Wave propagation in one-dimensional random media

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Abstract. Random media have material properties with such complicated spatial variations that they can only be described statistically. When looking at waves propagating in these media, we can only expect in general a statistical description of the wave. But sometimes there exists a deterministic result: the wave dynamics only depends on the statistics of the medium, and not on the particular realization of the medium. Such a phenomenon arises when the different scales present in the problem (wavelength, correlation length, and propagation distance) can be separated. In this lecture we restrict ourselves to one-dimensional wave problems that arise naturally in acoustics and geophysics.

Contents

1. Introduction

Wave propagation in linear random media has been studied for a long time by perturbation techniques when the random inhomogeneities are small. One finds that the mean amplitude decreases with distance traveled, since wave energy is converted to incoherent fluctuations. The fluctuating part of the field intensity is calculated approximatively from a transport equation, a linear radiative transport equation. This theory is well-established [34], although a complete mathematical theory is still lacking (for recent developments, see for instance [23]). However this theory is false in one-dimensional random media. This was first noted by Anderson [1], who claimed that random inhomogeneities trap wave energy in a finite region and do not allow it to spread as it would normally. This is the so-called wave localization phenomenon. It was first proved mathematically in [32]. Extensions and generalizations follow these pioneer works so that the problem is now well understood [18]. The mathematical statement is that the spectrum of the reduced wave equation is pure point with exponentially decaying eigenfunctions. However the authors did not give quantitative information associated with the wave propagation as no exact solution is available. In this lecture we are not interested in the study of the strongest form of Anderson localization. We actually address the simplest form of this problem: the wave transmission through a slab of random medium. It is now well-known that the transmission of the slab tends exponentially to zero as the length of the slab tends to infinity. Furstenberg first treated discrete versions of the transmission problem [30], and finally Kotani gave a proof of this result with minimal hypotheses [42]. The connection between the exponential decay of the transmission and the Anderson localization phenomenon is clarified in [22]. Once again, these works deal with qualitative properties. Quantitative information can be obtained only for some asymptotic limits: large or small wavenumbers, large or small variances of the fluctuations of the parameters of the medium, etc. A lot of work was devoted to the quantitative analysis of the transmission problem, in particular by Rytov, Tatarski, Klyatskin [39], and by Papanicolaou and its co-workers [40]. The tools for the quantitative analysis are limit theorems for stochastic equations developed by Khasminskii [37], by Papanicolaou-Stroock-Varadhan [54], and by Kushner [44].

There are three basic length scales in wave propagation phenomena: the typical wavelength λ, the typical propagation distance L, and the typical size of the inhomogeneities l_c. There is also a typical order of magnitude that characterizes the standard deviation σ of the fluctuations of the parameters of the medium. It is not always easy to identify the scale l_c, but we may think of l_c as a typical correlation length. When the standard deviation of the relative fluctuations is small $\sigma \ll 1$, then the most effective interaction of the waves with the random medium will occur when $l_c \sim \lambda$, that is, the wavelength is comparable to the correlation length. Such an interaction will be observable when the propagation distance L is large ($L \sim \lambda\sigma^{-2}$). This is the typical configuration in optics and in optical fibers.

Throughout this lecture we shall consider scales arising in acoustics and geophysics. The main differences with optics are that the fluctuations are not small. However, in geophysics, the typical wavelength of the pulse $\lambda \sim 150$ m is small compared to the probing depth $L \sim 10 - 50$ km, but large compared to the correlation length $l_c \sim 2 - 3$ m [69]. Accordingly we shall introduce a small parameter $0 < \varepsilon \ll 1$ and consider $l_c \sim \varepsilon^2$, $\lambda \sim \varepsilon$, and $L \sim 1$. The parameter ε is the ratio of the typical wavelength to propagation depth, as well as the ratio of correlation length to wavelength. This is a particularly interesting scaling limit mathematically because it is a high frequency limit with respect to the large scale variations of the medium, but it is a low frequency limit with respect to the fluctuations, whose effect acquires a canonical form independent of details. We shall study the asymptotic behavior of the transmitted and reflected waves in the framework introduced by Papanicolaou based on the separation of these scales.

The lecture is organized as follows. In Section 2 we present the method of averaging for stochastic processes that is an extension of the law of large numbers for the sums of independent random variables. These results provide the tools for the effective medium theory developed in Section 3. We give a review of the properties of Markov processes in Section 4. We propose limit theorems for ordinary differential equations driven by Markov processes that are applied in the following sections. Section 5 is devoted to the O'Doherty-Anstey problem, that is to say the spreading of a pulse traveling through a random medium. We compute the localization length of a monochromatic pulse in Section 6.1. We study the exponential localization phenomenon for a pulse in Section 6.2 and show that it is a self-averaging process. We study the statistics of the incoherent reflected waves in Section 7. Finally, we analyze time-reversal for waves in random media in Section 8.

An (excellent...) support for this lecture is the book [26]. The topics treated in these notes cover parts of Chapters 4-10.

2. Averages of stochastic processes

We begin by a brief review of the two main limit theorems for sums of independent random variables.

• The Law of Large Numbers: If $(X_i)_{i\in\mathbb{N}}$ is a sequence of independent identically distributed $\mathbb{R}$-valued random variables, with $\mathbb{E}[|X_1|] < \infty$, then the normalized partial sums

$$S_n = \frac{1}{n}\sum_{k=1}^n X_k$$

converge to the statistical average $\bar{X} = \mathbb{E}[X_1]$ with probability one (write a.s. for almost surely).

• The Central Limit Theorem: If $(X_i)_{i\in\mathbb{N}}$ is a sequence of independent and identically distributed $\mathbb{R}$-valued random variables, with $\bar{X} = \mathbb{E}[X_1]$ and $\mathbb{E}[X_1^2] < \infty$, then the normalized partial sums

$$\tilde{S}_n = \frac{1}{\sqrt{n}}\sum_{k=1}^n (X_k - \bar{X})$$

converge in distribution to a Gaussian random variable with mean 0 and variance $\sigma^2 = \mathbb{E}[(X_1 - \bar{X})^2]$. This distribution is denoted by $\mathcal{N}(0,\sigma^2)$. The above assertion means that, for any continuous bounded function f,

$$\mathbb{E}[f(\tilde{S}_n)] \stackrel{n\to\infty}{\longrightarrow} \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} f(x) \exp\left(-\frac{x^2}{2\sigma^2}\right) dx$$

or, for any interval $I \subset \mathbb{R}$,

$$\mathbb{P}\left(\tilde{S}_n \in I\right) \stackrel{n\to\infty}{\longrightarrow} \frac{1}{\sqrt{2\pi\sigma^2}} \int_I \exp\left(-\frac{x^2}{2\sigma^2}\right) dx$$

2.1. A toy model

Let us consider a particle moving on the line $\mathbb{R}$. Assume that it is driven by a random velocity field $\varepsilon F(t)$ where ε is a small dimensionless parameter, F is stepwise constant

$$F(t) = \sum_{i=1}^{\infty} F_i \mathbf{1}_{[i-1,i)}(t),$$

and F_i are independent and identically distributed random variables that are bounded, $\mathbb{E}[F_i] = \bar{F}$ and $\mathbb{E}[(F_i - \bar{F})^2] = \sigma^2$. The position of the particle starting from 0 at time $t = 0$ is:

$$X(t) = \varepsilon \int_0^t F(s) ds.$$

Clearly $X(t) \stackrel{\varepsilon\to 0}{\longrightarrow} 0$. The problem consists in finding the adequate asymptotic, that is to say the time scale which leads to a macroscopic motion of the particle.

Regime of the Law of Large Numbers. At the scale $t \to t/\varepsilon$, $X^\varepsilon(t) := X(\frac{t}{\varepsilon})$ reads as:

$$\begin{aligned}
X^\varepsilon(t) &= \varepsilon \int_0^{\frac{t}{\varepsilon}} F(s) ds \\
&= \varepsilon \left(\sum_{i=1}^{[\frac{t}{\varepsilon}]} F_i \right) + \varepsilon \int_{[\frac{t}{\varepsilon}]}^{\frac{t}{\varepsilon}} F(s) ds \\
&= \underset{\substack{\downarrow \\ t}}{\varepsilon \left[\frac{t}{\varepsilon}\right]} \times \underset{\substack{\text{a.s.} \downarrow \\ \mathbb{E}[F] = \bar{F}}}{\frac{1}{[\frac{t}{\varepsilon}]} \left(\sum_{i=1}^{[\frac{t}{\varepsilon}]} F_i \right)} + \underset{\substack{\text{a.s.} \downarrow \\ 0}}{\varepsilon \left(\frac{t}{\varepsilon} - \left[\frac{t}{\varepsilon}\right] \right) F_{[\frac{t}{\varepsilon}]}}
\end{aligned}$$

The convergence of $\displaystyle \frac{1}{\left[\frac{t}{\varepsilon}\right]}\left(\sum_{i=1}^{\left[\frac{t}{\varepsilon}\right]} F_i\right)$ is determined by the Law of Large Numbers. Thus the motion of the particle is ballistic in the sense that it has constant effective velocity:

$$X^\varepsilon(t) \stackrel{\varepsilon\to 0}{\longrightarrow} \bar{F}t.$$

However, in the case $\bar{F}=0$, the random velocity field seems to have no effect, which means that we have to consider a different scaling.

Regime of the Central Limit Theorem $\bar{F}=0$. At the scale $t \to t/\varepsilon^2$, $X^\varepsilon(t) = X(\frac{t}{\varepsilon^2})$ reads as:

$$\begin{aligned}
X^\varepsilon(t) &= \varepsilon \int_0^{\frac{t}{\varepsilon^2}} F(s)ds \\
&= \varepsilon \left(\sum_{i=1}^{\left[\frac{t}{\varepsilon^2}\right]} F_i\right) + \varepsilon \int_{\left[\frac{t}{\varepsilon^2}\right]}^{\frac{t}{\varepsilon^2}} F(s)ds \\
&= \underset{\substack{\downarrow \\ \sqrt{t}}}{\varepsilon\sqrt{\left[\frac{t}{\varepsilon^2}\right]}} \times \underset{\substack{\text{distribution}\ \downarrow \\ \mathcal{N}(0,\sigma^2)}}{\frac{1}{\sqrt{\left[\frac{t}{\varepsilon^2}\right]}}\left(\sum_{i=1}^{\left[\frac{t}{\varepsilon^2}\right]} F_i\right)} + \underset{\substack{\text{a.s.}\ \downarrow \\ 0}}{\varepsilon\left(\frac{t}{\varepsilon^2} - \left[\frac{t}{\varepsilon^2}\right]\right)} F_{\left[\frac{t}{\varepsilon^2}\right]}
\end{aligned}$$

The convergence of $\displaystyle \frac{1}{\sqrt{\left[\frac{t}{\varepsilon}\right]}}\left(\sum_{i=1}^{\left[\frac{t}{\varepsilon^2}\right]} F_i\right)$ is determined by the Central Limit Theorem. $X^\varepsilon(t)$ converges in distribution as $\varepsilon \to 0$ to the Gaussian statistics $\mathcal{N}(0,\sigma^2 t)$. The motion of the particle in this regime is diffusive.

2.2. Stationary and ergodic processes

A stochastic process $(F(t))_{t\geq 0}$ is an application from some probability space to a functional space. This means that for any fixed time t the quantity $F(t)$ is a random variable with values in $E=\mathbb{R}$ (or $\mathbb{C}$, or $\mathbb{C}^d$). Furthermore we shall only consider configurations where the functional space is either the set of the continuous functions $\mathcal{C}([0,\infty),E)$ equipped with the topology associated to the sup norm over the compact sets or the set of the càd-làg functions (right-continuous functions with left hand limits) equipped with the Skorokhod topology [9, 24]. This means that the realizations of the random process are either continuous or càd-làg functions. The statistical distribution of a stochastic process is characterized by its finite-dimensional distributions, that are moments of the form $\mathbb{E}\left[\phi(F(t_1),...,F(t_n))\right]$ for $n \in \mathbb{N}^*$, $t_1,...,t_n \geq 0$, and $\phi \in \mathcal{C}_b(E^n,\mathbb{R})$.

$(F(t))_{t\in\mathbb{R}^+}$ is a stationary stochastic process if the statistics of the process is invariant to a shift in the time origin: for any $t_0 \geq 0$,

$$(F(t_0+t))_{t\in\mathbb{R}^+} \overset{\text{distribution}}{=} (F(t))_{t\in\mathbb{R}^+}$$

It is a statistical steady state. A necessary and sufficient condition is that, for any $n \in \mathbb{N}^*$, for any $t_0, t_1, ..., t_n \in \mathbb{R}^+$, for any bounded continuous function $\phi \in \mathcal{C}_b(E^n, \mathbb{R})$, we have

$$\mathbb{E}\left[\phi(F(t_0+t_1), ..., F(t_0+t_n))\right] = \mathbb{E}\left[\phi(F(t_1), ..., F(t_n))\right]$$

Let us consider a stationary process such that $\mathbb{E}[|F(t)|] < \infty$. We set $\bar{F} = \mathbb{E}[F(t)]$. The ergodic theorem claims that the time average can be replaced by the statistical average under the so-called ergodic hypothesis [12].

Theorem 2.1. *If F satisfies the ergodic hypothesis, then*

$$\frac{1}{T}\int_0^T F(t)dt \overset{T\to\infty}{\longrightarrow} \bar{F} \qquad \mathbb{P} \ a.s.$$

The ergodic hypothesis requires that the orbit $(F(t))_t$ visits all of phase space. It is not easy to state and to understand (see Remark 2.3 below), although it seems an intuitive notion. The following example presents an example of a non-ergodic process.

Example 2.2. Let F_1 and F_2 be two ergodic processes (satisfying Theorem 2.1), and denote $\bar{F}_j = \mathbb{E}[F_j(t)]$, $j = 1, 2$. Assume $\bar{F}_1 \neq \bar{F}_2$. Now flip a coin independently of F_1 and F_2, whose result is $\chi = 1$ with probability 1/2 and 0 with probability 1/2. Let $F(t) = \chi F_1(t) + (1-\chi)F_2(t)$, which is a stationary process with mean $\bar{F} = \frac{1}{2}(\bar{F}_1 + \bar{F}_2)$. The time-averaged process satisfies

$$\begin{aligned}\frac{1}{T}\int_0^T F(t)dt &= \chi\left(\frac{1}{T}\int_0^T F_1(t)dt\right) + (1-\chi)\left(\frac{1}{T}\int_0^T F_2(t)dt\right)\\ &\overset{T\to\infty}{\longrightarrow} \chi\bar{F}_1 + (1-\chi)\bar{F}_2\end{aligned}$$

which is a random limit different from $\bar{F}$. The time-averaged limit depends on χ because F has been trapped in a part of phase space. The process $F(t)$ is not ergodic.

Remark 2.3 (Complement on the ergodic theory). Here we give a rigorous statement of an ergodic theorem (it is not necessary for the sequel). Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, that is:
- Ω is a non-empty set,
- $\mathcal{A}$ is a σ-algebra on Ω,
- $\mathbb{P} : \mathcal{A} \to [0, 1]$ is a probability (i.e. $\mathbb{P}(\Omega) = 1$ and $\mathbb{P}(\cup_j A_j) = \sum_j \mathbb{P}(A_j)$ for any numerable family of disjoint sets $A_j \in \mathcal{A}$).

Let $\theta_t : \Omega \to \Omega$, $t \geq 0$, be a measurable semi-group of shift operators (i.e. $\theta_t^{-1}(A) \in \mathcal{A}$ for any $A \in \mathcal{A}$ and $t \geq 0$, $\theta_0 = I_d$ and $\theta_{t+s} = \theta_t \circ \theta_s$ for any $t, s \geq 0$) that preserves the probability $\mathbb{P}$ (i.e. $\mathbb{P}(\theta_t^{-1}(A)) = \mathbb{P}(A)$ for any $A \in \mathcal{A}$ and $t \geq 0$).

The semi-group $(\theta_t)_{t\geq 0}$ is said to be ergodic if the invariant sets are negligible or of negligible complementary, i.e.

$$\theta_t^{-1}(A) = A \quad \forall t \geq 0 \implies \mathbb{P}(A) = 0 \text{ or } 1$$

We then have the following proposition.
Proposition. Let $f : (\Omega, \mathcal{A}, \mathbb{P}) \to \mathbb{R}$ and $F(t, \omega) = f(\theta_t(\omega))$.
1. F is a stationary random process.
2. if $f \in L^1(\mathbb{P})$ and $(\theta_t)_{t\geq 0}$ is ergodic, then

$$\frac{1}{T}\int_0^T F(t,\omega)dt \stackrel{T\to\infty}{\longrightarrow} \mathbb{E}[f] = \int_\Omega f d\mathbb{P} \quad \mathbb{P}-\text{a.s.}$$

2.3. Mean square theory

In this subsection we introduce a weaker form of the ergodic theorem, that holds true under a simple and explicit condition. Let $(F(t))_{t\geq 0}$ be a stationary process, $\mathbb{E}[F^2(0)] < \infty$. We introduce the autocorrelation function

$$R(\tau) = \mathbb{E}\left[(F(t) - \bar{F})(F(t+\tau) - \bar{F})\right]$$

By stationarity, R is an even function

$$R(-\tau) = \mathbb{E}\left[(F(t) - \bar{F})(F(t-\tau) - \bar{F})\right] = \mathbb{E}\left[(F(t'+\tau) - \bar{F})(F(t') - \bar{F})\right] = R(\tau)$$

By Cauchy-Schwarz inequality, R reaches its maximum at 0:

$$R(\tau) \leq \mathbb{E}\left[(F(t) - \bar{F})^2\right]^{1/2} \mathbb{E}\left[(F(t+\tau) - \bar{F})^2\right]^{1/2} = R(0) = \mathrm{Var}(F(0))$$

Proposition 2.4. *Assume that* $\int_0^\infty |R(\tau)|d\tau < \infty$. *Let* $S(T) = \frac{1}{T}\int_0^T F(t)dt$. *Then*

$$\mathbb{E}\left[(S(T) - \bar{F})^2\right] \stackrel{T\to\infty}{\longrightarrow} 0$$

more exactly

$$T\mathbb{E}\left[(S(T) - \bar{F})^2\right] \stackrel{T\to\infty}{\longrightarrow} 2\int_0^\infty R(\tau)d\tau$$

One should interpret the condition $\int_0^\infty |R(\tau)|d\tau < \infty$ as "the autocorrelation function $R(\tau)$ decays to 0 sufficiently fast as $\tau \to \infty$." This hypothesis is a mean square version of mixing: $F(t)$ and $F(t+\tau)$ are approximatively independent for long time lags τ. Mixing substitutes for independence in the law of large numbers. An example of mixing process is the piecewise constant process defined by:

$$F(s) = \sum_{k\in\mathbb{N}} f_k \mathbf{1}_{[L_k, L_{k+1})}(s)$$

with independent and identically distributed random variables f_k, $L_0 = 0$, $L_k = \sum_{j=1}^k l_j$ and independent exponential random variables l_j with mean 1. Here we have $R(\tau) = \mathrm{Var}(f_1)\exp(-|\tau|)$.

Proof. The proof consists in a straightforward calculation.

$$\begin{aligned}
\mathbb{E}\left[(S(T)-\bar{F})^2\right] &= \mathbb{E}\left[\frac{1}{T^2}\int_0^T dt_1 \int_0^T dt_2 (F(t_1)-\bar{F})(F(t_2)-\bar{F})\right] \\
&\overset{\text{symmetry}}{=} \frac{2}{T^2}\int_0^T dt_1 \int_0^{t_1} dt_2 R(t_1-t_2) \\
&\overset{\substack{\tau = t_1 - t_2 \\ h = t_2}}{=} \frac{2}{T^2}\int_0^T d\tau \int_0^{T-\tau} dh R(\tau) \\
&= \frac{2}{T^2}\int_0^T d\tau (T-\tau) R(\tau) = \frac{2}{T}\int_0^\infty d\tau R_T(\tau)
\end{aligned}$$

where $R_T(\tau) = R(\tau)(1-\tau/T)\mathbf{1}_{[0,T]}(\tau)$. By Lebesgue's convergence theorem:

$$T\mathbb{E}\left[(S(T)-\bar{F})^2\right] \overset{T\to\infty}{\longrightarrow} 2\int_0^\infty R(\tau)d\tau$$

□

Note that the $L^2(\mathbb{P})$ convergence implies convergence in probability as the limit is deterministic. Indeed, by Chebychev inequality, for any $\delta > 0$,

$$\mathbb{P}\left(|S(T)-\bar{F}| \geq \delta\right) \leq \frac{\mathbb{E}\left[(S(T)-\bar{F})^2\right]}{\delta^2} \overset{T\to\infty}{\longrightarrow} 0$$

2.4. Averaging theorem

Let us revisit our toy model and consider a more general model for the velocity field. Let $0 < \varepsilon \ll 1$ be a small parameter and X^ε satisfies:

$$\frac{dX^\varepsilon}{dt} = F(\frac{t}{\varepsilon}), \quad X^\varepsilon(0) = 0$$

where F is a stationary process with a decaying autocorrelation function such that $\int_0^\infty |R(\tau)|d\tau < \infty$. $F(t)$ is a process on its own natural time scale. $F(t/\varepsilon)$ is the speeded-up process. The solution is $X^\varepsilon(t) = \int_0^t F(s/\varepsilon)ds = t\frac{1}{T}\int_0^T F(s)ds$ where $T = t/\varepsilon \to \infty$ as $\varepsilon \to 0$. So $X^\varepsilon(t) \to \bar{F}t$ as $\varepsilon \to 0$, or else $X^\varepsilon \to \bar{X}$ solution of:

$$\frac{d\bar{X}}{dt} = \bar{F}, \quad \bar{X}(t=0) = 0$$

We can generalize this result to more general configurations.

Proposition 2.5. *[37, Khaminskii] Assume that, for each fixed value of $x \in \mathbb{R}^d$, $F(t,x)$ is a stochastic $\mathbb{R}^d$-valued process in t. Assume also that there exists a* *deterministic function $\bar{F}(x)$ such that*

$$\bar{F}(x) = \lim_{T\to\infty} \frac{1}{T}\int_{t_0}^{t_0+T} \mathbb{E}[F(t,x)]dt$$

with the limit independent of t_0. Let $\varepsilon > 0$ and X^ε be the solution of

$$\frac{dX^\varepsilon}{dt} = F(\frac{t}{\varepsilon}, X^\varepsilon), \quad X^\varepsilon(0) = 0$$

Define $\bar{X}$ as the solution of

$$\frac{d\bar{X}}{dt} = \bar{F}(\bar{X}), \quad \bar{X}(0) = 0$$

Then under mild technical hypotheses on F and $\bar{F}$, we have for any T:

$$\sup_{t\in[0,T]} \mathbb{E}\left[|X^\varepsilon(t) - \bar{X}(t)|\right] \stackrel{\varepsilon\to 0}{\longrightarrow} 0$$

Proof. The proof requires only elementary tools under the hypotheses:
1) F is stationary and $\mathbb{E}\left[\left|\frac{1}{T}\int_0^T F(t,x)dt - \bar{F}\right|\right] \stackrel{T\to\infty}{\longrightarrow} 0$ (to check this, we can use the mean square theory since $\mathbb{E}[|Y|] \leq \sqrt{\mathbb{E}[Y^2]}$).
2) For any t , $F(t,.)$ and $\bar{F}(.)$ are uniformly Lipschitz with a non-random Lipschitz constant c.
3) For any compact subset $K \subset \mathbb{R}^d$, $\sup_{t\in\mathbb{R}^+, x\in K} |F(t,x)| + |\bar{F}(x)| < \infty$.

We have

$$X^\varepsilon(t) = \int_0^t F(\frac{s}{\varepsilon}, X^\varepsilon(s))ds, \qquad \bar{X}(t) = \int_0^t \bar{F}(\bar{X}(s))ds$$

so the difference reads:

$$X^\varepsilon(t) - \bar{X}(t) = \int_0^t \Big(F(\frac{s}{\varepsilon}, X^\varepsilon(s)) - F(\frac{s}{\varepsilon}, \bar{X}(s))\Big)\, ds + g^\varepsilon(t)$$

where $g^\varepsilon(t) := \int_0^t F(\frac{s}{\varepsilon}, \bar{X}(s)) - \bar{F}(\bar{X}(s))ds$. Taking the modulus:

$$\begin{aligned} |X^\varepsilon(t) - \bar{X}(t)| &\leq \int_0^t \Big|F(\frac{s}{\varepsilon}, X^\varepsilon(s)) - F(\frac{s}{\varepsilon}, \bar{X}(s))\Big|\, ds + |g^\varepsilon(t)| \\ &\leq c\int_0^t |X^\varepsilon(s) - \bar{X}(s)|ds + |g^\varepsilon(t)| \end{aligned}$$

Taking the expectation and applying Gronwall's inequality:

$$\mathbb{E}\left[|X^\varepsilon(t) - \bar{X}(t)|\right] \leq e^{ct} \sup_{s\in[0,t]} \mathbb{E}[|g^\varepsilon(s)|]$$

It remains to show that the last term goes to 0 as $\varepsilon \to 0$. Let $\delta > 0$

$$\begin{aligned} g^\varepsilon(t) &= \sum_{k=0}^{[t/\delta]-1} \int_{k\delta}^{(k+1)\delta} \Big(F(\frac{s}{\varepsilon}, \bar{X}(s)) - \bar{F}(\bar{X}(s)) \Big)\, ds \\ &\quad + \int_{\delta[t/\delta]}^{t} \Big(F(\frac{s}{\varepsilon}, \bar{X}(s)) - \bar{F}(\bar{X}(s)) \Big)\, ds \end{aligned}$$

Denote $M_T = \sup_{t\in[0,T]} |\bar{X}(t)|$. As $K_T = \sup_{x\in[-M_T,M_T], t\in\mathbb{R}^+} |F(t,x)| + |\bar{F}(x)|$ is finite, the last term of the right-hand side is bounded by $2K_T\delta$. Furthermore F is Lipschitz, so that

$$\left| F(\frac{s}{\varepsilon}, \bar{X}(s)) - F(\frac{s}{\varepsilon}, \bar{X}(k\delta)) \right| \le c \left| \bar{X}(s) - \bar{X}(k\delta) \right| \le cK_T |s - k\delta|$$

We have similarly:

$$\left| \bar{F}(\bar{X}(s)) - \bar{F}(\bar{X}(k\delta)) \right| \le cK_T |s - k\delta|$$

Thus

$$\begin{aligned} |g^\varepsilon(t)| &\le \left| \sum_{k=0}^{[t/\delta]-1} \int_{k\delta}^{(k+1)\delta} \Big(F(\frac{s}{\varepsilon}, \bar{X}(k\delta)) - \bar{F}(\bar{X}(k\delta)) \Big)\, ds \right| \\ &\quad + 2cK_T \sum_{k=0}^{[t/\delta]-1} \int_{k\delta}^{(k+1)\delta} (s - k\delta) ds + 2K_T\delta \\ &\le \varepsilon \sum_{k=0}^{[t/\delta]-1} \left| \int_{k\delta/\varepsilon}^{(k+1)\delta/\varepsilon} \big(F(s, \bar{X}(k\delta)) - \bar{F}(\bar{X}(k\delta)) \big)\, ds \right| + 2K_T(ct+1)\delta \end{aligned}$$

Taking the expectation and the supremum:

$$\begin{aligned} \sup_{t\in[0,T]} \mathbb{E}[|g^\varepsilon(t)|] &\le \delta \sum_{k=0}^{[T/\delta]} \mathbb{E}\left[\left| \frac{\varepsilon}{\delta} \int_{k\delta/\varepsilon}^{(k+1)\delta/\varepsilon} \big(F(s, \bar{X}(k\delta)) - \bar{F}(\bar{X}(k\delta)) \big)\, ds \right| \right] \\ &\quad + 2K_T(cT+1)\delta \end{aligned}$$

Taking the limit $\varepsilon \to 0$:

$$\limsup_{\varepsilon\to 0} \sup_{t\in[0,T]} \mathbb{E}[|g^\varepsilon(t)|] \le 2K_T(cT+1)\delta$$

Letting $\delta \to 0$ completes the proof. □

A more elegant and quick proof of this proposition is given in Appendix B.

3. Effective medium theory

This section is devoted to the computation of the effective speed of an acoustic pulse traveling through a random medium. The material presented in this section is a shortened version of [26, Chapter 4].

3.1. Acoustic waves in random media

The acoustic pressure field $p^\varepsilon(t,z)$ and velocity field $u^\varepsilon(t,z)$ satisfy the continuity and momentum equations

$$\rho^\varepsilon \frac{\partial u^\varepsilon}{\partial t} + \frac{\partial p^\varepsilon}{\partial z} = 0 \tag{3.1}$$

$$\frac{\partial p^\varepsilon}{\partial t} + \kappa^\varepsilon \frac{\partial u^\varepsilon}{\partial z} = 0 \tag{3.2}$$

where ρ^ε is the material density and κ^ε is the bulk modulus of the medium.

Assume that $\rho^\varepsilon(z) = \rho(z/\varepsilon)$ and $\kappa^\varepsilon(z) = \kappa(z/\varepsilon)$ are stationary random functions of position on spatial scale of order ε, $0 < \varepsilon \ll 1$. They are piecewise smooth and also uniformly bounded such that $\|\rho\|_\infty \leq C$, $\|\rho^{-1}\|_\infty \leq C$, $\|\kappa\|_\infty \leq C$, and $\|\kappa^{-1}\|_\infty \leq C$ a.s..

Assume conditions so that the system admits a solution, for instance a Dirichlet condition at $z = 0$ of the type $u^\varepsilon(t, z = 0) = f(t)$ and $p^\varepsilon(t, z = 0) = g(t)$ with $f, g \in L^2$. We also assume for simplicity that the Fourier transforms $\hat{f}$ and $\hat{g}$ decay faster than any exponential (say for instance that f and g are Gaussian pulses).

