Wave propagation in random media

J. Garnier (Université Paris VII)
`http://www.proba.jussieu.fr/~garnier`

1. Wave propagation in one-dimensional random media: effective medium theory
   Homogenization theory
2. Wave propagation in one-dimensional random media: the coherent wave front, the incoherent wave fluctuations, time reversal.
   Diffusion approximation, asymptotic theory for random differential equations.
3. Wave propagation and time reversal in a random waveguide.
4. Wave propagation and time reversal in the parabolic regime.
   Semi-classical analysis of the Schrödinger equation with a random potential

What is a random medium ?

Problem: Wave propagation in a highly heterogeneous medium.

Stochastic modeling: the medium is a realization of a random medium (a set of possible media described statistically).
- takes into account the available data (mean, standard deviation of the fluctuations, ...)
- completes the modeling by a statistical description (Gaussian process, ...).

Statistical distribution of the random medium $\Longrightarrow$ statistical distribution of the wave (highly nonlinear problem).

What about a wave propagating in a "typical" realization ?
- Mean-field (or averaged) approach can be misleading.
- A complete statistical analysis is necessary.
- There exist statistically stable quantities.

Importance of scaled regimes and asymptotic theory.

Methodology

- Modeling:
  - Identification of the phenomena and equations.
  - Statistical description of the medium parameters.
  - Determination of the scales.
- Asymptotics:
  - Separation of scales.
  - Limit theorems.
- Limit problem:
  - Analysis of the physically relevant quantities.
  - Use of stochastic calculus.

The acoustic wave equations

The acoustic pressure $p(z,t)$ and velocity $u(z,t)$ satisfy the continuity and momentum equations

$$\rho \frac{\partial u}{\partial t} + \frac{\partial p}{\partial z} = 0$$
$$\frac{\partial p}{\partial t} + \kappa \frac{\partial u}{\partial z} = 0$$

where $\rho(z)$ is the material density,

$\kappa(z)$ is the bulk modulus of the medium.

Propagation in homogeneous medium

Linear hyperbolic system with ρ, κ constant.

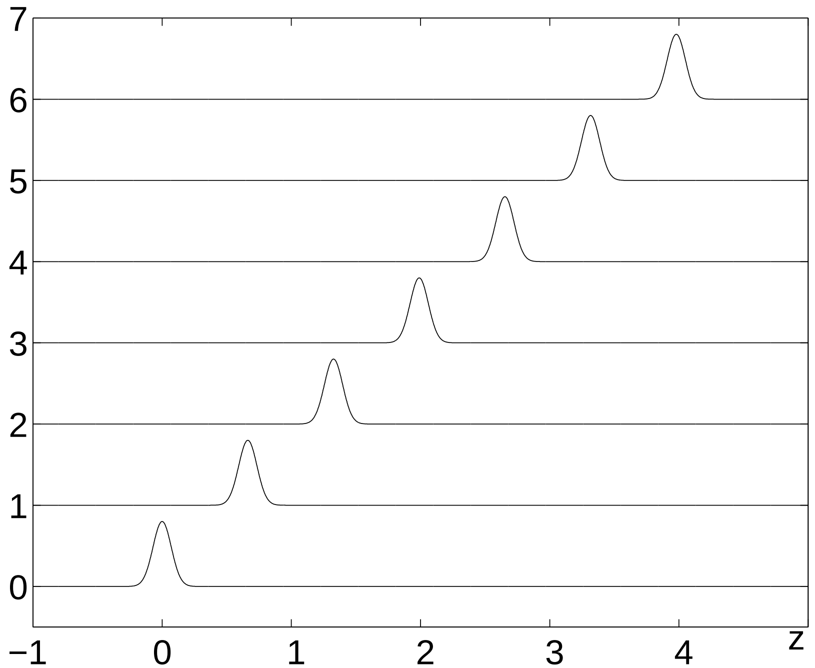
Impedance: $\zeta = \sqrt{\rho\kappa}$. Sound speed: $c = \sqrt{\kappa/\rho}$.

Right- and left-going modes:

$$A = \zeta^{1/2}u + \zeta^{-1/2}p, \qquad B = \zeta^{1/2}u - \zeta^{-1/2}p$$

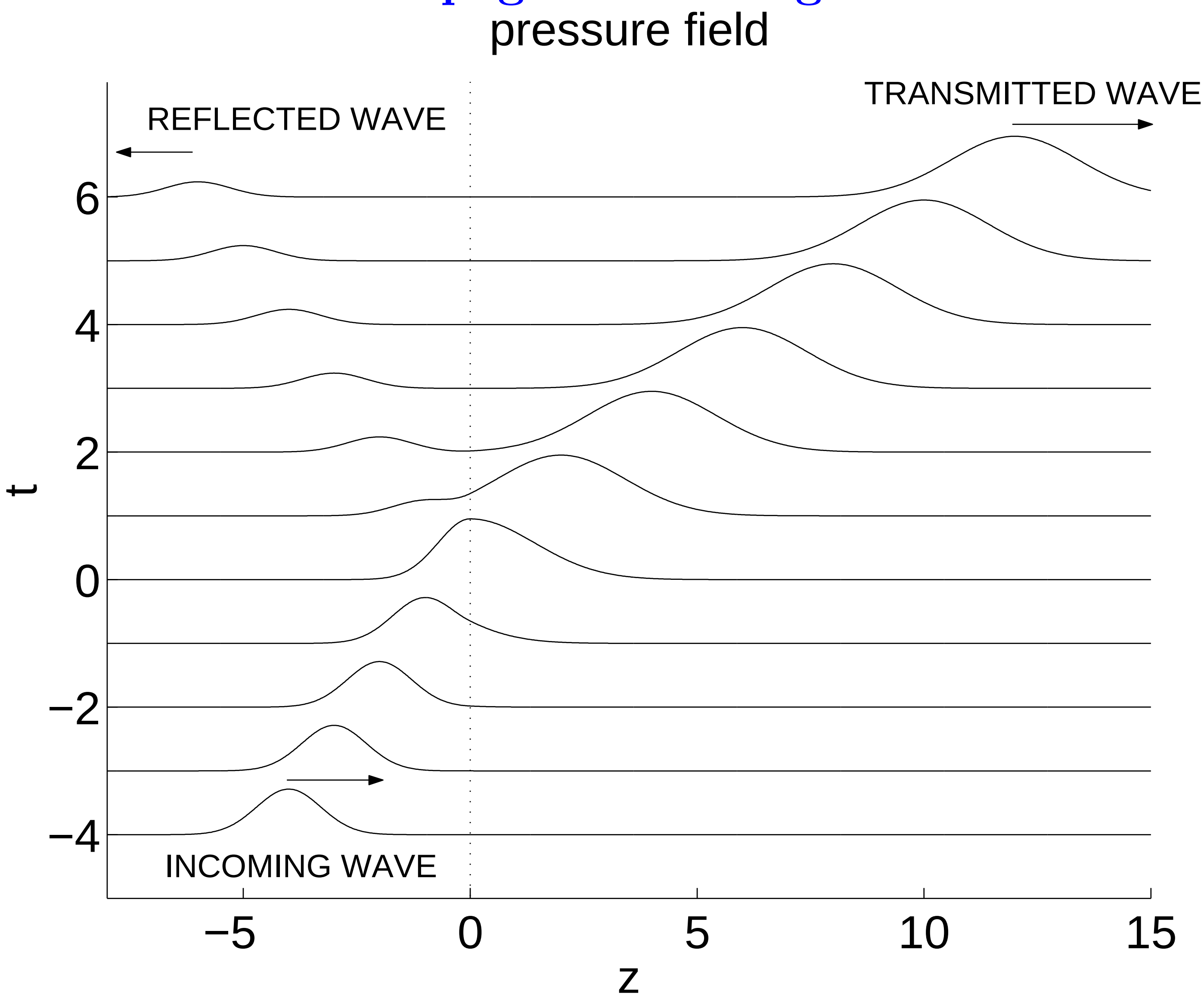
$$\frac{\partial A}{\partial t} + c\frac{\partial A}{\partial z} = 0, \qquad \frac{\partial B}{\partial t} - c\frac{\partial B}{\partial z} = 0$$

A: right-going wave $\qquad$ B: left-going wave.



Spatial profiles of the wave at different times for a pure right-going wave

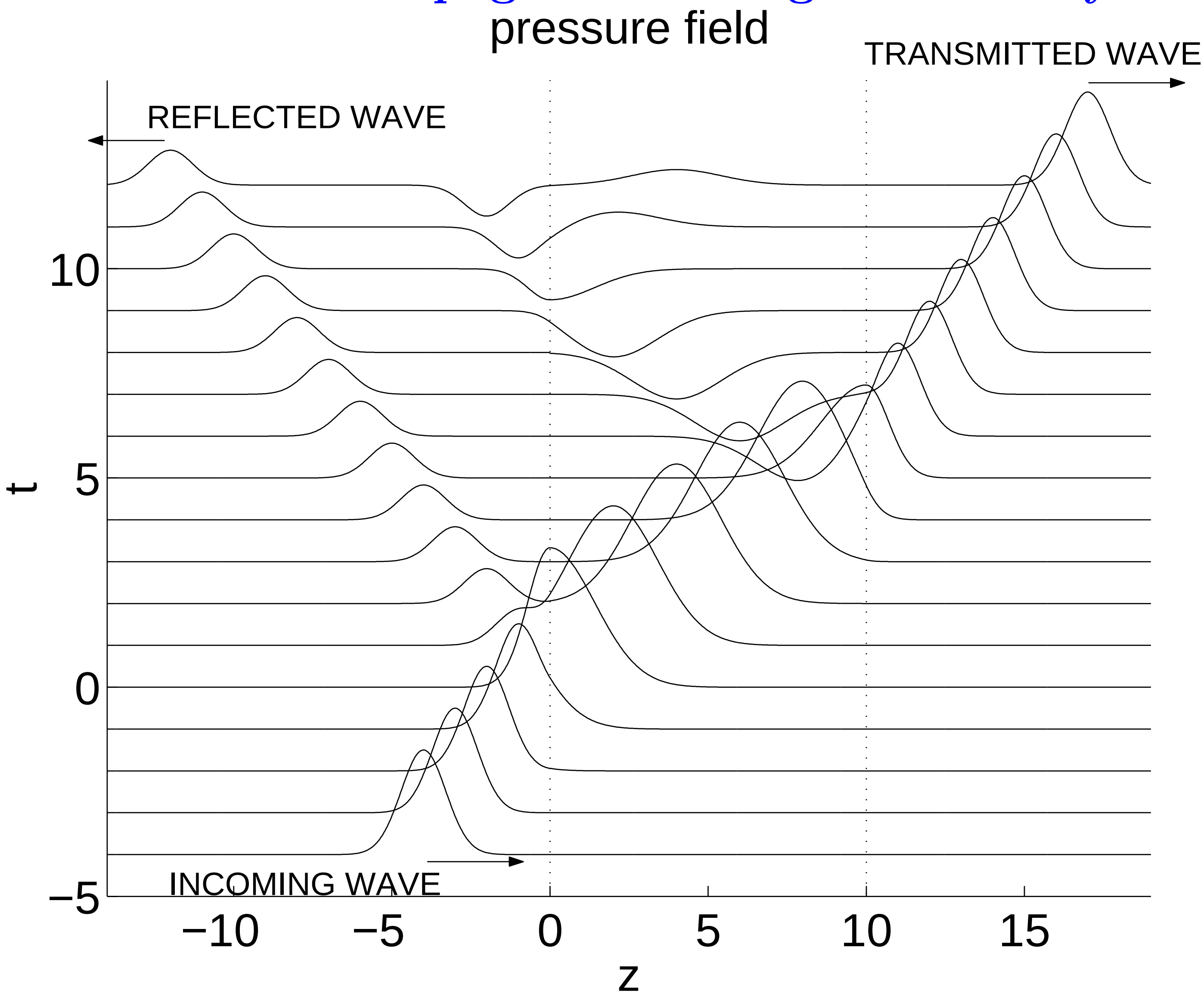
Propagation through an interface



Medium $z < 0$: $c = 1$, $\zeta = 1$.

