic idea is to adapt the topological gradient approach used for crack detection [4]: an image can be viewed as a piecewise smooth function and edges can be considered as a set of singularities. This can be applied to diffusive grey (or color) image restoration giving very promizing results [8]. An optimal material distribution is obtained at the first iteration. We also applied the topological gradient approach to the image classification problem. We show that it is possible to solve these image processing problems using topological optimization tools for the detection of edges, and for a nearly linear complexity.

2. TOPOLOGICAL GRADIENT

In this section, let Ω be an open bounded domain of $\mathbb{R}^2$ and $j(\Omega) = J(u_\Omega)$ be a cost function to me minimized, where u_Ω is the solution to a given Partial Differential Equations (PDE) problem defined in Ω.

For a small $\rho \geq 0$, let $\Omega_\rho = \Omega \backslash \overline{\omega_\rho}$ be the perturbed domain by the insertion of a small hole $\omega_\rho = x_0 + \rho\omega$, where $x_0 \in \Omega$ and ω is a fixed bounded domain of $\mathbb{R}^2$ containing the origin. The topological sensitivity theory provides an asymptotic expansion of j when ρ tends to zero. It takes the general form

$$j(\Omega_\rho) - j(\Omega) = f(\rho)\, G(x_0) + o\left(f(\rho)\right), \tag{1}$$

where $f(\rho)$ is an explicit positive function going to zero with ρ and $G(x_0)$ is called the topological gradient at point x_0. Then to minimize the criterion j, we have to insert small holes at points where G is negative. Using this gradient type information, it is possible to build fast algorithms. In most applications, a satisfying approximation of the optimal solution is reached at the first iteration of the optimization process. A topological sensitivity framework allowing to obtain such an expansion for general cost functions has been proposed in the work of Masmoudi [2, 4].

3. IMAGE RESTORATION

In this section, we use the topological gradient as a tool for detecting edges for image restoration. Let Ω be an open bounded domain of $\mathbb{R}^2$. For v a given function in $L^2(\Omega)$, the initial problem is defined on the safe domain and reads as follows: find $u \in H^1(\Omega)$ such that

$$\left\{ \begin{array}{lll} -div\,(c\nabla u) + u = v & \text{in} & \Omega, \\ \partial_n u = 0 & \text{on} & \partial\Omega, \end{array} \right. \tag{2}$$

where n denotes the outward unit normal to $\partial\Omega$ and c is a constant function.

For a given $x_0 \in \Omega$ and a small $\rho \geq 0$, let us now consider $\Omega_\rho = \Omega \backslash \overline{\sigma_\rho}$ the perturbed domain by the insertion of a crack $\sigma_\rho = x_0 + \rho\sigma(n)$, where $x_0 \in \Omega$, $\sigma(n)$ is a straight crack, and n a unit vector normal to the crack. Then, the new solution $u_\rho \in H^1(\Omega_\rho)$ satisfies

$$\begin{cases} -div\,(c\nabla u_\rho) + u_\rho = v & \text{in} \quad \Omega_\rho, \\ \partial_n u_\rho = 0 & \text{on} \quad \partial\Omega_\rho, \end{cases} \tag{3}$$

Edge detection is equivalent to look for a subdomain of Ω where the energy is small. So our goal is to minimize the energy norm outside edges

$$j(\rho) = J_\rho(u_\rho) = \int_{\Omega_\rho} \|\nabla u_\rho\|^2. \tag{4}$$

In our case, the cost function j has the following asymptotic expansion

$$j(\rho) - j(0) = \rho^2 G(x_0, n) + \circ(\rho^2), \tag{5}$$

with

$$\begin{aligned} G(x_0, n) \quad &= \quad -\pi(\nabla u_0(x_0).n)(\nabla v_0(x_0).n) \\ &\quad\quad -\pi|\nabla u_0(x_0).n|^2. \end{aligned} \tag{6}$$

and where v_0 is the solution to the adjoint problem

$$\begin{cases} -div(c\nabla v_0) + v_0 = -\partial_u J(u) & \text{in} \quad \Omega, \\ \partial_n v_0 = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{7}$$

The topological gradient could be written as

$$G(x, n) = < M(x)n, n >, \tag{8}$$

where $M(x)$ is the 2×2 symmetric matrix defined by

$$\begin{aligned} M(x) \quad &= \quad -\pi \frac{\nabla u_0(x)\nabla v_0(x)^T + \nabla v_0(x)\nabla u_0(x)^T}{2} \\ &\quad\quad -\pi\nabla u_0(x)\nabla u_0(x)^T. \end{aligned} \tag{9}$$

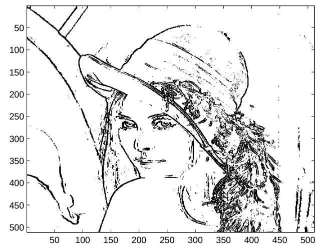
For a given x, $G(x, n)$ takes its minimal value when n is the eigenvector associated to the lowest eigenvalue λ_{min} of M. This value will be considered as the topological gradient associated to the optimal orientation of the crack $\sigma_\rho(n)$.