Note that the fluctuations of the medium parameters are not assumed to be small. This corresponds to typical situations in acoustics and geophysics. The estimation of the vertical correlation length of the inhomogeneities in the lithosphere from well-log data is considered in [69]. They found that $2-3$ m is a reasonable estimate of the correlation length of the fluctuations in sound speed. The typical pulse width is about 50 ms or, with a speed of 3 km/s, the typical wavelength is 150 m. So $\varepsilon = 10^{-2}$ is our framework.

We perform a Fourier analysis with respect to t. So we Fourier transform u^ε and p^ε

$$u^\varepsilon(t,z) = \frac{1}{2\pi}\int \hat{u}^\varepsilon(\omega,z)e^{-i\omega t}d\omega, \quad p^\varepsilon(t,z) = \frac{1}{2\pi}\int \hat{p}^\varepsilon(\omega,z)e^{-i\omega t}d\omega$$

so that we get a system of ordinary differential equations (ODE):

$$\frac{dX^\varepsilon}{dz} = F(\frac{z}{\varepsilon}, X^\varepsilon),$$

where

$$X^\varepsilon = \begin{pmatrix} \hat{p}^\varepsilon \\ \hat{u}^\varepsilon \end{pmatrix}, \qquad F(z,X) = \mathbf{M}(z)X, \qquad \mathbf{M}(z) = i\omega \begin{pmatrix} 0 & \rho(z) \\ \frac{1}{\kappa(z)} & 0 \end{pmatrix}$$

Note first that a straightforward estimate shows that $|X^\varepsilon(\omega,z)| \leq |X_0(\omega)| \exp(C\omega z)$.

3.2. Homogenization

We now apply the method of averaging. We get that $X^\varepsilon(\omega, z)$ converges in $L^1(\mathbb{P})$ to $\bar{X}(\omega, z)$ solution of

$$\frac{d\bar{X}}{dz} = \bar{\mathbf{M}}\bar{X}, \quad \bar{\mathbf{M}} = i\omega \begin{pmatrix} 0 & \bar{\rho} \\ \frac{1}{\bar{\kappa}} & 0 \end{pmatrix}, \quad \bar{\rho} = \mathbb{E}[\rho], \quad \bar{\kappa} = \left(\mathbb{E}[\kappa^{-1}]\right)^{-1}$$

We now come back to the time domain. We introduce the deterministic "effective medium" with parameters $\bar{\rho}$, $\bar{\kappa}$ and the solution $(\bar{p}, \bar{u})$ of

$$\begin{aligned} \bar{\rho}\frac{\partial \bar{u}}{\partial t} + \frac{\partial \bar{p}}{\partial z} = 0 \\ \frac{\partial \bar{p}}{\partial t} + \bar{\kappa}\frac{\partial \bar{u}}{\partial z} = 0 \end{aligned}$$

The parameters are constant, so $\bar{p}$ satisfies the closed form equation $\frac{\partial^2 \bar{p}}{\partial t^2} - \bar{c}^2 \frac{\partial^2 \bar{p}}{\partial z^2} = 0$, which is the standard wave equation with the effective wave speed $\bar{c} = \sqrt{\bar{\kappa}/\bar{\rho}}$.

We now compare $p^\varepsilon(t, z)$ with $\bar{p}(t, z)$:

$$\begin{aligned} \mathbb{E}\left[|p^\varepsilon(t,z) - \bar{p}(t,z)|\right] &= \frac{1}{2\pi}\mathbb{E}\left[\left|\int e^{-i\omega t}(\hat{p}^\varepsilon(\omega,z) - \hat{\bar{p}}(\omega,z))d\omega\right|\right] \\ &\le \frac{1}{2\pi}\int \mathbb{E}\left[|\hat{p}^\varepsilon(\omega,z) - \hat{\bar{p}}(\omega,z)|\right] d\omega \end{aligned}$$

The dominated convergence theorem then gives the convergence in $L^1(\mathbb{P})$ of p^ε to $\bar{p}$ in the time domain. Thus the effective speed of the acoustic wave $(p^\varepsilon, u^\varepsilon)$ as $\varepsilon \to 0$ is $\bar{c}$.

Example: bubbles in water. Air and water are characterized by the following parameters:
$\rho_a = 1.2\ 10^3$ g/m^3, $\kappa_a = 1.4\ 10^8$ g/s^2/m, $c_a = 340$ m/s.
$\rho_w = 1.0\ 10^6$ g/m^3, $\kappa_w = 2.0\ 10^{18}$ g/s^2/m, $c_w = 1425$ m/s.
If we consider a pulse whose frequency content is in the range 10 Hz - 30 kHz, then the wavelengths lie in the range 1 cm - 100 m. The bubble sizes are much smaller, so the effective medium theory can be applied. Let us denote by ϕ the volume fraction of air. The averaged density and bulk modulus are

$$\bar{\rho} = \mathbb{E}[\rho] = \phi\rho_a + (1-\phi)\rho_w = \begin{cases} 9.9\ 10^5 \text{ g/m}^3 & \text{if } \phi = 1\% \\ 9\ 10^5 \text{ g/m}^3 & \text{if } \phi = 10\% \end{cases}$$

$$\bar{\kappa} = \left(\mathbb{E}[\kappa^{-1}]\right)^{-1} = \left(\frac{\phi}{\kappa_a} + \frac{1-\phi}{\kappa_w}\right)^{-1} = \begin{cases} 1.4\ 10^{10} \text{ g/s}^2\text{/m} & \text{if } \phi = 1\% \\ 1.4\ 10^9 \text{ g/s}^2\text{/m} & \text{if } \phi = 10\% \end{cases}$$

Accordingly $\bar{c} = 120$ m/s if $\phi = 1\%$ and $\bar{c} = 37$ m/s if $\phi = 10$ %.

The above example demonstrates that the average velocity may be much smaller than the minimum of the velocities of the medium components. However it cannot happen in such a configuration that the velocity be larger than the maximum (or the essential supremum) of the velocities of the medium components. Indeed,

$$\mathbb{E}[c^{-1}] = \mathbb{E}\left[\kappa^{-1/2}\rho^{1/2}\right] \leq \mathbb{E}[\kappa^{-1}]^{1/2}\mathbb{E}[\rho]^{1/2} = \bar{c}^{-1}$$

Thus $\bar{c} \leq \mathbb{E}[c^{-1}]^{-1} \leq \text{ess sup}(c)$.

3.3. Bibliographic notes

The theory and the results presented in these notes rely heavily on modeling and analysis with separation of scales, which has been developed in the past thirty-five years. The main probabilistic tool for the homogenization theory of the equations considered in these notes is the law of large numbers or, more generally, the ergodic theorem. We introduce this basic result in Section 2. We refer to the book of Breiman [12] for a more complete introduction to probabilistic tools at the level used in these lecture notes. In Section 2 we reformulate homogenization as an averaging theorem for random differential equations. Such averaging theorems were first given by Khasminskii [37]. A review of different averaging techniques can be found in the book by Holmes [33]. Multi-dimensional homogenization theory for periodic media is extensively treated by Milton [48] and Bensoussan–Lions–Papanicolaou [7]. A review of results on homogenization for random media is presented in [52]. Acoustic waves in bubbly liquids were analyzed in [17]. Electromagnetic waves in composite materials are discussed in [63].

4. Diffusion-approximation

In this section we give a brief presentation of the asymptotic analysis of random differential equations in the form that they have in models of wave propagation in one-dimensional random media. The section serves two purposes: It provides an introduction to Markovian models of random media. It gives a treatment of the theory of diffusion approximations for random differential equations in a form that can be readily used for the asymptotic analysis of reflected and transmitted waves in one-dimensional random media. This section is a shortened version of the extensive treatment proposed in [26, Chapter 6]. In a first lecture, Subsection 4.2 can be skipped.

4.1. Markov processes

A stochastic process $(X_t)_{t\geq 0}$ with state space E is a Markov process if $\forall 0 \leq s < t$ and $B \in \mathcal{B}(E)$ (the σ-algebra of Borel sets of E)

$$\mathbb{P}(X_t \in B | X_u, u \leq s) = \mathbb{P}(X_t \in B | X_s)$$

"the state X_s at time s contains all relevant information for calculating probabilities of future events". The rigorous definition needs σ-algebras of measurable sets. This is a generalization to the stochastic case of the dynamical deterministic systems without memory of the type $\frac{dx}{dt} = f(t, x(t))$.

The distribution of X_t starting from x at time s is the transition probability

$$\mathbb{P}(X_t \in B | X_s = x) = P(s, x; t, B) = \int_{y \in B} P(s, x; t, dy)$$

Definition 4.1. A transition probability P is a function from $\mathbb{R}^+ \times E \times \mathbb{R}^+ \times \mathcal{B}(E)$ such that
1) $P(s, x; t, A)$ is measurable in x for fixed $s \in \mathbb{R}^+$, $t \in \mathbb{R}^+$, $A \in \mathcal{B}(E)$,
2) $P(s, x; t, A)$ is a probability measure in A for fixed $s \in \mathbb{R}^+$, $t \in \mathbb{R}^+$, $x \in E$,
3) P satisfies the Chapman Kolmogorov equation:

$$P(s, x; t, A) = \int_E P(s, x; \tau, dz) P(\tau, z; t, A) \quad \forall 0 \leq s < \tau < t$$

Note that the Chapman-Kolmogorov equation can be deduced heuristically from a disintegration formula:

$$\mathbb{P}(X_t \in A | X_s = x) = \int_E \mathbb{P}(X_t \in A | X_s = x, X_\tau = z) \mathbb{P}(X_\tau \in dz | X_s = x)$$

and the application of the Markov property.

A Markov process is temporally homogeneous if the transition probability depends only on $t - s$: $P(s, x; t, A) = P(0, x; t - s, A)$. This function is denoted by $P_t(x, A)$. From now on, we only consider homogeneous Markov processes.

We can now define the family of operators defined on the space of measurable bounded functions $L^\infty(E)$:

$$T_t f(x) = \mathbb{E}[f(X_t) | X_0 = x] = \int_E P_t(x, dy) f(y)$$

Proposition 4.2. *1)* $T_0 = I_d$
2) $\forall s, t \leq 0$ $T_{s+t} = T_s T_t$
3) T_t *is a contraction* $\|T_t f\|_\infty \leq \|f\|_\infty$ $\forall f \in L^\infty(E)$.

Proof. The second point follows from the Chapman-Kolmogorov relations, and the third one from the fact that $P_t(x, \cdot)$ is a probability:

$$|T_t f(x)| \leq \int_E P_t(x, dy) \|f\|_\infty = \|f\|_\infty$$

□

The family (T_t) is a semi-group, but it is not a group as T_t does not possess an inverse. We shall consider Feller processes defined as follows. We denote by C the set of continuous functions from E to $\mathbb{R}$ (if E is not compact, we should define C as the set of continuous function f such that $\lim_{|x|\to\infty} f(x)$ exists and equals 0). We say that the process is Feller if, for any $h > 0$, T_h maps C into C and $\|T_h f - f\|_\infty \to 0$ as $h \searrow 0$. This continuity property goes from the family operators (T_t) to the process itself. The exact assertion claims that this is true for a modification of the process. A stochastic process $\tilde{X}$ is called a modification of a process X if $\mathbb{P}(X_t = \tilde{X}_t) = 1$ for all t.

Proposition 4.3. *Let (X_t) be a Feller Markov process. Then (X_t) has a modification whose realizations are càd-làg functions.*

From now on we shall always consider such a modification of the Markov process. The generator of a Markov process is the operator

$$Q := \lim_{h\searrow 0} \frac{T_h - I_d}{h}$$

It is defined on the subset of C such that the above limit exists in C.

Proposition 4.4. *Let $f \in \mathrm{dom}(Q)$. The function $u(t,x) := T_t f(x)$ belongs to $\mathrm{dom}(Q)$ for any t, it is differentiable with respect to t, and it satisfies the Kolmogorov equation*

$$\frac{\partial u}{\partial t} = Qu, \quad t \geq 0, \qquad u(t=0,x) = f(x) \tag{4.1}$$

Proof. We consider the increment

$$\frac{u(t+h,x) - u(t,x)}{h} = T_t \frac{T_h - I_d}{h} f(x)$$

Since $f \in \mathrm{dom}(Q)$, and T_t maps C into C, we get the existence and the value of the limit

$$\lim_{h\searrow 0} \frac{u(t+h,x) - u(t,x)}{h} = T_t Q f(x)$$

which proves that $u(t,x)$ is differentiable with respect to t and $\partial_t u = T_t Q f$. We can also write

$$\frac{u(t+h,x) - u(t,x)}{h} = \frac{T_h - I_d}{h} T_t f(x) = \frac{T_h - I_d}{h} u(t,x)$$

Since we have just shown that the left-hand side converges in C, this proves that $u(t,\cdot)$ belongs to $\mathrm{dom}(Q)$, and the limit is $\partial_t u = Qu$. □

Example 4.5 (A two-state Markov process). Let $(\tau_j)_{j\geq 1}$ be a sequence of independent and identically distributed random variables with exponential distribution with parameter 1:

$$\mathbb{P}(\tau_1 \geq t) = e^{-t}$$

Let $T_0 = 0$ and $T_n = \sum_{j=1}^n \tau_j$, $n \geq 1$.
Let us define $N_t = \sum_{n\geq 0} \mathbf{1}_{T_n \leq t}$. The process $(N_t)_{t\geq 0}$ is a Poisson point process:
- it is right-continuous, non-decreasing, and takes values in $\mathbb{N}$.
- $(N_t)_{t\geq 0}$ has independent increments, $N_{s+t} - N_s$ has the same distribution as N_t, which is the Poisson distribution $\mathbb{P}(N_t = k) = e^{-t}t^k/k!$, $k \in \mathbb{N}$.

Let $X_0 \in \{-1, 1\}$ and set

$$X_t = X_0(-1)^{N_t}, \quad t \geq 0$$

The process $(X_t)_{t\geq 0}$ takes values in $E = \{-1, 1\}$. In fact, the process $(X_t)_{t\geq 0}$ is stepwise constant, it takes alternatively the values ± 1, and the time intervals are independent with the common exponential distribution with mean 1.

The functions $f \in L^\infty(E)$ are vectors in $\mathbb{R}^2$. The semigroup $(T_t)_{t\geq 0}$ is a family of 2×2 matrices:

$$\begin{aligned} T_t &= \begin{pmatrix} \mathbb{P}(X_t = 1|X_0 = 1) & \mathbb{P}(X_t = 1|X_0 = -1) \\ \mathbb{P}(X_t = -1|X_0 = 1) & \mathbb{P}(X_t = -1|X_0 = -1) \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} + \frac{1}{2}e^{-2t} & \frac{1}{2} - \frac{1}{2}e^{-2t} \\ \frac{1}{2} - \frac{1}{2}e^{-2t} & \frac{1}{2} + \frac{1}{2}e^{-2t} \end{pmatrix} \end{aligned}$$

The generator is a matrix:

$$Q = \lim_{h\to 0} \frac{T_h - I_d}{h} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

In this simple example, the analysis of the distribution of $(X_t)_{t\geq 0}$ is reduced to linear algebra.

The following Proposition is an application of the previous results to the case of an ordinary differential equation driven by a Feller Markov process.

Proposition 4.6. *Let $(q_t)_{t\geq 0}$ be a S-valued Feller process with generator Q and X be the solution of:*

$$\frac{dX}{dt} = F(q_t, X(t)), \qquad X(0) = x$$

where $F : S \times \mathbb{R}^d \to \mathbb{R}^d$ is a bounded Borel function such that $x \mapsto F(q, x)$ has bounded continuous derivatives uniformly with respect to $q \in S$.

1. *$Y = (q, X)$ is a Markov process with generator:*

$$\mathcal{L} = Q + \sum_{j=1}^d F_j(q, x) \frac{\partial}{\partial x_j}$$

The domain of the generator is the subset of functions $f \in L^\infty(\mathbb{R}^d \times S)$ such that f has bounded continuous derivatives in x and belongs to dom(Q) *for any x.*

2. For any function $f \in \mathrm{dom}(\mathcal{L})$, the process

$$M_f(t) := f(Y(t)) - \int_0^t \mathcal{L} f(Y(s)) ds$$

is a martingale, which means that, for any $0 \le s \le t$:

$$\mathbb{E}[M_f(t) | \mathcal{F}_s] = M_f(s)$$

where $\mathcal{F}_t = \sigma(q_s, 0 \le s \le t)$.

Proof. Step 1. $(X(t))$ is $\mathcal{F}_t$-measurable, (and so is $Y(t) = (X(t), q_t)$).
We introduce $X^{(0)} = x$ and for $n \ge 0$

$$X^{(n+1)}(t) = x + \int_0^t F(q_s, X^{(n)}(s)) ds$$

We check by a straightforward inductive argument that $X^{(n)}(t)$ is $\mathcal{F}_t$-measurable for all n. We also prove that $X^{(n)}$ is uniformly convergent to X. This shows that $X(t) = \lim_{n\to\infty} X^{(n)}(t)$ is $\mathcal{F}_t$-measurable.

Step 2. $Y = (X, q)$ is Markov, i.e. for any bounded Borel function f we have

$$\mathbb{E}[f(Y(t+h)) \mid \mathcal{F}_t] = \mathbb{E}[f(Y(t+h)) \mid Y(t)]$$

We introduce the family of processes $Z_{h,x}$ that satisfy

$$\frac{dZ_{h,x}}{dh} = F(q_{t+h}, Z_{h,x}), \qquad Z_{0,x} = x$$

Note that $Z_{h,X(t)} = X(t+h)$ and, for any $x \in \mathbb{R}^d$, $Z_{h,x}$ is $\sigma(q_s, t \le s \le t+h)$-measurable. Therefore

$$\begin{aligned}
\mathbb{E}[f(Y(t+h)) \mid \mathcal{F}_t, X(t) = x] &= \mathbb{E}[f(q_{t+h}, Z_{h,X(t)}) \mid \mathcal{F}_t, X(t) = x] \\
&= \mathbb{E}[f(q_{t+h}, Z_{h,x}) \mid \mathcal{F}_t, X(t) = x] \\
&= \mathbb{E}[f(q_{t+h}, Z_{h,x}) \mid q_t, X(t) = x] \\
&= \mathbb{E}[f(Y(t+h)) \mid q_t, X(t) = x]
\end{aligned}$$

Step 3. Computation of the infinitesimal generator. Let $f : S \times \mathbb{R}^d \to \mathbb{R}$ be a smooth bounded test function (i.e. $q \mapsto f(q,x) \in \mathrm{dom}(Q)$ $\forall x \in \mathbb{R}^d$, and $x \mapsto f(q,x)$ is bounded with bounded continuous derivatives).
Let $y = (q, x) \in S \times \mathbb{R}^d$.

$$\mathbb{E}\left[f(Y(t+h)) \mid Y(t) = y\right] - f(y) = A_h + B_h \tag{4.2}$$

where

$$\begin{aligned}
A_h &= \mathbb{E}\left[f(q_{t+h}, X(t+h)) - f(q_{t+h}, X(t)) \mid Y(t) = (q,x)\right] \\
B_h &= \mathbb{E}\left[f(q_{t+h}, X(t)) - f(q_t, X(t)) \mid Y(t) = (q,x)\right]
\end{aligned}$$

We first consider A_h:

$$|f(q_{t+h}, X(t+h)) - f(q_{t+h}, X(t)) - \nabla f(q_{t+h}, X(t)).(X(t+h) - X(t))| \leq Ch^2$$

We have

$$X(t+h) - X(t) = \int_t^{t+h} F(q_s, X(s))ds$$

so that

$$\left| X(t+h) - X(t) - \int_t^{t+h} F(q_s, X(t))ds \right| \leq C \int_t^{t+h} |X(s) - X(t)|ds \leq Ch^2$$

Accordingly

$$\left| f(q_{t+h}, X(t+h)) - f(q_{t+h}, X(t)) - \nabla f(q_{t+h}, X(t)). \int_t^{t+h} F(q_s, X(t))ds \right| \leq Ch^2$$

and

$$\left| A_h - \mathbb{E}\left[\nabla f(q_{t+h}, x). \int_t^{t+h} F(q_s, x)ds \mid q_t = q \right] \right| \leq Ch^2$$

We have

$$\mathbb{E}\left[\nabla f(q_{t+h}, x).F(q_s, x) \mid q_t = q\right] = T_{s-t}\left[T_{t+h-s}(\nabla f(\cdot, x)).F(\cdot, x)\right](q)$$

$\|T_t\|_\infty \leq 1$ and q is assumed to be Feller so that $T_{a_n}(f_n) \to f$ if $f_n \to f$ and $a_n \to 0$. This shows

$$\mathbb{E}\left[\nabla f(q_{t+h}, x).F(q_s, x) \mid q_t = q\right] \stackrel{h \to 0}{\longrightarrow} \nabla f(q, x).F(q, x)$$

which implies

$$\frac{A_h}{h} \stackrel{h \to 0}{\longrightarrow} \nabla f(q, x).F(q, x) \tag{4.3}$$

We now consider B_h:

$$\frac{B_h}{h} = \frac{1}{h}\mathbb{E}\left[f(q_{t+h}, x) - f(q_t, x) \mid q_t = q\right] \stackrel{h \to 0}{\longrightarrow} Qf(q, x) \tag{4.4}$$

Substituting (4.3) and (4.4) into (4.2) yields the result.

Step 4. M_f is a martingale.

$$\begin{aligned}
\mathbb{E}[M_f(t)|\mathcal{F}_s] &= M_f(s) + \mathbb{E}\left[f(Y(t)) - f(Y(s)) - \int_s^t \mathcal{L}f(Y(u))du|Y(s) \right] \\
&= M_f(s) + T_{t-s}f(Y(s)) - f(Y(s)) - \int_s^t T_{t-u}\mathcal{L}f(Y(s))du \\
&= M_f(s) + T_{t-s}f(Y(s)) - f(Y(s)) - \int_0^{t-s} T_u\mathcal{L}f(Y(s))du
\end{aligned}$$

The function $(t, y) \mapsto T_t f(y)$ satisfies the Kolmogorov equation, which shows that the last three terms of the r.h.s. cancel. $\square$

4.2. Feller processes

In this section we consider a temporally homogeneous Feller process. The distribution $\mathbf{P}_x$ of the Markov process starting from $x \in E$ is described by the probability transition. Indeed, by the Chapman-Kolmogorov equation, for any $n \in \mathbb{N}^*$, for any $0 < t_1 < t_2 < ... < t_n < \infty$ and for any $A_1, ..., A_n \in \mathcal{B}(E)$, we have:

$$\begin{aligned} &\mathbf{P}_x(X_{t_1} \in A_1, ..., X_{t_n} \in A_n) \\ &= \int_{A_1} ... \int_{A_n} P_{t_1}(x, dx_1) P_{t_2-t_1}(x_1, dx_2) ... P_{t_n-t_{n-1}}(x_{n-1}, dx_n) \end{aligned}$$

We shall denote by $\mathbf{E}_x$ the expectation with respect to $\mathbf{P}_x$. We have in particular $\mathbf{E}_x[f(X_t)] = T_t f(x)$.

The Markov processes have been extensively studied and classified. This classification is based upon the notions of recurrence and transience. From this classification simple conditions for the ergodicity of the process can be deduced.

The main hypothesis that will insure most of the forthcoming results is that the transition probability has a positive, continuous density function:

Hypothesis D. *There exists a Borel measure μ on E supported by E, and a strictly positive function $p_t(x, y)$ continuous in $(t, x, y) \in \mathbb{R}^{+*} \times E^2$ such that the transition probability $P_t(x, dy)$ equals $p_t(x, y)\mu(dy)$.*

The two-state process of Example 4.5 satisfies Hypothesis D with $\mu =$ the counting measure on $E = \{-1, 1\}$.

4.2.1. Recurrent and transient properties The Feller process is called recurrent (in the sense of Harris) if

$$\mathbf{P}_x\left(\int_0^\infty \mathbf{1}_A(X_t)dt = \infty\right) = 1$$

for every $x \in E$ and $A \in \mathcal{B}(E)$ such that $\mu(A) > 0$. This means that the time spent by the process in any subset is infinite, or else that it comes back an infinite number of times in any subset.

The Feller process is called transient if

$$\sup_{x \in E} \mathbf{E}_x\left[\int_0^\infty \mathbf{1}_A(X_t)dt\right] < \infty$$

for every compact subset A of E. This means that the process spends only a finite time in any compact subset, so that it goes to infinity. The classification result holds as follows.

Proposition 4.7. *A Feller process that satisfies Hypothesis D is either recurrent or transient.*

The proof can be found in [60]. It consists in 1) introducing a suitable Markov process with discrete time parameters, 2) showing a similar transient-recurrent dichotomy for this process, and 3) applying this result to the Feller process.

4.2.2. Invariant measures Let π be a Borel measure on E. It is called an invariant measure of the Feller semigroup (T_t) if

$$\int_E T_t f(x)\pi(dx) = \int_E f(x)\pi(dx)$$

for every $t \geq 0$ and for any nonnegative function f with compact support. The following proposition (whose proof can be found for instance in [43]) shows that a recurrent process possesses a unique invariant measure.

Proposition 4.8. *Let (X_t) be a Feller process satisfying Hypothesis D. If it is recurrent, then there exists an invariant measure π which is unique up to a multiplicative constant and mutually absolutely continuous with respect to the reference Borel measure μ.*

In the case where the state space is compact, the process cannot be transient, hence it is recurrent and possesses an invariant measure, which has to be a bounded Borel measure.

Corollary 4.9. *Let (X_t) be a Feller process with compact state space satisfying Hypothesis D. Then it is recurrent and has a unique invariant probability measure.*

4.2.3. Ergodic properties The ergodic theorems that we are going to state are based on the following proposition.

Proposition 4.10. *Let (X_t) be a recurrent Feller process satisfying Hypothesis D. For every pair of probabilities π_1 and π_2, we have*

$$\lim_{t\to\infty} \|(\pi_1 - \pi_2)P_t\| = 0$$

where $\pi_1 P_t(.) = \int_E \pi_1(dx)P_t(x,.)$ and $\|.\|$ is here the norm of the total variations.

The proposition means that the process forgets its initial distribution as $t \to \infty$. Applying the proposition with $\pi_1 = \delta_x$ and $\pi_2 =$ the invariant measure of the process is the key to the proof of the ergodic theorem for Markov processes:

Proposition 4.11. *Let (X_t) be a Feller process satisfying Hypothesis D.*
1) If it is recurrent and has an invariant probability measure π, then the process is ergodic and for any $f \in L^\infty(E)$ and for any $x \in E$,

$$T_t f(x) \stackrel{t\to\infty}{\longrightarrow} \int_E f(y)\pi(dy)$$

2) If it is recurrent and has an infinite invariant measure, then for any $f \in L^\infty(E)$ such that $\int_E |f(y)|\pi(dy) < \infty$ and for any $x \in E$:

$$T_t f(x) \stackrel{t\to\infty}{\longrightarrow} 0$$

3) If it is transient, then for any $f \in L^\infty(E)$ with compact support and for any x:

$$T_t f(x) \stackrel{t\to\infty}{\longrightarrow} 0$$

The ergodic theorem has also a pathwise version:

Proposition 4.12. *Let (X_t) be a Feller process satisfying Hypothesis D.*
1) If it is recurrent and has an invariant probability measure π, then for any $f \in L^\infty(E)$ and for any $x \in E$,

$$\frac{1}{t}\int_0^t f(X_t) \stackrel{t\to\infty}{\longrightarrow} \int_E f(y)\pi(dy) \quad \mathbf{P}_x \textit{ almost surely}$$

2) If it is recurrent and has an infinite invariant measure, then for any $f \in L^\infty(E)$ such that $\int_E |f(y)|\pi(dy) < \infty$ and for any $x \in E$:

$$f(X_t) \stackrel{t\to\infty}{\longrightarrow} 0 \qquad \textit{in } L^1(\mathbf{P}_x)$$

3) If it is transient, then for any $f \in L^\infty(E)$ with compact support and for any x:

$$f(X_t) \stackrel{t\to\infty}{\longrightarrow} 0 \quad \mathbf{P}_x \textit{ almost surely}$$

The two-state process of Example 4.5 is ergodic and its invaraint probability meaure on $E = \{-1, 1\}$ is the uniform distribution $\pi(-1) = \pi(1) = 1/2$.

4.2.4. Resolvent equations and potential kernels We consider a Feller process satisfying Hypothesis D. We set, for $\alpha > 0$,

$$u_\alpha(x, y) = \int_0^\infty e^{-\alpha t} p_t(x, y) dt$$

which is a strictly lower semicontinuous function. The family of operators $(U_\alpha)_{\alpha>0}$

$$U_\alpha(x) = \int_E u_\alpha(x, y) f(y)\mu(dy) = \int_0^\infty e^{-\alpha t} T_t f(x) dt$$

is called the resolvent of the semigroup (T_t). It satisfies the resolvent equation:

$$(\alpha - \beta)U_\alpha U_\beta f + U_\alpha f - U_\beta f = 0 \tag{4.5}$$

for any $\alpha, \beta > 0$.

A kernel U (i.e. a family of Borel measures $\{U(x, .), x \in E\}$ such that $U(x, A)$ is measurable with respect to x for all $A \in \mathcal{B}(E)$) is called a potential kernel if it satisfies:

$$(I - \alpha U_\alpha)U f = U_\alpha f \tag{4.6}$$

for every $\alpha > 0$ and f such that $Uf \in \mathcal{C}_b(E)$. The function Uf is called a potential of f. Note that Eq. (4.6) is the limit form of Eq. (4.5) as $\beta \to 0$.