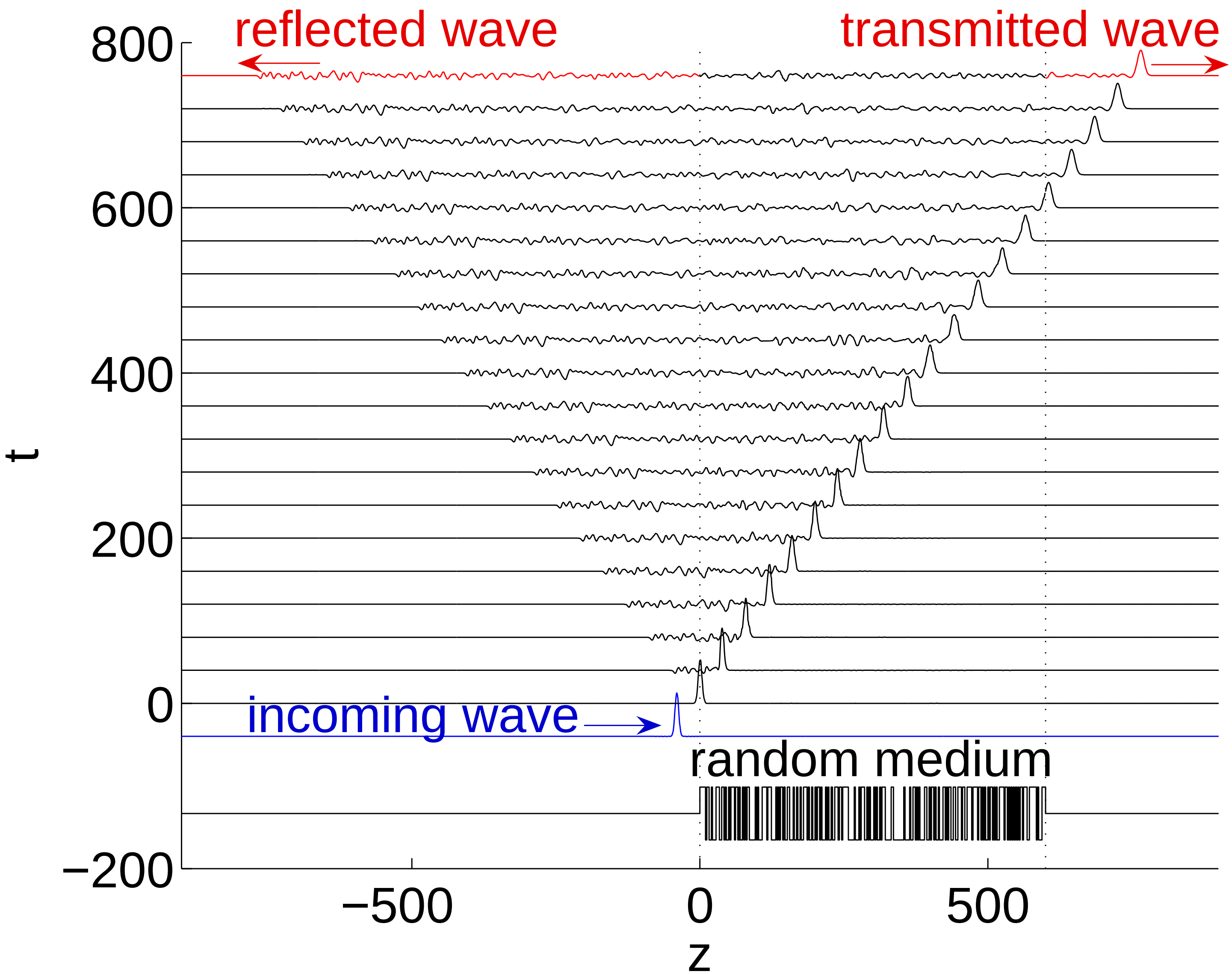
Medium $z > 0$: $c = 2$, $\zeta = 2$.

Propagation through a thick layer



Medium $\left\{ \begin{matrix} z < 0 \\ z > 10 \end{matrix} \right\}$: $c = 1$, $\zeta = 1$.   Medium $0 < z < 10$: $c = 2$, $\zeta = 2$.

A numerical experiment in random medium



Random medium: stack of thin layers composed of two materials.

The three scales

The acoustic pressure $p(z,t)$ and velocity $u(z,t)$ satisfy the continuity and momentum equations

$$\rho \frac{\partial u}{\partial t} + \frac{\partial p}{\partial z} = 0$$
$$\frac{\partial p}{\partial t} + \kappa \frac{\partial u}{\partial z} = 0$$

where $\rho(z)$ is the material density,

$\kappa(z)$ is the bulk modulus of the medium.

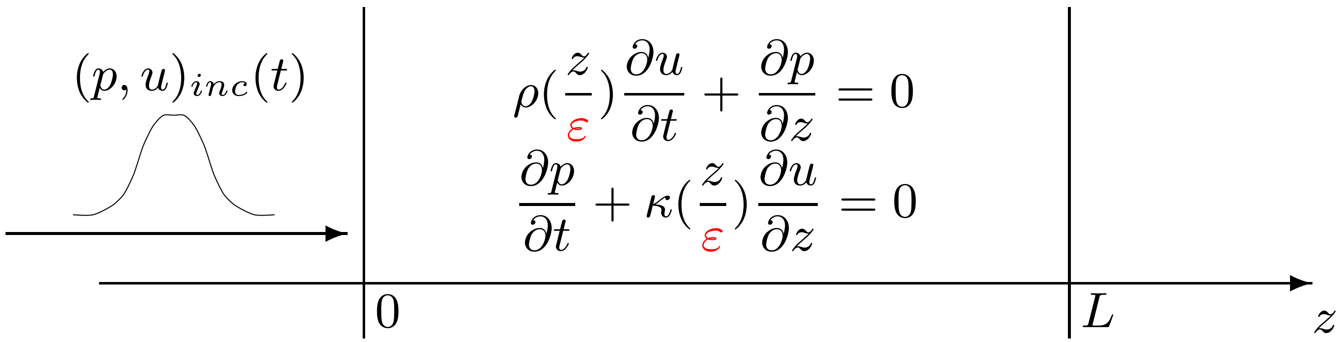
Three scales:

l_c: correlation radius of the random process ρ and κ.

λ: typical wavelength of the incoming pulse.

L: propagation distance.

Effective medium theory $L \sim \lambda \gg l_c$



Model: $\rho = \rho(z/\varepsilon)$ and $\kappa = \kappa(z/\varepsilon)$, where $0 < \varepsilon \ll 1$ and ρ, κ are random functions.

Perform a Fourier transform with respect to t:

$$u(t,z) = \frac{1}{2\pi}\int \hat{u}(\omega,z)e^{-i\omega t}d\omega, \quad p(t,z) = \frac{1}{2\pi}\int \hat{p}(\omega,z)e^{-i\omega t}d\omega$$

so that we get a system of ordinary differential equations:

$$\frac{dX^\varepsilon}{dz} = F(\frac{z}{\varepsilon}, X^\varepsilon),$$

where

$$X^\varepsilon = \begin{pmatrix} \hat{p} \\ \hat{u} \end{pmatrix}, \quad F(z,X) = -i\omega \begin{pmatrix} 0 & \rho(z) \\ \frac{1}{\kappa(z)} & 0 \end{pmatrix} X$$

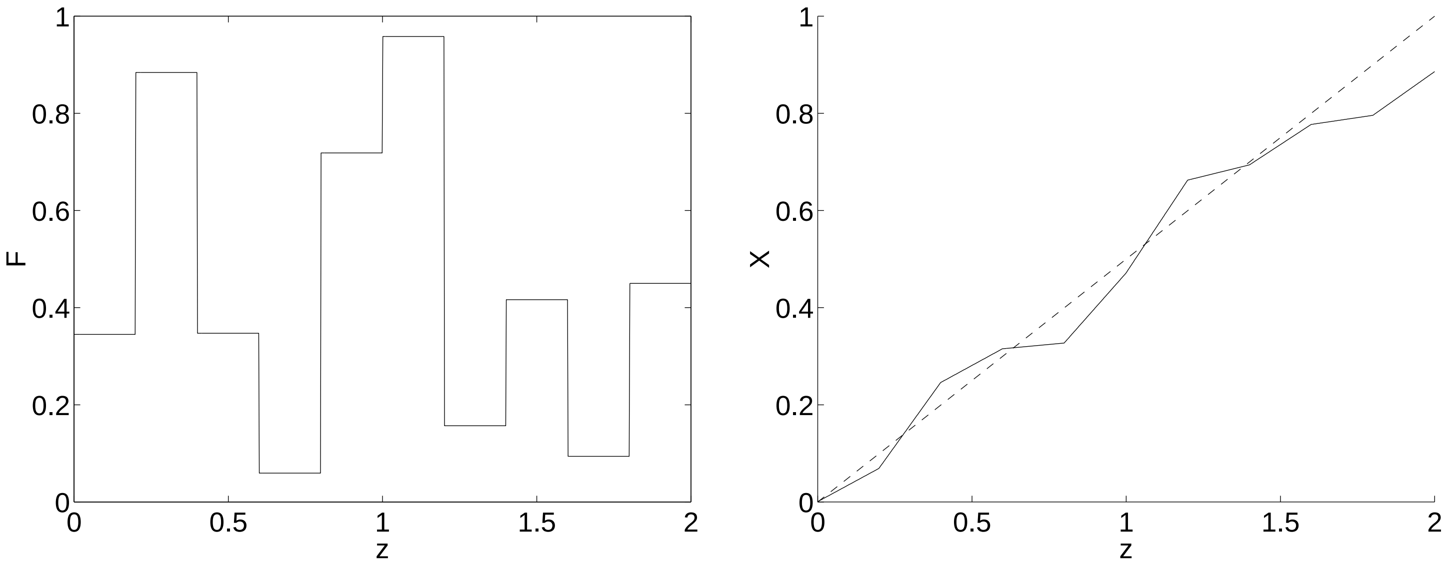
Method of averaging: Toy model

Let $X^\varepsilon(z) \in \mathbb{R}$ be the solution of

$$\frac{dX^\varepsilon}{dz} = F(\frac{z}{\varepsilon})$$

with $F(z) = \sum_{i=1}^\infty F_i \mathbf{1}_{[i-1,i)}(z)$, F_i independent random variables $\mathbb{E}[F_i] = \bar{F}$ and $\mathbb{E}[(F_i - \bar{F})^2] = \sigma^2$.