Our algorithm consists in inserting small heterogeneities in regions where the topological gradient is smaller than a given threshold $\alpha < 0$. These regions are the edges ω_ρ of the image. Our method can be interpreted as a linear isotropic diffusion scheme. The algorithm is as follows

- Initialization : $c = c_0$.
- Calculation of u_0 and v_0 : solutions of the direct (3) and adjoint (7) problems.

(a) Original image.

(b) Noised image (SNR=17).



(c) Identified edges by the topological gradient.

(d) Restored image (SNR=29).

Fig. 1. Application of our restoration algorithm to a 512×512 image.

- Computation of the 2×2 matrix M and its lowest eigenvalue λ_{min} at each point of the domain.
- Set

$$c_1 = \begin{cases} \varepsilon & if\ x \in \Omega \text{ such that} \lambda_{min} < \alpha < 0,\ \varepsilon > 0 \\ c_0 & elsewhere. \end{cases} \tag{10}$$

- Calculation of u_1 solution to problem (3) using c_1.

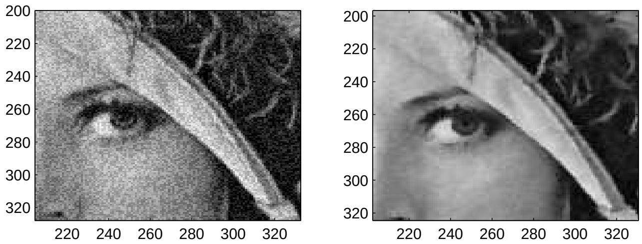
From the numerical point of view, it is more convenient to simulate the cracks by a small value of c.

We present in figure 1 numerical tests. The first image (a) shows the original image. The second image (b) shows the perturbed image, which is obtained with an additive gaussian noise, with a SNR equal to 17. Then figure 1-(c) shows the identified edges of the image by the topological gradient, and finally, the image (d) is the restored image using our algorithm. The SNR of the restored image is 29. One should notice that this result is performed in only one iteration.

Figure 2 shows a zoom of the previous images, i.e. the noised image and the restored image. One can see that the edges are very well preserved, and the quality of the restored image is very good.

4. A RESTORATION-BASED PREPROCESSING ALGORITHM FOR IMAGE CLASSIFICATION

Inspired by the work of Aubert *et al.* [9, 10] in which the authors propose a classification model coupled with a restora-



(a) Zoom of the noised image. (b) Zoom of the restored image.

Fig. 2. Zoom of the noised and restored images.

tion process, we propose in this section to extend the topological gradient approach applied to image restoration problem [8] for the regularized classificaton problem. It consists firstly in an iteration of the topological asymptotic analysis for the image smoothering and secondly in thresholding the restored image for its classification.

We still consider the following equation, which is the restoration equation:

$$\begin{cases} -div(c\nabla u) + u = u_0 & \text{in } \Omega, \\ \partial_n u = 0 & \text{in } \Gamma = \partial\Omega, \end{cases} \tag{11}$$

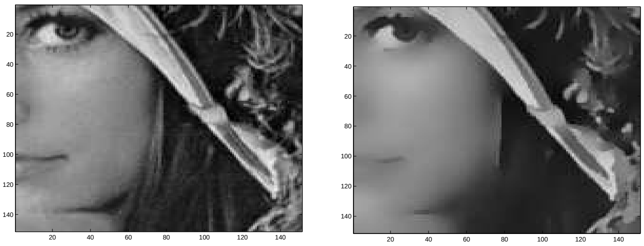
but with $c = \dfrac{1}{\varepsilon}$ in Ω_ρ and $c = \varepsilon$ in σ_ρ. σ_ρ still represents the contours of the image. As ε is supposed to be a small positive real number, if we are on a contour, $c = \varepsilon$ and then u and u_0 are almost the same. But otherwise, $c = \dfrac{1}{\varepsilon}$ and then the p.d.e. is nearly equivalent to $\Delta u = 0$, which will provide a really smooth image.

If we consider the same algorithm as in the previous section (but with this new definition of c_1), we find the contours of the image, and smooth the image everywhere else. The idea is then to simply threshold the smothered image: each pixel is assigned to its closest class.