If the process is transient, then

$$U(x, A) := \int_0^\infty P_t(x, A) dt \tag{4.7}$$

is a potential kernel. Indeed if $f \in L^\infty(E)$ with compact support, $Uf \in \mathcal{C}_b(E)$ and satisfies $Uf = \lim_{\beta\to 0} U_\beta f$. Therefore it satisfies (4.6) since (U_α) satisfies the resolvent equation (4.5).

If the process is recurrent and has invariant measure π, then (4.7) diverges as soon as $\pi(A) > 0$. However it is possible to construct a kernel W such that $Wf \in \mathcal{C}_b(E)$ and satisfies (4.6) if $f \in L^\infty(E)$ with compact support satisfying $\int_E f(y)\pi(dy) = 0$. Such a kernel is called a recurrent potential kernel.

Proposition 4.13. *Let (X_t) be a recurrent Feller process satisfying Hypothesis D. There exists a recurrent potential W. Assume further that the process has an invariant probability measure π. If $f \in L^\infty(E)$ with compact support satisfies $\int_E f(y)\pi(dy) = 0$, then $\int_0^t T_s f(x)ds$ is bounded in (t,x) and*

$$\int_0^t T_s f(x)ds \stackrel{t\to\infty}{\longrightarrow} Wf(x) - \int_E Wf(y)\pi(dy)$$

Note that in the recurrent case with invariant probability measure, the recurrent potential kernel exists and is unique if one adds the condition that $\int_E Wf(y)d\pi(dy) = 0$ for all $f \in L^\infty(E)$ with compact support satisfying $\int_E f(y)\pi(dy) = 0$. Furthermore, if Wf belongs to the domain of the infinitesimal generator of the semigroup (T_t), then the potential kernel satisfies $QWf = (\alpha - U_\alpha^{-1})Wf = -f$. We can then state the important following corollary.

Corollary 4.14. *Let (X_t) be a recurrent Feller process with an invariant probability measure π satisfying Hypothesis D. If $f \in L^\infty(E)$ with compact support satisfies $\int_E f(y)\pi(dy) = 0$, and if Wf belongs to the domain of the infinitesimal generator of the semigroup (T_t), then Wf is a solution of the Poisson equation $Qu = -f$.*

In the case of the two-state process of Example 4.5, the Poisson equation reads as the linear system

$$Qu = -f \Longleftrightarrow \left\{ \begin{array}{l} -u(-1) + u(1) = -f(-1) \\ u(-1) - u(1) = -f(1) \end{array} \right.$$

It has a solution if and only if $f(-1) + f(1) = 0$, that is, $\int_E f(y)\pi(dy) = 0$ where $\pi(-1) = \pi(1) = 1/2$ is the invariant probability measure. Then

$$u = \int_0^\infty T_t f dt = \left(\begin{array}{cc} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{array} \right) \left(\begin{array}{c} f(-1) \\ f(1) \end{array} \right) = \frac{1}{2} \left(\begin{array}{c} f(-1) \\ f(1) \end{array} \right) = \frac{1}{2} f$$

is the unique solution such that $\int_E u(y)\pi(dy) = 0$, that is, $u(-1) + u(1) = 0$.

4.3. Diffusion Markov processes

Definition 4.15. Let P be a transition probability. It is associated to a diffusion process if:

1) $\forall x \in \mathbb{R}^d$, $\forall \varepsilon > 0$, $\displaystyle\int_{|y-x|>\varepsilon} P_t(x,dy) = o(t)$,
2) $\forall x \in \mathbb{R}^d$, $\forall \varepsilon > 0$, $\displaystyle\int_{|y-x|\leq\varepsilon} (y_i - x_i) P_t(x,dy) = b_i(s,x)t + o(t)$ for $i = 1,...,d$,
3) $\forall x \in \mathbb{R}^d$, $\forall \varepsilon > 0$, $\displaystyle\int_{|y-x|\leq\varepsilon} (y_i - x_i)(y_j - x_j) P_t(x,dy) = a_{ij}(x)t + o(t)$ for $i,j = 1,...,d$.

The functions b_i characterize the drift of the process, while a_{ij} describe the diffusive properties of the diffusion process. We are going to see that these functions completely characterize a diffusion Markov process with some additional technical hypotheses.

We shall assume the following hypotheses:

(H)
a_{ij} are of class $\mathcal{C}^2$ with bounded derivatives,
b_i are of class $\mathcal{C}^1$ with bounded derivatives,
a satisfies the strong ellipticity condition:
There exists some $\gamma > 0$ such that, for any x,
$\displaystyle\sum_{ij} a_{ij}(x)\xi_i\xi_j \geq \gamma \sum_i \xi_i^2.$

We introduce the second-order differential operator L:

$$Lf(x) = \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2 f(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x) \frac{\partial f(x)}{\partial x_i}.$$

Proposition 4.16. *Under Hypotheses H:*
1) There exists a unique Green's function $p_t(x,y)$ from $\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R}^d$ to $\mathbb{R}$ such that
$p_t(x,y) > 0$ $\forall t > 0$, $x, y \in \mathbb{R}^d$,
p is continuous on $\mathbb{R}^d \times \mathbb{R}^{+*} \times \mathbb{R}^d$,
p is $\mathcal{C}^2$ in x and $\mathcal{C}^1$ in t,
as a function of t and x, p satisfies $\frac{\partial p}{\partial t} = Lp$,
$\forall x$, $\int_{\mathbb{R}^d} p_t(x,y) f(y) dy \to f(x)$ as $t \to 0^+$ for any continuous and bounded function f.
2) There exists a unique positive function $\bar{p} \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R})$ such that $L^*\bar{p} \equiv 0$ up to a multiplicative constant. If $\bar{p}$ has finite mass, then we choose the normalization $\int_{\mathbb{R}^d} \bar{p}(y) dy = 1$.

We refer to [28] for the proof. This proposition is obtained by means of PDE tools. The first point is the key for the proof of the following proposition that describes a class of Markov processes satisfying Hypothesis D.

Proposition 4.17. *Under Hypotheses H, there exists a unique diffusion Markov process with drift b and diffusive matrix a. It is Feller and it satisfies hypothesis D with μ =the Lebesgue measure. It has continuous sample paths. L is its infinitesimal generator. The Green's function p is the transition probability density:*

$$P_t(x,A) = \int_A p_t(x,y) dy$$

which satisfies the Kolmogorov backward equation as a function of t and x:

$$\frac{\partial p}{\partial t} = Lp, \quad p_{t=0}(x,y) = \delta(x-y)$$

and the Kolmogorov forward equation as a function of t and y:

$$\frac{\partial p}{\partial t} = L^* p, \quad p_{t=0}(x,y) = \delta(x-y)$$

where L^* is the adjoint operator of L:

$$L^* f(x) = \sum_{i,j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j} \left(a_{ij}(x) f(x)\right) - \sum_{i=1}^{d} \frac{\partial}{\partial x_i} \left(b_i(x) f(x)\right).$$

Note that the Kolmogorov backward equation is the same as Eq. (4.1). The Kolmogorov forward equation is also known as the Fokker-Planck equation.

Remark: A diffusion process can be either transient or recurrent. A criterion that insures recurrence is that there exists $K > 0$, $\alpha \geq -1$, $r > 0$ such that

$$\sum_{j=1}^{d} b_j(x) \frac{x_j}{|x|} \leq -r|x|^{\alpha} \quad \forall |x| \geq K$$

(r should be large enough in the case $\alpha = -1$). In such a case the drift b plays the role of a trapping force that pushes the process to the origin [68].

The following proposition expresses the Fredholm alternative for the operator L in the ergodic case. It can be seen as a consequence of Proposition 4.13 and Corollary 4.14 in the framework of diffusion processes.

Proposition 4.18. *Under Hypotheses H, if the invariant measure has finite mass, then the diffusion process is ergodic. If moreover $f \in \mathcal{C}^2$ with compact support satisfies $\int_{\mathbb{R}^d} f(y)\bar{p}(y)dy = 0$ then there exists a unique function χ which satisfies $L\chi = -f$ and the centering condition $\int_{\mathbb{R}^d} \chi(y)\bar{p}(y)dy \equiv 0$. It is given by $\int_0^\infty T_s f(x) ds = \int_0^\infty \int_{\mathbb{R}^d} p_s(x,y) f(y) dy ds$.*

Example 1: The Brownian motion.
The d-dimensional Brownian motion is the $\mathbb{R}^d$-valued homogeneous Markov process with infinitesimal generator:

$$Q = \frac{1}{2}\Delta$$

where Δ is the standard Laplacian operator. The probability transition density $p_t(x,y)$ has the Gaussian density with mean x and covariance matrix $t\mathbf{I}$:

$$p_t(x,y) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|y-x|^2}{2t}\right)$$

An explicit calculation demonstrates that the Brownian motion is recurrent in dimensions 1 and 2 and transient in higher dimensions. The Brownian motion possesses an invariant measure which is simply the Lebesgue measure over $\mathbb{R}^d$ which has infinite mass. Whatever the dimension is, the Brownian motion is not ergodic, it escapes to infinity. It will not be a suitable process for describing a stationary ergodic medium. However if we add a trapping potential the Brownian motion becomes ergodic, as shown in the next example.

Example 2: The Ornstein-Uhlenbeck process.
It is a $\mathbb{R}$-valued homogeneous Markov process defined as the solution of the stochastic differential equation $dX_t = -\lambda X_t + dW_t$ that admits the closed form expression:

$$X_t = X_0 e^{-\lambda t} + \int_0^t e^{-\lambda(t-s)} dW_s$$

where W is a standard one-dimensional Brownian motion. It describes the evolution of the position of a diffusive particle trapped in a quadratic potential. It is the homogeneous Markov process with infinitesimal generator:

$$Q = \frac{1}{2}\frac{\partial^2}{\partial x^2} - \lambda x \frac{\partial}{\partial x}$$

The probability transition $p_t(x, y)$ has a Gaussian density with mean $xe^{-\lambda t}$ and variance $\sigma^2(t)$:

$$p_t(x, y) = \frac{1}{\sqrt{2\pi\sigma(t)^2}} \exp\left(-\frac{(y - xe^{-\lambda t})^2}{2\sigma^2(t)}\right), \quad \sigma^2(t) = \frac{1 - e^{-2\lambda t}}{2\lambda}$$

It is a recurrent ergodic process whose invariant probability measure has a density with respect to the Lebesgue measure:

$$\bar{p}(y) = \sqrt{\frac{\lambda}{\pi}} \exp\left(-\lambda y^2\right) \tag{4.8}$$

Note that it may be more comfortable in some circumstances to deal with a process with compact state space. For instance the process $Y_t = \arctan(X_t)$ with (X_t) an Ornstein-Uhlenbeck process is one of the models that can be used to describe the fluctuations of the parameters of a random medium.

4.4. The Poisson equation and the Fredholm alternative

We consider in this section an ergodic Feller Markov process with infinitesimal generator Q. The probability transitions converge to the invariant probability measure by the ergodic theorem. The resolution of the Poisson equation $Qu = f$ requires fast enough mixing. A set of hypotheses for rapid convergence is stated in the following proposition due to Doeblin [8]:

Proposition 4.19. *Assume that the Markov process has a compact state space E, it is Feller, and there exists $t_0 > 0$, $c > 0$ and a probability ν over E such that $P_t(x, A) \geq c\nu(A)$ for all $t \geq t_0$, $x \in E$, $A \in \mathcal{B}(E)$. Then there exists a unique invariant probability $\bar{p}$ and two positive numbers $c_1 > 0$ and $\delta > 0$ such that*

$$\sup_{x \in E} \sup_{A \in \mathcal{B}(E)} |P_t(x, A) - \bar{p}(A)| \leq c_1 e^{-\delta t}$$

A Feller process with compact state space E satisfying Hypothesis D possesses nice mixing properties. Indeed, if $t_0 > 0$, then $\delta := \inf_{x,y\in E} p(0,x;t_0,y)$ is positive as E is compact and p is continuous. Denoting by ν the uniform distribution over E we have $P_{t_0}(x,A) \geq \delta\nu(A)$ for all $A \in \mathcal{B}(E)$. This property also holds true for any time $t_0 + t$, $t \geq 0$, as the Chapman-Kolmogorov relation implies $P_{t+t_0}(x,A) = \int_E P_t(x,dz)P_{t_0}(z,A) \geq \delta \int_E \nu(A)P_t(x,dz) = \delta\nu(A)$. This proves the following corollary:

Corollary 4.20. *If a Feller process with compact state space satisfies Hypothesis D, then it fulfills the hypotheses and conclusion of Proposition 4.19.*

We end up the section by revisiting the problem of the Poisson equation in terms of the Fredholm alternative. We consider an ergodic Feller Markov process satisfying the Fredholm alternative. We first investigate the null spaces of the generator Q. Considering $T_t 1 = 1$, we have $Q1 = 0$, so that $1 \in \mathrm{Null}(Q)$. As a consequence $\mathrm{Null}(Q^*)$ is at least one-dimensional. As the process is ergodic $\mathrm{Null}(Q^*)$ is exactly one-dimensional. In other words there exists a unique invariant probability measure which satisfies $Q^*\bar{p} = 0$. In such conditions the probability transition converges to $\bar{p}$ as $t \to \infty$. The spectrum of Q^* gives the rate of forgetfulness, i.e. the mixing rate. For instance the existence of a spectral gap

$$\inf_{f, \int_E f d\bar{p} = 0} \frac{-\int_E fQf d\bar{p}}{\int_E f^2 d\bar{p}} > 0$$

insures an exponential convergence of $P_t(x,.)$ to $\bar{p}$. We now investigate the solutions of the Poisson equation $Qu = f$. Of course Q is not invertible since it has a nontrivial null space $\{1\}$. $\mathrm{Null}(Q^*)$ has dimension 1 and is generated by the invariant probability measure $\bar{p}$. By the Fredholm alternative, the Poisson equation admits a solution if f satisfies the orthogonality condition $f \perp \mathrm{Null}(Q^*)$ which means that f has zero mean $\mathbb{E}[f(X_0)] = 0$ where $\mathbb{E}$ is the expectation with respect to the distribution of the Markov process starting with the invariant measure $\bar{p}$:

$$\mathbb{E}[f(X_t)] = \int_E \bar{p}(dx)\mathbf{E}_x[f(X_t)]$$

In such a case, a particular solution of the Poisson equation $Qu = f$ is

$$u_0(x) = -\int_0^\infty ds T_s f(x)$$

Remember that the following expressions are equivalent:

$$T_s f(x) = \int_E P_s(x,dy)f(y) = \mathbf{E}_x[f(X_s)] = \mathbb{E}[f(X_s)|X_0 = x]$$

The convergence of the integral needs fast enough mixing. Also we have formally (i.e. if the integrals are finite)

$$Qu_0 = -\int_0^\infty ds QT_s f = -\int_0^\infty ds \frac{dT_s}{ds} f = -\left[T_s f\right]_0^\infty = f - \mathbb{E}[f(X_0)] = f$$

Note that another solution of the Poisson equation belongs to Null(Q) and is therefore a constant. Here we have $\mathbb{E}[u_0(X_0)] = 0$ as $\mathbb{E}[f(X_s)] = \mathbb{E}[f(X_0)] = 0$, so that we can claim that $-\int_0^\infty dsT_s : \mathcal{D} \to \mathcal{D}$ is the inverse of Q restricted to $\mathcal{D} = (\mathrm{Null}(Q^*))^\perp$.

4.5. Diffusion-approximation for Markov processes

Proposition 4.21. *Let us consider the system:*

$$\frac{dX^\varepsilon}{dt}(t) = \frac{1}{\varepsilon} F\left(X^\varepsilon(t), q(\frac{t}{\varepsilon^2}), \frac{t}{\varepsilon}\right), \qquad X^\varepsilon(0) = x_0 \in \mathbb{R}^d. \tag{4.9}$$

Assume that
1) q is a Markov, stationary, ergodic process on a compact space with generator Q, satisfying the Fredholm alternative.
2) F satisfies the centering condition: $\mathbb{E}[F(x, q(0), \tau)] = 0$ for all x and τ where $\mathbb{E}[.]$ denotes the expectation with respect to the invariant probability measure of q.
3) Assume that F is of class $\mathcal{C}^2$ and has bounded partial derivatives in x.
4) Assume that F is periodic with respect to τ with period T_0.

If $\varepsilon \to 0$ then the processes $(X^\varepsilon(t))_{t\geq 0}$ converge in distribution to the Markov diffusion process X with generator:

$$\mathcal{L}f(x) = \int_0^\infty du \left\langle \mathbb{E}\left[F(x, q(0), .).\nabla \left(F(x, q(u), .).\nabla f(x)\right)\right]\right\rangle_\tau . \tag{4.10}$$

where $\langle . \rangle_\tau$ stands for an averaging over a period in τ.

Remark 1: The infinitesimal generator also reads:

$$\mathcal{L} = \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(x) \frac{\partial}{\partial x_i}$$

with

$$a_{ij}(x) = \int_0^\infty du \left\langle \mathbb{E}\left[F_i(x, q(0), .) F_j(x, q(u), .)\right]\right\rangle_\tau .$$

$$b_i(x) = \sum_{j=1}^d \int_0^\infty du \left\langle \mathbb{E}\left[F_j(x, q(0), .) \frac{\partial F_i}{\partial x_j}(x, q(u), .)\right]\right\rangle_\tau .$$

Remark 2: We can also consider the case when F depends continuously on the macroscopic time variable t in (4.9). We then get the same result with the limit process described as a time-inhomogeneous Markov process with generator $\mathcal{L}_t$ defined as above.

Remark 3: The periodicity condition 4) can be removed if we assume instead of 4):

4bis) Assume that the limits

$$\lim_{T\to\infty} \frac{1}{T}\int_{t_0}^{T+t_0} d\tau \mathbb{E}\left[F_i(x,q(0),\tau)F_j(x,q(u),\tau)\right]$$
$$\lim_{T\to\infty} \frac{1}{T}\int_{t_0}^{T+t_0} d\tau \mathbb{E}\left[F_j(x,q(0),\tau)\frac{\partial F_i}{\partial x_j}(x,q(u),\tau)\right]$$

exist uniformly with respect to x in a compact, are independent on t_0 and are integrable with respect to u. We then denote

$$\begin{aligned} a_{ij}(x) &= \int_0^\infty du \left(\lim_{T\to\infty}\frac{1}{T}\int_0^T d\tau \mathbb{E}\left[F_i(x,q(0),\tau)F_j(x,q(u),\tau)\right]\right) \\ b_i(x) &= \sum_{j=1}^d \int_0^\infty du \left(\lim_{T\to\infty}\frac{1}{T}\int_0^T d\tau \mathbb{E}\left[F_j(x,q(0),\tau)\frac{\partial F_i}{\partial x_j}(x,q(u),\tau)\right]\right)\end{aligned}$$

and assume that a and b are smooth enough so that there exists a unique diffusion process with generator $\mathcal{L}$ (assume Hypothese H for instance).

In the following we give a formal proof of Proposition 4.21 which contains the key points. The strategy of the complete and rigorous proof is based of the theory of martingales and is sketched out in Appendix B. It also relies on the perturbed test function method.

The process $\bar{X}^\varepsilon(.) := (X^\varepsilon(.), q(./\varepsilon^2))$ is Markov with generator

$$\mathcal{L}^\varepsilon = \frac{1}{\varepsilon^2}Q + \frac{1}{\varepsilon}F(x,q,\frac{t}{\varepsilon}).\nabla.$$

The backward Kolmogorov equation for this process can be written as follows:

$$\frac{\partial V^\varepsilon}{\partial t} = \mathcal{L}^\varepsilon V^\varepsilon, \quad t\geq 0, \tag{4.11}$$

and we consider initial conditions that do not depend on q:

$$V^\varepsilon(t=0,q,x) = f(x)$$

where f is a smooth test function. We will solve (4.11) asymptotically as $\varepsilon\to 0$ by assuming the multiple scale expansion:

$$V^\varepsilon = \sum_{n=0}^\infty \varepsilon^n \, V_n(t,q,x,\tau)|_{\tau=t/\varepsilon} \tag{4.12}$$

To expand (4.11) in multiple scales we must replace $\frac{\partial}{\partial t}$ by $\frac{\partial}{\partial t}+\frac{1}{\varepsilon}\frac{\partial}{\partial \tau}$. Thus Eq. (4.11) becomes

$$\frac{\partial V^\varepsilon}{\partial t} = \frac{1}{\varepsilon^2}QV^\varepsilon + \frac{1}{\varepsilon}F.\nabla V^\varepsilon + \frac{1}{\varepsilon}\frac{\partial V^\varepsilon}{\partial \tau}, \quad t\geq 0, \tag{4.13}$$

Substitution of (4.12) into (4.13) yields a hierarchy of equations:

$$\begin{aligned} QV_0 &= 0 && (4.14) \\ QV_1 + F.\nabla V_0 + \frac{\partial V_0}{\partial \tau} &= 0 && (4.15) \\ QV_2 + F.\nabla V_1 + \frac{\partial V_1}{\partial \tau} - \frac{\partial V_0}{\partial t} &= 0 && (4.16) \end{aligned}$$

Taking into account the ergodicity of q Eq. (4.14) implies that V_0 does not depend on q. Taking the expectation of (4.15) the equation can be reduced to $\frac{\partial V_0}{\partial \tau} = 0$ which shows that V_0 does not depend on τ and V_1 should satisfy:

$$QV_1 = -F(x, q, \tau).\nabla V_0(t, x) \tag{4.17}$$

We have assumed that q is ergodic and satisfies the Fredholm alternative, so Q has an inverse on the subspace of the functions that have mean zero with respect to the invariant probability measure $\mathbb{P}$. The right-hand side of Eq. (4.17) belongs to this subspace, so we can solve this equation for V_1:

$$V_1(t, x, q, \tau) = -Q^{-1} F(x, q, \tau).\nabla V_0(t, x) \tag{4.18}$$

where $-Q^{-1} = \int_0^\infty dt T_t$. We now substitute (4.18) into (4.16) and take the expectation with respect to $\mathbb{P}$ and the averaging over a period in τ. We then see that V_0 must satisfy:

$$\frac{\partial V_0}{\partial t} = \left\langle \mathbb{E}\left[F.\nabla(-Q^{-1}F.\nabla V_0)\right]\right\rangle_\tau$$

This is the solvability condition for (4.16) and it is the limit backward Kolmogorov equation for the process X^ε, which takes the form:

$$\frac{\partial V_0}{\partial t} = \mathcal{L} V_0$$

with the limit infinitesimal generator:

$$\mathcal{L} = \int_0^\infty dt \left\langle \mathbb{E}\left[F.\nabla(T_t F.\nabla\right]\right\rangle_\tau$$

Using the probabilistic interpretation of the semigroup T_t we can express $\mathcal{L}$ as (4.10).

4.6. Bibliographic notes

In this section we have presented a self-contained summary of the basic tools of the theory of stochastic processes needed for modeling one-dimensional random media and for carrying out asymptotic analysis in various scaling limits. For an introduction to Markov processes we refer to the book by Breiman [12]. An advanced treatment of the theory of Markov processes, associated semigroups, and limit theorems is in the book by Ethier and Kurtz [24]. The martingale approach to diffusions and limit theorems is in the book of Stroock and Varadhan [65]. An introduction to stochastic calculus with Brownian motions can be found in the book by Oksendal [50] and a more advanced treatment in the book by Karatzas and Shreve [35]. The first diffusion-approximation results for random differential equations were given by Khasminskii in 1966 [37, 38]. The martingale approach to limit theorems for random differential equations was presented by Papanicolaou–Stroock–Varadhan in 1976 [54] and in Blankenship–Papanicolaou [10], including the perturbed-test-function method that is used extensively in these notes. Similar methods are used in homogenization [7] and in stochastic stability and control [44]. We also refer to a recent series of papers by Pardoux and Veretennikov [56] for an extended treatment of Poisson equations and diffusion approximation. We have only considered Markovian models of random equations here for simplicity. The results, however, can be extended to a large class of mixing processes as is done in [36] and in the books by Ethier–Kurtz [24] and by Kushner [44].

5. Spreading of a pulse traveling through a random medium

We are interested in the following question: how the shape of a pulse has been modified when it emerges from a one-dimensional random medium ? This analysis takes place in the general framework, based on separation of scales, introduced by G. Papanicolaou and his co-authors (see for instance [15] for the one-dimensional case or [4] for the three-dimensional case). We consider here the problem of acoustic propagation when the incident pulse wavelength is long compared to the correlation length of the random inhomogeneities but short compared to the size of the slab.

5.1. The boundary-value problem

The O'Doherty-Anstey theory states that a pulse traveling through a one-dimensional random medium retains its shape up to a low spreading; furthermore, its shape is deterministic when observed from the point of view of an observer traveling at the same random speed as the wave while it is stochastic when the observer's speed is the mean speed of the wave. The main result of this section consists in a complete description of the asymptotic law of the emerging pulse: we prove a limit theorem which shows that the pulse spreads in a deterministic way and we identify the statistical distribution of the random time delay. For simplicity we present the proof in the one-dimensional case with no macroscopic variations of the medium and the noise only appearing in the bulk modulus of the medium. We refer to [19, 26] for the result for one-dimensional media with macroscopic variations.

We consider an acoustic wave traveling in a one-dimensional random medium located in the region $0 \leq z \leq L$, satisfying the linear conservation laws:

$$\left\{ \begin{array}{ll} \rho^\varepsilon(z) \dfrac{\partial u^\varepsilon}{\partial t}(t,z) + \dfrac{\partial p^\varepsilon}{\partial z}(t,z) & = 0 \\ \dfrac{\partial p^\varepsilon}{\partial t}(t,z) + \kappa^\varepsilon(z) \dfrac{\partial u^\varepsilon}{\partial z}(t,z) & = 0 \end{array} \right. \tag{5.1}$$

Here $u^\varepsilon(t,z)$ and $p^\varepsilon(t,z)$ are respectively the speed and pressure of the wave, whereas $\rho^\varepsilon(z)$ and $\kappa^\varepsilon(z)$ are the density and bulk modulus of the medium. In our simplified model we suppose that the medium parameters are given by

$$\begin{array}{rcl} \dfrac{1}{\kappa^\varepsilon(z)} & = & \left\{ \begin{array}{ll} 1 + \eta(z/\varepsilon^2) & \text{for } \; z \in [0,L], \\ 1 & \text{for } \; z \in (-\infty,0) \cup (L,\infty), \end{array} \right. \\ \rho^\varepsilon(z) & = & 1 \;\; \text{for all } \; z, \end{array}$$

where $\eta(z/\varepsilon^2)$ is the rapidly varying random coefficient describing the inhomogeneities. Since these coefficients are positive we suppose that $|\eta|$ is less than a constant strictly less than 1. Furthermore we assume that $\eta(z)$ is stationary, centered and mixing enough. We may think for instance that $\eta(z) = f(X_z)$ where $(X_z)_{z \geq 0}$ is the Ornstein-Uhlenbeck process and f is a smooth function from $\mathbb{R}$ to $[-\delta, \delta]$, $\delta < 1$, satisfying $\int f(y)\bar{p}(y)dy$ where $\bar{p}$ is the invariant probability density (4.8) of the Ornstein Uhlenbeck process.

In order to specify our boundary conditions we introduce the right- and left-going waves $A^\varepsilon = u^\varepsilon + p^\varepsilon$ and $B^\varepsilon = u^\varepsilon - p^\varepsilon$ which satisfy the following system of equations:

$$\frac{\partial}{\partial z} \left(\begin{array}{c} A^\varepsilon \\ B^\varepsilon \end{array} \right) = \left(\left(\begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right) + \frac{1}{2}\eta(\frac{z}{\varepsilon^2}) \left(\begin{array}{cc} -1 & 1 \\ -1 & 1 \end{array} \right) \right) \frac{\partial}{\partial t} \left(\begin{array}{c} A^\varepsilon \\ B^\varepsilon \end{array} \right) \tag{5.2}$$

The slab of medium we are considering is located in the region $0 \leq z \leq L$ and at $t = 0$ an incident pulse is generated at the interface $z = 0$ between the random medium and the outside homogeneous medium. According to previous works [15] or [4] we choose a pulse which is broad compared to the size of the random inhomogeneities but short compared to the macroscopic scale of the medium. There is no wave entering the medium at $z = L$ (see Fig. 1). Therefore the system (5.2) is complemented with the boundary conditions:

$$A^\varepsilon(t,0) = f(\frac{t}{\varepsilon}), \qquad B^\varepsilon(t,L) = 0 \tag{5.3}$$

where f is a function whose Fourier transform $\hat{f}$ belongs to $L^1 \cap L^2$.

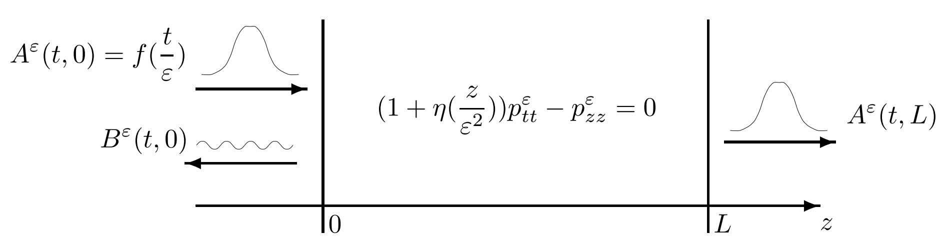


Figure 1: Spreading of a pulse.