($z \mapsto t$, particle in a random velocity field)

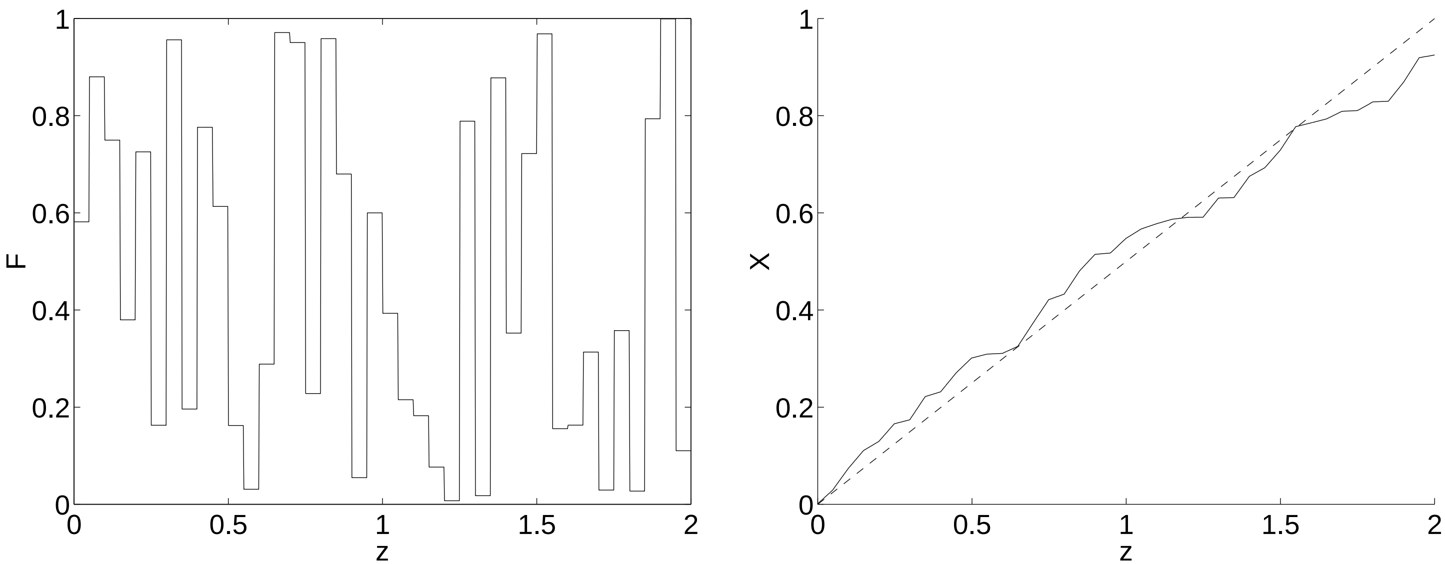


$\varepsilon = 0.2$

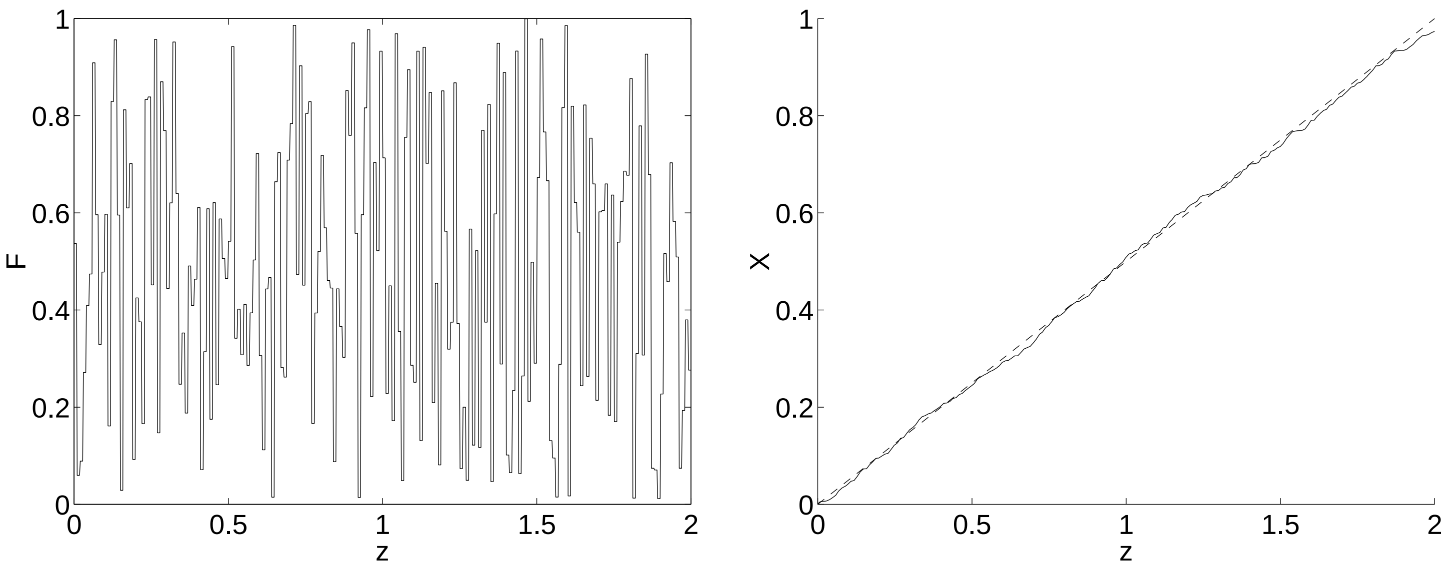
$$
\begin{aligned}
X^{\varepsilon}(z) \quad &= \varepsilon \int_0^{\frac{z}{\varepsilon}} F(s)ds = \varepsilon \left(\sum_{i=1}^{\left[\frac{z}{\varepsilon}\right]} F_i \right) + \varepsilon \int_{\left[\frac{z}{\varepsilon}\right]}^{\frac{z}{\varepsilon}} F(s)ds \\
&= \quad \underbrace{\varepsilon \left[\frac{z}{\varepsilon}\right]}_{\substack{{\color{blue}\varepsilon \to 0} \downarrow \\ z}} \quad \times \quad \underbrace{\frac{1}{\left[\frac{z}{\varepsilon}\right]} \left(\sum_{i=1}^{\left[\frac{z}{\varepsilon}\right]} F_i \right)}_{\substack{\text{a.s.} \downarrow (LLN) \\ \mathbb{E}[F(z)] = \bar{F}}} \quad + \quad \underbrace{\varepsilon \left(\frac{z}{\varepsilon} - \left[\frac{z}{\varepsilon}\right] \right) F_{\left[\frac{z}{\varepsilon}\right]}}_{\substack{\text{a.s.} \downarrow \\ 0}}
\end{aligned}
$$

Thus:

$$
X^{\varepsilon}(z) \xrightarrow{\varepsilon \to 0} \bar{X}(z), \qquad {\color{magenta}\frac{d\bar{X}}{dz} = \bar{F}}.
$$



$$\varepsilon = 0.05$$



$$\varepsilon = 0.01$$

Random process

• Real random variable X = random number

Example: $X \sim \mathcal{U}(0,1)$ is a random number that can take any value in $(0,1)$ with equiprobability.

Distribution of X characterized by moments of the form $\mathbb{E}[\phi(X)]$ where $\phi \in \mathcal{C}_b(\mathbb{R},\mathbb{R})$.

Example: $X \sim \mathcal{U}(0,1) \mapsto \mathbb{E}[\phi(X)] = \int_0^1 \phi(x)dx$.

Example: $X \sim \mathcal{E}(1) \mapsto \mathbb{E}[\phi(X)] = \int_0^\infty \phi(x)e^{-x}dx$.

Example: $X \sim \mathcal{N}(0,1) \mapsto \mathbb{E}[\phi(X)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \phi(x)e^{-\frac{x^2}{2}}dx$.

• Stochastic process $(F(z))_{z\geq 0}$ = random function = random "variable" taking values in a functional space, e.g. $\mathcal{C}([0,\infty),\mathbb{R}^d)$.

A realization of the process = a function from $[0,\infty)$ to $\mathbb{R}^d$.

Distribution of $(F(z))_{z\geq 0}$ characterized by moments of the form $\mathbb{E}[\phi(F)]$, where $\phi \in \mathcal{C}_b(E,\mathbb{R})$.

In fact, moments of the form $\mathbb{E}[\phi(F(z_1),...,F(z_n))]$, for any n, $z_1,...,z_n \geq 0$, $\phi \in \mathcal{C}_b(\mathbb{R}^n,\mathbb{R})$, are sufficient to characterize the distribution.

Example: Gaussian process.

Gaussian process

• Real Gaussian process $(F(z))_{z\geq 0}$ characterized by its first two moments $m(z_1) = \mathbb{E}[F(z_1)]$ and $c(z_1, z_2) = \mathbb{E}[F(z_1)F(z_2)]$.

Any linear combination $F_\lambda = \sum_{i=1}^n \lambda_i F(z_i)$ has Gaussian distribution

$$\mathbb{E}[\phi(F_\lambda)] = \frac{1}{\sqrt{2\pi}\sigma_\lambda} \int_{-\infty}^{\infty} \phi(x) \exp\Big(- \frac{(x-m_\lambda)^2}{2\sigma_\lambda^2}\Big) dx$$

$$\text{where } m_\lambda = \sum_{i=1}^n \lambda_i \mathbb{E}[F(z_i)] \quad \sigma_\lambda^2 = \sum_{i,j=1}^n \lambda_i\lambda_j \mathbb{E}[F(z_i)F(z_j)] - m_\lambda^2$$

• Simulation: in order to simulate $(F(z_1), ..., F(z_n))$:

- compute the mean vector $M_i = \mathbb{E}[F(z_i)]$ and the covariance matrix $C_{ij} = \mathbb{E}[F(z_i)F(z_j)] - \mathbb{E}[F(z_i)]\mathbb{E}[F(z_j)]$.
- generate a random vector $X = (X_1, ..., X_n)$ of n independent Gaussian random variables with mean 0 and variance 1.
- compute $Y = M + C^{1/2}X$. The vector Y has the distribution of $(F(z_1), ..., F(z_n))$.

Brownian motion

• Brownian motion $(W_z)_{z\geq 0}$ (starting from 0)= real Gaussian process with mean 0 and autocorrelation function

$$\mathbb{E}[W_z W_{z'}] = z \wedge z'$$

The realizations of the Brownian motion are continuous but not differentiable.

The increments of the Brownian motion are independent:
if $z_n \geq z_{n-1} \geq ... \geq z_1 \geq z_0 = 0$, then $(W_{z_n} - W_{z_{n-1}}, ..., W_{z_2} - W_{z_1}, W_{z_1})$ are independent Gaussian random variables with mean 0 and variance

$$\mathbb{E}\Big[(W_{z_j} - W_{z_{j-1}})^2\Big] = z_j - z_{j-1}$$

• Simulation: in order to simulate $(W_h, W_{2h}, ..., W_{nh})$:
- generate a random vector $X = (X_1, ..., X_n)$ of n independent Gaussian random variables with mean 0 and variance 1.
- compute $Y_j = \sqrt{h}\sum_{i=1}^{j} X_i$. The vector Y has the distribution of $(W_h, W_{2h}, ..., W_{nh})$.

Stationary random process

- $(F(z))_{z\in\mathbb{R}^+}$ is stationary if $(F(z+z_0))_{z\in\mathbb{R}^+}$ has the same distribution as $(F(z))_{z\in\mathbb{R}^+}$ for any $z_0 \geq 0$.

Sufficient and necessary condition:

$$\mathbb{E}[\phi(F(z_1),...,F(z_n))] = \mathbb{E}[\phi(F(z_0+z_1),...,F(z_0+z_n))]$$

for any n, $z_0,...,z_n \geq 0$, $\phi \in \mathcal{C}_b(\mathbb{R}^n,\mathbb{R})$.

Example: Gaussian process $F(z)$ with mean zero $\mathbb{E}[F(z)] = 0\ \forall z$ and autocorrelation function $\mathbb{E}[F(z')F(z'+z)] = c(z)$.

- Spectral representation (of stationary Gaussian process):

$$F(z) = \int e^{ikz}\sqrt{\hat{c}(k)}dW_k$$

with W_k complex Brownian motion, i.e.:
- $W_k = (W_k^{(1)} + iW_k^{(2)})/\sqrt{2}$ for $k \geq 0$
- $W_k = (W_{-k}^{(1)} - iW_{-k}^{(2)})/\sqrt{2}$ for $k \geq 0$
- $W_k^{(1)}$ and $W_k^{(2)}$ independent real Brownian motions.

Formally: "$\dot{W}_k = \frac{dW_k}{dk}$" complex Gaussian white noise, i.e.
$\dot{W}_k$ Gaussian, $\mathbb{E}\left[\dot{W}_k\right] = 0$, $\mathbb{E}\left[\dot{W}_k\overline{\dot{W}_{k'}}\right] = \delta(k-k')$.