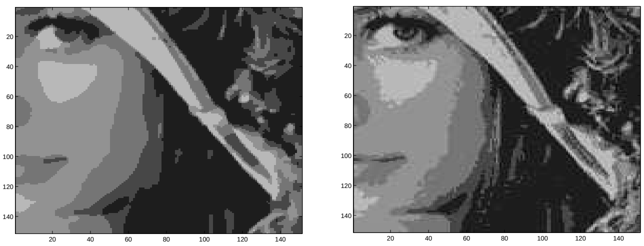
Figure 3 shows the original image (a), and the smooth image provided by our improved restoration algorithm (b). Then, we simply assign each pixel of this image to its closest class, and we obtain image (c). In this experiment, we used 5 classes, and their values are $C = \{29; 71; 117; 146; 184\}$. For comparison, figure 3-(d) shows the result of a thresholding on the original image, which can be seen as the result of an unregularized classification. We can see that we obtain very smooth contours on the classified image.

5. COMPUTATIONAL COST

As the first resolution of the direct problem is performed with a constant value of c, it is possible to largely accelerate the computation by using the DCT (Discrete Cosine Transform)



(a) Original image. (b) Smooth image using our algorithm.



(c) Classified image (5 classes). (d) Unregularized classified image.

Fig. 3. Application of our restoration-based preprocessing algorithm for image classification to a 150×150 image.

method. If we consider the following cosine basis

$$\phi_{m,n} = \delta_{m,n} \cos(m\pi x) \cos(n\pi y)$$

where $\delta_{m,n}$ are appropriate normalisation coefficients, equation (2) is equivalent to

$$\sum_{m,n} \left(1 + c(m\pi)^2 + c(n\pi)^2\right) u_{m,n} \phi_{m,n} = \sum_{m,n} u_{0_{m,n}} \phi_{m,n}, \tag{12}$$

where $\left(u_{0_{m,n}}\right)$ represents DCT coefficients of the original image u_0. It is then straightforward to identify (u_{nm}), the DCT coefficients of u in (12), and then to compute u using an inverse DCT. The complexity of this resolution is then $\mathcal{O}\left(N \log(N)\right)$ where N is the size of the image (i.e. the number of pixels). Then, for the second resolution of the direct problem with a non constant c, the DCT solver is used as a preconditioner to the conjugate gradient algorithm. This works very well because c is close to a constant; it is equal to a constant except on the edges of the image.

One can see on figure 4 the computational cost of the algorithms we presented in the previous sections, versus the size of the image. The curve with crosses corresponds to the computational cost when we use a Gauss elimination method for solving the different partial differential equations, whereas the curve with circles corresponds to a preconditioned conjugate gradient approach with the discrete cosine transform (presented in the previous paragraph). One can see on this

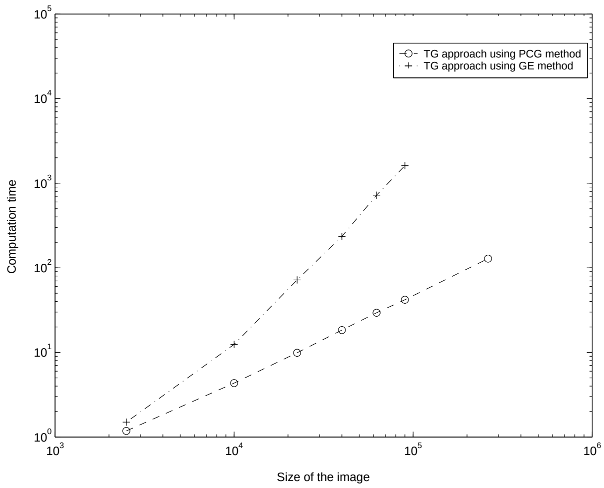


Fig. 4. Computational cost versus the size of the image for the topological gradient algorithm using a Gauss Elimination (GE) method (cross) or a Preconditioned Conjugate Gradient (PCG) method (circle).

figure that the GE approach has a nearly quadratic complexity whereas the PCG approach is nearly linear. As the equations are the same in our different applications, this figure confirms the theoretical complexity of our algorithms: both the restoration and classification processes we presented in the previous sections are performed in only one iteration, and with a $\mathcal{O}\left(N\log(N)\right)$ complexity.

6. CONCLUSION

An application of the topological asymptotic expansion for image restoration with edge detection has been presented in this work. To make this method relevant with real life applications, we have used the Discrete Cosine Transform as a preconditioner for the conjugate gradient method. The results obtained are very promising especially with computation time and number of iterations. The extension of this method to other problems in image processing such as classification have also been presented, and the results are still obtained very quickly.

This algorithm can be extended to three-dimensional images (such as movies for example), but also to color images. Finally, the idea of using the topological derivative may be extended to the inpainting and segmentation problems.

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