5.2. Asymptotic analysis of the transmitted pulse

We are interested in the transmitted pulse $A^\varepsilon(t,L)$ around the arrival time $t=L$ and in the same scale as the entering pulse $A^\varepsilon(t,0)$; therefore the quantity of interest is the windowed signal $A^\varepsilon(L+\varepsilon s, L)_{s\in(-\infty,\infty)}$ which will be given by the following centered and rescaled quantities:

$$a^\varepsilon(s,z) = A^\varepsilon(z+\varepsilon s, z), \qquad b^\varepsilon(s,z) = B^\varepsilon(-z+\varepsilon s, z) \tag{5.4}$$

The solution of (5.2)+(5.3) takes place in an infinite-dimensional space because of the variable t. So we perform the scaled Fourier transforms:

$$\hat{a}^\varepsilon(\omega,z) = \int e^{i\omega s} a^\varepsilon(s,z)ds, \qquad \hat{b}^\varepsilon(\omega,z) = \int e^{i\omega s} b^\varepsilon(s,z)ds$$

In the frequency domain, with the change of variables (5.4), Eq. (5.2) becomes:

$$\frac{d}{dz}\begin{pmatrix} \hat{a}^\varepsilon \\ \hat{b}^\varepsilon \end{pmatrix} = \frac{i\omega}{2\varepsilon}\eta\left(\frac{z}{\varepsilon^2}\right)\begin{pmatrix} 1 & -e^{-2i\omega\frac{z}{\varepsilon}} \\ e^{2i\omega\frac{z}{\varepsilon}} & -1 \end{pmatrix}\begin{pmatrix} \hat{a}^\varepsilon \\ \hat{b}^\varepsilon \end{pmatrix} \tag{5.5}$$

with the boundary conditions

$$\hat{a}^\varepsilon(\omega,0) = \hat{f}(\omega), \qquad \hat{b}^\varepsilon(\omega,L) = 0 \tag{5.6}$$

We obtain the following representation for the transmitted pulse:

$$A^\varepsilon(L+\varepsilon s, L) = a^\varepsilon(s,L) = \frac{1}{2\pi}\int e^{-i\omega s}\hat{a}^\varepsilon(\omega,L)d\omega \tag{5.7}$$

The 2×2 propagator matrix $\mathbf{P}^\varepsilon(\omega,z)$ is the solution of equation (5.5) with the initial condition $\mathbf{P}^\varepsilon(\omega,0)=\mathbf{I}$. The process $(\hat{a}^\varepsilon(\omega,z),\hat{b}^\varepsilon(\omega,z))$ can be deduced from $(\hat{a}^\varepsilon(\omega,0),\hat{b}^\varepsilon(\omega,0))$ through the identity:

$$\begin{pmatrix} \hat{a}^\varepsilon(\omega,z) \\ \hat{b}^\varepsilon(\omega,z) \end{pmatrix} = \mathbf{P}^\varepsilon(\omega,z)\begin{pmatrix} \hat{a}^\varepsilon(\omega,0) \\ \hat{b}^\varepsilon(\omega,0) \end{pmatrix} \tag{5.8}$$

The structure of the propagator matrix can be exhibited. If $(\hat{\alpha}^\varepsilon,\hat{\beta}^\varepsilon)^T$ is a solution of (5.5) with the initial condition $\hat{\alpha}^\varepsilon(0)=1$, $\hat{\beta}^\varepsilon(0)=0$, then $(\overline{\hat{\beta}^\varepsilon},\overline{\hat{\alpha}^\varepsilon})^T$ is another solution linearly independent from the previous one [1] and we can thus write $\mathbf{P}^\varepsilon(\omega,z)$ as:

$$\mathbf{P}^\varepsilon(\omega,z) = \begin{pmatrix} \hat{\alpha}^\varepsilon(\omega,z) & \overline{\hat{\beta}^\varepsilon}(\omega,z) \\ \hat{\beta}^\varepsilon(\omega,z) & \overline{\hat{\alpha}^\varepsilon}(\omega,z) \end{pmatrix} \tag{5.9}$$

The trace of the matrix appearing in the linear equation (5.5) being 0, we deduce that the determinant of $\mathbf{P}^\varepsilon(\omega,z)$ is constant and equal to 1 which implies that $|\hat{\alpha}^\varepsilon(\omega,z)|^2 - |\hat{\beta}^\varepsilon(\omega,z)|^2 = 1$ for every z. Using (5.8) and (5.9) at $z=L$ and

[1] Here the bar $\bar{\cdot}$ stands for complex conjugation

boundary conditions (5.6) we deduce that:

$$\hat{a}^\varepsilon(\omega, L) = \frac{1}{\overline{\hat{\alpha}^\varepsilon}(\omega, L)}\hat{f}(\omega), \qquad \hat{b}^\varepsilon(\omega, 0) = -\frac{\hat{\beta}^\varepsilon(\omega, L)}{\overline{\hat{\alpha}^\varepsilon}(\omega, L)}\hat{f}(\omega) \tag{5.10}$$

In particular we have the following relation of conservation of energy:

$$|\hat{a}^\varepsilon(\omega, L)|^2 + |\hat{b}^\varepsilon(\omega, 0)|^2 = |\hat{f}(\omega)|^2 \tag{5.11}$$

Lemma 5.1. *The transmitted pulse* $((a^\varepsilon(s, L))_{-\infty<s<\infty})_{\varepsilon>0}$ *is a tight (i.e. weakly compact) family in the space of continuous trajectories equipped with the sup norm.*

Proof. We must show that, for any $\delta > 0$, there exists a compact subset K of the space of continuous bounded functions such that:

$$\sup_{\varepsilon>0} \mathbb{P}(a^\varepsilon(\cdot, L) \in K) \geq 1 - \delta$$

On the one hand (5.11) yields that $a^\varepsilon(s, L)$ is uniformly bounded by:

$$|a^\varepsilon(s, L)| \leq \frac{1}{2\pi}\int |\hat{f}(\omega)| d\omega$$

On the other hand the modulus of continuity

$$M^\varepsilon(\delta) = \sup_{|s_1 - s_2| \leq \delta} |a^\varepsilon(s_1, L) - a^\varepsilon(s_2, L)|$$

is bounded by

$$M^\varepsilon(\delta) \leq \int \sup_{|s_1 - s_2| \leq \delta} |1 - \exp(i\omega(s_1 - s_2))||\hat{f}(\omega)| d\omega$$

which goes to zero as δ goes to zero uniformly with respect to ε. □

Moreover the finite-dimensional distributions will be characterized by the moments

$$\mathbb{E}[a^\varepsilon(s_1, L)^{p_1} \dots a^\varepsilon(s_k, L)^{p_k}] \tag{5.12}$$

for every real numbers $s_1 < \dots < s_k$ and every integers $p_1, \dots, p_k$.

Let us first address the first moment. Using the representation (5.7) and (5.10) the expectation of $a^\varepsilon(s, L)$ reads:

$$\mathbb{E}[a^\varepsilon(s, L)] = \frac{1}{2\pi}\int e^{-i\omega s}\hat{f}(\omega)\mathbb{E}[\hat{\alpha}^\varepsilon(\omega, L)^{-1}] d\omega$$

We fix some ω and denote $X_1^\varepsilon(z) = \mathrm{Re}(\hat{\alpha}^\varepsilon(z, \omega))$, $X_2^\varepsilon(z) = \mathrm{Im}(\hat{\alpha}^\varepsilon(z, \omega))$, $X_3^\varepsilon(z) = \mathrm{Re}(\hat{\beta}^\varepsilon(z, \omega))$ and $X_4^\varepsilon(z) = \mathrm{Im}(\hat{\beta}^\varepsilon(z, \omega))$. The $\mathbb{R}^4$-valued process X^ε satisfies the linear differential equation

$$\frac{dX^\varepsilon(z)}{dz} = \frac{1}{\varepsilon}\mathbf{F}_\omega\left(\eta(\frac{z}{\varepsilon^2}), \frac{z}{\varepsilon}\right) X^\varepsilon(z), \tag{5.13}$$

with the initial conditions $X_1^\varepsilon(0) = 1$ and $X_{j'}^\varepsilon(0) = 0$ if $j' = 2, 3, 4$, where

$$\mathbf{F}_\omega(\eta, h) = \frac{\omega\eta}{2} \begin{pmatrix} 0 & -1 & -\sin(2\omega h) & \cos(2\omega h) \\ 1 & 0 & -\cos(2\omega h) & -\sin(2\omega h) \\ -\sin(2\omega h) & -\cos(2\omega h) & 0 & 1 \\ \cos(2\omega h) & -\sin(2\omega h) & -1 & 0 \end{pmatrix}$$

Applying the approximation-diffusion theorem 4.21, we get that X^ε converges in distribution to a Markov diffusion process X characterized by an infinitesimal generator denoted by $\mathcal{L}$:

$$\mathcal{L} = \sum_{i,j=1}^{4} a_{ij}(X) \frac{\partial^2}{\partial X_i \partial X_j} + \sum_{i=1}^{4} b_i(X) \frac{\partial}{\partial X_i}$$

whose entries b_i are all vanishing and:

$$\begin{aligned} a_{11} &= \frac{\gamma\omega^2}{8}\left(X_2^2 + \frac{X_3^2 + X_4^2}{2}\right), \\ a_{22} &= \frac{\gamma\omega^2}{8}\left(X_1^2 + \frac{X_3^2 + X_4^2}{2}\right), \\ a_{12} &= a_{21} = \frac{\gamma\omega^2}{8}\left(-X_1 X_2\right), \end{aligned}$$

where

$$\gamma = \int_{-\infty}^{\infty} \mathbb{E}[\eta(0)\eta(z)]dz = 2\int_0^{\infty} \mathbb{E}[\eta(0)\eta(z)]dz$$

The expectation $\phi(z) = \mathbb{E}[(X_1(z) - iX_2(z))^{-1}]$ satisfies the equation

$$\frac{d\phi}{dz} = \mathcal{L}\phi = -\frac{\gamma\omega^2}{4}\phi, \quad \phi(0) = 1.$$

The solution of this ODE is: $\phi(L) = \exp(-\gamma\omega^2 L/4)$. The expectation of $\hat{\alpha}^\varepsilon(\omega, L)^{-1}$ thus converges to $\phi(L)$ (remember that $|\hat{\alpha}^\varepsilon| \geq 1$). By Lebesgue's theorem (remember that $\hat{f} \in L^1$) the expectation of $a^\varepsilon(s, L)$ converges to:

$$\mathbb{E}[a^\varepsilon(s, L)] \overset{\varepsilon \to 0}{\longrightarrow} \frac{1}{2\pi} \int e^{-i\omega s} \hat{f}(\omega) \exp(-\gamma\omega^2 L/4) d\omega$$

Let us now consider the general moment (5.12). Using the representation (5.7) for each factor a^ε, these moments can be written as multiple integrals over $n =$

$\sum_{j=1}^{k} p_j$ frequencies:

$$\begin{aligned}\mathbb{E}\left[a^\varepsilon(s_1,L)^{p_1}...a^\varepsilon(s_k,L)^{p_k}\right] &= \frac{1}{(2\pi)^n}\int\cdots\int \prod_{\substack{1\leq j\leq k\\ 1\leq l\leq p_j}} \hat{f}(\omega_{j,l})e^{-i\omega_{j,l}s_j} \\ &\quad\times\mathbb{E}\left[\prod_{\substack{1\leq j\leq k\\ 1\leq l\leq p_j}} \hat{\alpha}^\varepsilon(\omega_{j,l},L)^{-1}\right]\prod_{\substack{1\leq j\leq k\\ 1\leq l\leq p_j}} d\omega_{j,l}\end{aligned}$$

The dependency in ε and in the randomness only appears through the quantity $\mathbb{E}[\hat{\alpha}^\varepsilon(\omega_1,L)^{-1}\ldots\hat{\alpha}^\varepsilon(\omega_n,L)^{-1}]$. Our problem is now to find the limit, as ε goes to 0, of these moments for n distinct frequencies. In other words we want to study the limit in distribution of $(\hat{\alpha}^\varepsilon(\omega_1,L),\ldots,\hat{\alpha}^\varepsilon(\omega_n,L))$ which results once again from the application of a diffusion-approximation theorem. We define the n-dimensional propagator as the $2n\times 2n$ matrix

$$\mathbf{P}^\varepsilon(\omega_1,\omega_2,\ldots,\omega_n,z) = \oplus_{j=1}^N \mathbf{P}^\varepsilon(\omega_j,z)$$

which satisfies an equation similar to (5.5) with $\mathbf{P}^\varepsilon(\omega,z=0)=\mathbf{I}$.

In order to be allowed to apply the diffusion-approximation theorem, we have to take care to consider separately the real and imaginary parts of each coefficient $\hat{\alpha}^\varepsilon$ and $\hat{\beta}^\varepsilon$, so that we actually deal with a system with $4n$ linear differential equations. Denoting $X^\varepsilon_{4j+1}(z)=\mathrm{Re}(\hat{\alpha}^\varepsilon(\omega_j,z))$, $X^\varepsilon_{4j+2}(z)=\mathrm{Im}(\hat{\alpha}^\varepsilon(\omega_j,z))$, $X^\varepsilon_{4j+3}(z)=\mathrm{Re}(\hat{\beta}^\varepsilon(\omega_j,z))$ and $X^\varepsilon_{4j+4}(z)=\mathrm{Im}(\hat{\beta}^\varepsilon(\omega_j,z))$, $j=1,\ldots,n$, the $\mathbb{R}^{4n}$-valued process X^ε satisfies the linear differential equation

$$\frac{dX^\varepsilon(z)}{dz} = \frac{1}{\varepsilon}\mathbf{F}\left(\eta(\frac{z}{\varepsilon^2}),\frac{z}{\varepsilon}\right)X^\varepsilon(z), \tag{5.14}$$

with the initial conditions $X^\varepsilon_{4j+j'}(0)=1$ if $j'=1$, $X^\varepsilon_{4j+j'}(0)=0$ if $j'=2,3,4$, where

$$\mathbf{F}(\eta,h) = \oplus_{j=1}^n \mathbf{F}_{\omega_j}(\eta,h)$$

Applying the approximation-diffusion theorem 4.21, we get that X^ε converges in distribution to a Markov diffusion process X characterized by an infinitesimal generator denoted by $\mathcal{L}$:

$$\mathcal{L} = \sum_{i,i'=1}^n\sum_{j,j'=1}^4 a_{4i+j,4i'+j'}(X)\frac{\partial^2}{\partial X_{4i+j}\partial X_{4i'+j'}}$$

$$\begin{aligned}a_{4i+1,4i+1} &= \frac{\gamma\omega_i^2}{8}\left(X_{4i+2}^2+\frac{X_{4i+3}^2+X_{4i+4}^2}{2}\right),\\ a_{4i+2,4i+2} &= \frac{\gamma\omega_i^2}{8}\left(X_{4i+1}^2+\frac{X_{4i+3}^2+X_{4i+4}^2}{2}\right),\\ a_{4i+1,4i+2} &= a_{4i+2,4i+1} = \frac{\gamma\omega_i^2}{8}\left(-X_{4i+1}X_{4i+2}\right),\end{aligned}$$

and for $i \neq i'$:

$$\begin{aligned}
a_{4i+1,4i'+1} &= \frac{\gamma\omega_i\omega_{i'}}{8}\left(X_{4i+2}X_{4i'+2}\right), \\
a_{4i+2,4i'+2} &= \frac{\gamma\omega_i\omega_{i'}}{8}\left(X_{4i+1}X_{4i'+1}\right), \\
a_{4i+1,4i'+2} &= a_{4i'+2,4i+1} = \frac{\gamma\omega_i\omega_{i'}}{8}\left(-X_{4i+2}X_{4i'+1}\right).
\end{aligned}$$

The quantity of interest $\mathbb{E}[\hat{\alpha}^\varepsilon(\omega_1,z)^{-1}\dots\hat{\alpha}^\varepsilon(\omega_n,z)^{-1}]$ is denoted by $\phi^\varepsilon(z)$. An application of the infinitesimal generator to the expectation

$$\mathbb{E}\left[\prod_{j=1}^{n}(X_{4j+1} - iX_{4j+2})^{-1}\right]$$

gives the following equation for $\phi(z) = \lim_{\varepsilon\to 0}\phi^\varepsilon(z)$:

$$\frac{d\phi(z)}{dz} = -\frac{2\gamma\sum_k \omega_k^2 + \gamma\sum_{k\neq l}\omega_k\omega_l}{8}\phi(z)$$

with the initial condition $\phi(0) = 1$. This linear equation has a unique solution but instead of solving it and computing explicitly our moments one can easily see that it is also satisfied by $\tilde{\phi}(z) = \mathbb{E}\left[\prod_{k=1}^{n}\tilde{T}(\omega_k,z)\right]$ where

$$\tilde{T}(\omega,z) = \exp(i\frac{\omega\sqrt{\gamma}}{2}W_z - \frac{\omega^2\gamma}{8}z)$$

and (W_z) is a standard one-dimensional Brownian motion (W_z is a Gaussian random variable with zero-mean and variance z). Therefore $\phi(L) = \tilde{\phi}(L)$ and using (5.7) the limit in law of $a^\varepsilon(s,L)$ is equal to $(2\pi)^{-1}\int e^{-i\omega s}\hat{f}(\omega)\tilde{T}(\omega,L)d\omega$. Interpreting $\frac{\omega\sqrt{\gamma}}{2}W_L$ as a random phase and $\exp(-\frac{\omega^2\gamma}{8}L)$ as the Fourier transform of the centered Gaussian density with variance $\frac{\gamma L}{4}$ denoted by $G_{\frac{\gamma L}{4}}$:

$$G_{\frac{\gamma L}{4}}(s) = \frac{\sqrt{2}}{\sqrt{\pi\gamma L}}\exp\left(-\frac{2s^2}{\gamma L}\right)$$

we get the main result of this section:

Proposition 5.2. *The process $(a^\varepsilon(s,L))_{s\in(-\infty,\infty)}$ converges in distribution in the space of the continuous functions to $(\bar{a}(s,L))_{s\in(-\infty,\infty)}$*

$$\bar{a}(s,L) = f * G_{\frac{\gamma L}{4}}\left(s - \frac{\sqrt{\gamma}}{2}W_L\right) \tag{5.15}$$

which means that the initial pulse f spreads in a deterministic way through the convolution by a Gaussian density and a random Gaussian centering appears through the Brownian motion W_L.

$\bar{a}$ is the asymptotic pulse front. The energy of $\bar{a}$ is non-random and given by:

$$\mathcal{E}_{coh} = \int |f * G_{\frac{\gamma L}{4}}(s)|^2 ds$$

If $f(t)$ is narrowband around a high-carrier frequency:

$$f(t) = \cos(\omega_0 t)\exp(-t^2\delta\omega^2), \quad \delta\omega \ll \omega_0$$

then it is found that the energy of the front decays exponentially:

$$\begin{aligned}
\mathcal{E}_{coh}(L) &= \mathcal{E}_{coh}(0)\frac{1}{\sqrt{1+\gamma\delta\omega^2 L}}\exp\left(-\frac{\gamma\omega_0^2 L}{4(1+\gamma\delta\omega^2 L)}\right) \\
&\simeq \mathcal{E}_{coh}(0)\exp\left(-\frac{\gamma\omega_0^2 L}{4}\right)
\end{aligned} \tag{5.16}$$

We end this section by the following remarks:

- The previous analysis has been done at L fixed. It is not difficult to generalize it to the convergence in distribution of $a^\varepsilon(s, L)$ as a process in s and L (see [19] for details). The limit is again given by (5.15) which means that the random centering of the spread pulse follows the trajectory of a Brownian motion as the pulse travels into the medium.

- In the ε-scale, the energy entering the medium at $z = 0$ is equal to $\int |f(s)|^2 ds$. The energy exiting the medium at $z = L$, in a coherent way around time $t = L$ in the ε-scale, is equal to $\int |f * G_{\frac{\gamma L}{4}}(s)|^2 ds$ which is strictly less than $\int |f(s)|^2 ds$. We may ask the following question: do we have a part of the missing energy exiting the medium in a coherent way somewhere else or at a different time? In other words what is the limit in distribution of $A^\varepsilon(L + t_0 + \varepsilon s, L)$ for $t_0 \neq 0$ (energy exiting at $z = L$) or $B^\varepsilon(t_0 + \varepsilon s, 0)$ (energy reflected at $z = 0$). A similar analysis shows that these two processes (in s) vanish as ε goes to 0 (see [19] for details). This means that there is no other coherent energy in the ε-scale exiting the slab $[0, L]$.

5.3. Bibliographic notes

The stabilization of the wave front in one-dimensional random media was first noted by O'Doherty and Anstey in a geophysical context [49]. A time-domain integral equation approach to pulse stabilization is given in [13, 16]. The frequency-domain approach presented in this section follows [19]. The use of the martingale representation for the transmission coefficient is new. An approach using the Riccati equation of Section 7 is in [46]. Generalizations to three-dimensional randomly layered media are presented in [26, Chapter 14].

6. Energy transmission through a random medium

6.1. Transmission of monochromatic waves

This section is devoted to the propagation of monochromatic waves. This is a very natural approach since any wave can be described as the superposition of such elementary wavetrains by Fourier transform. Let $\hat{p}^\varepsilon(z)$ be the amplitude at $z \in \mathbb{R}$ of a monochromatic wave $p^\varepsilon(t,z) = \exp(-i\omega^\varepsilon t)\hat{p}^\varepsilon(z)$ traveling in the one-dimensional medium described in Fig. 2. The medium is homogeneous outside the slab $[0,L]$ and the wave p^ε obeys the wave equation $p^\varepsilon_{tt} - p^\varepsilon_{zz} = 0$. Accordingly $\hat{p}^\varepsilon$ satisfies

$$\hat{p}^\varepsilon_{zz} + {\omega^\varepsilon}^2 \hat{p}^\varepsilon = 0$$

so that it has the form

$$\hat{p}^\varepsilon(z) = e^{i\omega^\varepsilon z} - R^\varepsilon_\omega e^{-i\omega^\varepsilon z} \quad \text{for} \quad z \le 0 \tag{6.1}$$

and

$$\hat{p}^\varepsilon(z) = T^\varepsilon_\omega e^{i\omega^\varepsilon z} \quad \text{for} \quad z \ge L. \tag{6.2}$$

The complex-valued random variables R^ε_ω and T^ε_ω are the reflection and transmission coefficients, respectively. They depend on the particular realization of η^ε, the wavenumber ω^ε and the length of the slab L.
Inside the slab $[0,L]$ the perturbation is nonzero. It is the realization of a random, stationary, ergodic, and zero-mean process η^ε. The dimensionless parameter $\varepsilon > 0$ characterizes the scale of the fluctuations of the random medium as well as the wavelength of the wave. We assume that:

$$\eta^\varepsilon(z) = \eta(\frac{z}{\varepsilon^2}), \quad \omega^\varepsilon = \frac{\omega}{\varepsilon}$$

which means that the correlation length of the medium $\sim \varepsilon^2$ is much smaller than the wavelength $\sim \varepsilon$ which is much smaller than the length of the medium ~ 1.

The scalar field $\hat{p}^\varepsilon$ satisfies, for $z \in [0,L]$:

$$\hat{p}^\varepsilon_{zz} + {\omega^\varepsilon}^2(1+\eta^\varepsilon(z))\hat{p}^\varepsilon = 0. \tag{6.3}$$

The continuity of $\hat{p}^\varepsilon$ and $\hat{p}^\varepsilon_z$ at $z=0$ and $z=L$ implies that the solution $\hat{p}^\varepsilon$ satisfies the two-point boundary conditions:

$$i\omega^\varepsilon \hat{p}^\varepsilon + \hat{p}^\varepsilon_z = 2i\omega^\varepsilon \text{ at } z=0, \qquad i\omega^\varepsilon \hat{p}^\varepsilon - \hat{p}^\varepsilon_z = 0 \text{ at } z=L. \tag{6.4}$$

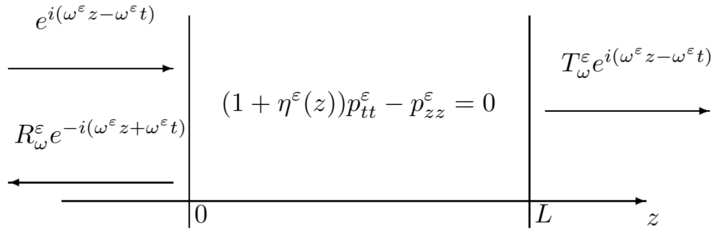


Figure 2: Scattering of a monochromatic pulse.

The following statements hold true when the perturbation η is a stationary process that has finite moments of all orders and is rapidly mixing. We may think for instance that η is a Markov, stationary, ergodic process on a compact space satisfying the Fredholm alternative.

Proposition 6.1. *There exists a length L^ε_{loc} such that, with probability one:*

$$\lim_{L\to\infty}\frac{1}{L}\ln|T^\varepsilon_\omega(L)|^2=-\frac{1}{L^\varepsilon_{loc}}. \tag{6.5}$$

This length can be expanded as powers of ε:

$$\frac{1}{L^\varepsilon_{loc}}=\frac{\gamma\omega^2}{4}+O(\varepsilon),\qquad \gamma:=2\int_0^\infty \mathbb{E}[\eta(0)\eta(z)]dz. \tag{6.6}$$

Proof. The study of the exponential behavior of the power transmission coefficient $|T^\varepsilon_\omega|^2$ can be divided into two steps. First the localization length is shown to be equal to the inverse of the Lyapunov exponent associated to the random oscillator $v_{zz}+\omega^{\varepsilon 2}(1+\eta^\varepsilon(z))v=0$. Second the expansion of the Lyapunov exponent of the random oscillator is computed.

We first transform the boundary value problem (6.3)+(6.4) into an initial value problem. This step is similar to the analysis carried out in Section 5. Inside the perturbed slab we expand $\hat{p}^\varepsilon$ in the form

$$\hat{p}^\varepsilon(\omega,z)=\frac{1}{2}\left(\hat{a}^\varepsilon(\omega,z)e^{i\omega^\varepsilon z}-\hat{b}^\varepsilon(\omega,z)e^{-i\omega^\varepsilon z}\right), \tag{6.7}$$

where $\hat{a}^\varepsilon$ and $\hat{b}^\varepsilon$ are respectively the right-going and left-going modes:

$$\hat{a}^\varepsilon=\frac{i\omega^\varepsilon\hat{p}^\varepsilon+\hat{p}^\varepsilon_z}{i\omega^\varepsilon}e^{-i\omega^\varepsilon z},\qquad \hat{b}^\varepsilon=\frac{-i\omega^\varepsilon\hat{p}^\varepsilon+\hat{p}^\varepsilon_z}{i\omega^\varepsilon}e^{i\omega^\varepsilon z}$$

The process $(\hat{a}^\varepsilon,\hat{b}^\varepsilon)$ is solution of

$$\frac{d}{dz}\begin{pmatrix}\hat{a}^\varepsilon\\ \hat{b}^\varepsilon\end{pmatrix}=\mathbf{H}^\varepsilon(\omega,z)\begin{pmatrix}\hat{a}^\varepsilon\\ \hat{b}^\varepsilon\end{pmatrix}, \tag{6.8}$$

$$\begin{aligned}\mathbf{H}^\varepsilon(\omega,z)&=\frac{i\omega^\varepsilon}{2}\eta^\varepsilon(z)\begin{pmatrix}1 & -e^{-2i\omega^\varepsilon z}\\ e^{2i\omega^\varepsilon z} & -1\end{pmatrix}\\ &=\frac{i\omega}{2\varepsilon}\eta(\frac{z}{\varepsilon^2})\begin{pmatrix}1 & -e^{-2i\omega\frac{z}{\varepsilon}}\\ e^{2i\omega\frac{z}{\varepsilon}} & -1\end{pmatrix}.\end{aligned} \tag{6.9}$$

The boundary conditions (6.4) read in terms of $\hat{a}^\varepsilon$ and $\hat{b}^\varepsilon$:

$$\hat{a}^\varepsilon(\omega,0)=2, \quad \hat{b}^\varepsilon(\omega,L)=0. \tag{6.10}$$

We introduce the propagator $\mathbf{P}^\varepsilon$, i.e. the fundamental matrix solution of the linear system of differential equations: $\frac{d}{dz}\mathbf{P}^\varepsilon=\mathbf{H}^\varepsilon\mathbf{P}^\varepsilon$, $\mathbf{P}^\varepsilon(0)=\mathbf{I}$. From symmetries in Eq. (6.8), $\mathbf{P}^\varepsilon$ is of the form

$$\mathbf{P}^\varepsilon(\omega,z)=\left(\begin{array}{cc}\hat{\alpha}^\varepsilon(\omega,z) & \overline{\hat{\beta}^\varepsilon}(\omega,z)\\ \hat{\beta}^\varepsilon(\omega,z) & \overline{\hat{\alpha}^\varepsilon}(\omega,z)\end{array}\right), \tag{6.11}$$

where $(\hat{\alpha}^\varepsilon,\hat{\beta}^\varepsilon)^T$ is solution of (6.8) with the initial conditions:

$$\hat{\alpha}^\varepsilon(\omega,0)=1, \quad \hat{\beta}^\varepsilon(\omega,0)=0. \tag{6.12}$$

The modes $\hat{A}^\varepsilon$ and $\hat{B}^\varepsilon$ can be expressed in terms of the propagator:

$$\left(\begin{array}{c}\hat{a}^\varepsilon(\omega,z)\\ \hat{b}^\varepsilon(\omega,z)\end{array}\right)=\mathbf{P}^\varepsilon(\omega,z)\left(\begin{array}{c}\hat{a}^\varepsilon(\omega,0)\\ \hat{b}^\varepsilon(\omega,0)\end{array}\right). \tag{6.13}$$

From the identity (6.13) applied for $z=L$ and the boundary conditions (6.10) we can deduce that

$$\hat{b}^\varepsilon(\omega,0)=-(2\hat{\beta}^\varepsilon/\overline{\hat{\alpha}^\varepsilon})(\omega,L), \quad \hat{a}^\varepsilon(\omega,L)=(2/\overline{\hat{\alpha}^\varepsilon})(\omega,L),$$

and from (6.7), (6.1-6.2), we obtain

$$R^\varepsilon_\omega(L)=-(\hat{\beta}^\varepsilon/\overline{\hat{\alpha}^\varepsilon})(\omega,L), \quad T^\varepsilon_\omega(L)=(1/\overline{\hat{\alpha}^\varepsilon})(\omega,L). \tag{6.14}$$

The power transmission coefficient $|T^\varepsilon_\omega|^2$ is equal to $1/|\hat{\alpha}^\varepsilon|^2(\omega,L)$. We introduce the slow process $v^\varepsilon(\omega,z):=\hat{\alpha}^\varepsilon(\omega,\varepsilon z)e^{i\omega^\varepsilon\varepsilon z}+\hat{\beta}^\varepsilon(\omega,\varepsilon z)e^{-i\omega^\varepsilon\varepsilon z}$. It satisfies the equation

$$v^\varepsilon_{zz}+\omega^2(1+\eta(\frac{z}{\varepsilon}))v^\varepsilon=0$$

with the initial condition $v^\varepsilon(0)=1$, $v^\varepsilon_z(0)=-i\omega$. Let us introduce the quantity $r^\varepsilon(\omega,z):=|v^\varepsilon|^2+|v^\varepsilon_z|^2/\omega^2$. A straightforward calculation shows that

$$r^\varepsilon(\omega,z)=1+2|\hat{\alpha}^\varepsilon|^2(\omega,\varepsilon z)$$

By Eq. (6.14) we get a relation between r^ε and T^ε_ω:

$$r^\varepsilon(\omega,z)=1+2|T^\varepsilon_\omega(\varepsilon z)|^{-2} \tag{6.15}$$

If $\gamma^\varepsilon(\omega)$ denotes the Lyapunov exponent that governs the exponential growth of $r^\varepsilon(\omega,z)$:

$$\gamma^\varepsilon(\omega)=\lim_{z\to\infty}\frac{1}{z}\ln r^\varepsilon(\omega,z)$$

then Eq. (6.15) insures that $|T^\varepsilon_\omega(L)|^2$ will decay as $\exp(-\gamma^\varepsilon(\omega)L/\varepsilon)$ as soon as $\gamma^\varepsilon(\omega)>0$. In Appendix A the existence of the Lyapunov exponent $\gamma^\varepsilon(\omega)$ is proved, and its expansion with respect to ε is derived. $\square$

Note that γ is a nonnegative real number since it is proportional to the power spectral density of the stationary random process η (Wiener-Khintchine theorem [47, p. 141]). The existence and positivity of the exponent $1/L_{loc}^{\varepsilon}$ can be obtained with minimal hypotheses. Kotani [42] established that a sufficient condition is that η is a stationary, ergodic process that is bounded with probability one and nondeterministic. The expansion of the localization length requires some more hypotheses about the mixing properties of η. A discussion and sufficient hypotheses are proposed in Appendix A.