Simulation of $F(z_0), ..., F(z_{n-1})$, $z_i = ih$

- First generate a random vector $X = (X_1, ..., X_n)$ of n independent Gaussian random variables with mean 0 and variance 1.
- Second apply the discrete Fourier transform (FFT):

$$\hat{X}_u = \sum_{j=0}^{n-1} X_j e^{i2\pi ju/n}$$

- Third multiply (filter) $\tilde{X}_u = \hat{X}_u \sqrt{\hat{C}_u}$, where

$$\hat{C}_u = \sum_{j=0}^{n-1} [c(jh) + c(n-j)h)] e^{i2\pi ju/n}$$

Note that $\hat{C}_u = \overline{\hat{C}_u} = \hat{C}_{n-u}$.

- Forth apply the discrete inverse Fourier transform (IFFT):

$$\check{X}_l = \frac{1}{n} \sum_{u=0}^{n-1} \tilde{X}_u e^{-i2\pi lu/n}$$

- Result: $(\check{X}_0, ..., \check{X}_{n-1})$ is a real-valued, Gaussian vector, with zero-mean and covariance $\mathbb{E}[\check{X}_l \check{X}_{l'}] = c\Big((l-l')h\Big) + c\Big((n-(l-l'))h\Big)$

$\hookrightarrow$ periodic version of $(F(0), ..., F((n-1)h))$

Ergodic theory

Ergodic Theorem. If F satisfies the ergodic hypothesis, then

$$\frac{1}{Z}\int_0^Z F(z)dz \xrightarrow{Z\to\infty} \bar{F} \quad \text{a.s., where } \bar{F} = \mathbb{E}[F(0)] = \mathbb{E}[F(z)]$$

Ergodic hypothesis = "the orbit $(F(z))_{z\geq 0}$ visits all of phase space".

Ergodic theorem = "the spatial average is equivalent to the statistical average".

Counter-example for the ergodic hypothesis:
Let F_1 and F_2 be stationary, both satisfy the ergodic theorem, $\bar{F}_j = \mathbb{E}[F_j(z)]$, $j = 1, 2$, with $\bar{F}_1 \neq \bar{F}_2$.
Flip a coin (independently of F_j) $\to$ random variable $\chi = 0$ or 1 with probability $1/2$.
Let $F(z) = \chi F_1(z) + (1-\chi)F_2(z)$.
F is a stationary process with mean $\bar{F} = \frac{1}{2}(\bar{F}_1 + \bar{F}_2)$.

$$\begin{aligned}\frac{1}{Z}\int_0^Z F(z)dz \quad &= \quad \chi\left(\frac{1}{Z}\int_0^Z F_1(z)dz\right) + (1-\chi)\left(\frac{1}{Z}\int_0^Z F_2(z)dz\right) \\ &\xrightarrow{Z\to\infty} \quad \chi\bar{F}_1 + (1-\chi)\bar{F}_2\end{aligned}$$

which is a random limit different from $\bar{F}$.
The limit depends on χ because F has been trapped in a part of phase space.

Mean square theory

Let F be a stationary process, $\mathbb{E}[F(0)^2] < \infty$. Its autocorrelation function is:

$$R(z) = \mathbb{E}\left[(F(z_0) - \bar{F})(F(z_0+z) - \bar{F})\right]$$

- R is independent of z_0 by stationarity of F.
- $|R(z)| \leq R(0)$ by Cauchy-Schwarz:

$$|R(z)| \leq \mathbb{E}\left[(F(0) - \bar{F})^2\right]^{1/2} \mathbb{E}\left[(F(z) - \bar{F})^2\right]^{1/2} = R(0)$$

- R is an even function $R(-z) = R(z)$:

$$\begin{aligned} R(-z) &= \mathbb{E}\left[(F(z_0 - z) - \bar{F})(F(z_0) - \bar{F})\right] \\ &\overset{z_0=z}{=} \mathbb{E}\left[(F(0) - \bar{F})(F(z) - \bar{F})\right] = R(z) \end{aligned}$$

Proposition. Assume $\int_0^\infty |R(z)| dz < \infty$. Let $S(Z) = \frac{1}{Z}\int_0^Z F(z)dz$. Then

$$\mathbb{E}\left[(S(Z) - \bar{F})^2\right] \xrightarrow{Z\to\infty} 0$$

Corollary. For any $\delta > 0$

$$\mathbb{P}\left(|S(Z) - \bar{F}| > \delta\right) \leq \frac{\mathbb{E}\left[(S(Z) - \bar{F})^2\right]}{\delta^2} \xrightarrow{Z\to\infty} 0$$

We can show that

$$Z\mathbb{E}\left[(S(Z)-\bar{F})^2\right] \xrightarrow{Z\to\infty} 2\int_0^\infty R(z)dz$$

Proof:

$$\begin{aligned}
\mathbb{E}\left[(S(Z)-\bar{F})^2\right] &= \mathbb{E}\left[\frac{1}{Z^2}\int_0^Z dz_1\int_0^Z dz_2(F(z_1)-\bar{F})(F(z_2)-\bar{F})\right] \\
&= \frac{2}{Z^2}\int_0^Z dz_1\int_0^{z_1} dz_2 R(z_1-z_2) \\
&= \frac{2}{Z^2}\int_0^Z dz\int_0^{Z-z} dh R(z) \\
&= \frac{2}{Z}\int_0^Z \frac{Z-z}{Z}R(z)dz
\end{aligned}$$

Thus, denoting $R_Z(z)=\frac{Z-z}{Z}R(z)\mathbf{1}_{[0,Z]}(z)$, and using Lebesgue's theorem:

$$Z\mathbb{E}\left[(S(Z)-\bar{F})^2\right] = 2\int_0^\infty R_Z(z)dz \xrightarrow{Z\to\infty} 2\int_0^\infty R(z)dz$$

Let F be a stationary zero-mean random process. Denote

$$S_k(Z) = \frac{1}{\sqrt{Z}} \int_0^Z e^{ikz} F(z) dz$$

We can show similarly

$$\mathbb{E}[|S_k(Z)|^2] \xrightarrow{Z \to \infty} 2 \int_0^\infty R(z) \cos(kz) dz = \int_{-\infty}^\infty R(z) e^{ikz} dz$$

Simplified form of Bochner's theorem: If F is a stationary process, then the Fourier transform of its autocorrelation function is nonnegative.

Averaging

Let us consider $F(z,x)$, $z \in \mathbb{R}^+$, $x \in \mathbb{R}^d$, such that:

1) for all $x \in \mathbb{R}^d$, $F(z,x) \in \mathbb{R}^d$ is a stochastic process in z.

2) there is a deterministic function $\bar{F}(x)$ such that

$$\bar{F}(x) = \lim_{Z \to \infty} \frac{1}{Z} \int_{z_0}^{z_0+Z} \mathbb{E}[F(z,x)]dz$$

(limit independent of z_0).

Let $\varepsilon \ll 1$ and X^ε be the solution of

$$\frac{dX^\varepsilon}{dz} = F(\frac{z}{\varepsilon}, X^\varepsilon), \quad X^\varepsilon(0) = 0$$

Let us define $\bar{X}$ solution of

$$\frac{d\bar{X}}{dz} = \bar{F}(\bar{X}), \quad \bar{X}(0) = 0$$

With some mild technical assumptions we have for any Z:

$$\sup_{z \in [0,Z]} \mathbb{E}\left[|X^\varepsilon(z) - \bar{X}(z)|\right] \xrightarrow{\varepsilon \to 0} 0$$

The proof can be obtained with elementary calculations with the hypotheses :

1) F is stationary. For all x, $\mathbb{E}\left[\left|\frac{1}{Z}\int_0^Z F(z,x)dz - \bar{F}(x)\right|\right] \stackrel{Z\to\infty}{\longrightarrow} 0$

2) For all z, $F(z,.)$ and $\bar{F}(.)$ are Lipschitz with a deterministic constant c.

3) For any compact $K \subset \mathbb{R}^d$, $\sup_{z\in\mathbb{R}^+, x\in K} |F(z,x)| + |\bar{F}(x)| < \infty$.

Remark: 1) is satisfied if for any x, the autocorrelation function $R_x(z)$ of $z \mapsto F(z,x)$ is integrable $\int |R_x(z)|dz < \infty$.

We have:

$$X^\varepsilon(z) = \int_0^z F(\frac{s}{\varepsilon}, X^\varepsilon(s))ds, \qquad \bar{X}(z) = \int_0^z \bar{F}(\bar{X}(s))ds$$

so the error can be written:

$$X^\varepsilon(z) - \bar{X}(z) = \int_0^z \Big(F(\frac{s}{\varepsilon}, X^\varepsilon(s)) - F(\frac{s}{\varepsilon}, \bar{X}(s))\Big)\, ds + g^\varepsilon(z)$$

where $g^\varepsilon(z) := \displaystyle\int_0^z F(\frac{s}{\varepsilon}, \bar{X}(s)) - \bar{F}(\bar{X}(s))ds$.

$$
\begin{aligned}
|X^\varepsilon(z) - \bar{X}(z)| &\leq \int_0^z \left| F(\frac{s}{\varepsilon}, X^\varepsilon(s)) - F(\frac{s}{\varepsilon}, \bar{X}(s)) \right| ds + |g^\varepsilon(z)| \\
&\leq c \int_0^t |X^\varepsilon(s) - \bar{X}(s)| ds + |g^\varepsilon(z)|
\end{aligned}
$$

Take the expectation and apply Gronwall

$$
\mathbb{E}\left[|X^\varepsilon(z) - \bar{X}(z)|\right] \leq e^{ct} \sup_{s\in[0,z]} \mathbb{E}[|g^\varepsilon(s)|]
$$

It remains to show that the last term goes to 0 as $\varepsilon \to 0$.
Let $\delta > 0$

$$
\begin{aligned}
g^\varepsilon(z) &= \sum_{k=0}^{[z/\delta]-1} \int_{k\delta}^{(k+1)\delta} \left(F(\frac{s}{\varepsilon}, \bar{X}(s)) - \bar{F}(\bar{X}(s)) \right) ds \\
&\quad + \int_{\delta[z/\delta]}^{z} \left(F(\frac{s}{\varepsilon}, \bar{X}(s)) - \bar{F}(\bar{X}(s)) \right) ds
\end{aligned}
$$

Denote $M_Z = \sup_{z\in[0,Z]} |\bar{X}(z)|$. Since F is Lipschitz and $K_Z = \sup_{x\in[-M_Z,M_Z]} |\bar{F}(x)|$ is finite:

$$
\left| F(\frac{s}{\varepsilon}, \bar{X}(s)) - F(\frac{s}{\varepsilon}, \bar{X}(k\delta)) \right| \leq c \left| \bar{X}(s) - \bar{X}(k\delta) \right| \leq c\bar{K}_Z |s - k\delta|
$$

Denoting $K_Z = \sup_{z\in\mathbb{R}^+, x\in[-M_Z,M_Z]} |F(z,x)|$:

$$\left|\bar{F}(\bar{X}(s)) - \bar{F}(\bar{X}(k\delta))\right| \leq cK_Z|s-k\delta|$$

Thus

$$\begin{aligned}
|g^\varepsilon(z)| &\leq \sum_{k=0}^{[z/\delta]-1} \left| \int_{k\delta}^{(k+1)\delta} \left(F(\frac{s}{\varepsilon}, \bar{X}(s)) - \bar{F}(\bar{X}(s))\right) ds \right| \\
&\quad + \left| \int_{\delta[z/\delta]}^{z} \left(F(\frac{s}{\varepsilon}, \bar{X}(s)) - \bar{F}(\bar{X}(s))\right) ds \right| \\
&\leq \left| \sum_{k=0}^{[z/\delta]-1} \int_{k\delta}^{(k+1)\delta} \left(F(\frac{s}{\varepsilon}, \bar{X}(k\delta)) - \bar{F}(\bar{X}(k\delta))\right) ds \right| \\
&\quad + c(\bar{K}_Z + K_Z) \sum_{k=0}^{[z/\delta]-1} \int_{k\delta}^{(k+1)\delta} (s-k\delta) ds + (\bar{K}_Z + K_Z)\delta \\
&\leq \varepsilon \sum_{k=0}^{[z/\delta]-1} \left| \int_{k\delta/\varepsilon}^{(k+1)\delta/\varepsilon} \left(F(s, \bar{X}(k\delta)) - \bar{F}(\bar{X}(k\delta))\right) ds \right| \\
&\quad + (\bar{K}_Z + K_Z)(cz+1)\delta
\end{aligned}$$

Take the expectation and the supremum :

$$\begin{aligned}\sup_{z\in[0,Z]} \mathbb{E}[|g^\varepsilon(z)|] \quad &\leq \quad \delta \sum_{k=0}^{[Z/\delta]} \mathbb{E}\left[\left|\frac{\varepsilon}{\delta}\int_{k\delta/\varepsilon}^{(k+1)\delta/\varepsilon} \big(F(s,\bar{X}(k\delta)) - \bar{F}(\bar{X}(k\delta))\big)\,ds\right|\right] \\ &\quad + (\bar{K}_Z + K_Z)(cZ+1)\delta\end{aligned}$$

Take the limit $\varepsilon \to 0$:

$$\limsup_{\varepsilon\to 0} \sup_{t\in[0,Z]} \mathbb{E}[|g^\varepsilon(t)|] \leq (\bar{K}_Z + K_Z)(cZ+1)\delta$$

Let $\delta \to 0$.