Note also that the localization length of a monochromatic wave with frequency ω_0 is equal to the length that governs the exponential decay of the coherent transmitted part of a narrowband pulse with carrier frequency ω_0 (compare (6.6) and (5.16)).

For a finite L it is possible to give the complete statistical description of the power transmission coefficient in the limit $\varepsilon \to 0$.

Proposition 6.2. *The power transmission coefficient $|T_\omega^\varepsilon(L)|^2$ converges in distribution as a continuous process in L to the Markov process $\mathcal{T}_\omega(L)$ whose infinitesimal generator is:*

$$\mathcal{L}_\omega = \frac{1}{4}\gamma\omega^2\mathcal{T}^2(1-\mathcal{T})\frac{\partial^2}{\partial\mathcal{T}^2} - \frac{1}{4}\gamma\omega^2\mathcal{T}^2\frac{\partial}{\partial\mathcal{T}}. \tag{6.16}$$

Proof. The power transmission coefficient $|T_\omega^\varepsilon|^2$ can be expressed in terms of a random variable that is the solution of a Ricatti equation. Indeed, as a byproduct of the proof of Proposition 6.1 we find that $|T_\omega^\varepsilon|^2 = 1 - |\Gamma_\omega^\varepsilon|^2$ where $\Gamma_\omega^\varepsilon(L) = \hat{\beta}^\varepsilon/\hat{\alpha}^\varepsilon(\omega, L)$ and $(\hat{\alpha}^\varepsilon, \hat{\beta}^\varepsilon)$ are defined as the solutions of Eqs. (6.8)+(6.12). Differentiating $\hat{\beta}^\varepsilon/\hat{\alpha}^\varepsilon$ with respect to L yields that the coefficient $\Gamma_\omega^\varepsilon$ satisfies a closed-form nonlinear equation:

$$\frac{d\Gamma_\omega^\varepsilon}{dL} = \frac{i\omega}{2\varepsilon}\eta(\frac{L}{\varepsilon^2})\left(e^{2i\omega\frac{L}{\varepsilon}} - 2\Gamma_\omega^\varepsilon + e^{-2i\omega\frac{L}{\varepsilon}}\Gamma_\omega^{\varepsilon\,2}\right), \quad \Gamma_\omega^\varepsilon(0) = 0. \tag{6.17}$$

One then consider the process $X^\varepsilon := (r^\varepsilon, \psi^\varepsilon) := (|\Gamma_\omega^\varepsilon|^2, \arg(\Gamma_\omega^\varepsilon))$ which satisfies:

$$\frac{dX^\varepsilon}{dL}(L) = \frac{1}{\varepsilon}F\left(\eta(\frac{L}{\varepsilon^2}), X^\varepsilon(L), \frac{L}{\varepsilon}\right),$$

where F is defined by:

$$F(\eta, r, \psi, l) = \frac{\omega\eta}{2}\begin{pmatrix} -2\sin(\psi - 2\omega l)(r^{3/2} - r^{1/2}) \\ -2 + \cos(\psi - 2\omega l)(r^{1/2} + r^{-1/2}) \end{pmatrix}$$

One then applies the diffusion-approximation theorem 4.21 to the process $(r^\varepsilon, \psi^\varepsilon)$ which gives the result. $\square$

In particular the expectation of the power transmission coefficient $|T_\omega^\varepsilon(L)|^2$ converges to $\bar{\mathcal{T}}_\omega(L) := \mathbb{E}[\mathcal{T}_\omega(L)]$ (see [26, Section 7.1]):

$$\bar{\mathcal{T}}_\omega(L) = \frac{4}{\sqrt{\pi}}\exp\left(-\frac{\gamma\omega^2 L}{16}\right)\int_0^\infty dx \frac{x^2 e^{-x^2}}{\cosh(\sqrt{\gamma\omega^2 L/(2\sqrt{2})}x)}. \tag{6.18}$$

This shows that:

$$\frac{1}{L}\ln \bar{\mathcal{T}}_\omega(L) \overset{L\gg 1}{\simeq} -\frac{\gamma\omega^2}{16} \tag{6.19}$$

We get actually that, for any $n \in \mathbb{N}^*$:

$$\mathbb{E}[\mathcal{T}_\omega(L)^n] \overset{L\gg 1}{\simeq} \frac{c_n(\omega)}{L^{3/2}} \exp\left(-\frac{\gamma\omega^2 L}{16}\right).$$

By comparing Eq. (6.19) with Eqs. (6.5)+(6.6) we can see that the exponential behavior of the expectation of the power transmission coefficient is different from the sample behavior of the power transmission coefficient.

This is a quite common phenomenon when studying systems driven by random processes. We now give some heuristic arguments to complete the discussion. Let us set $\mathcal{T}_\omega(L) = 2/(1+\rho(L))$. ρ is a Markov process with infinitesimal generator:

$$\mathcal{L} = \frac{1}{4}\gamma\omega^2(\rho^2-1)\frac{\partial^2}{\partial\rho^2} + \frac{1}{2}\gamma\omega^2\rho\frac{\partial}{\partial\rho}$$

Applying standard tools of stochastic analysis (Itô's formula) we can represent the process ρ as the solution of the stochastic differential equation:

$$d\rho = \frac{\sqrt{\gamma}}{\sqrt{2}}\omega\sqrt{\rho^2-1}dW_L + \frac{\gamma}{2}\omega^2\rho dL, \quad \rho(0) = 1.$$

The long-range behavior is determined by the drift so that $\rho \gg 1$ and:

$$d\rho \simeq \frac{\sqrt{\gamma}}{\sqrt{2}}\omega\rho dW_L + \frac{\gamma}{2}\omega^2\rho dL$$

which can be solved as:

$$\rho(L) \sim \exp\left(\frac{\sqrt{\gamma}}{\sqrt{2}}\omega W_L + \frac{\gamma}{4}\omega^2 L\right).$$

Please note that these identities are just heuristic ! As $L \gg 1$, with probability very close to 1, we have $W_L \sim \sqrt{L}$ which is negligible compared to L, so $\rho(L) \sim \exp\left(\gamma\omega^2 L/4\right)$ and $\mathcal{T}_\omega(L) \sim \exp\left(-\gamma\omega^2 L/4\right) = \exp(-L/L_{loc})$.
But, if $\sqrt{\gamma/2}\omega W_L < -\gamma\omega^2 L/4$, then $\rho \lesssim 1$ and $\mathcal{T}_\omega \sim 1$! This is a very rare event, its probability is only $\mathbb{P}(\sqrt{\gamma/2}\omega W_L < -\gamma\omega^2 L/4) = \mathbb{P}(W_1 < -\sqrt{\gamma\omega^2 L}/(2\sqrt{2})) \sim \exp(-\gamma\omega^2 L/16)$. But this set of rare events (=realizations of the random medium) imposes the values of the moments of the transmission coefficient.

Thus the expectation of the power transmission coefficient is imposed by exceptional realizations of the medium. Apparently the "right" localization length is the "sample" one (6.6), in the sense that it is the one that will be observed for a typical realization of the medium. Actually we shall see that this holds true only for purely monochromatic waves.

6.2. Transmission of pulses

We consider an incoming wave from the left:

$$p_{inc}^{\varepsilon}(t,z) = \frac{1}{2\pi}\int \hat{f}^{\varepsilon}(\omega)\exp i\left(\omega z - \omega t\right) d\omega, \quad z \le 0, \tag{6.20}$$

where $\hat{f}^{\varepsilon} \in L^2 \cap L^1$ and contains frequencies of order ε^{-1} (i.e. wavelengths of order ε):

$$\hat{f}^{\varepsilon}(\omega) = \sqrt{\varepsilon}\hat{f}(\varepsilon\omega) \Longleftrightarrow f^{\varepsilon}(t) = \frac{1}{\sqrt{\varepsilon}} f(\frac{t}{\varepsilon})$$

The amplitude of the incoming pulse is normalized so that its energy is independent of ε:

$$\mathcal{E}_{inc} := \int |p_{inc}^{\varepsilon}(t,0)|^2 dt = \frac{1}{2\pi}\int |\hat{f}^{\varepsilon}(\omega)|^2 d\omega = \frac{1}{2\pi}\int |\hat{f}(\omega)|^2 d\omega$$

The total field in the region $z \le 0$ thus consists of the superposition of the incoming wave p_{inc}^{ε} and the reflected wave:

$$p_{ref}^{\varepsilon}(t,z) = -\frac{1}{2\pi\sqrt{\varepsilon}}\int \hat{f}(\omega) R_{\omega}^{\varepsilon}(L)\exp i\left(-\omega\frac{z}{\varepsilon} - \omega\frac{t}{\varepsilon}\right) d\omega, \quad z \le 0,$$

where $R_{\omega}^{\varepsilon}(L)$ is the reflection coefficient for the frequency ω/ε. The field in the region $z \ge L$ consists only of the transmitted wave that is right going:

$$p_{tr}^{\varepsilon}(t,z) = \frac{1}{2\pi\sqrt{\varepsilon}}\int \hat{f}(\omega) T_{\omega}^{\varepsilon}(L)\exp i\left(\omega\frac{z}{\varepsilon} - \omega\frac{t}{\varepsilon}\right) d\omega, \quad z \ge L, \tag{6.21}$$

where $T_{\omega}^{\varepsilon}(L)$ is the transmission coefficient for the frequency ω/ε. Inside the slab the wave has the general form:

$$p^{\varepsilon}(t,z) = \frac{1}{2\pi}\int \hat{p}^{\varepsilon}(\omega,z)\exp\left(-i\omega\frac{t}{\varepsilon}\right) d\omega, \quad 0 \le z \le L,$$

where $\hat{p}^{\varepsilon}$ satisfies the reduced wave equation:

$$\hat{p}_{zz}^{\varepsilon} + (1+\eta^{\varepsilon}(z))\hat{p}^{\varepsilon} = 0, \quad 0 \le z \le L.$$

The total transmitted energy is:

$$\mathcal{E}_T^{\varepsilon}(L) = \int |p_{tr}^{\varepsilon}(t,L)|^2 dt = \frac{1}{2\pi}\int |\hat{f}(\omega)|^2 |T_{\omega}^{\varepsilon}(L)|^2 d\omega.$$

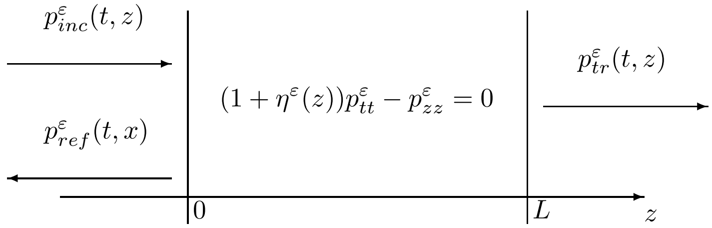


Figure 3: Scattering of a pulse.

We first compute the two-frequency correlation function. The following Lemma is an extension of Proposition 6.2.

Lemma 6.3. *Let $\omega_1 = \omega - h\varepsilon^a/2$ and $\omega_2 = \omega + h\varepsilon^a/2$.*
1. If $a = 0$, then the power transmission coefficients $(|T^\varepsilon_{\omega_1}(L)|^2, |T^\varepsilon_{\omega_2}(L)|^2)$ converge in distribution to $(\mathcal{T}_{\omega-h/2}(L), \mathcal{T}_{\omega+h/2}(L))$ where $\mathcal{T}_{\omega-h/2}(L)$ and $\mathcal{T}_{\omega+h/2}(L)$ are two independent Markov processes whose infinitesimal generators are respectively $\mathcal{L}_{\omega-h/2}$ and $\mathcal{L}_{\omega+h/2}$ defined by (6.16).
2. If $a = 1$, then the power transmission coefficients $(|T^\varepsilon_{\omega_1}(L)|^2, |T^\varepsilon_{\omega_2}(L)|^2)$ converge in distribution to $(\mathcal{T}_1(L), \mathcal{T}_2(L))$ where $(\mathcal{T}_1(L), \mathcal{T}_2(L), \theta(L))$ is the Markov process whose infinitesimal generator is:

$$\begin{aligned}
\mathcal{L} \quad = \quad & \frac{\gamma\omega^2}{4}\mathcal{T}_1^2(1-\mathcal{T}_1)\frac{\partial^2}{\partial \mathcal{T}_1^2} - \frac{\gamma\omega^2}{4}\mathcal{T}_1^2\frac{\partial}{\partial \mathcal{T}_1} + \frac{\gamma\omega^2}{4}\mathcal{T}_1^2(1-\mathcal{T}_2)\frac{\partial^2}{\partial \mathcal{T}_2^2} - \frac{\gamma\omega^2}{4}\mathcal{T}_2^2\frac{\partial}{\partial \mathcal{T}_2} \\
& + \frac{\gamma}{2}\omega^2\cos(\theta)\sqrt{(1-\mathcal{T}_1)(1-\mathcal{T}_2)}\mathcal{T}_2\mathcal{T}_1\frac{\partial^2}{\partial \mathcal{T}_1 \partial \mathcal{T}_2} \\
& + 2h\frac{\partial}{\partial\theta} + \frac{\gamma\omega^2}{8}\left(\frac{(2-\mathcal{T}_1)^2}{1-\mathcal{T}_1} + \frac{(2-\mathcal{T}_2)^2}{1-\mathcal{T}_2} - 2\frac{(2-\mathcal{T}_1)(2-\mathcal{T}_2)}{\sqrt{(1-\mathcal{T}_1)(1-\mathcal{T}_2)}}\cos(\theta)\right)\frac{\partial^2}{\partial\theta^2} \\
& + \frac{\gamma\omega^2}{8}\frac{\sqrt{1-\mathcal{T}_1}\mathcal{T}_1(2-\mathcal{T}_2)}{\sqrt{1-\mathcal{T}_2}}\sin(\theta)\frac{\partial^2}{\partial \mathcal{T}_1\partial\theta} \\
& + \frac{\gamma\omega^2}{8}\frac{\sqrt{1-\mathcal{T}_2}\mathcal{T}_2(2-\mathcal{T}_1)}{\sqrt{1-\mathcal{T}_1}}\sin(\theta)\frac{\partial^2}{\partial \mathcal{T}_2\partial\theta} \qquad (6.22)
\end{aligned}$$

starting from $\mathcal{T}_1(0) = 1$, $\mathcal{T}_2(0) = 1$, and $\theta(0) = 0$.

Proof. The most interesting case is $a = 1$, since this is the correct scaling that describes the correlation of the transmission coefficients at two nearby frequencies. Let us denote $|T^\varepsilon_{\omega_j}(L)|^2 = 1 - |\Gamma^\varepsilon_j(L)|^2$ for $j = 1, 2$, where $\Gamma^\varepsilon_j(L) = \hat{\beta}^\varepsilon/\hat{\alpha}^\varepsilon(\omega_j, L)$. We then introduce the four-dimensional process

$$X^\varepsilon := (r^\varepsilon_1, \psi^\varepsilon_1, r^\varepsilon_2, \psi^\varepsilon_2) := (|\Gamma^\varepsilon_1|^2, \arg(\Gamma^\varepsilon_1), |\Gamma^\varepsilon_2|^2, \arg(\Gamma^\varepsilon_2))$$

which satisfies:

$$\frac{dX^\varepsilon}{dL}(L) = \frac{1}{\varepsilon}F\left(\eta(\frac{L}{\varepsilon^2}), X^\varepsilon(L), \frac{L}{\varepsilon}, L\right),$$

where F is defined by:

$$F(\eta, r_1, \psi_1, r_2, \psi_2, l, L) = \frac{\omega\eta}{2} \begin{pmatrix} -2\sin(\psi_1 - 2\omega l - hL)(r_1^{3/2} - r_1^{1/2}) \\ -2 + \cos(\psi_1 - 2\omega l - hL)(r_1^{1/2} + r_1^{-1/2}) \\ -2\sin(\psi_2 - 2\omega l + hL)(r_2^{3/2} - r_2^{1/2}) \\ -2 + \cos(\psi_2 - 2\omega l + hL)(r_2^{1/2} + r_2^{-1/2}) \end{pmatrix}$$

Applying the diffusion-approximation theorem 4.21 to the process X^ε establishes that X^ε converges to a non-homogeneous Markov process $X = (r_1, \psi_1, r_2, \psi_2)$ whose infinitesimal generator (that depends on L) can be computed explicitly. By introducing $\theta := \psi_1 - \psi_2 - 2hL$ it turns out that the process (r_1, r_2, θ) is a homogeneous Markov process whose infinitesimal generator is given by (6.22). □

This Lemma shows that the transmission coefficients corresponding to two nearby frequencies ω_1 and ω_2 are uncorrelated as soon as $|\omega_1 - \omega_2| \gg \varepsilon$. Once this result is known, it is easy to derive the asymptotic behavior of the pulse transmittivity.

Proposition 6.4. *The transmittivity $\mathcal{E}_T^\varepsilon(L)$ converges in probability to $\mathcal{E}_T(L)$:*

$$\mathcal{E}_T(L) = \frac{1}{2\pi} \int |\hat{f}(\omega)|^2 \bar{\mathcal{T}}_\omega(L) d\omega,$$

where $\bar{\mathcal{T}}_\omega(L)$ is the asymptotic value (6.18) of the mean power transmission coefficient.

Proof. Proposition 6.2 gives the limit value of the expectation of $|T_\omega^\varepsilon(L)|^2$ for one frequency ω, so that:

$$\mathbb{E}\left[\mathcal{E}_T^\varepsilon(L)\right] \stackrel{\varepsilon\to 0}{\longrightarrow} \frac{1}{2\pi} \int |\hat{f}(\omega)|^2 \bar{\mathcal{T}}_\omega(L) d\omega.$$

Then one considers the second moment:

$$\mathbb{E}\left[\mathcal{E}_T^\varepsilon(L)^2\right] = \frac{1}{4\pi^2} \int\int |\hat{f}(\omega)|^2 |\hat{f}(\omega')|^2 \mathbb{E}\left[|T_\omega^\varepsilon(L)|^2 |T_{\omega'}^\varepsilon(L)|^2\right] d\omega d\omega'.$$

The computation of this moment requires to study the two-frequency process $(|T_\omega^\varepsilon(L)|, |T_{\omega'}^\varepsilon(L)|)$ for $\omega \neq \omega'$. Applying Lemma 6.3 one finds that $|T_\omega^\varepsilon(L)|$ and $|T_{\omega'}^\varepsilon(L)|$ are asymptotically uncorrelated as soon as $\omega \neq \omega'$, so that

$$\begin{aligned} \mathbb{E}\left[\mathcal{E}_T^\varepsilon(L)^2\right] &\stackrel{\varepsilon\to 0}{\longrightarrow} \frac{1}{4\pi^2} \int\int |\hat{f}(\omega)|^2 |\hat{f}(\omega')|^2 \bar{\mathcal{T}}_\omega(L) \bar{\mathcal{T}}_{\omega'}(L) \\ &= \left(\frac{1}{2\pi} \int |\hat{f}(\omega)^2 \bar{\mathcal{T}}_\omega(L) d\omega\right)^2, \end{aligned}$$

which proves the convergence of $\mathcal{E}_T^\varepsilon(L)$ to $\mathcal{E}_T(L)$ in $L^2(\mathbb{P})$:

$$\mathbb{E}\left[\left(\mathcal{E}_T^\varepsilon(L) - \mathcal{E}_T(L)\right)^2\right] = \mathbb{E}\left[\mathcal{E}_T^\varepsilon(L)^2\right] - 2\mathbb{E}\left[\mathcal{E}_T^\varepsilon(L)\right]\mathcal{E}_T(L) + \mathcal{E}_T(L)^2 \stackrel{\varepsilon\to 0}{\longrightarrow} 0$$

By the Chebychev inequality this implies a convergence in probability:

$$\text{For any } \delta > 0,\ \mathbb{P}\left(|\mathcal{E}_T^\varepsilon(L) - \mathcal{E}_T(L)| > \delta\right) \leq \frac{\mathbb{E}\left[\left(\mathcal{E}_T^\varepsilon(L) - \mathcal{E}_T(L)\right)^2\right]}{\delta^2} \stackrel{\varepsilon \to 0}{\longrightarrow} 0$$

□

Let us assume that the incoming wave is narrowband, that is to say that the spectral content $\hat{f}$ is concentrated around the carrier wavenumber ω_0 and has narrow bandwidth (smaller than 1, but larger than ε). Then $\mathcal{E}_T(L)$ decays exponentially with the length of the slab as:

$$\frac{1}{L} \ln \mathcal{E}_T(L) \stackrel{L \gg 1}{\simeq} -\frac{\gamma \omega_0^2}{16}$$

Note that this is the typical behavior of the *expected* value of the power transmission coefficient of a monochromatic wave with wavenumber ω_0. In the time domain the localization process is self-averaging ! This self-averaging is implied by the asymptotic decorrelation of the transmission coefficients at different frequencies. Actually $T_\omega^\varepsilon(L)$ and $T_{\omega'}^\varepsilon(L)$ are correlated only if $|\omega - \omega'| \leq \varepsilon$.

We can now describe the energy content of the transmitted wave. It consists of a coherent part described by the O'Doherty-Anstey theory, with coherent energy that decays quickly as $\exp(-\gamma\omega_0^2 L/4)$. There is also an incoherent part in the transmitted wave, whose energy decays as $\exp(-\gamma\omega_0^2 L/16)$. Thus the incoherent wave contains most of the energy of the total transmitted wave in the regime $\gamma\omega_0^2 L \geq 1$.

6.3. Bibliographic notes

The presentation of power transmission through a slab of random medium of Section 6.1 follows the treatment in [40]. A more physical approach to this problem is given in Klyatskin's book [39]. The self-averaging property presented in Section 6.2 does not seem to be well known. The analytical reason for the phenomenon is the decorrelation of the power transmission coefficients at distinct frequencies, which is well known [15].

The treatment of wave localization in Section 6.1 is limited to the analysis of the exponential decay of the power transmission coefficient. The phenomenon of wave localization was discovered in 1958 by Anderson [1] in connection with electron waves in semiconductors. The mathematical theory was developed only twenty years later, starting with the paper of Goldsheid–Molchanov–Pastur [32]. Since that paper there has been a great deal of research published on the subject, in particular in the one-dimensional case, for which the theory of products of random matrices is available. We cite here some books that also contain additional references: Carmona–Lacroix [18], Pastur–Figotin [58], and the review papers by Van Tiggelen, Lacroix, and Klein in the proceedings [25].

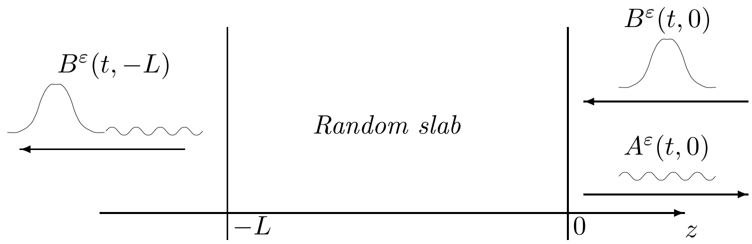


Figure 4: The new scattering problem.

7. Incoherent wave fluctuations

This section is a self-contained statistical analysis of the time- and frequency-domain properties of the incoherent waves reflected by a one-dimensional random medium. We have shown in the previous section that the energy of the transmitted wave front decays exponentially with the size of the random medium, which implies that the incoherent waves carry most of the energy. The analysis of the incoherent waves is therefore important in many applications, especially in time reversal, as we will see in the next section. We have also seen that the total transmitted energy decays exponentially with the size of the random medium, but that the decay rate is slower than that of the wave front. We will therefore focus attention on the incoherent reflected waves. For an extensive treatment we refer to [26, Chapter 9].

7.1. The reflected wave

It turns out that it is more convenient to reformulate the scattering problem as shown in Figure 4 in order to deal with forward equations. The incident wave comes from the right, and the reflected wave exits into the homogeneous half-space on the right also.

We consider the acoustic equations (3.1-3.2). We assume that the medium parameters are

$$\begin{aligned} \frac{1}{\kappa^\varepsilon(z)} &= \begin{cases} 1+\eta(z/\varepsilon^2) & \text{for } \ z \in [-L,0], \\ 1 & \text{for } \ z \in (-\infty,-L)\cup(0,\infty), \end{cases} \\ \rho^\varepsilon(z) &= 1 \ \text{ for all } \ z, \end{aligned}$$

and an incoming left-going wave impinges on the interface $z=0$. The pulse width is of order ε, and its amplitude is scaled so that it has energy of order one. It is given by

$$\frac{1}{\sqrt{\varepsilon}} f\left(\frac{t}{\varepsilon}\right),$$

where f is square-integrable, so that

$$\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{\varepsilon}}f\left(\frac{t}{\varepsilon}\right)\right]^2 dt = \int_{-\infty}^{\infty} f(u)^2 du < \infty.$$

As in previous sections, we introduce the right- and left-going modes

$$A^\varepsilon(t,z) = u^\varepsilon(t,z) + p^\varepsilon(t,z), \qquad B^\varepsilon(t,z) = u^\varepsilon(t,z) - p^\varepsilon(t,z).$$

We consider these modes in coordinates moving with the effective speed 1 and on the time scale of the incoming pulse,

$$a^\varepsilon(s,z) = A^\varepsilon(\varepsilon s + z, z), \qquad b^\varepsilon(s,z) = B^\varepsilon(\varepsilon s - z, z)\,.$$

In the Fourier domain the modes satisfy the differential equations

$$\frac{d}{dz}\begin{pmatrix} \hat{a}^\varepsilon \\ \hat{b}^\varepsilon \end{pmatrix} = \frac{i\omega}{2\varepsilon}\eta\left(\frac{z}{\varepsilon^2}\right)\begin{pmatrix} 1 & -e^{-2i\omega z/\varepsilon} \\ e^{2i\omega z/\varepsilon} & -1 \end{pmatrix}\begin{pmatrix} \hat{a}^\varepsilon \\ \hat{b}^\varepsilon \end{pmatrix}. \tag{7.1}$$

The modes also satisfy boundary conditions corresponding to a left-going wave impinging at $z = 0$ and the radiation condition at $z = -L$,

$$\hat{b}^\varepsilon(\omega, 0) = \frac{1}{\sqrt{\varepsilon}}\hat{f}(\omega), \qquad \hat{a}^\varepsilon(\omega, -L) = 0. \tag{7.2}$$

We first transform the boundary value problem (7.1–7.2) into an initial value problem. This step is similar to the analysis carried out in Section 5. We introduce the propagator $\mathbf{P}^\varepsilon_\omega(-L,z)$, that is, the fundamental solution matrix of the linear system of differential equations (7.1) with initial condition $\mathbf{P}^\varepsilon_\omega(-L, z = -L) = \mathbf{I}$. From symmetries in (7.1), $\mathbf{P}^\varepsilon_\omega$ is of the form

$$\mathbf{P}^\varepsilon_\omega(-L,z) = \begin{pmatrix} \hat{\alpha}^\varepsilon_\omega(-L,z) & \overline{\hat{\beta}^\varepsilon_\omega(-L,z)} \\ \hat{\beta}^\varepsilon_\omega(-L,z) & \overline{\hat{\alpha}^\varepsilon_\omega(-L,z)} \end{pmatrix}, \tag{7.3}$$

where $(\hat{\alpha}^\varepsilon_\omega, \hat{\beta}^\varepsilon_\omega)^T$ is a solution of (7.1) with the initial conditions

$$\hat{\alpha}^\varepsilon_\omega(-L, z = -L) = 1, \qquad \hat{\beta}^\varepsilon_\omega(-L, z = -L) = 0. \tag{7.4}$$

The modes $\hat{a}^\varepsilon$ and $\hat{b}^\varepsilon$ can be expressed in terms of the propagator as

$$\begin{pmatrix} \hat{a}^\varepsilon(\omega,z) \\ \hat{b}^\varepsilon(\omega,z) \end{pmatrix} = \mathbf{P}^\varepsilon_\omega(-L,z)\begin{pmatrix} \hat{a}^\varepsilon(\omega,-L) \\ \hat{b}^\varepsilon(\omega,-L) \end{pmatrix}. \tag{7.5}$$

We can now define the transmission and reflection coefficients $T^\varepsilon_\omega(-L,z)$ and $R^\varepsilon_\omega(-L,z)$, respectively, for a slab $[-L,z]$ by (see Figure 5)

$$\mathbf{P}^\varepsilon_\omega(-L,z)\begin{pmatrix} 0 \\ T^\varepsilon_\omega(-L,z) \end{pmatrix} = \begin{pmatrix} R^\varepsilon_\omega(-L,z) \\ 1 \end{pmatrix}. \tag{7.6}$$

In terms of the propagator entries they are given by

$$R^\varepsilon_\omega(-L,z) = \frac{\overline{\hat{\beta}^\varepsilon_\omega(-L,z)}}{\overline{\hat{\alpha}^\varepsilon_\omega(-L,z)}}, \qquad T^\varepsilon_\omega(-L,z) = \frac{1}{\overline{\hat{\alpha}^\varepsilon_\omega(-L,z)}}. \tag{7.7}$$

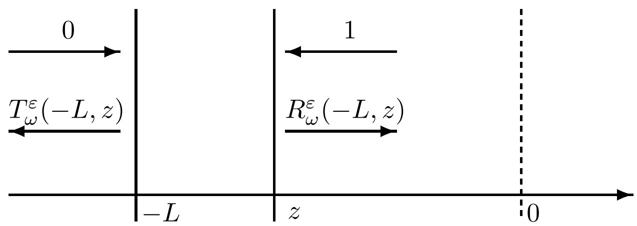


Figure 5: Reflection and transmission coefficients.

By (7.2) and (7.5) applied at $z=0$, the reflected and transmitted mode amplitudes can be expressed in terms of the reflection and transmission coefficients as

$$\hat{a}^\varepsilon(\omega,0) = \frac{1}{\sqrt{\varepsilon}}\hat{f}(\omega)R^\varepsilon_\omega(-L,0)\,, \qquad \hat{b}^\varepsilon(\omega,-L) = \frac{1}{\sqrt{\varepsilon}}\hat{f}(\omega)T^\varepsilon_\omega(-L,0)\,.$$

We want to derive a closed equation for the reflection coefficient. By differentiating $R^\varepsilon_\omega(-L,z)$ and $T^\varepsilon_\omega(-L,z)$ with respect to z, we have

$$\begin{aligned} \frac{dR^\varepsilon_\omega}{dz} &= \overline{\frac{1}{\hat{\alpha}^\varepsilon_\omega}\frac{d\hat{\beta}^\varepsilon_\omega}{dz} - \frac{\hat{\beta}^\varepsilon_\omega}{(\hat{\alpha}^\varepsilon_\omega)^2}\frac{d\hat{\alpha}^\varepsilon_\omega}{dz}}, \\ \frac{dT^\varepsilon_\omega}{dz} &= -\overline{\frac{1}{(\hat{\alpha}^\varepsilon_\omega)^2}\frac{d\hat{\alpha}^\varepsilon_\omega}{dz}}. \end{aligned}$$

From the equations (7.1) satisfied by $(\hat{\alpha}^\varepsilon_\omega, \hat{\beta}^\varepsilon_\omega)$, we get

$$\frac{dR^\varepsilon_\omega}{dz} = -\frac{i\omega}{2\varepsilon}\eta\left(\frac{z}{\varepsilon^2}\right)\left(e^{-2i\omega z/\varepsilon} - 2R^\varepsilon_\omega + (R^\varepsilon_\omega)^2 e^{2i\omega z/\varepsilon}\right), \tag{7.8}$$

$$\frac{dT^\varepsilon_\omega}{dz} = \frac{i\omega}{2\varepsilon}\eta\left(\frac{z}{\varepsilon^2}\right)\left(1 - R^\varepsilon_\omega e^{2i\omega z/\varepsilon}\right)T^\varepsilon_\omega\,. \tag{7.9}$$

The initial conditions for these nonlinear differential equations are

$$R^\varepsilon_\omega(-L,z=-L) = 0, \qquad T^\varepsilon_\omega(-L,z=-L) = 1,$$

at $z=-L$. This is because the medium is homogeneous for $z<-L$, and left-going waves simply travels at constant speed to the left. Equation (7.8) is the Riccati equation for the reflection coefficient, and (7.9) is the associated linear equation for the transmission coefficient, which depends on the reflection coeffcient. From these equations it is easy to check the conservation of energy relation

$$|R^\varepsilon_\omega(-L,z)|^2 + |T^\varepsilon_\omega(-L,z)|^2 = 1, \tag{7.10}$$

which is the same as (5.11).

The reflected wave at $z = 0$ admits the following representation in terms of the reflection coefficient:

$$\begin{aligned} A^\varepsilon(t,0) &= a^\varepsilon\left(\frac{t}{\varepsilon},0\right) \\ &= \frac{1}{2\pi}\int \hat{a}^\varepsilon(\omega,0)e^{-i\frac{\omega t}{\varepsilon}}d\omega \\ &= \frac{1}{2\pi\sqrt{\varepsilon}}\int R^\varepsilon_\omega(-L,0)\hat{f}(\omega)e^{-i\frac{\omega t}{\varepsilon}}d\omega. \end{aligned} \tag{7.11}$$

The statistical description of the reflected wave is thus closely related to the statistical distribution of the reflection coefficient. In this section we will focus attention on:

1. The mean amplitude $\mathbb{E}[A^\varepsilon(t,0)]$, which describes the coherent reflected wave.

2. The mean intensity $\mathbb{E}[A^\varepsilon(t,0)^2]$, which describes the energy distribution of the reflected wave in the time domain.

3. The correlation function $c^\varepsilon_t(s) = \mathbb{E}[A^\varepsilon(t+\varepsilon s,0)A^\varepsilon(t,0)]$, which describes the time fluctuations in a time window of size of the order of ε.

We will also give the complete statistical distribution of the reflected wave. These results will be derived from the integral representation (7.11).

The mean amplitude is

$$\mathbb{E}[A^\varepsilon(t,0)] = \frac{1}{2\pi\sqrt{\varepsilon}}\int \mathbb{E}[R^\varepsilon_\omega(-L,0)]\hat{f}(\omega)e^{-i\frac{\omega t}{\varepsilon}}d\omega. \tag{7.12}$$

Higher-order moments of the reflected wave involve an expansion in multiple integrals and moments of products of reflection coefficients. Let us consider the second moment, that is, the mean intensity. Since $A^\varepsilon(t,0)$ is real-valued,

$$\begin{aligned} A^\varepsilon(t,0)^2 &= A^\varepsilon(t,0)\overline{A^\varepsilon(t,0)} \\ &= \frac{1}{4\pi^2\varepsilon}\left(\int R^\varepsilon_{\omega_1}(-L,0)\hat{f}(\omega_1)e^{-\frac{i\omega_1 t}{\varepsilon}}d\omega_1\right)\left(\int \overline{R^\varepsilon_{\omega_2}(-L,0)}\overline{\hat{f}(\omega_2)}e^{\frac{i\omega_2 t}{\varepsilon}}d\omega_2\right) \\ &= \frac{1}{4\pi^2\varepsilon}\int\int R^\varepsilon_{\omega_1}(-L,0)\overline{R^\varepsilon_{\omega_2}(-L,0)}\hat{f}(\omega_1)\overline{\hat{f}(\omega_2)}e^{i\frac{(\omega_2-\omega_1)t}{\varepsilon}}d\omega_1\,d\omega_2. \end{aligned}$$

By taking the expectation we obtain an expression of the mean intensity in terms of the frequency autocorrelation function of the reflection coefficient:

$$\begin{aligned} \mathbb{E}[A^\varepsilon(t,0)^2] = \frac{1}{4\pi^2\varepsilon}\int\int &\mathbb{E}\left[R^\varepsilon_{\omega_1}(-L,0)\overline{R^\varepsilon_{\omega_2}(-L,0)}\right] \\ &\times\hat{f}(\omega_1)\overline{\hat{f}(\omega_2)}e^{i\frac{(\omega_2-\omega_1)t}{\varepsilon}}d\omega_1\,d\omega_2. \end{aligned}$$

The presence of the fast phase $(\omega_2-\omega_1)t/\varepsilon$ suggests the change of variables

$$\omega_1 = \omega + \varepsilon h/2\,, \qquad \omega_2 = \omega - \varepsilon h/2\,,$$

which leads to the representation

$$\mathbb{E}[A^\varepsilon(t,0)^2] = \frac{1}{4\pi^2}\int\int \mathbb{E}\left[R^\varepsilon_{\omega+\varepsilon h/2}(-L,0)\overline{R^\varepsilon_{\omega-\varepsilon h/2}(-L,0)}\right] \times \hat{f}(\omega+\varepsilon h/2)\overline{\hat{f}(\omega-\varepsilon h/2)}e^{-iht}d\omega\, dh. \tag{7.13}$$

This shows that the correlation between the reflection coefficients at two nearby frequencies plays an important role. We shall thus carry out in the next subsection a careful analysis of the distribution of the reflection coefficient at two nearby frequencies in the asymptotic limit $\varepsilon \to 0$.

7.2. Statistics of the reflected wave in the frequency domain

7.2.1. Moments of the reflection coefficient We aim at computing the moments of the reflection coefficient. In view of (7.12) and (7.13), we are particularly interested in the first and second moments. However, the Riccati equation (7.8) satisfied by the reflection coefficient is nonlinear. As a result, we need to introduce a complete family of moments in order to get a closed system of equations. We introduce for $p, q \in \mathbb{N}$,

$$U^\varepsilon_{p,q}(\omega,h,z) = \left(R^\varepsilon_{\omega+\varepsilon h/2}(-L,z)\right)^p \left(\overline{R^\varepsilon_{\omega-\varepsilon h/2}(-L,z)}\right)^q. \tag{7.14}$$

The moments of interest to us are the first moment

$$\mathbb{E}\left[R^\varepsilon_\omega(-L,0)\right] = \mathbb{E}\left[U^\varepsilon_{1,0}(\omega,0,0)\right] \tag{7.15}$$

and the two-frequency autocorrelation function

$$\mathbb{E}\left[R^\varepsilon_{\omega+\varepsilon h/2}(-L,z)\overline{R^\varepsilon_{\omega-\varepsilon h/2}(-L,z)}\right] = \mathbb{E}\left[U^\varepsilon_{1,1}(\omega,h,z)\right]. \tag{7.16}$$

Using the Riccati equation (7.8) satisfied by R^ε_ω, we see that the family $(U^\varepsilon_{p,q})_{p,q\in\mathbb{N}}$ satisfies

$$\begin{aligned}\frac{\partial U^\varepsilon_{p,q}}{\partial z} &= i\omega\eta^\varepsilon(p-q)U^\varepsilon_{p,q} + \frac{i\omega}{2}\eta^\varepsilon e^{\frac{2i\omega z}{\varepsilon}}\left(qe^{-ihz}U^\varepsilon_{p,q-1} - pe^{ihz}U^\varepsilon_{p+1,q}\right)\\ &\quad + \frac{i\omega}{2}\eta^\varepsilon e^{-\frac{2i\omega z}{\varepsilon}}\left(qe^{ihz}U^\varepsilon_{p,q+1} - pe^{-ihz}U^\varepsilon_{p-1,q}\right), \qquad -L \le z \le 0,\end{aligned}$$

starting from

$$U^\varepsilon_{p,q}(\omega,h,z=-L) = \mathbf{1}_0(p)\mathbf{1}_0(q).$$

Here $\mathbf{1}_0(p) = 1$ if $p = 0$ and is 0 otherwise, and we have set

$$\eta^\varepsilon(z) = \frac{1}{\varepsilon}\eta\left(\frac{z}{\varepsilon^2}\right).$$

The system of random ordinary differential equations for $U^\varepsilon_{p,q}$ has a form that is almost suitable for the application of a diffusion-approximation theorem. One major problem is that we need an infinite-dimensional version of these theorems.

This requires a weak formulation and the introduction of an appropriate space of test functions. We refer to [55] for the details, and here we simply apply the result as if it were in a finite-dimensional context. Another problem is the presence of slow components of the form $\exp(\pm ihz)$. We first remove these terms by taking a shifted and scaled Fourier transform with respect to h:

$$V_{p,q}^{\varepsilon}(\omega,\tau,z) = \frac{1}{2\pi}\int e^{-ih(\tau-(p+q)z)} U_{p,q}^{\varepsilon}(\omega,h,z)dh. \tag{7.17}$$

The system of equations satisfied by $(V_{p,q}^{\varepsilon})_{p,q\in\mathbb{N}}$ is

$$\begin{aligned}\frac{\partial V_{p,q}^{\varepsilon}}{\partial z} &= -(p+q)\frac{\partial V_{p,q}^{\varepsilon}}{\partial \tau} + i\omega\eta^{\varepsilon}(p-q)V_{p,q}^{\varepsilon} + \frac{i\omega}{2}\eta^{\varepsilon}e^{\frac{2i\omega z}{\varepsilon}}\left(qV_{p,q-1}^{\varepsilon} - pV_{p+1,q}^{\varepsilon}\right)\\ &\quad+\frac{i\omega}{2}\eta^{\varepsilon}e^{-\frac{2i\omega z}{\varepsilon}}\left(qV_{p,q+1}^{\varepsilon} - pV_{p-1,q}^{\varepsilon}\right),\end{aligned} \tag{7.18}$$

starting from

$$V_{p,q}^{\varepsilon}(\omega,\tau,z=-L) = \delta(\tau)\mathbf{1}_0(p)\mathbf{1}_0(q).$$

We now apply the limit theorem B.2. This establishes that the process $(V_{p,q}^{\varepsilon})_{p,q\in\mathbb{N}}$ converges in distribution as $\varepsilon \to 0$ to a diffusion process $(V_{p,q})_{p,q\in\mathbb{N}}$. The infinitesimal generator is quite cumbersome, but the limit diffusion process can be identified as the solution of the Itô stochastic differential equation

$$\begin{aligned}dV_{p,q} &= -(q+p)\frac{\partial V_{p,q}}{\partial \tau}dz + i\sqrt{\gamma}\omega(p-q)V_{p,q}dW_0(z)\\ &\quad+\frac{i\sqrt{\gamma}\omega}{2\sqrt{2}}\left(qV_{p,q-1} - pV_{p+1,q} + qV_{p,q+1} - pV_{p-1,q}\right)dW_1(z)\\ &\quad+\frac{\sqrt{\gamma}\omega}{2\sqrt{2}}\left(qV_{p,q-1} - pV_{p+1,q} - qV_{p,q+1} + pV_{p-1,q}\right)dW_2(z)\\ &\quad+\frac{\gamma\omega^2}{4}\left[pq(V_{p+1,q+1} + V_{p-1,q-1} - 2V_{p,q}) - 3(p-q)^2V_{p,q}\right]dz,\end{aligned} \tag{7.19}$$

where W_j, $j=0,1,2$, are three independent Brownian motions and $\gamma = 2\int_0^\infty \mathbb{E}[\eta(0)\eta(z)]dz$ is the integrated covariance of the process η. The form of these stochastic differential equations (7.19) can be derived from (7.18) by replacing the integrals of $\eta^{\varepsilon}(z)$, $\eta^{\varepsilon}(z)\cos(2\omega z/\varepsilon)$ and $\eta^{\varepsilon}(z)\sin(2\omega z/\varepsilon)$ by the three independent Brownian motions $\sqrt{\gamma}W_0$, $\sqrt{\gamma/2}W_1$, and $\sqrt{\gamma/2}W_2$. The last line in (7.19) is the Itô-Stratonovich correction.

Taking the expectation of the stochastic differential equation (7.19) yields a closed system satisfied by the moments

$$\begin{aligned}\frac{\partial \mathbb{E}[V_{p,q}]}{\partial z} &= -(q+p)\frac{\partial \mathbb{E}[V_{p,q}]}{\partial \tau} - \frac{3\gamma\omega^2}{4}(p-q)^2\mathbb{E}[V_{p,q}]\\ &\quad+\frac{\gamma\omega^2}{4}pq\left(\mathbb{E}[V_{p+1,q+1}] + \mathbb{E}[V_{p-1,q-1}] - 2\mathbb{E}[V_{p,q}]\right).\end{aligned}$$

We now proceed with the computation of the moments.

Consider first the family of moments $f_p(\omega,\tau,z)=\mathbb{E}[V_{p+1,p}(\omega,\tau,z)]$, $p\in\mathbb{N}$. It satisfies the closed system

$$\frac{\partial f_p}{\partial z}=-(2p+1)\frac{\partial f_p}{\partial \tau}+\frac{\gamma\omega^2}{4}\left[p(p+1)(f_{p+1}+f_{p-1}-2f_p)-3f_p\right],$$

starting from $f_p(\omega,\tau,z=-L)=0$. This is a linear system of transport equations starting from a zero initial condition. As a result, the solution is $f_p\equiv 0$ for all p. From $f_0=0$ we see therefore that $\mathbb{E}[V^\varepsilon_{1,0}(\omega,\tau,0)]$ converges to zero as $\varepsilon\to 0$, so that $\mathbb{E}[U^\varepsilon_{1,0}(\omega,h,0)]$ also converges to zero as $\varepsilon\to 0$. Note that the last implication is rigorous in the weak formulation described in [55]. As a consequence, the first moment (7.15) converges to zero:

$$\mathbb{E}[R^\varepsilon_\omega(-L,0)]\stackrel{\varepsilon\to 0}{\longrightarrow}0. \tag{7.20}$$

This result can be generalized as follows. For a fixed positive integer n_0, consider the family of moments $f_p(\omega,\tau,z)=\mathbb{E}[V_{p+n_0,p}(\omega,\tau,z)]$, $p\in\mathbb{N}$. Proceeding as above, the family of functions $(f_p(\omega,\tau,z))_{p\in\mathbb{N}}$ is a solution of a system of transport equations with zero initial conditions. Thus $f_p\equiv 0$, and consequently

$$\mathbb{E}[U^\varepsilon_{p,q}(\omega,h,0)]\stackrel{\varepsilon\to 0}{\longrightarrow}0, \tag{7.21}$$

for $p\neq q$.

Consider now the diagonal family of moments $g_p(\omega,\tau,z)=\mathbb{E}[V_{p,p}(\omega,\tau,z)]$, $p\in\mathbb{N}$. It satisfies the closed system

$$\frac{\partial g_p}{\partial z}=-2p\frac{\partial g_p}{\partial \tau}+\frac{\gamma\omega^2}{4}p^2(g_{p+1}+g_{p-1}-2g_p),$$

starting from $g_p(\omega,\tau,z=-L)=\delta(\tau)\mathbf{1}_0(p)$. This is a linear system of transport equations that admits a nontrivial solution. We have thus identified the limits of the expectations $\mathbb{E}[V^\varepsilon_{p,p}(\omega,\tau,z)]$, $p\in\mathbb{N}$. They converge to $\mathcal{W}_p(\omega,\tau,-L,z)$, which obey the closed system of transport equations

$$\frac{\partial \mathcal{W}_p}{\partial z}+2p\frac{\partial \mathcal{W}_p}{\partial \tau} = (\mathcal{L}_\omega\mathcal{W})_p\,,\quad z\geq -L\,,\ \tau\in\mathbb{R}\,,\ p\in\mathbb{N}\,, \tag{7.22}$$

$$(\mathcal{L}_\omega\phi)_p = \frac{1}{L_{\mathrm{loc}}(\omega)}p^2\left(\phi_{p+1}+\phi_{p-1}-2\phi_p\right)\,, \tag{7.23}$$

starting from

$$\mathcal{W}_p(\omega,\tau,-L,z=-L)=\delta(\tau)\mathbf{1}_0(p)\,.$$

Here $L_{\mathrm{loc}}(\omega)$ is the localization exhibited in Proposition 6.1

$$L_{\mathrm{loc}}(\omega)=\frac{4}{\gamma\omega^2}\,.$$

Using (7.16) and (7.17), we get the limit of the autocorrelation function of the reflection coefficient

$$\begin{aligned}\mathbb{E}\left[R^{\varepsilon}_{\omega+\varepsilon h/2}(-L,0)\overline{R^{\varepsilon}_{\omega-\varepsilon h/2}(-L,0)}\right] &= \mathbb{E}\left[U^{\varepsilon}_{11}(\omega,h,0)\right] \\ &= \int \mathbb{E}\left[V^{\varepsilon}_{11}(\omega,\tau,0)\right]e^{ih\tau}d\tau \\ &\stackrel{\varepsilon\to 0}{\longrightarrow} \int \mathcal{W}_1(\omega,\tau,-L,0)e^{ih\tau}d\tau. \quad (7.24)\end{aligned}$$

More generally, we get

$$\mathbb{E}[U^{\varepsilon}_{p,p}(\omega,h,0)] \stackrel{\varepsilon\to 0}{\longrightarrow} \int \mathcal{W}_p(\omega,\tau,-L,0)e^{ih\tau}d\tau. \tag{7.25}$$

We can summarize the results of this section in the following proposition.

Proposition 7.1. *The expectation of the product of two reflection coefficients at two nearby frequencies,*

$$\mathbb{E}\left[\left(R^{\varepsilon}_{\omega+\varepsilon h/2}(-L,0)\right)^p\left(\overline{R^{\varepsilon}_{\omega-\varepsilon h/2}(-L,0)}\right)^q\right],$$

has the following limit as $\varepsilon\to 0$:
(1) If $p\neq q$, then it converges to 0.
(2) If $p=q$, then it converges to

$$\int \mathcal{W}_p(\omega,\tau,-L,0)e^{ih\tau}d\tau,$$

where $\mathcal{W}_p(\omega,\tau,-L,z)$ is the solution of the system of transport equations (7.22).

It is possible to generalize this proposition to arbitrary moments, by using the same method. We obtain the following proposition [26, Section 9.2].

Proposition 7.2. *The expectation of the product of $2n$ reflection coefficients*

$$\mathbb{E}\left[\prod_{j=1}^{n} R^{\varepsilon}_{\omega_j+\varepsilon h_j/2}(-L,0)\overline{R^{\varepsilon}_{\omega_j-\varepsilon h_j/2}(-L,0)}\right],$$

where n is a positive integer, $(\omega_j)_{1\leq j\leq n}\in\mathbb{R}^n$ are all distinct, and $(h_j)_{1\leq j\leq n}\in\mathbb{R}^n$, converges as $\varepsilon\to 0$ to the limit

$$\prod_{j=1}^{n}\int e^{ih_j\tau_j}\mathcal{W}_1(\omega_j,\tau_j,-L,0)d\tau_j,$$

where $\mathcal{W}_1$ is the solution of the system of transport equations (7.22).

If there is one or several unmatched frequencies in the product of reflection coefficients, then the limit of the moment is zero.

7.2.2. Probabilistic representation of the transport equations In this section we give a probabilistic representation of the solution to the transport equations

(7.22) in terms of a jump Markov process. This representation is helpful because it leads to explicit solutions in some particular cases, and in the general case it provides an efficient Monte Carlo method for numerical simulations.

We introduce the jump Markov process $(N_z)_{z\geq -L}$ with state space $\mathbb{N}$ and infinitesimal generator $\mathcal{L}_\omega$ given by (7.23). The construction of the jump process is as follows. When it reaches the state $n>0$, a random clock with exponential distribution and parameter $2n^2/L_{\rm loc}(\omega)$ starts running. When the clock strikes, the process jumps to $n+1$ or $n-1$ with probability $1/2$. Zero is an absorbing state. Define the process

$$\frac{\partial S_z}{\partial z} = -2N_z,$$

with $S_{-L}=s$. The pair $(N_z,S_z)_{z\geq -L}$ is Markovian with generator

$$\mathcal{L}_\omega - 2n\frac{\partial}{\partial s}.$$

The probabilistic representation of the solution of the Kolmogorov equation

$$\frac{\partial u}{\partial z} = \left(\mathcal{L}_\omega - 2n\frac{\partial}{\partial s}\right)u, \qquad z>-L, \qquad u(n,s,z=-L)=u_0(n,s), \tag{7.26}$$

is

$$\begin{aligned} u(n,s,z) &= \mathbb{E}\left[u_0\left(N_z,S_z\right) \mid N_{-L}=n, S_{-L}=s\right] \\ &= \mathbb{E}\left[u_0\left(N_z, s-2\int_{-L}^z N_{z'}dz'\right) \mid N_{-L}=n\right]. \end{aligned} \tag{7.27}$$

The solution of the transport equations (7.22) is exactly of the form (7.26), so we can use the probabilistic representation in terms of the jump Markov process $(N_z)_{z\geq -L}$. Taking $u_0(n,\tau)=\mathbf{1}_0(n)\delta(\tau)$, we obtain $u(p,\tau,0)=\mathcal{W}_p(\omega,\tau,-L,0)$, which gives

$$\int_{\tau_0}^{\tau_1}\mathcal{W}_p(\omega,\tau,-L,0)d\tau = \mathbb{P}\left(N_0=0\ ,\ 2\int_{-L}^0 N_{z'}dz'\in[\tau_0,\tau_1] \mid N_{-L}=p\right), \tag{7.28}$$

after integrating in τ between τ_0 and τ_1.

From this probabilistic representation of the solution $\mathcal{W}_p$ of the system of transport equations (7.22), we deduce the following hyperbolicity property. If $\tau_1<2L$, then the only paths that can contribute to the probability (7.28) should satisfy

$$2\int_{-L}^0 N_z dz \leq \tau_1 < 2L,$$

and thus N_z, which takes only integer values, has to vanish before reaching 0. We recall that zero is an absorbing state, so that the process stays at zero afterwards. As a result, $\mathcal{W}_p(\omega,\tau,-L,0)$ does not depend on the value of L for $L\geq \tau/2$. This result, derived from the probabilistic representation of the transport equations, is consistent with the hyperbolic nature of the acoustic wave equations in the homogenized medium with finite speed of propagation.

We can give another application of the probabilistic representation (7.28). If we take $h = 0$ in (7.24), then we obtain

$$\mathbb{E}\left[|R_\omega^\varepsilon(-L,0)|^2\right] \stackrel{\varepsilon \to 0}{\longrightarrow} \int \mathcal{W}_1(\omega,\tau,-L,0)d\tau.$$

We thus have a simple probabilistic representation of the limit of the mean-square reflection coefficient

$$\mathbb{E}\left[|R_\omega^\varepsilon(-L,0)|^2\right] \stackrel{\varepsilon \to 0}{\longrightarrow} \mathbb{P}(N_0 = 0 \mid N_{-L} = 1).$$

This can be used to deduce an explicit integral representation of this limiting moment. We have that

$$\lim_{\varepsilon \to 0} \mathbb{E}\left[|R_\omega^\varepsilon(-L,0)|^2\right] = \int \mathcal{W}_1(\omega,\tau,-L,0)d\tau = 1 - \bar{\mathcal{T}}_\omega(L)\,, \tag{7.29}$$

where $\bar{\mathcal{T}}_\omega(L)$ is given by (6.18). We know that

$$\lim_{L\to\infty}\lim_{\varepsilon \to 0} \mathbb{E}\left[|R_\omega^\varepsilon(-L,0)|^2\right] = 1\,,$$

which means total reflection by the random half-space, and implies

$$\lim_{L\to\infty} \int \mathcal{W}_1(\omega,\tau,-L,0)d\tau = 1\,. \tag{7.30}$$

7.2.3. Explicit solution for a random half-space In the limit $L \to \infty$, in which the random slab occupies the full half-space $z \le 0$, we can compute explicitly the solution of the transport equations (7.22). For this we shift the probabilistic representation (7.28) of the solution:

$$\int_{\tau_0}^{\tau_1} \mathcal{W}_p(\omega,\tau,-L,0)d\tau = \mathbb{P}\left(N_L = 0\ ,\ 2\int_0^L N_{z'}dz' \in [\tau_0,\tau_1] \mid N_0 = p\right),$$

where $(N_z)_{z\ge 0}$ is a jump Markov process with state space $\mathbb{N}$ and infinitesimal generator $\mathcal{L}_\omega$ given by (7.23). The process $(N_z)_{z\ge 0}$ behaves like a symmetric random walk on the set of positive integers. It is a well known result from probability theory that it will eventually reach the state 0, and since 0 is an absorbing state, $N_z = 0$ for z large enough. Therefore, the random variable

$$S_\infty = 2\int_0^\infty N_z dz$$

is well defined. As a result

$$\int_{\tau_0}^{\tau_1} \mathcal{W}_p(\omega,\tau,-L,0)d\tau \stackrel{L \to \infty}{\longrightarrow} \mathbb{P}\left(S_\infty \in [\tau_0,\tau_1] \mid N_0 = p\right).$$

The probability density function P_p^∞ of the random variable S_∞ (with the initial condition $N_0 = p$) satisfies the system of differential equations

$$\frac{\partial P_p^\infty}{\partial \tau} = \frac{\gamma\omega^2}{8} p\left(P_{p+1}^\infty - 2P_p^\infty + P_{p-1}^\infty\right),$$

with $P_0^\infty(\tau) = \delta(\tau)$ and P_p^∞ does not have a Dirac mass at $\tau = 0$. The solution of this system is

$$P_p^\infty(\omega, \tau) = \frac{\partial}{\partial \tau}\left[\left(\frac{\gamma\omega^2\tau}{8+\gamma\omega^2\tau}\right)^p \mathbf{1}_{[0,\infty)}(\tau)\right]$$

Therefore, the solution for the system of transport equations (7.22) has the limit, as $L \to \infty$,

$$\lim_{L\to\infty} \mathcal{W}_p(\omega, \tau, -L, 0) = P_p^\infty(\tau) \ . \tag{7.31}$$

In particular, the function $\mathcal{W}_1$ that appears in the limit expression (7.24) of the autocorrelation function of the reflection coefficient has the limit, as $L \to \infty$,

$$\lim_{L\to\infty} \mathcal{W}_1(\omega, \tau, -L, 0) = \frac{8\gamma\omega^2}{(8+\gamma\omega^2\tau)^2}\mathbf{1}_{[0,\infty)}(\tau) \tag{7.32}$$

By integrating the right-hand side in (7.32) with respect to τ, we recover the result (7.30), which implies total reflection by the random half-space.