Method of averaging: Khasminskii theorem

$$\frac{dX^\varepsilon}{dz} = F(\frac{z}{\varepsilon}, X^\varepsilon), \quad X^\varepsilon(0) = x_0$$

$x \mapsto F(x,z)$ and $x \mapsto \bar{F}(x)$ are Lipschitz, $z \mapsto F(z,x)$ is stationary and ergodic

$$\bar{F}(x) = \mathbb{E}[F(z,x)]$$

Let $\bar{X}$ be the solution of

$$\frac{d\bar{X}}{dz} = \bar{F}(\bar{X}), \quad \bar{X}(0) = x_0$$

Theorem: for any $Z > 0$,

$$\sup_{z \in [0,Z]} \mathbb{E}\left[|X^\varepsilon(z) - \bar{X}(z)|\right] \xrightarrow{\varepsilon \to 0} 0$$

[1] R. Z. Khasminskii, Theory Probab. Appl. **11** (1966), 211-228.

Equations for the Fourier components of the wave:

$$\frac{dX^\varepsilon}{dz} = F(\frac{z}{\varepsilon}, X^\varepsilon),$$

where

$$X^\varepsilon = \begin{pmatrix} \hat{p} \\ \hat{u} \end{pmatrix}, \qquad F(z, X) = -i\omega \begin{pmatrix} 0 & \rho(z) \\ \frac{1}{\kappa(z)} & 0 \end{pmatrix} X$$

Apply the method of averaging $\Longrightarrow$ $X^\varepsilon(\omega, z)$ converges in $L^1(\mathbb{P})$ to $\bar{X}(\omega, z)$

$$\frac{d\bar{X}}{dz} = -i\omega \begin{pmatrix} 0 & \bar{\rho} \\ \frac{1}{\bar{\kappa}} & 0 \end{pmatrix} \bar{X}, \quad \bar{\rho} = \mathbb{E}[\rho], \quad \bar{\kappa} = \left(\mathbb{E}[\kappa^{-1}]\right)^{-1}$$

$\hookrightarrow$ deterministic "effective medium" with parameters $\bar{\rho}$, $\bar{\kappa}$.

Let $(\bar{p}, \bar{u})$ be the solution of the homogeneous effective system

$$\bar{\rho}\frac{\partial \bar{u}}{\partial t} + \frac{\partial \bar{p}}{\partial z} = 0$$
$$\frac{\partial \bar{p}}{\partial t} + \bar{\kappa}\frac{\partial \bar{u}}{\partial z} = 0$$

The propagation speed of $(\bar{p}, \bar{u})$ is $\bar{c} = \sqrt{\bar{\kappa}/\bar{\rho}}$.

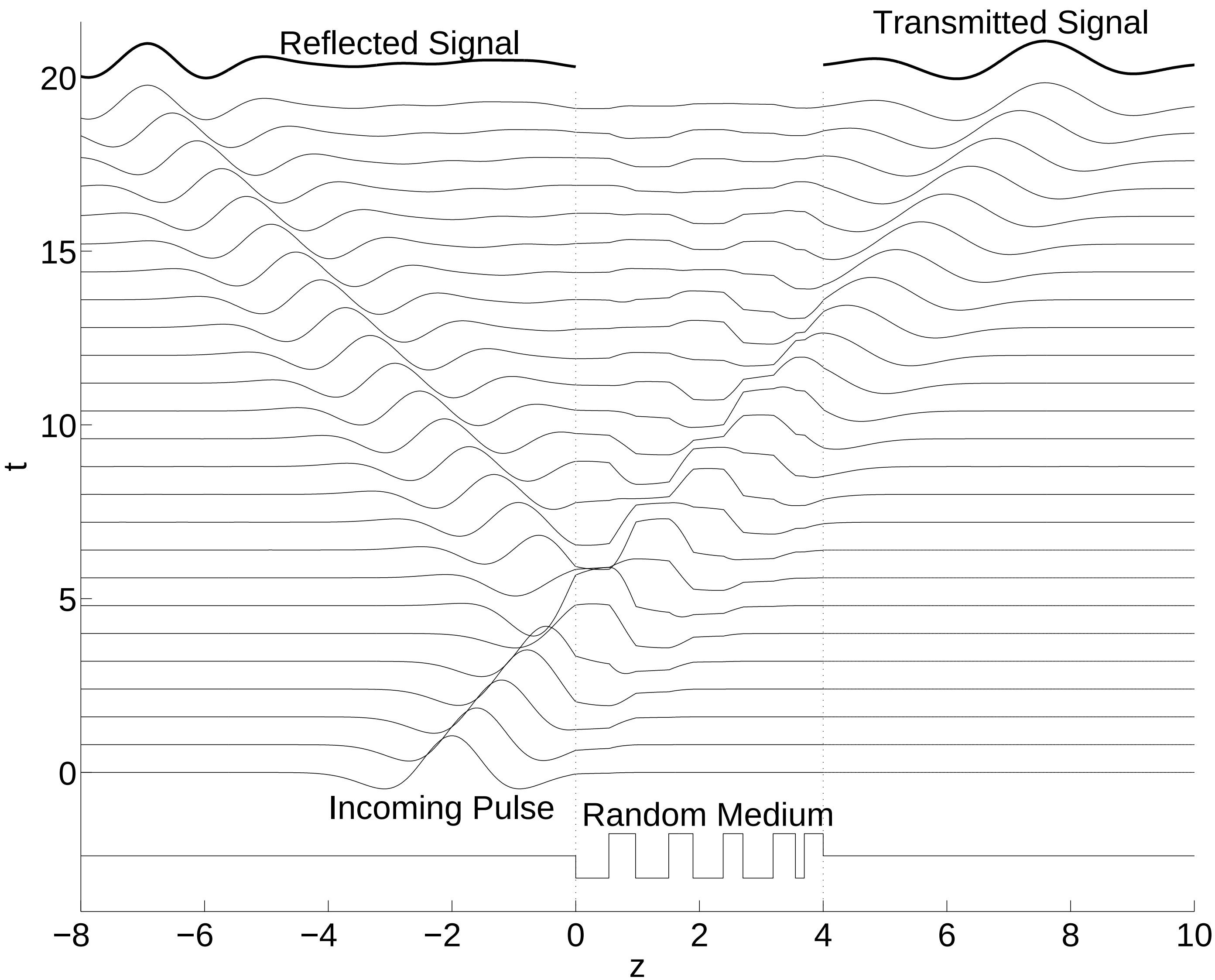
Compare $p^\varepsilon(t, z)$ with $\bar{p}(t, z)$:

$$\begin{aligned} \mathbb{E}\left[|p^\varepsilon(t,z) - \bar{p}(t,z)|\right] &= \frac{1}{2\pi}\mathbb{E}\left[\left|\int e^{-i\omega t}(\hat{p}^\varepsilon(\omega,z) - \hat{\bar{p}}(\omega,z))d\omega\right|\right] \\ &\leq \frac{1}{2\pi}\int \mathbb{E}\left[|\hat{p}^\varepsilon(\omega,z) - \hat{\bar{p}}(\omega,z)|\right] d\omega \end{aligned}$$

The dominated convergence theorem then gives the convergence in $L^1(\mathbb{P})$ of p^ε to $\bar{p}$ in the time domain.
$\hookrightarrow$ the effective speed of the acoustic wave $(p^\varepsilon, u^\varepsilon)$ as $\varepsilon \to 0$ is $\bar{c}$.

This analysis is just a small piece of the homogenization theory.
cf book *The theory of composites* by G. Milton.

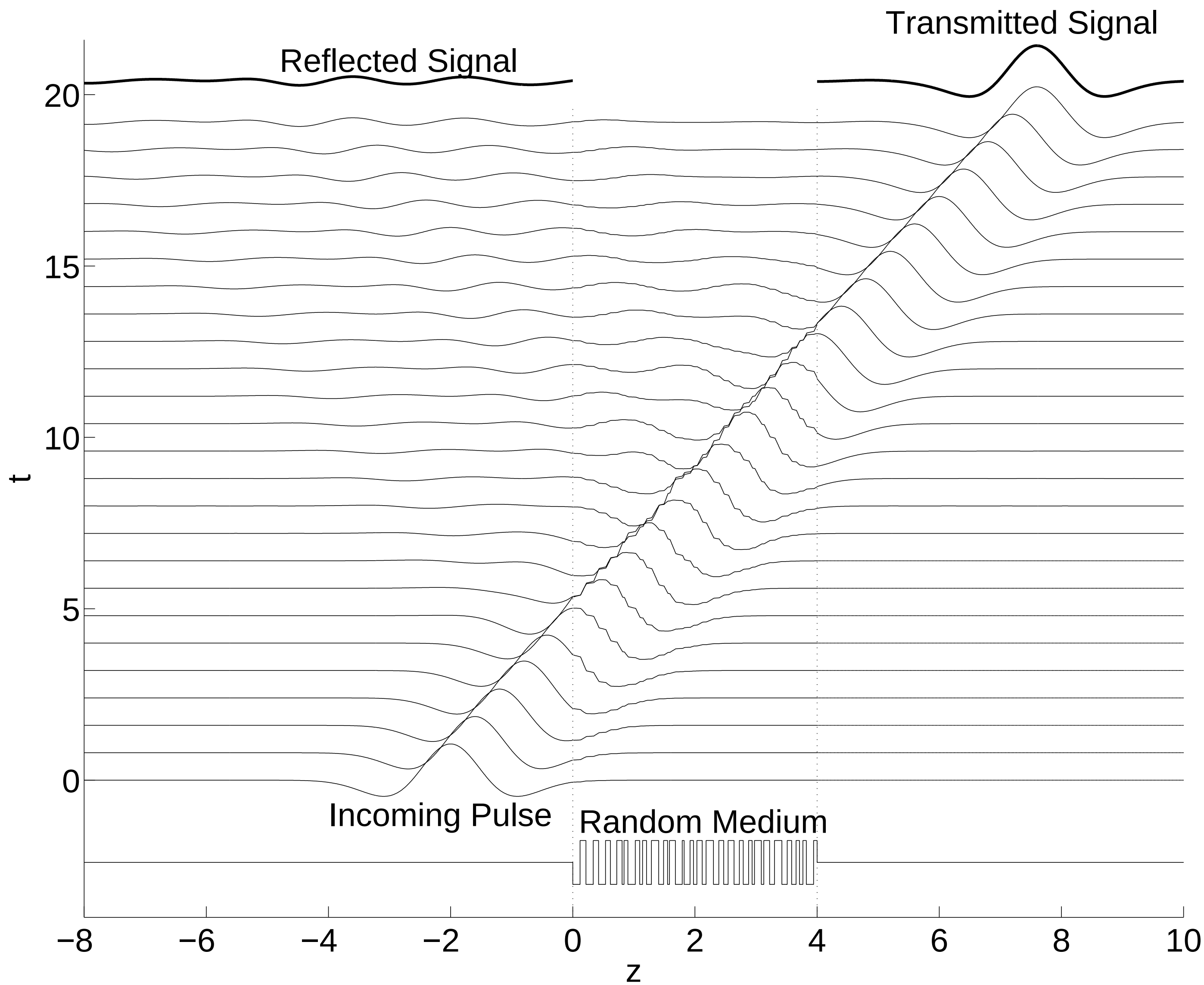
Propagation through a stack of random layers



Sizes of the layers: i.i.d. with uniform distribution over $[0.2, 0.6]$ (mean 0.4).