7.3. Statistics of the reflected wave in the time domain

7.3.1. Mean amplitude By (7.12) the mean amplitude of the reflected wave is

$$\mathbb{E}[A^\varepsilon(t,0)] = \frac{1}{2\pi\sqrt{\varepsilon}} \int \mathbb{E}[R_\omega^\varepsilon(-L,0)]\hat{f}(\omega)e^{-i\frac{\omega t}{\varepsilon}}d\omega. \tag{7.33}$$

By (7.20), we know that $\mathbb{E}[R_\omega^\varepsilon(-L,0)] \to 0$ as $\varepsilon \to 0$. In fact, in we follow the proof of the diffusion-approximation theorem, then we get that $\mathbb{E}[R_\omega^\varepsilon(-L,0)]$ converges to 0 with an error of order ε, which neutralizes the singular factor $1/\sqrt{\varepsilon}$, and thus we get the expected result

$$\mathbb{E}[A^\varepsilon(t,0)] \stackrel{\varepsilon\to 0}{\longrightarrow} 0. \tag{7.34}$$

The mean amplitude is vanishing in the limit $\varepsilon \to 0$, which means that the reflected wave is incoherent.

7.3.2. Mean intensity We consider the representation (7.13) of the mean intensity:

$$\mathbb{E}[A^\varepsilon(t,0)^2] = \frac{1}{4\pi^2}\int\int \mathbb{E}[U_{11}^\varepsilon(\omega,h,0)]\hat{f}(\omega+\varepsilon h/2)\overline{\hat{f}(\omega-\varepsilon h/2)}e^{-iht}d\omega\, dh.$$

From (7.24) we know the limit of the expectation that appears in this integral, so that we can write

$$\mathbb{E}[A^\varepsilon(t,0)^2] \stackrel{\varepsilon\to 0}{\longrightarrow} I(t),$$

with

$$
\begin{aligned}
I(t) &= \frac{1}{4\pi^2}\int\int\int \mathcal{W}_1(\omega,\tau,-L,0)|\hat{f}(\omega)|^2 e^{ih(\tau-t)}dh\,d\tau\,d\omega \\
&= \frac{1}{4\pi^2}\int\int \mathcal{W}_1(\omega,\tau,-L,0)|\hat{f}(\omega)|^2 2\pi\delta(\tau-t)d\tau\,d\omega \\
&= \frac{1}{2\pi}\int \mathcal{W}_1(\omega,t,-L,0)|\hat{f}(\omega)|^2 d\omega. \qquad (7.35)
\end{aligned}
$$

Integrating (7.35) with respect to t we obtain the total reflected energy. Using (7.29), we have the explicit expression

$$\int I(t)dt = \frac{1}{2\pi}\int \left[1-\bar{\mathcal{T}}_\omega(L)\right]|\hat{f}(\omega)|^2 d\omega,$$

where $\bar{\mathcal{T}}_\omega(L)$ is given by (6.18).

For L large enough, the reflected intensity at a given time t does not depend on L. This is a simple consequence of the hyperbolicity of the acoustic wave equation with a bounded speed of propagation. This has also been pointed out in Section 7.2.2, where we have shown with the probabilistic representation of $\mathcal{W}_1(\omega,\tau,-L,0)$ that it does not depend on L for $L \geq \tau/2$. In particular, the transmitted intensity (7.35) does not depend on L for L large enough, and therefore it is equal to its limit as $L \to \infty$. In the case of the random half-space analyzed in Section 7.2.3 we have the explicit formula (7.32) for $\mathcal{W}_1$, leading to

$$I^\infty(t) = \frac{1}{2\pi}\int \frac{2/L_{\mathrm{loc}}(\omega)}{(2+t/L_{\mathrm{loc}}(\omega))^2}|\hat{f}(\omega)|^2 d\omega.$$

where $L_{\mathrm{loc}}(\omega) = 4/(\gamma\omega^2)$ is the localization length exhibited in Proposition 6.1. As noted in Section 7.2.3, the total reflected energy equals the total incident energy,

$$\int I^\infty(t)dt = \frac{1}{2\pi}\int |\hat{f}(\omega)|^2 d\omega = \int f(t)^2 dt\,,$$

which confirms that the wave has been completely reflected by the random medium, as predicted by the localization theory.

When the incident signal is a narrowband pulse with carrier frequency ω_0 and energy $E_0 = \int f(t)^2 dt$, the mean reflected intensity is approximately

$$I^\infty(t) = \frac{2E_0/L_{\mathrm{loc}}(\omega_0)}{(2+t/L_{\mathrm{loc}}(\omega_0))^2} = \frac{E_0/t_0}{(1+t/t_0)^2}\ , \qquad (7.36)$$

where $t_0 = 2L_{\mathrm{loc}}(\omega_0)$. This slow power law decay as t^{-2} is typical of one-dimensional random media that produce reflections that continue for a long time. Half the reflected energy is captured in the time interval $[0,t_0]$. The rough picture is that the wave penetrates into the medium up to the distance $L_{\mathrm{loc}}(\omega_0)$, and then it is scattered back, which takes $t_0 = 2L_{\mathrm{loc}}(\omega_0)$ time.

7.3.3. Autocorrelation and time-domain localization We now consider the local time autocorrelation function of the reflected signal at a fixed time t with lag

εs, on the scale of the incident pulse, which is defined by

$$c_t^\varepsilon(s) = \mathbb{E}[A^\varepsilon(t,0)A^\varepsilon(t+\varepsilon s,0)].$$

Using the integral representation (7.11), we have

$$c_t^\varepsilon(s) = \frac{1}{4\pi^2}\int\int \mathbb{E}[U_{11}^\varepsilon(\omega,h,0)]\hat{f}(\omega+\varepsilon h/2)\overline{\hat{f}(\omega-\varepsilon h/2)}e^{-iht+i\omega s+i\varepsilon hs}d\omega\, dh.$$

Taking the limit $\varepsilon \to 0$, using the finite energy of the pulse and Proposition 7.1, we have that

$$c_t^\varepsilon(s) \overset{\varepsilon\to 0}{\longrightarrow} c_t(s),$$

where c_t is given by

$$\begin{aligned} c_t(s) &= \frac{1}{4\pi^2}\int\int\int \mathcal{W}_1(\omega,\tau,-L,0)|\hat{f}(\omega)|^2 e^{ih(\tau-t)}e^{i\omega s}d\tau\, d\omega\, dh \\ &= \frac{1}{2\pi}\int \mathcal{W}_1(\omega,t,-L,0)|\hat{f}(\omega)|^2 e^{i\omega s}d\omega. \end{aligned} \tag{7.37}$$

We see therefore that the local power spectral density of the reflected wave around time t is $\mathcal{W}_1(\omega,t,-L,0)|\hat{f}(\omega)|^2$.

In the case of a random half-space, $\mathcal{W}_1$ is given by (7.32), and so

$$\mathcal{W}_1^\infty(\omega,t,0) = \frac{2/L_{\mathrm{loc}}(\omega)}{(2+t/L_{\mathrm{loc}}(\omega))^2}\mathbf{1}_{[0,\infty)}(t)\,.$$

For a fixed time t the maximum of this quantity over ω is attained at $\omega^*(t)$, where

$$t = \frac{2}{L_{\mathrm{loc}}(\omega^*(t))}\,, \tag{7.38}$$

or

$$\omega^*(t) = \sqrt{\frac{2t}{\gamma}}\,.$$

We interpret this as follows. Assuming that $|\hat{f}(\omega)|$ is flat over its bandwidth, then the maximum of the local power spectral density of the reflected signal at time t is at $\omega^*(t)$, which is defined by (7.38). This is the frequency for which waves travel to a distance equal to the localization length $L_{\mathrm{loc}}(\omega^*)$ and back. This provides a time-domain interpretation of the localization length as the distance from which the most scattered energy is carried by the reflected waves.

It is also possible to show that the sequence of processes $(A^\varepsilon(t+\varepsilon s,0))_{-\infty<s<\infty}$, with t fixed, converges as $\varepsilon \to 0$ in distribution to a Gaussian process. This is done by showing that for any smooth test function $g(s)$, the sequence of random variables

$$A_{t,g}^\varepsilon = \int A^\varepsilon(t+\varepsilon s,0)g(s)ds$$

converges in distribution to a Gaussian random variable as $\varepsilon \to 0$. This requires to compute the limiting moments of $A^\varepsilon_{t,g}$. The final result is given in the following proposition [26, Section 9.3].

Proposition 7.3. *The reflected wave around some time t, on the scale ε,*

$$A^\varepsilon(t + \varepsilon s, 0),$$

converges as $\varepsilon \to 0$ as a process in s to $(\mathcal{A}_t(s))_{-\infty < s < \infty}$, which is a stationary Gaussian process with mean zero and autocorrelation function

$$\mathbb{E}[\mathcal{A}_t(s')\mathcal{A}_t(s'+s)] = \frac{1}{2\pi} \int \mathcal{W}_1(\omega, t, -L, 0)|\hat{f}(\omega)|^2 e^{i\omega s} d\omega \,.$$

Here $\mathcal{W}_1$ is the solution of the system of transport equations (7.22).

7.4. Bibliographic notes

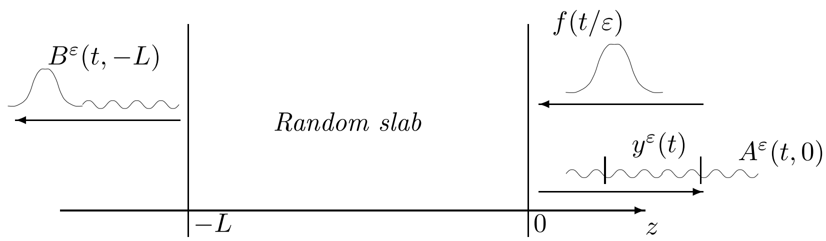
The statistics of the incoherent waves presented here in a self-contained way have been derived in the series of papers [4, 5, 14, 15, 16, 40, 41, 53, 55, 61, 62, 71, 70]. The analysis of the transmission coefficient can be found in [26, Chapter 9].

8. Time reversal

In this section we introduce the concept of time reversal of waves. We consider the case of time reversal in reflection, in which a source emits a pulse at one end of a one-dimensional slab, and a time-reversal mirror (TRM) placed at the same location records the reflected signal. The mirror then reemits a part of the recorded signal trace in the reverse direction of time, so that what is recorded last is sent first (last-in-first-out at the mirror). This is in contrast to a standard mirror, which corresponds to first-in-first-out. This basic time reversal setup is illustrated in Figure 6. The remarkable properties of time reversal in random media are (i) the **refocusing** (or recompression) of the wave field at a given deterministic time (Section 8.2) (ii) the **statistical stability** of the refocused pulse (Section 8.3). We will see that the degree or quality of refocusing and stability depends on how much of the reflected signal is recorded. This section is a shortened version of [26, Chapter 10].

8.1. Time-reversal setup

We again consider a random slab $(-L, 0)$ embedded in a homogeneous medium with no background discontinuities. A pulse of the form $f(t/\varepsilon)$ incoming from the



(a) A left-going pulse $f(t/\varepsilon)$ is on impinging the random slab $(-L,0)$ and it generates a reflected signal $A^\varepsilon(t,0)$. The time-reversal-mirror (TRM), used in a passive mode, records a segment $y^\varepsilon(t)$ of the reflected signal.

(b) The TRM is used as an active device that sends back in the medium the signal $y^\varepsilon(t_1-t)$. We observe the new reflected signal $A^\varepsilon_{\rm new}(t,0)$.

Figure 6: Setup for a time reversal in reflection (TRR) experiment.

right homogeneous half-space is scattered by the random slab. We have seen in (7.11) that the reflected wave $A^\varepsilon(t,0)$ has the form:

$$A^\varepsilon(t,0) = \frac{1}{2\pi}\int R^\varepsilon_\omega(-L,0)\hat{f}(\omega)e^{-\frac{i\omega t}{\varepsilon}}d\omega, \tag{8.1}$$

where $R^\varepsilon_\omega(-L,0)$ is the reflection coefficient defined by (7.7). The first step in the time-reversal consists in recording the reflected signal at $z=0$. It turns out that as $\varepsilon \to 0$, the interesting asymptotic regime arises when we record the signal up to a large time of order one, which we denote by t_1 (with $t_1 > 0$). A segment of the recorded signal with support of order one is clipped using a cutoff function $G(t)$. We denote the recorded part of the wave by y^ε so that

$$y^\varepsilon(t) = A^\varepsilon(t,0)G(t).$$

We then time-reverse this segment of signal about t_1 and send it back into the same medium as shown in Figure 6. This means that we have a new scattering problem defined by the same acoustic equations, but with the new incoming signal

$$f^\varepsilon_{\rm new}(t) = y^\varepsilon(t_1-t) = A^\varepsilon(t_1-t,L)G(t_1-t),$$

which corresponds to a left-going wave incoming from the right homogeneous half-space. Since we are dealing with real-valued signals, we can write

$$A^\varepsilon(t,0) = \overline{A^\varepsilon(t,0)} \quad = \quad \frac{1}{2\pi}\int \overline{R^\varepsilon_\omega(-L,0)}\;\overline{\hat{f}(\omega)}e^{\frac{i\omega t}{\varepsilon}}\,d\omega\,,$$

so that the scaled Fourier transform of the new incoming signal has the form

$$\begin{aligned}
\hat{f}^\varepsilon_{\rm new}(\omega) &= \int e^{\frac{i\omega t}{\varepsilon}} A^\varepsilon(t_1-t,0)G(t_1-t)\,dt \\
&= \varepsilon\int e^{i\omega s}A^\varepsilon(t_1-\varepsilon s,0)G(t_1-\varepsilon s)\,ds \\
&= \varepsilon\int e^{i\omega s}\left\{\frac{1}{2\pi}\int e^{-i\omega' s}\;\overline{R^\varepsilon_{\omega'}(-L,0)}\;\overline{\hat{f}(\omega')}e^{\frac{i\omega' t_1}{\varepsilon}}\,d\omega'\right\}\;G(t_1-\varepsilon s)\,ds \\
&= \frac{\varepsilon}{2\pi}\int\;\overline{R^\varepsilon_{\omega'}(-L,0)}\;\overline{\hat{f}(\omega')}\left\{\int e^{i(\omega'-\omega)(-s)}\;G(t_1-\varepsilon s)\;ds\right\}e^{\frac{i\omega' t_1}{\varepsilon}}\,d\omega' \\
&= \frac{1}{2\pi}\int\;\overline{R^\varepsilon_{\omega'}(-L,0)}\;\overline{\hat{f}(\omega')}\;\overline{\hat{G}\left(\frac{\omega-\omega'}{\varepsilon}\right)}e^{\frac{i\omega t_1}{\varepsilon}}\,d\omega'\,.
\end{aligned}$$

The new incoming signal is scattered by the random slab and gives rise to a reflected wave $A^\varepsilon_{\rm new}(t,0)$ at $z=0$ and a transmitted wave $B^\varepsilon_{\rm new}(t,-L)$ at $z=-L$. The reflected signal observed in the time domain around the observation time $t_{\rm obs}$ on the scale ε is given by

$$S^\varepsilon_L(t_{\rm obs}+\varepsilon s) \quad := \quad A^\varepsilon_{\rm new}(t_{\rm obs}+\varepsilon s,0) = \frac{1}{2\pi\varepsilon}\int e^{-i\omega(s+\frac{t_{\rm obs}}{\varepsilon})}R^\varepsilon_\omega(-L,0)\hat{f}^\varepsilon_{\rm new}(\omega)\,d\omega\,.$$

Substituting the expression of $\hat{f}^\varepsilon_{\rm new}$ into this equation gives the integral representation of the reflected signal

$$S_L^\varepsilon(t_{\rm obs}+\varepsilon s) = \frac{1}{(2\pi)^2\varepsilon}\int\int e^{-i\omega_1 s} e^{i\frac{\omega_1(t_1-t_{\rm obs})}{\varepsilon}}\overline{\hat{f}(\omega_2)}\,\overline{\hat{G}\left(\frac{\omega_1-\omega_2}{\varepsilon}\right)} \times\overline{R^\varepsilon_{\omega_2}(-L,0)}R^\varepsilon_{\omega_1}(-L,0)\,d\omega_1\,d\omega_2\,.$$

Motivated by the scaled argument in $\hat{G}$ we use the change of variables $\omega_1 = \omega+\varepsilon h/2$, $\omega_2=\omega-\varepsilon h/2$ and get

$$S_L^\varepsilon(t_{\rm obs}+\varepsilon s) = \frac{1}{(2\pi)^2}\int\int e^{-i\omega s} e^{i\frac{\omega(t_1-t_{\rm obs})}{\varepsilon}} e^{ih(t_1-t_{\rm obs})/2-i\varepsilon hs/2}\overline{\hat{f}(\omega-\varepsilon h/2)} \times\overline{\hat{G}\,(h)}\,\overline{R^\varepsilon_{\omega-\varepsilon h/2}(-L,0)}R^\varepsilon_{\omega+\varepsilon h/2}(-L,0)\,dh\,d\omega\,. \tag{8.2}$$

We will analyze the behavior of this reflected signal in the limit $\varepsilon\to 0$.

8.2. Time-reversal refocusing

We first observe that the signal (8.2), recorded at $z=0$, vanishes in the limit $\varepsilon\to0$ if the time of observation $t_{\rm obs}$ is not the time of recording t_1. Indeed, the rapid phase $\exp(i\omega(t_1-t_{\rm obs})/\varepsilon)$ averages out the integral except when $t_{\rm obs}=t_1$. This means that

refocusing can be observed only at the time $t_{\rm obs}=t_1$.

In other words, an observer located at $z=0$ detects no coherent signal at any time different from t_1. The observed small incoherent wave fluctuations vanish in the limit $\varepsilon\to0$. This is what is called time-reversal refocusing, and the precise description of the refocused pulse observed at time t_1 is carried out in the next section. The refocused pulse at time $t_{\rm obs}=t_1$ has the form

$$S_L^\varepsilon(t_1+\varepsilon s) = \frac{1}{(2\pi)^2}\int\int e^{-i\omega s-i\varepsilon hs/2}\overline{\hat{f}(\omega-\varepsilon h/2)}\,\overline{\hat{G}\,(h)} \times R^\varepsilon_{\omega+\varepsilon h/2}(-L,0)\overline{R^\varepsilon_{\omega-\varepsilon h/2}(-L,0)}\,dh\,d\omega\,. \tag{8.3}$$

Note that the product of reflection coefficients that appears in this integral has been analyzed extensively in Section 7.

8.3. The limiting refocused pulse

The uniform boundedness of the reflection coefficient, which follows from the conservation of energy as given in (7.10), implies that the finite-dimensional distributions of the process $S_L^\varepsilon(t_1+\varepsilon\cdot)$ will be characterized by the moments

$$\mathbb{E}[S_L^\varepsilon(t_1+\varepsilon s_1)^{p_1}\cdots S_L^\varepsilon(t_1+\varepsilon s_k)^{p_k}] \tag{8.4}$$

for all real number in the range $s_1 < \cdots < s_k$ and all integer $p_1, \ldots, p_k$.

8.3.1. First Moment We start by considering the first moment. Using the representation (8.3), the expected value of $S_L^\varepsilon(t_1 + \varepsilon s)$ is

$$\begin{aligned}\mathbb{E}[S_L^\varepsilon(t_1+\varepsilon s)] &= \frac{1}{(2\pi)^2}\int\int e^{-i\omega s}e^{-i\varepsilon hs/2}\overline{\hat{f}(\omega-\varepsilon h/2)}\,\overline{\hat{G}(h)} \\ &\quad\times\mathbb{E}\left[R^\varepsilon_{\omega+\varepsilon h/2}(-L,0)\overline{R^\varepsilon_{\omega-\varepsilon h/2}(-L,0)}\right]\,dh\,d\omega\,.\end{aligned}$$

Taking the limit $\varepsilon \to 0$ and applying Proposition 7.1 gives

$$\begin{aligned}\mathbb{E}[S_L^\varepsilon(t_1+\varepsilon s)] &\overset{\varepsilon\to 0}{\longrightarrow} \frac{1}{(2\pi)^2}\int\int e^{-i\omega s}\overline{\hat{f}(\omega)\hat{G}(h)}\left[\int e^{ih\tau}\mathcal{W}_1(\omega,\tau,-L,0)\,d\tau\right]dh\,d\omega \\ &= \frac{1}{(2\pi)^2}\int\int e^{-i\omega s}\overline{\hat{f}(\omega)}\left[\int\overline{\hat{G}(h)}e^{ih\tau}dh\right]\mathcal{W}_1(\omega,\tau,-L,0)\,d\tau\,d\omega \\ &= \frac{1}{2\pi}\int\int e^{-i\omega s}\overline{\hat{f}(\omega)}G(\tau)\mathcal{W}_1(\omega,\tau,-L,0)\,d\tau\,d\omega\,,\end{aligned}$$

where the quantity $\mathcal{W}_1(\omega,\tau,-L,0)$ is obtained by solving the system of transport equations (7.22). We have also used the fact that G is real-valued.

8.3.2. Higher Order Moments Let us now consider the general moments (8.4). Using the representation (8.3) for each factor $S_L^\varepsilon(t_1+\varepsilon s_j)$, these moments can be written as multiple integrals over $p = \sum_{j=1}^k p_j$ frequencies:

$$\begin{aligned}&\mathbb{E}\left[\prod_{j=1}^k S_L^\varepsilon(t_1+\varepsilon s_j)^{p_j}\right] \\ &= \frac{1}{(2\pi)^{2p}}\int\cdots\int\prod_{\substack{1\le j\le k\\1\le l\le p_j}}\overline{\hat{f}(\omega_{j,l})}e^{-i\omega_{j,l}s_j}e^{-i\varepsilon h_{j,l}s_j/2}\overline{\hat{G}(h_{j,l})} \\ &\times\mathbb{E}\left[\prod_{\substack{1\le j\le k\\1\le l\le p_j}}R^\varepsilon_{\omega_{j,l}+\varepsilon h_{j,l}/2}(-L,0)\overline{R^\varepsilon_{\omega_{j,l}-\varepsilon h_{j,l}/2}(-L,0)}\right]\prod_{\substack{1\le j\le k\\1\le l\le p_j}}d\omega_{j,l}\,dh_{j,l}\,.\end{aligned}$$

The important quantity is the expectation of the product of reflection coefficients, whose limit as $\varepsilon \to 0$ is given by Proposition 7.2. As a result, taking the limit

$\varepsilon \to 0$ gives

$$\mathbb{E}\left[\prod_{j=1}^{k} S_L^\varepsilon(t_1+\varepsilon s_j)^{p_j}\right] \stackrel{\varepsilon\to 0}{\longrightarrow} \frac{1}{(2\pi)^p}\int\cdots\int \prod_{\substack{1\le j\le k\\ 1\le l\le p_j}} \mathcal{W}_1(\omega_{j,l},\tau_{j,l},-L,0)$$

$$\prod_{\substack{1\le j\le k\\ 1\le l\le p_j}} \overline{\hat{f}(\omega_{j,l})}e^{-i\omega_{j,l}s_j}G(\tau_{j,l})\,d\omega_{j,l}\,d\tau_{j,l}$$

$$= \prod_{1\le j\le k}\left(\frac{1}{2\pi}\int \mathcal{W}_1(\omega,\tau,-L,0)\overline{\hat{f}(\omega)}e^{-i\omega s_j}G(\tau)d\omega\,d\tau\right)^{p_j}.$$

This shows that the expectation of a product of terms $S_\varepsilon^L(t_1+\varepsilon s)$ converges to the product of the limits of the expectations:

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\prod_{j=1}^{k} S_L^\varepsilon(t_1+\varepsilon s_j)^{p_j}\right] \quad = \quad \prod_{j=1}^{k}\lim_{\varepsilon\to 0}\mathbb{E}\left[S_L^\varepsilon(t_1+\varepsilon s_j)^{p_j}\right].$$

This result is in dramatic contrast to the statistical description of the reflected wave before time reversal in terms of a Gaussian process (see Proposition 7.3). We have therefore shown that the finite-dimensional distributions of $(S_\varepsilon^L(t_1+\varepsilon s))_{s\in(-\infty,\infty)}$ converge to those of the deterministic function

$$\frac{1}{2\pi}\int \mathcal{W}_1(\omega,\tau,-L,0)\overline{\hat{f}(\omega)}e^{-i\omega s}G(\tau)\,d\omega\,d\tau\,.$$

8.3.3. Tightness We have characterized the limiting refocused pulse in terms of its finite-dimensional time distributions. In fact, a tightness argument shows that this limit holds in the sense of the convergence in distribution for continuous processes. This is done by showing that the sequence of processes $S_L^\varepsilon(t_1+\varepsilon\cdot)$, $\varepsilon>0$, is precompact in the space of continuous functions (see [44]). On the one hand, the conservation of energy relation yields that $|R_\omega^\varepsilon|\le 1$ and $S_L^\varepsilon(t_1+\varepsilon s)$ is uniformly bounded by

$$|S_L^\varepsilon(t_1+\varepsilon s)| \le \frac{1}{(2\pi)^2}\int|\hat{f}(\omega)|\,d\omega\times\int|\hat{G}(h)|\,dh\,. \tag{8.5}$$

On the other hand, the modulus of continuity

$$M^\varepsilon(\delta) \quad = \sup_{|s_1-s_2|\le\delta}|S_L^\varepsilon(t_1+\varepsilon s_1)-S_L^\varepsilon(t_1+\varepsilon s_2)|$$

is bounded by

$$M^\varepsilon(\delta)\le\frac{1}{(2\pi)^2}\int\sup_{|s_1-s_2|\le\delta}|1-\exp(i\omega(s_1-s_2))||\hat{f}(\omega)|d\omega\times\int|\hat{G}(h)|\,dh\,,$$

which goes to zero as δ goes to zero uniformly with respect to ε. As a result, the refocused pulse $((S_L^\varepsilon(t_1+\varepsilon s))_{-\infty<s<\infty})_{\varepsilon>0}$ is a tight (i.e., weakly compact) family in the space of continuous trajectories equipped with the supremum norm.

8.3.4. Convergence of the refocused pulse We have just shown the tightness of the process $\left(S_L^\varepsilon(t_1+\varepsilon s)\right)_{s\in(-\infty,\infty)}$ as well as the convergence of its finite-dimensional distributions. Accordingly, we have shown that this process converges in probability as $\varepsilon \to 0$ to the deterministic function

$$S_L(s) \quad = \quad \frac{1}{2\pi}\int \Lambda_{\mathrm{TRR}}^L(\omega,\tau)\overline{\hat{f}(\omega)}e^{-i\omega s}G(\tau)\,d\omega\,d\tau\,,$$

where $\Lambda_{\mathrm{TRR}}^L(\omega,\tau) = \mathcal{W}_1(\omega,\tau,-L,0)$ is given by (7.22) and TRR stands for "time reversal in reflection." We summarize this result in the following proposition.

Proposition 8.1. *The refocused signal* $\left(S_L^\varepsilon(t_1+\varepsilon s)\right)_{s\in(-\infty,\infty)}$ *converges in probability as* $\varepsilon \to 0$ *to the deterministic pulse shape*

$$S_L(s) = \left(f(-\,\cdot) * K_{\mathrm{TRR}}(\cdot)\right)(s), \tag{8.6}$$

where the Fourier transform of the refocusing kernel K_{TRR} *is given by*

$$\hat{K}_{\mathrm{TRR}}(\omega) = \int G(\tau)\Lambda_{\mathrm{TRR}}^L(\omega,\tau)\,d\tau\,, \tag{8.7}$$

and the refocusing density $\Lambda_{\mathrm{TRR}}^L(\omega,\tau) = \mathcal{W}_1(\omega,\tau,-L,0)$ *is given by the system (7.22).*

If the medium is homogeneous, that is, $\gamma = 0$, then the refocusing kernel is zero. Indeed, in this case nothing is recorded by the TRM, since the initial pulse simply travels to the left without scattering. If the medium is random, $\gamma > 0$, then we get the striking result that we observe a refocused pulse whose shape does not depend on the particular realization of the medium, but only on its statistical distribution through the parameter γ. This is the statistical stability property of the refocused pulse. In the next paragraph we examine a particular case in which an explicit formula can be derived for the refocusing kernel.