Medium parameters $\rho \equiv 1$, $1/\kappa_a = 0.2$, $1/\kappa_b = 1.8$.

Propagation through a stack of random layers



Sizes of the layers: i.i.d. with uniform distribution over $[0.04, 0.12]$ (mean 0.08).

Example: bubbles in water

$\rho_a = 1.2\ 10^3$ g/m^3, $\kappa_a = 1.4\ 10^8$ g/s^2/m, $c_a = 340$ m/s.
$\rho_w = 1.0\ 10^6$ g/m^3, $\kappa_w = 2.0\ 10^{12}$ g/s^2/m, $c_w = 1425$ m/s.
If the typical pulse frequency is 10 Hz - 30 kHz, then the typical wavelength is 1 cm - 100 m. The bubble sizes are much smaller $\Longrightarrow$ the effective medium theory can be applied.

$$\bar{\rho} = \mathbb{E}[\rho] = \phi \rho_a + (1-\phi)\rho_w = \begin{cases} 9.9\ 10^5\ \mathrm{g/m^3} & \text{if } \phi = 1\% \\ 9\ 10^5\ \mathrm{g/m^3} & \text{if } \phi = 10\% \end{cases}$$

$$\bar{\kappa} = \left(\mathbb{E}[\kappa^{-1}]\right)^{-1} = \left(\frac{\phi}{\kappa_a} + \frac{1-\phi}{\kappa_w}\right)^{-1} = \begin{cases} 1.4\ 10^{10}\ \mathrm{g/s^2/m} & \text{if } \phi = 1\% \\ 1.4\ 10^{9}\ \mathrm{g/s^2/m} & \text{if } \phi = 10\% \end{cases}$$

where ϕ = volume fraction of air.
Thus, $\bar{c} = 120$ m/s if $\phi = 1\%$ and $\bar{c} = 37$ m/s if $\phi = 10$ %.

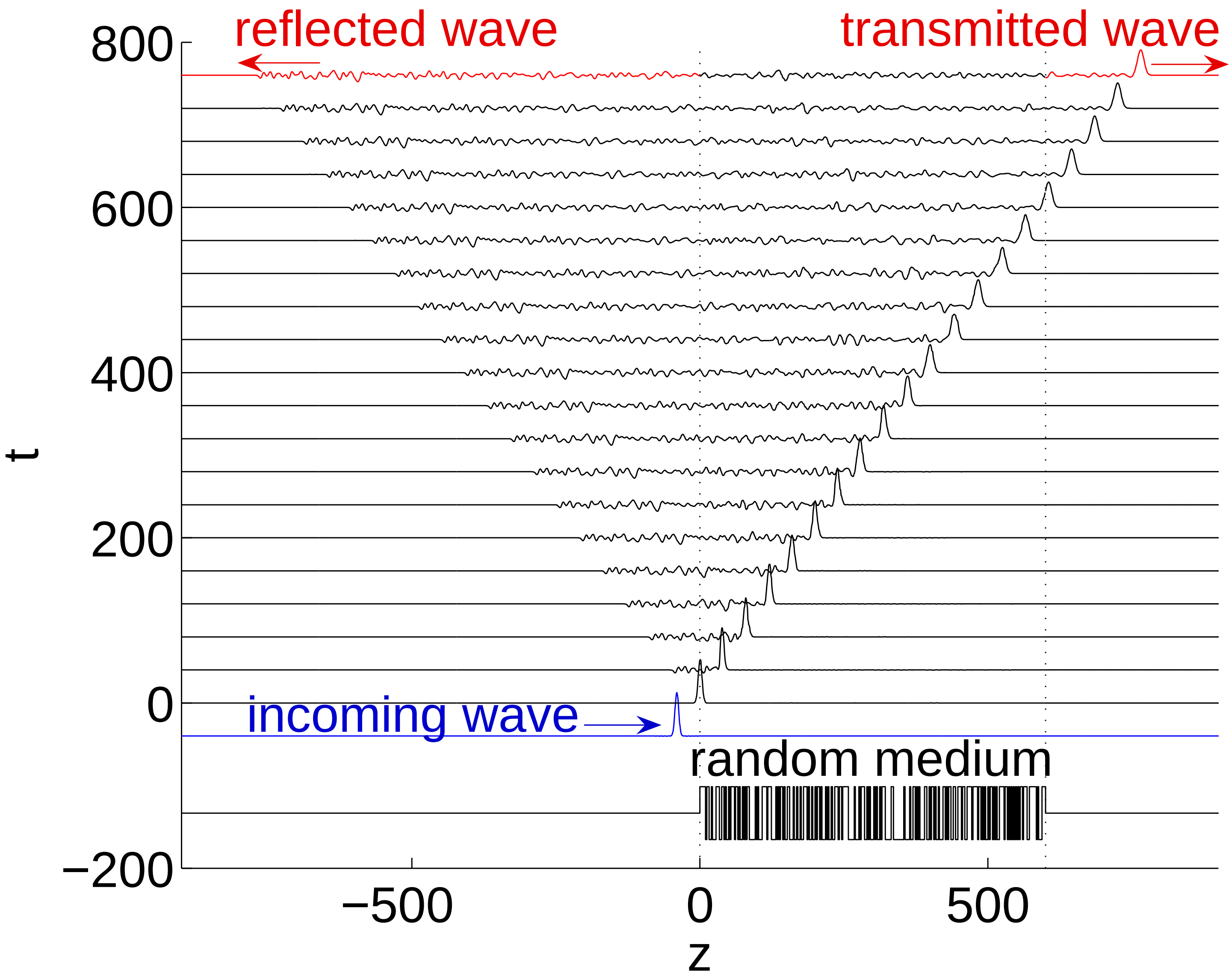
$\hookrightarrow$ the average sound speed $\bar{c}$ can be much smaller than ess inf(c).

The converse is impossible:

$$\mathbb{E}[c^{-1}] = \mathbb{E}\left[\kappa^{-1/2}\rho^{1/2}\right] \leq \mathbb{E}[\kappa^{-1}]^{1/2}\mathbb{E}[\rho]^{1/2} = \bar{c}^{-1}$$

Thus $\bar{c} \leq \mathbb{E}[c^{-1}]^{-1} \leq$ ess sup(c).

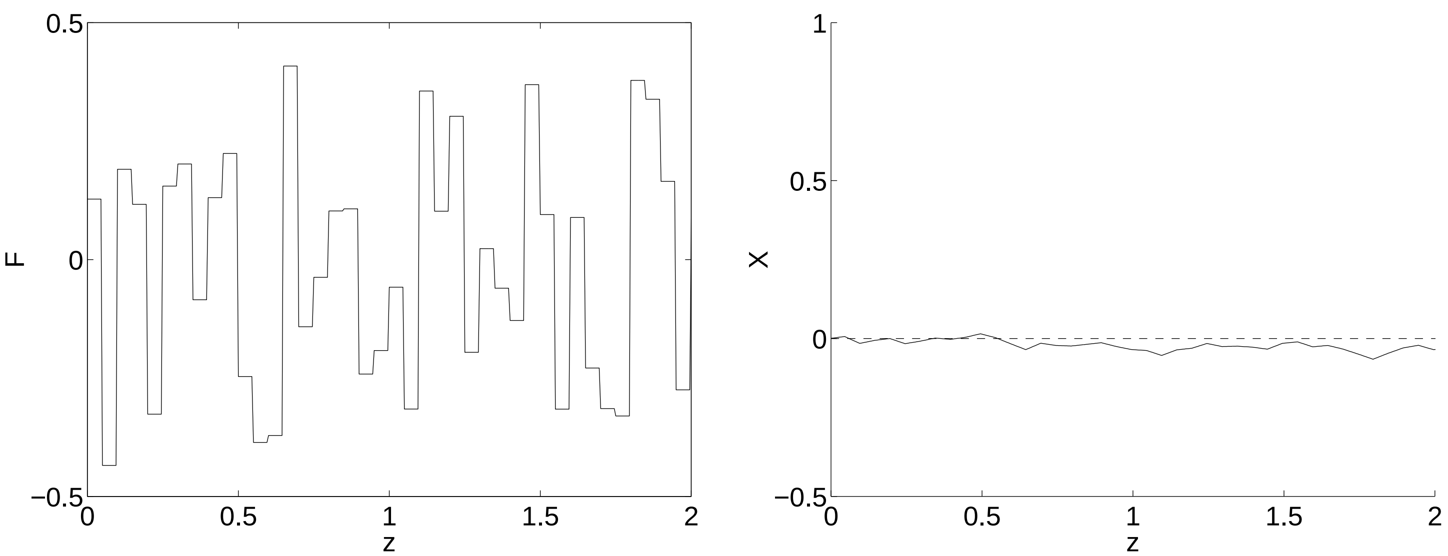
Long distance propagation



Toy model

$$\frac{dX^\varepsilon}{dz} = F(\frac{z}{\varepsilon})$$

with $F(z) = \sum_{i=1}^{\infty} F_i \mathbf{1}_{[i-1,i)}(z)$, F_i independent random variables $\mathbb{E}[F_i] = \bar{F} = 0$ and $\mathbb{E}[(F_i - \bar{F})^2] = \sigma^2$.



$\varepsilon = 0.05$

For any $z \in [0, Z]$, we have

$$X^\varepsilon(z) \xrightarrow{\varepsilon \to 0} \bar{X}(z), \qquad \frac{d\bar{X}}{dz} = \bar{F} = 0.$$

No macroscopic evolution is noticeable.

$\to$ it is necessary to look at larger z to get an effective behavior

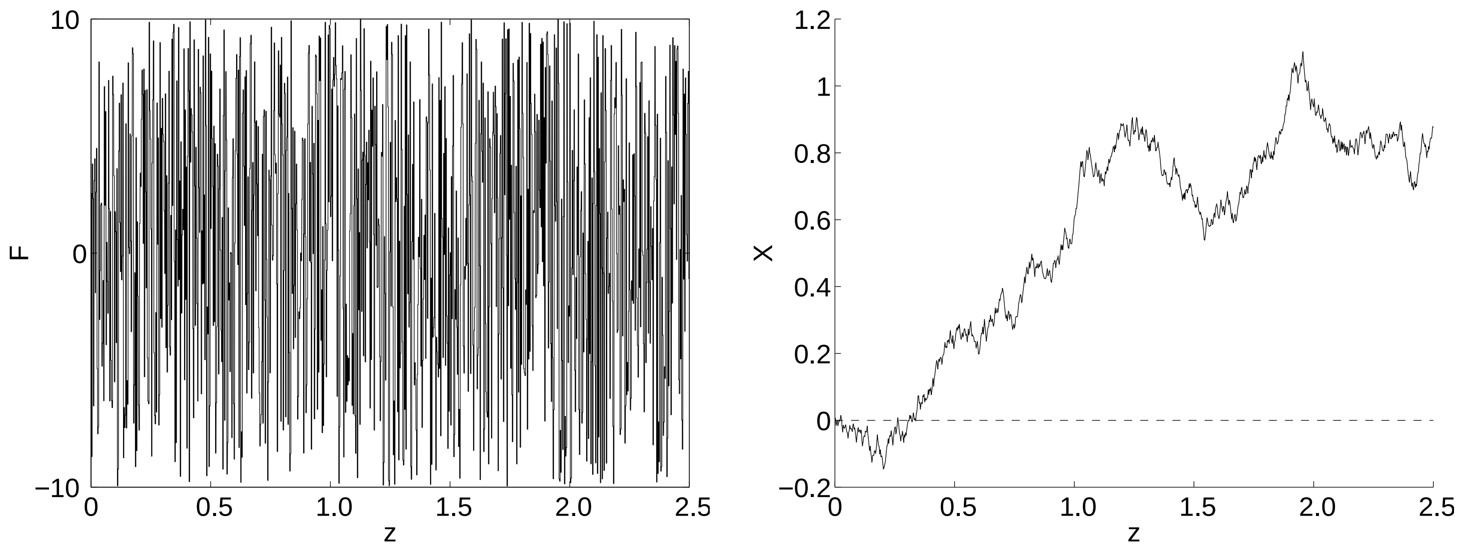
$$z \mapsto \frac{z}{\varepsilon}, \qquad \tilde{X}^\varepsilon(z) = X^\varepsilon(\frac{z}{\varepsilon})$$

$$\frac{d\tilde{X}^\varepsilon}{dz} = \frac{1}{\varepsilon} F(\frac{z}{\varepsilon^2})$$

Diffusion-approximation: Toy model

$$\frac{dX^\varepsilon}{dz} = \frac{1}{\varepsilon} F\left(\frac{z}{\varepsilon^2}\right)$$

with $F(z) = \sum_{i=1}^{\infty} F_i \mathbf{1}_{[i-1,i)}(z)$, F_i independent random variables $\mathbb{E}[F_i] = 0$ and $\mathbb{E}[F_i^2] = \sigma^2$.



$\varepsilon = 0.05$

$$X^{\varepsilon}(z) = \varepsilon \int_0^{\frac{z}{\varepsilon^2}} F(s)ds = \varepsilon \left(\sum_{i=1}^{\left[\frac{z}{\varepsilon^2}\right]} F_i \right) + \varepsilon \int_{\left[\frac{z}{\varepsilon^2}\right]}^{\frac{z}{\varepsilon^2}} F(s)ds$$

$$= \underset{\substack{\varepsilon \to 0 \downarrow \\ \sqrt{z}}}{\varepsilon \sqrt{\left[\frac{z}{\varepsilon^2}\right]}} \times \underset{\substack{\text{law} \downarrow (CLT) \\ \mathcal{N}(0,\sigma^2)}}{\frac{1}{\sqrt{\left[\frac{z}{\varepsilon^2}\right]}} \left(\sum_{i=1}^{\left[\frac{z}{\varepsilon^2}\right]} F_i \right)} + \underset{\substack{\text{a.s.} \downarrow \\ 0}}{\varepsilon \left(\frac{z}{\varepsilon^2} - \left[\frac{z}{\varepsilon^2}\right] \right) F_{\left[\frac{z}{\varepsilon^2}\right]}}$$

Thus: $X^{\varepsilon}(z)$ converges in distribution as $\varepsilon \to 0$ to the Gaussian statistics $\mathcal{N}(0, \sigma^2 z)$.

With some more work: The process $(X^{\varepsilon}(z))_{z \in \mathbb{R}^+}$ converges in distribution to a Brownian motion $\sigma W(z)$.

Next goal: determine the limit $\lim_{\varepsilon \to 0} X^\varepsilon(z)$ where

$$\frac{dX^\varepsilon}{dz} = \frac{1}{\varepsilon} F\left(\frac{z}{\varepsilon^2}, X^\varepsilon(z)\right)$$

when

$$\mathbb{E}[F(z, x)] = 0 \quad \forall x$$

Markov process

A stochastic process Y_z with state space S is Markov if $\forall 0 \leq s \leq z$ and $f \in L^\infty(S)$

$$\mathbb{E}[f(Y_z)|Y_u, u \leq s] = \mathbb{E}[f(Y_z)|Y_s]$$

"the state Y_s at time s contains all relevant information for calculating probabilities of future events".

The processus is stationary if $\forall z \geq s \geq 0$, $\mathbb{E}[f(Y_z)|Y_s] = \mathbb{E}[f(Y_{z-s})|Y_0]$.

Define the family of operators on $L^\infty(S)$:

$$T_z f(y) = \mathbb{E}[f(Y_z)|Y_0 = y]$$

Proposition.
1) $T_0 = I_d$
2) $\forall s, z \leq 0$, $T_{z+s} = T_z T_s$
3) T_z is a contraction $\|T_z f\|_\infty \leq \|f\|_\infty$.

Proof of 2):

$$\begin{aligned} T_{z+s}f(y) &= \mathbb{E}[f(Y_{z+s})|Y_0 = y] = \mathbb{E}[\mathbb{E}[f(Y_{z+s})|Y_u,\, u \leq z]|Y_0 = y] \\ &= \mathbb{E}[\mathbb{E}[f(Y_{z+s})|Y_z]|Y_0 = y] = \mathbb{E}[T_s f(Y_z)|Y_0 = y] \\ &= T_z T_s f(y) \end{aligned}$$

Feller process: T_z is strongly continuous from $\mathcal{C}_0$ to $\mathcal{C}_0$ (for any $f \in \mathcal{C}_0$, $\|T_z f - f\|_\infty \xrightarrow{z \to 0} 0$).

The generator of the Markov process is:

$$Q := \lim_{z \searrow 0} \frac{T_z - I_d}{z}$$

It is defined on a subset of $\mathcal{C}^0$, supposed to be dense.

Proposition. If $f \in \mathrm{Dom}(Q)$, then the function $u(z,y) = T_z f(y)$ satisfies the Kolmogorov equation

$$\frac{\partial u}{\partial z} = Qu, \qquad u(z=0,y) = f(y)$$

Proof.

$$\frac{u(z+h,y) - u(z,y)}{h} = \frac{T_{z+h}f(y) - T_z f(y)}{h} = T_z \frac{T_h - I_d}{h} f(y) \xrightarrow{h \to 0} T_z Q f(y)$$

because $f \in \mathrm{Dom}(Q)$ and T_z is continuous. This shows that u is differentiable and $\partial_z u = T_z Q f$. Besides

$$\frac{T_h - I_d}{h} T_z f(y) = \frac{T_{z+h}f(y) - T_z f(y)}{h} = \frac{u(z+h,y) - u(z,y)}{h}$$

has a limit as $h \to 0$, which shows that $T_z f \in \mathrm{Dom}(Q)$ and $\partial_z u = Q T_z f = Qu$.

Example: Brownian motion

W_z: Gaussian process with independent increments

$$\mathbb{E}[(W_{z+h} - W_z)^2] = h$$

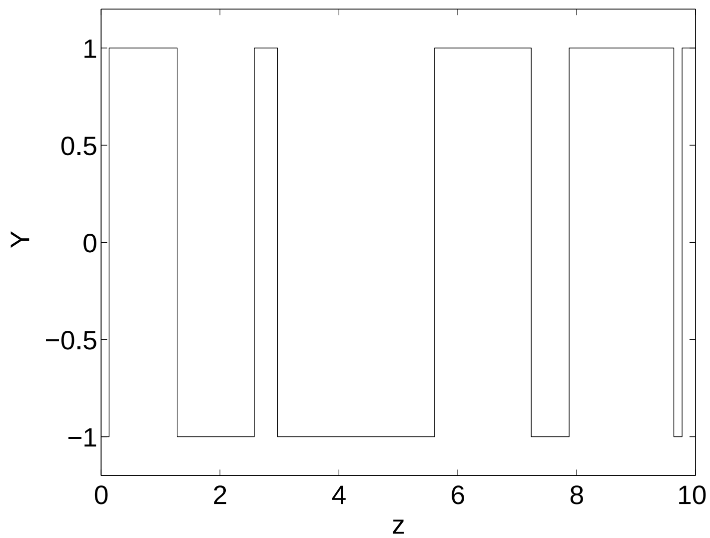
The semi-group T_z is the heat kernel:

$$\begin{aligned} T_z f(x) &= \mathbb{E}[f(x + W_z^{(0)})] = \int f(x+w) \frac{1}{\sqrt{2\pi z}} \exp\left(-\frac{w^2}{2z}\right) dz \\ &= \int f(y) \frac{1}{\sqrt{2\pi z}} \exp\left(-\frac{(y-x)^2}{2z}\right) dy \end{aligned}$$

It is a Markov process with the generator:

$$Q = \frac{1}{2} \frac{\partial^2}{\partial x^2}$$

Example: Two-state Markov process



The process Y_z takes values in $S = \{-1, 1\}$.

The time intervals are independent with the common exponential distribution with mean 1.

Functions $f \in L^\infty(S)$ are vectors. The semigroup $(T_z)_{z\geq 0}$ is a family of matrices:

$$T_z = \begin{pmatrix} \mathbb{P}(Y_z = 1 | Y_0 = 1) & \mathbb{P}(Y_z = 1 | Y_0 = -1) \\ \mathbb{P}(Y_z = -1 | Y_0 = 1) & \mathbb{P}(Y_z = -1 | Y_0 = -1) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + \frac{1}{2} e^{-2z} & \frac{1}{2} - \frac{1}{2} e^{-2z} \\ \frac{1}{2} - \frac{1}{2} e^{-2z} & \frac{1}{2} + \frac{1}{2} e^{-2z} \end{pmatrix}$$

The generator is a matrix:

$$Q = \lim_{h \to 0} \frac{T_h - I}{h} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

Martingale property

For any function $f \in \mathrm{Dom}(Q)$, the process

$$M_f(z) := f(Y_z) - \int_0^z Qf(Y_u)du$$

is a martingale.

Denoting $\mathcal{F}_s = \sigma(Y_u, 0 \leq u \leq s)$,

$$\begin{aligned} \mathbb{E}[M_f(z)|\mathcal{F}_s] &= M_f(s) + \mathbb{E}\left[f(Y_z) - f(Y_s) - \int_s^z Qf(Y_u)du|Y_s\right] \\ &= M_f(s) + T_{z-s}f(Y_s) - f(Y_s) - \int_s^z T_{u-s}Qf(Y_s)du \\ &= M_f(s) + T_{z-s}f(Y_s) - f(Y_s) - \int_0^{z-s} T_uQf(Y_s)du \end{aligned}$$

The function $T_z f(y)$ satisfies the Kolmogorov equation, which shows that the last three terms of the r.h.s. cancel:

$$\mathbb{E}[M_f(z)|\mathcal{F}_s] = M_f(s)$$

Reciprocal: If Q is non-degenerate, and M_f is a martingale for all test functions f, then Y is a Markov process with generator Q.