8.3.5. The refocusing kernel for a half-space We consider the case of a random half-space, that is, $L \to \infty$. We have computed explicitly the solution for the system of transport equations in this case (see (7.31)). We thus get a closed-form expression for the refocusing local spectral density $\Lambda_{\mathrm{TRR}}^\infty$ in this case

$$\Lambda_{\mathrm{TRR}}^\infty(\omega,\tau) = \frac{8\gamma\omega^2}{\left(8+\gamma\omega^2\tau\right)^2} = \frac{2/L_{\mathrm{loc}}(\omega)}{\left(2+\tau/L_{\mathrm{loc}}(\omega)\right)^2}\,, \tag{8.8}$$

where $L_{\mathrm{loc}}(\omega) = 4/(\gamma\omega^2)$ is the localization length exhibited in Proposition 6.1. If we also assume that $G(t) = \mathbf{1}_{[0,t_1]}(t)$, then by computing the integral in (8.7) we find that the refocusing kernel is

$$\hat{K}_{\mathrm{TRR}}(\omega) \quad = \quad \frac{\gamma\omega^2 t_1}{8+\gamma\omega^2 t_1} = \frac{t_1/L_{\mathrm{loc}}(\omega)}{2+t_1/L_{\mathrm{loc}}(\omega)}\,.$$

Note that if we assume that we record everything at the mirror ($t_1 = \infty$ and $G \equiv 1$), then $\hat{K}_{\mathrm{TRR}}(\omega) = 1$. This is of course expected: the pulse has been completely scattered back by the random half-space due to localization, as seen in Section 7. We have sent back everything that has been recorded, so we get a perfect refocusing as a result of the time-reversibility of the wave equation.

If $t_1 < \infty$, then the kernel $\hat{K}_{\rm TRR}$ has the form of a high-pass filter with cutoff frequency

$$\omega_c^2 = \frac{8}{\gamma t_1}.$$

Frequencies above ω_c are recovered in the refocused pulse but frequencies below ω_c are lost. The reason is that even though the medium is completely reflecting because of the localization effect, time does play a role. High frequencies have a very short localization length, given by $L_{\rm loc}(\omega) = 4/(\gamma\omega^2)$, so that they are scattered back very quickly by the medium. Low frequencies have a large localization length, so they can penetrate deep into the medium, and it takes more time for them to be reflected. We saw in Section 7.3.2 that they spend an average time on the order of $2L_{\rm loc}(\omega)$ in the medium. As a result, if this time is larger than t_1, then they are not recorded by the TRM during the recording time window. The relation $2L_{\rm loc}(\omega) \le t_1$ gives the bandwidth of the refocusing kernel $|\omega| \le \omega_c$.

8.4. Bibliographic notes

The reflected signal and its spectral content have been studied in the regime of separation of scales in [4], [5], [14], [15], [16]. Refocusing and self-averaging for time reversal in reflection in the one-dimensional case was derived in 1997 by Clouet and Fouque in the article [20]. An iterative time-reversal method to estimate higher moments is also presented in that reference.

Appendix A. The random harmonic oscillator

The random harmonic oscillator:

$$y_{tt} + (\kappa + \sigma\eta(\frac{t}{\varepsilon}))y = 0 \tag{A.1}$$

with $\eta(t)$ a random process arises in many physical contexts such as solid state physics [45, 29, 57], vibrations in mechanical and electrical circuits [64, 67], and wave propagation in one-dimensional random media [39, 4]. The dimensionless parameter $\varepsilon > 0$ (resp. σ) characterizes the correlation length (resp. the amplitude) of the random fluctuations.

A.1. Lyapunov exponents

The sample Lyapunov exponent governs the exponential growth of the modulation:

$$G := \lim_{t\to\infty} \frac{1}{t} \ln r(t), \qquad r(t) = \sqrt{|y(t)|^2 + |y_t(t)|^2}. \tag{A.2}$$

Note that G could be random since η is random. So it should be relevant to study the mean and fluctuations of the Lyapunov exponent. For this purpose we shall analyze the normalized Lyapunov exponent which governs the exponential growth of the p-th moment of the modulation:

$$G_p := \lim_{t\to\infty} \frac{1}{pt} \ln \mathbb{E}\left[r(t)^p\right], \tag{A.3}$$

where $\mathbb{E}$ stands for the expectation with respect to the distribution of the process η. If the process were deterministic, then we would have $G_p = G$ for every p. But due to randomness this may not hold true since we can not invert the nonlinear power function "$|.|^p$" and the linear statistical averaging "$\mathbb{E}[.]$". The random matrix products theory applies to the problem (A.1). For instance let us assume that the random process η is piecewise constant over intervals $[n, n+1)$ and take random values on the successive intervals. Under appropriate assumptions on the laws of the values taken by η, it is proved in Ref. [6, Theorem 4] that there exists an analytic function $g(p)$ such that:

$$\lim_{t\to\infty} \frac{1}{t} \ln \mathbb{E}\left[|r(t)|^p\right] = g(p), \tag{A.4}$$

$$\lim_{t\to\infty} \frac{1}{t} \ln r(t) = g'(0) \text{ almost surely}, \tag{A.5}$$

$$\frac{\ln r(t) - tg'(0)}{\sqrt{t}} \stackrel{dist.}{\longrightarrow} \mathcal{N}(0, g''(0)). \tag{A.6}$$

Moreover the convergence is uniform for $r(t=0)$ with unit modulus, and the function $p \mapsto g(p)/p$ is monotone increasing. This proves in particular that $G = g'(0)$ is non-random. In case of non-piecewise constant processes η, various versions of the above theorem exist which yield the same conclusion [30, 66, 11, 2]. Unfortunately the expression of $g(p)$ is very intricate, even for very simple random processes η. In the following sections we shall derive closed form expressions for the expansion of the sample Lyapunov exponent with respect to a small parameter.

We shall assume in the following that the driving random process $\eta(t)$ is built from a Markov process $m(t)$ by a smooth bounded function $\eta(t) = c(m(t))$. As a consequence η may be non-Markovian itself.

A.2. Small perturbations

We shall assume here that the perturbation is slow $\varepsilon = 1$, but weak $\sigma \ll 1$. There are two cases that should be distinguished: the case in which $\kappa = \sigma$ and the case in which $\kappa = 1$. Introducing polar coordinates $(r(t), \psi(t))$ as $y(t) = r(t)\cos(\psi(t))$

and $y_t(t) = r(t)\sin(\psi(t))$ the system (A.1) is equivalent to:

$$r(t) = r_0 \exp\left(\int_0^t q(\psi(s), m(s))ds\right), \tag{A.7}$$

$$\psi_t(t) = h(\psi(t), m(t)), \tag{A.8}$$

with $q(\psi, m) = q_0(\psi) + \sigma q_1(\psi, m)$ and $h(\psi, m) = h_0(\psi) + \sigma h_1(\psi, m)$:

$$\text{if } \kappa = 1 \qquad \begin{cases} q_0(\psi) = 0, & q_1(\psi, m) = -c(m)\sin(\psi)\cos(\psi), \\ h_0(\psi) = -1, & h_1(\psi, m) = -c(m)\cos^2(\psi) \end{cases}$$

$$\text{if } \kappa = \sigma \qquad \begin{cases} q_0(\psi) = 0, & q_1(\psi, m) = -c(m)\sin(\psi)\cos(\psi), \\ h_0(\psi) = 0, & h_1(\psi, m) = -1 - c(m)\cos^2(\psi). \end{cases}$$

Let us assume that the process m is an ergodic Markov process with infinitesimal generator Q on a manifold $\mathbb{M}$ with invariant probability $\pi(dm)$. From Eq. (A.8) (ψ, m) is a Markov process on the state space $\mathbb{S}^1 \times \mathbb{M}$ where $\mathbb{S}^1$ denotes the circumference of the unit circle with infinitesimal generator: $\mathcal{L} = Q + h(\psi, m)\frac{\partial}{\partial\psi}$ and with invariant measure $\bar{p}(\psi, m)d\psi\pi(dm)$ where $\bar{p}$ can be obtained as the solution of $\mathcal{L}^*\bar{p} = 0$. According to the theorem of Crauel [21] the long-time behavior of $r(t)$ can be expressed in terms of the Lyapunov exponent G which is given by:

$$G = \int_{\mathbb{S}^1\times\mathbb{M}} q(\psi, m)\bar{p}(\psi, m)d\psi\pi(dm). \tag{A.9}$$

This result and the following ones hold true in particular under condition *H1* [59] or *H2* [3]:

H1 $\mathbb{M}$ is a finite set and Q is a finite-dimensional matrix which generates a continuous parameter irreducible, time-reversible Markov chain.

H2 $\mathbb{M}$ is a compact manifold. Q is a self-adjoint elliptic diffusion operator on $\mathbb{M}$ with zero an isolated, simple eigenvalue.

We have considered simple situations where $Q^* = Q$. Note that the result can be generalized. For instance one can also work with the class of the ϕ-mixing processes with $\phi \in L^{1/2}$ (see [44, pp. 82-83]). The Lyapunov exponent G can be estimated in case of small noise using the technique introduced by Pinsky [59] under *H1* and Arnold et al. [3] under *H2*. In the following we assume *H2*. The invariant probability measure is then simply the uniform distribution over $\mathbb{M}$.

We shall assume from now on that $\sigma \ll 1$ and we look for an expansion of G with respect to $\sigma \ll 1$. The strategy follows closely the one developed in Ref. [3]. We first divide the generator $\mathcal{L}$ into the sum $\mathcal{L} = \mathcal{L}_0 + \sigma\mathcal{L}_1$ with:

$$\mathcal{L}_0 = Q + h_0(\psi)\frac{\partial}{\partial\psi}, \qquad \mathcal{L}_1 = h_1(\psi, m)\frac{\partial}{\partial\psi}.$$

As shown in [3] the probability density $\bar{p}$ can be expanded as $\bar{p} = \bar{p}_0 + \sigma\bar{p}_1 + \sigma^2\bar{p}_2 + ...$ where $\bar{p}_0$, $\bar{p}_1$, and $\bar{p}_2$ satisfy $\mathcal{L}_0^*\bar{p}_0 = 0$ and $\mathcal{L}_0^*\bar{p}_1 + \mathcal{L}_1^*\bar{p}_0 = 0$, $\mathcal{L}_0^*\bar{p}_2 + \mathcal{L}_1^*\bar{p}_1 = 0$,... For once the expansion of p is known, it can be used in (A.9) to give the expansion of G at order 2 with respect to σ:

$$G = \int_{\mathbb{S}^1\times\mathbb{M}} \left(q_0\bar{p}_0 + \sigma(q_1\bar{p}_0 + q_0\bar{p}_1) + \sigma^2(q_1\bar{p}_1 + q_0\bar{p}_2)\right)(\psi, m)d\psi\pi(dm) + O(\sigma^3). \tag{A.10}$$

If $\kappa = \sigma$. $\bar{p}_0$ satisfies $Q^*\bar{p}_0 = 0$. Thus $\bar{p}_0$ is the density of the uniform density on $\mathbb{S}^1\times\mathbb{M}$: $\bar{p}_0 \equiv (2\pi)^{-1}$. For $\bar{p}_1$ we have the equation $Q^*\bar{p}_1 = -\mathcal{L}_1^*\bar{p}_0 = \partial_\psi(h_1\bar{p}_0)$. Since $Q = Q^*$ this equation is of Poisson type $Q\bar{p}_1 = \partial_\psi(h_1\bar{p}_0)$. Note that $\partial_\psi(h_1\bar{p}_0)$ has zero-mean with respect to the invariant probability $\pi(dm)$ of Q, so the Poisson equation admits a solution $\bar{p}_1$. Let $p(0, m_0; t, m)$ be the transition probability density of the process $m(t)$. It is defined by the equation $\frac{\partial p}{\partial t} = Q^*p$, $p(0, m_0; t = 0, m) = \delta(m - m_0)$. In terms of p we can solve the equation for $\bar{p}_1$ to obtain:

$$\bar{p}_1(\psi, m_0) = -\frac{1}{2\pi}\int_0^\infty dt \int_{\mathbb{M}} \partial_\psi h_1(\psi, m)p(0, m_0; t, m)\pi(dm).$$

Hence

$$\begin{aligned} G &= \sigma^2 \int_{\mathbb{S}^1\times\mathbb{M}} q_1\bar{p}_1(\psi, m_0)d\psi\pi(dm_0) + O(\sigma^3) \\ &= -\frac{\sigma^2}{2\pi}\int_{\mathbb{S}^1} d\psi \int_{\mathbb{M}} \pi(dm_0)\int_{\mathbb{M}} \pi(dm)\int_0^\infty dt q_1(\psi, m_0)\partial_\psi h_1(\psi, m)p(0, m_0; t, m) \\ &\quad +O(\sigma^3). \end{aligned}$$

Taking into account that the autocorrelation function reads:

$$\mathbb{E}[m(0)m(t)] = \int_{\mathbb{M}} \pi(dm_0)\int_{\mathbb{M}} \pi(dm)c(m)c(m_0)p(0, m_0; t, m),$$

this can be simplified to give $G = \sigma^2\gamma_0/8$ with

$$\gamma_0 = 2\int_0^\infty ds\mathbb{E}[c(m(0))c(m(s))] = 2\int_0^\infty ds\mathbb{E}[\eta(0)\eta(s)]. \tag{A.11}$$

If $\kappa = 1$. Since h_0 is constant $= -1$, $\bar{p}_0$ is the uniform density on $\mathbb{S}^1\times\mathbb{M}$: $\bar{p}_0 \equiv (2\pi)^{-1}$. Further $\bar{p}_1$ satisfies $\mathcal{L}_0^*\bar{p}_1 = -\mathcal{L}_1^*\bar{p}_0 = \partial_\psi(h_1\bar{p}_0)$. Note that $\partial_\psi(h_1\bar{p}_0)$ has zero-mean with respect to the invariant probability $\bar{p}_0\pi(dm)d\psi$ of $\mathcal{L}_0^*$, so the Poisson equation admits a solution $\bar{p}_1$. In terms of the transition probability p it is given by:

$$\bar{p}_1(\psi, m) = -\frac{1}{2\pi}\int_0^\infty dt \int_{\mathbb{M}} \partial_\psi h_1(\psi + t)p(0, m_0; t, m)\pi(dm)$$

Substituting into (A.10) we obtain that $G = \sigma^2\gamma_1/8 + O(\sigma^3)$, where γ_1 is nonnegative and proportional to the power spectral density of the process m evaluated at 2-frequency:

$$\gamma_1 = 2\int_0^\infty ds\cos(2s)\mathbb{E}[c(m(0))c(m(s))] = 2\int_0^\infty ds\cos(2s)\mathbb{E}[\eta(0)\eta(s)]. \tag{A.12}$$

A.3. Fast perturbations

We consider the random harmonic oscillator:

$$y_{tt} + \omega^2 (1 + \eta(\frac{t}{\varepsilon}))y = 0 \tag{A.13}$$

Proposition A.1. *The Lyapunov exponent of the harmonic oscillator (A.13) can be expanded as powers of ε:*

$$G = \frac{\varepsilon \omega^2 \gamma_0}{8} + O(\varepsilon^2)$$

Proof. We introduce the rescaled process $\tilde{y}(t) := y(\varepsilon t)$ that satisfies:

$$\tilde{y}_{tt} + \omega^2 \varepsilon^2 (1 + \eta(t))\tilde{y} = 0$$

Applying the above results establishes that the Lyapunov exponent of $\tilde{y}$ is $\tilde{G} = \varepsilon^2 \omega^2 \gamma_0 / 8 + O(\varepsilon^3)$ which gives the desired result. □

A.4. Bibliographic notes

The theory of random dynamical systems, including Lyapunov exponents and the multiplicative ergodic theory, is presented in the book by Arnold [2]. The asymptotic analysis of the Lyapunov exponent of the random harmonic oscillator can be found in [3, 59].

Appendix B. Diffusion-approximation theorems

In this appendix we give the scheme for the rigorous proof of the diffusion approximation stated in Proposition 4.21. We first consider a simple case without periodic modulation.

Proposition B.1. *Let us consider the system:*

$$\frac{dX^\varepsilon}{dt}(t) = \frac{1}{\varepsilon} F\left(X^\varepsilon(t), q(\frac{t}{\varepsilon^2})\right), \qquad X^\varepsilon(0) = x_0 \in \mathbb{R}^d.$$

Assume that q is a Markov, stationary, ergodic process on a compact space with generator Q, satisfying the Fredholm alternative. F satisfies the centering condition: $\mathbb{E}[F(x, q(0))] = 0$ where $\mathbb{E}[.]$ denotes the expectation with respect to the invariant probability measure of q. Instead of technical sharp conditions, assume also that F is smooth and has bounded partial derivatives in x. Then the continuous processes $(X^\varepsilon(t))_{t \geq 0}$ converge in distribution to the Markov diffusion process

X with generator:

$$\mathcal{L}f(x) = \int_0^\infty du \mathbb{E}\left[F(x, q(0)).\nabla\left(F(x, q(u)).\nabla f(x)\right)\right].$$

Proof. For an extended version of the proof and sharp conditions we refer to [44, 31] and [26, Chapter 6]. The process $\bar{X}^\varepsilon(.) := (X^\varepsilon(.), q(./\varepsilon^2))$ is Markov with generator

$$\mathcal{L}^\varepsilon = \frac{1}{\varepsilon^2}Q + \frac{1}{\varepsilon}F(x, q).\nabla.$$

This implies that, for any smooth function f, the process $f(\bar{X}^\varepsilon(t)) - f(\bar{X}^\varepsilon(s)) - \int_s^t \mathcal{L}^\varepsilon f(\bar{X}^\varepsilon(u))du$ is a martingale. The proof consists in demonstrating the convergence of the corresponding martingale problems. It is based on the so-called perturbed test function method.

Step 1. Perturbed test function method. $\forall f \in \mathcal{C}_b^\infty$, $\forall K$ *compact subset of* $\mathbb{R}^d$*, there exists a family* f^ε *such that:*

$$\sup_{x\in K, q} |f^\varepsilon(x, q) - f(x)| \stackrel{\varepsilon\to 0}{\longrightarrow} 0, \qquad \sup_{x\in K, q} |\mathcal{L}^\varepsilon f^\varepsilon(x, q) - \mathcal{L}f(x)| \stackrel{\varepsilon\to 0}{\longrightarrow} 0. \tag{B.1}$$

Define $f^\varepsilon(x, q) = f(x) + \varepsilon f_1(x, q) + \varepsilon^2 f_2(x, q)$. Applying $\mathcal{L}^\varepsilon$ to f^ε, one gets:

$$\mathcal{L}^\varepsilon f^\varepsilon = \frac{1}{\varepsilon}\left(Qf_1 + F(x, q).\nabla f(x)\right) + \left(Qf_2 + F.\nabla f_1(x, q)\right) + O(\varepsilon).$$

One then defines the corrections f_j as follows:

1. $f_1(x, q) = -Q^{-1}\left(F(x, q).\nabla f(x)\right)$. This function is well-defined since Q has an inverse on the subspace of centered functions (Fredholm alternative). It also admits the representation:

$$f_1(x, q) = \int_0^\infty du \mathbb{E}[F(x, q(u)).\nabla f(x) \mid q(0) = q].$$

2. $f_2(x, q) = -Q^{-1}\left(F.\nabla f_1(x, q) - \mathbb{E}[F.\nabla f_1(x, q)]\right)$ is well defined since the argument of Q^{-1} has zero-mean. It thus remains: $\mathcal{L}^\varepsilon f^\varepsilon = \mathbb{E}[F.\nabla f_1(x, q)] + O(\varepsilon)$ which proves (B.1).

Step 2. Convergence of martingale problems. One first establishes the tightness of the process X^ε in the space of the càd-làg functions equipped with the Skorohod topology by checking a standard criterion (see [44, Section 3.3]). Second one considers a subsequence $\varepsilon_p \to 0$ such that $X^{\varepsilon_p} \to X$. One takes $t_1 < ... < t_n < s < t$ and $h_1, ..., h_n \in \mathcal{C}_b^\infty$:

$$\mathbb{E}\left[\left(f^\varepsilon(X^\varepsilon(t), q(\frac{t}{\varepsilon^2})) - f^\varepsilon(X^\varepsilon(s), q(\frac{s}{\varepsilon^2})) - \right.\right.$$
$$\left.\left. - \int_s^t \mathcal{L}^\varepsilon f^\varepsilon(X^\varepsilon(u), q(\frac{u}{\varepsilon^2}))du\right) h_1(X^\varepsilon(t_1))...h_n(X^\varepsilon(t_n))\right] = 0$$

Taking the limit $\varepsilon_p \to 0$:

$$\mathbb{E}\left[\left(f(X(t)) - f(X(s)) - \int_s^t \mathcal{L}f(X(u))du\right) h_1(X(t_1))...h_n(X(t_n))\right] = 0$$

which shows that X is solution of the martingale problem associated to $\mathcal{L}$. This problem is well-posed (in the sense that it has a unique solution), which proves the result. $\square$

Proposition B.2. *Let us consider the system:*

$$\frac{dX^\varepsilon}{dt}(t) = \frac{1}{\varepsilon}F\left(X^\varepsilon(t), q(\frac{t}{\varepsilon^2}), \frac{t}{\varepsilon}\right), \qquad X^\varepsilon(0) = x_0 \in \mathbb{R}^d.$$

We assume the same hypotheses as in Proposition B.1. We assume also that $F(x,q,\tau)$ is periodic with respect to τ with period T_0 and F satisfies the centering condition: $\mathbb{E}[F(x,q(0),\tau)] = 0$ for all x and τ. Then the continuous processes $(X^\varepsilon(t))_{t\geq 0}$ converge in distribution to the Markov diffusion process X with generator:

$$\mathcal{L}f(x) = \int_0^\infty du \left\langle \mathbb{E}\left[F(x,q(0),.).\nabla\left(F(x,q(u),.).\nabla f(x)\right)\right]\right\rangle_\tau,$$

where $\langle . \rangle_\tau$ stands for an averaging over a period in τ.

Proof. The proof is similar to the one of Proposition B.1. The key consists in building a suitable family of perturbed functions from a given test function.
$\forall f \in \mathcal{C}_b^\infty$, *$\forall K$ compact subset of $\mathbb{R}^d$, there exists a family f^ε such that:*

$$\sup_{x\in K,q,\tau} |f^\varepsilon(x,q,\tau) - f(x)| \stackrel{\varepsilon\to 0}{\longrightarrow} 0, \qquad \sup_{x\in K,q,\tau} |\mathcal{L}^\varepsilon f^\varepsilon(x,q,\tau) - \mathcal{L}f(x)| \stackrel{\varepsilon\to 0}{\longrightarrow} 0. \quad \text{(B.2)}$$

Let us introduce $\tau(t) := t \bmod T_0$. The process $\bar{X}^\varepsilon(.) := (X^\varepsilon(.), q(./\varepsilon^2), \tau(./\varepsilon))$ is Markov with generator

$$\mathcal{L}^\varepsilon = \frac{1}{\varepsilon^2}Q + \frac{1}{\varepsilon}F(x,q,\tau).\nabla + \frac{1}{\varepsilon}\frac{\partial}{\partial\tau}.$$

Let $f \in \mathcal{C}_b$. We define $f_1 = f_{11} + f_{12}$ where f_{11} is the same term as in the absence of a periodic component:

$$f_{11}(x,q,\tau) = -Q^{-1}(F(x,q,\tau).\nabla f(x))$$

while f_{12} does not depend on q so that $Qf_{12} = 0$:

$$f_{12}(x,\tau) = -\int_0^\tau \left(\mathbb{E}[F(x,q(0),s).\nabla f_{11}(x,q(0),s)] - \bar{f}_1(x)\right) ds$$

where $\bar{f}_1(x) = \frac{1}{T_0}\int_0^{T_0} \mathbb{E}[F(x,q(0),u).\nabla f_{11}(x,q(0),u)]du$. Note that f_{12} is uniformly bounded because of the correction $\bar{f}_1$. We finally define:

$$\begin{aligned} f_2(x,q,\tau) &= -Q^{-1}\Bigg(F(x,q,\tau).\nabla f_1(x,q,\tau) + \frac{\partial f_1}{\partial \tau} \\ &\qquad -\mathbb{E}[F(x,q,\tau).\nabla f_1(x,q,\tau)] - \mathbb{E}[\frac{\partial f_1}{\partial \tau}]\Bigg) \end{aligned}$$

Now we set:

$$f^\varepsilon(x,q,h) = f(x) + \varepsilon f_1(x,q,\tau) + \varepsilon^2 f_2(x,q,\tau)$$

Applying the infinitesimal generator $\mathcal{L}^\varepsilon$ we get:

$$\mathcal{L}^\varepsilon f^\varepsilon = \mathbb{E}[F(x,q,\tau).\nabla f_1(x,q,\tau)] + \mathbb{E}[\frac{\partial f_1}{\partial \tau}] + O(\varepsilon)$$

which simplifies into:

$$\mathcal{L}^\varepsilon f^\varepsilon(x,q,\tau) = \bar{f}_1(x) + O(\varepsilon)$$

□

Proposition B.3. *Let us consider the system:*

$$\frac{dX^\varepsilon}{dt}(t) = \frac{1}{\varepsilon}F\left(X^\varepsilon(t), q(\frac{t}{\varepsilon^2}), \frac{t}{\varepsilon^2}\right), \qquad X^\varepsilon(0) = x_0 \in \mathbb{R}^d.$$

We assume the same hypotheses as in Proposition B.1. We assume also that $F(x,q,\tau)$ is periodic with respect to τ with period T_0 and F satisfies the centering condition: $\langle \mathbb{E}[F(x,q(0),.)]\rangle_\tau = 0$ for all x, where $\langle . \rangle_\tau$ stands for an averaging over a period in τ. Then the continuous processes $(X^\varepsilon(t))_{t\geq 0}$ converge in distribution to the Markov diffusion process X with generator:

$$\mathcal{L}f(x) = \int_0^\infty du \, \langle \mathbb{E}\left[F(x,q(0),.).\nabla\left(F(x,q(u),.+u).\nabla f(x)\right)\right]\rangle_\tau .$$

Proof. The proof is similar to the one of Proposition B.1. The key consists in building a family of perturbed functions satisfying (B.2) for any given test function. The process $\bar{X}^\varepsilon(.) := (X^\varepsilon(.), q(./\varepsilon^2), \tau(./\varepsilon^2))$ is Markov with generator

$$\mathcal{L}^\varepsilon = \frac{1}{\varepsilon^2}Q + \frac{1}{\varepsilon}F(x,q,\tau).\nabla + \frac{1}{\varepsilon^2}\frac{\partial}{\partial \tau}.$$

Define $f^\varepsilon(x,q,\tau) = f(x) + \varepsilon f_1(x,q,\tau) + \varepsilon^2 f_2(x,q,\tau)$. Applying $\mathcal{L}^\varepsilon$ to f^ε, one gets:

$$\begin{aligned} \mathcal{L}^\varepsilon f^\varepsilon &= \frac{1}{\varepsilon}\left(\left(Q + \frac{\partial}{\partial \tau}\right) f_1 + F(x,q,\tau).\nabla f(x)\right) \\ &\quad + \left(\left(Q + \frac{\partial}{\partial \tau}\right) f_2 + F(x,q,\tau).\nabla f_1(x,q,\tau)\right) + O(\varepsilon). \end{aligned}$$

One then defines the correction f_1 as:

$$f_1(x,q,\tau) = -\left(Q + \frac{\partial}{\partial\tau}\right)^{-1} \left(F(x,q,\tau).\nabla f(x)\right).$$

This function is well-defined although $\left(Q + \frac{\partial}{\partial\tau}\right)$ does not possess an inverse. Indeed $(q(t),\tau(t))_{t\geq 0}$ is an ergodic process which satisfies the Fredholm alternative. Its invariant probability measure over $S\otimes[0,T_0]$ is $\mathbb{P}(dq)\times\frac{1}{T_0}\mathbf{1}_{[0,T_0]}(\tau)d\tau$. As a consequence $\left(Q + \frac{\partial}{\partial\tau}\right)$ has an inverse on the subspace of functions which have zero-mean with respect to the invariant measure. This inverse admits the representation:

$$f_1(x,q,\tau) = \int_0^\infty du\mathbb{E}[F(x,q(u),\tau+u).\nabla f(x)|q(0)=q].$$

The second correction f_2 is defined as usual by

$$f_2(x,q,\tau) = -\left(Q + \frac{\partial}{\partial\tau}\right)^{-1} \left(F.\nabla f_1(x,q,\tau) - \langle\mathbb{E}[F.\nabla f_1(x,q,.)]\rangle_\tau\right).$$

This function is well defined since the argument of $\left(Q + \frac{\partial}{\partial\tau}\right)^{-1}$ has zero-mean. It thus remains: $\mathcal{L}^\varepsilon f^\varepsilon = \langle\mathbb{E}[F(x,q,.).\nabla f_1(x,q,.)]\rangle_\tau + O(\varepsilon)$ which proves (B.2). $\square$

We refer to [31] and [26, Chapter 6] for other multi-scaled versions of these propositions. We complete this appendix by revisiting the averaging theorem stated in Proposition 2.5. Consider the random differential equation

$$\frac{dX^\varepsilon}{dt} = F\left(X^\varepsilon(t), q\left(\frac{t}{\varepsilon}\right)\right) \ , \quad X^\varepsilon(0) = x_0$$

where we do not assume that $F(x,q)$ is centered. We denote its mean by $\bar{F}(x) = \mathbb{E}[F(x,q(0))]$. Then $(X^\varepsilon(\cdot), q(\cdot/\varepsilon))$ is a Markov process with generator

$$\mathcal{L}^\varepsilon = \frac{1}{\varepsilon}Q + F(x,q).\nabla$$

Let $f(x)$ be a test function. Define $f^\varepsilon(x,q) = f(x) + \varepsilon f_1(x,q)$ where f_1 solves the Poisson equation

$$Qf_1(x,q) + \left[F(x,q).\nabla f(x) - \bar{F}(x).\nabla f(x)\right] = 0$$

We get $\mathcal{L}^\varepsilon f^\varepsilon(x,q) = \bar{F}(x).\nabla f(x) + O(\varepsilon)$. Therefore the processes $X^\varepsilon(t)$ converge to the solution of the martingale problem associated with the generator $\mathcal{L}f(x) = \bar{F}(x)\cdot\nabla f(x)$. The solution is the deterministic process $\bar{X}(t)$

$$\frac{d\bar{X}}{dt} = \bar{F}(\bar{X}(t)) \ , \quad \bar{X}(0) = x_0.$$

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