Ordinary differential equation driven by a Feller process

Proposition. Let Y be a S-valued Feller process with generator Q and X be the solution of:

$$\frac{dX}{dz} = F(Y_z, X(z)), \qquad X(0) = x \in \mathbb{R}^d$$

where $F : S \times \mathbb{R}^d \to \mathbb{R}^d$ is a bounded Borel function such that $x \mapsto F(y, x)$ has bounded derivatives uniformly with respect to $y \in S$. Then $\tilde{X} = (Y, X)$ is a Markov process with generator:

$$\mathcal{L} = Q + \sum_{j=1}^{d} F_j(y, x) \frac{\partial}{\partial x_j}$$

Formal proof. Let f be a test function.

$$\begin{aligned}
&\frac{d}{dz}\mathbb{E}[f(Y_z, X(z))|Y_0 = y, X(0) = x] \\
&\quad = \mathbb{E}[Qf(Y_z, X(z))|Y_0 = y, X(0) = x] \\
&\qquad + \mathbb{E}[\nabla_x f(Y_z, X(z))F(Y_z, X(z))|Y_0 = y, X(0) = x] \\
&\quad = \mathbb{E}[\mathcal{L}f(Y_z, X(z))|Y_0 = y, X(0) = x]
\end{aligned}$$

Ergodic Markov process

Ergodicity is related to the **null spaces of Q and Q^*.**
With some additional hypotheses (irreducibility):
A Markov process is ergodic iff there is a unique invariant probability measure $\mathbb{P}$ satisfying $Q^*\mathbb{P} = 0$, i.e.

$$\int T_z f(y) d\mathbb{P}(y) = \int f(y) d\mathbb{P}(y) \Longleftrightarrow \mathbb{E}_{\mathbb{P}}[f(Y_z)] = \mathbb{E}_{\mathbb{P}}[f(Y_0)]$$

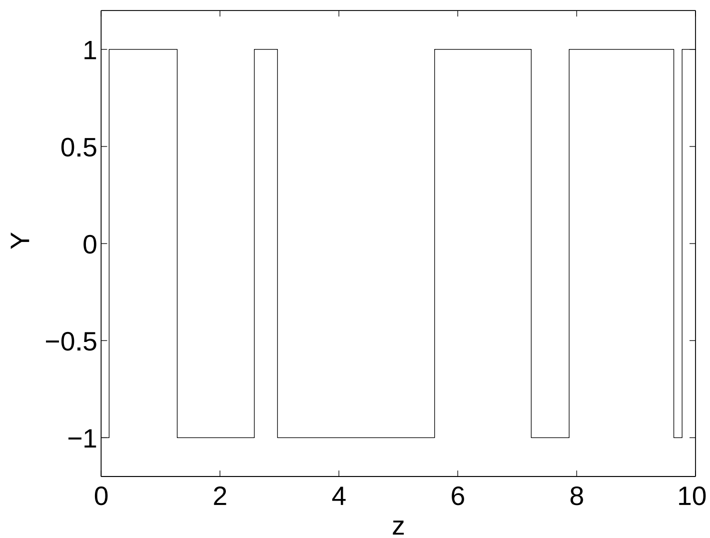
Ergodicity: $T_z f(y)$ converges to $\mathbb{E}_{\mathbb{P}}[f(Y_0)]$ as $z \to \infty$. The spectrum of Q gives the convergence (mixing) rate. The existence of a spectral gap

$$\inf_{f, \int f d\mathbb{P}=0} \frac{-\int f Q f d\mathbb{P}}{\int f^2 d\mathbb{P}} > 0$$

ensures the exponential convergence of $T_z f(y)$ to $\mathbb{E}_{\mathbb{P}}[f(Y_0)]$.

Also: Since $T_z 1 = 1$, we have $Q1 = 0$, so that $1 \in \mathrm{Null}(Q)$.
Thus $\mathrm{Null}(Q^*)$ is at least one-dimensional.
A Markov process is ergodic iff $\mathrm{Null}(Q) = \mathrm{Span}(\{1\})$

Example: Two-state Markov process



The process Y_z takes values in $S = \{-1, 1\}$.

The time intervals are independent with the common exponential distribution with mean 1.

The semigroup $(T_z)_{z\geq 0}$ is a family of matrices:

$$T_z = \begin{pmatrix} \mathbb{P}(Y_z = 1 | Y_0 = 1) & \mathbb{P}(Y_z = 1 | Y_0 = -1) \\ \mathbb{P}(Y_z = -1 | Y_0 = 1) & \mathbb{P}(Y_z = -1 | Y_0 = -1) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + \frac{1}{2} e^{-2z} & \frac{1}{2} - \frac{1}{2} e^{-2z} \\ \frac{1}{2} - \frac{1}{2} e^{-2z} & \frac{1}{2} + \frac{1}{2} e^{-2z} \end{pmatrix}$$

The generator is a matrix:

$$Q = \lim_{h \to 0} \frac{T_h - I}{h} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

It is ergodic. The invariant probability ($Q^T \bar{p} = 0$) is the uniform probability $\bar{p} = (1/2, 1/2)^T$ over S.

Example: Brownian motion

W_z: Gaussian process with independent increments

$$\mathbb{E}[(W_{z+h} - W_z)^2] = h$$

The semi-group T_z is the heat kernel:

$$T_z f(x) = \int f(y) p_z(x,y) dy\,, \qquad p_z(x,y) = \frac{1}{\sqrt{2\pi z}} \exp\left(-\frac{(y-x)^2}{2z}\right)$$

It is a Markov process with the generator:

$$Q = \frac{1}{2}\frac{\partial^2}{\partial x^2}$$

It is not ergodic.

Example: Ornstein-Uhlenbeck process

Solution of the stochastic differential equation $dX(z) = -\lambda X(z)dz + dW_z$:

$$X(z) = X_0 e^{-\lambda z} + \int_0^z e^{-\lambda(z-s)} dW_s$$

where W_z is a Brownian motion, $\lambda > 0$.
(if $z \mapsto t$, this process describes the motion of a particle in a quadratic potential)
The semi-group T_z is

$$T_z f(x) = \int f(y) p_z(x,y) dy$$

$y \mapsto p_z(x,y)$ is a Gaussian density with mean $xe^{-\lambda z}$ and variance $\sigma^2(z)$:

$$p_z(x,y) = \frac{1}{\sqrt{2\pi\sigma(z)^2}} \exp\left(-\frac{(y - xe^{-\lambda z})^2}{2\sigma^2(z)}\right), \quad \sigma^2(z) = \frac{1 - e^{-2\lambda z}}{2\lambda}$$

The generator is:

$$Q = \frac{1}{2}\frac{\partial^2}{\partial x^2} - \lambda x \frac{\partial}{\partial x}$$

$X(z)$ is ergodic. Its invariant probability density ($Q^*\bar{p} = 0$) is

$$\bar{p}(y) = \sqrt{\frac{\lambda}{\pi}} \exp\left(-\lambda y^2\right)$$

Diffusion processes

• Let σ and b be $\mathcal{C}^1(\mathbb{R},\mathbb{R})$ functions with bounded derivatives.
Let W_z be a Brownian motion.
The solution $X(z)$ of the 1D stochastic differential equation:

$$dX(z) = \sigma(X(z))dW_z + b(X(z))dz$$

is a Markov process with the generator

$$Q = \frac{1}{2}\sigma^2(x)\frac{\partial^2}{\partial x^2} + b(x)\frac{\partial}{\partial x}$$

• Let $\sigma \in \mathcal{C}^1(\mathbb{R}^n,\mathbb{R}^m)$ and $b \in \mathcal{C}^1(\mathbb{R}^n,\mathbb{R}^n)$ with bounded derivatives.
Let W_z be a m-dimensional Brownian motion.
The solution $X(z)$ of the stochastic differential equation:

$$dX(z) = \sigma(X(z))dW_z + b(X(z))dz$$

is a Markov process with the generator

$$Q = \frac{1}{2}\sum_{ij} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i(x)\frac{\partial}{\partial x_i}$$

with $a = \sigma\sigma^T$.

Poisson equation $Qu = f$

Let us consider an ergodic Markov process with generator Q.

$\text{Null}(Q^*)$ has dimension 1 and is spanned by the invariant probability $\mathbb{P}$.

By Fredholm alternative, the Poisson equation has a solution iff $f \perp \text{Null}(Q^*)$, i.e. $\int f d\mathbb{P} = 0$ or $\mathbb{E}[f(Y_0)] = 0$ where $\mathbb{E}$ is the expectation w.r.t. the invariant probability $\mathbb{P}$.

Proposition. *If* $\mathbb{E}[f(Y_0)] = 0$, *a solution of* $Qu = f$ *is*

$$u(y) = -\int_0^\infty T_z f(y) dz$$

Remember that $T_z f(y) = \mathbb{E}[f(Y_z)|Y_0 = y]$.

Proof.

$$u(y) = -\int_0^\infty T_z f(y) dz = -\int_0^\infty \{T_z f(y) - \mathbb{E}[f(\mathrm{Y}_0)]\} \, dz$$

The convergence of this integral requires some mixing.

Using Kolmogorov equation

$$Qu = -\int QT_z f dz = -\int_0^\infty \frac{dT_z f}{dz} dz = -\left[T_z f\right]_0^\infty = f - \mathbb{E}[f(Y_0)] = f$$

Moreover $\mathbb{E}[u(Y_0)] = 0$ because $\mathbb{E}[f(Y_z)] = \mathbb{E}[f(Y_0)] = 0$.

Finally:
$\left[-\int_0^\infty dz T_z\right] : \mathcal{D} \to \mathcal{D}$ *is the inverse of* Q *on* $\mathcal{D} = (\mathrm{Null}(Q^*))^\perp$.

Diffusion-approximation

$$\frac{dX^\varepsilon}{dz}(z) = \frac{1}{\varepsilon} F\left(Y(\frac{z}{\varepsilon^2}), X^\varepsilon(z)\right), \qquad X^\varepsilon(0) = x_0 \in \mathbb{R}^d.$$

Y stationary and ergodic, F centered: $\mathbb{E}[F(Y(0), x)] = 0$.

Theorem: *The processes $(X^\varepsilon(z))_{z\geq 0}$ converge in distribution in $\mathbf{C}^0([0,\infty), \mathbb{R}^d)$ to the diffusion (Markov) process X with generator $\mathcal{L}$.*

$$\mathcal{L}f(x) = \int_0^\infty du \mathbb{E}\left[F(Y(0), x).\nabla\left(F(Y(u), x).\nabla f(x)\right)\right].$$

$$\mathcal{L} = \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^d b_j(x) \frac{\partial}{\partial x_j}$$

with

$$\begin{aligned} a_{ij}(x) &= \int_0^\infty du \mathbb{E}\left[F_i(Y(0), x) F_j(Y(u), x)\right] \\ b_j(x) &= \sum_{i=1}^d \int_0^\infty du \mathbb{E}\left[F_i(Y(0), x) \partial_{x_i} F_j(Y(u), x)\right] \end{aligned}$$

Formal proof. Assume that Y is Markov, with generator Q, ergodic (+ technical conditions for the Fredholm alternative).

The joint process $\tilde{X}^\varepsilon(z) := (Y(z/\varepsilon^2), X^\varepsilon(z))$ is Markov with

$$\mathcal{L}^\varepsilon = \frac{1}{\varepsilon^2}Q + \frac{1}{\varepsilon}F(y,x).\nabla$$

The Kolmogorov backward equation for this process is

$$\frac{\partial U^\varepsilon}{\partial z} = \mathcal{L}^\varepsilon U^\varepsilon \tag{1}$$

Let us take an initial condition at $z=0$ independent of y:

$$U^\varepsilon(z=0,y,x) = f(x)$$

where f is a smooth test function. We solve (1) as $\varepsilon \to 0$ by assuming the multiple scale expansion:

$$U^\varepsilon = \sum_{n=0}^{\infty} \varepsilon^n U_n(z,y,x) \tag{2}$$

Then Eq. (1) becomes

$$\frac{\partial U^\varepsilon}{\partial z} = \frac{1}{\varepsilon^2}QU^\varepsilon + \frac{1}{\varepsilon}F.\nabla U^\varepsilon \tag{3}$$

We obtain a hierarchy of equations:

$$QU_0 = 0 \tag{4}$$

$$QU_1 + F.\nabla U_0 = 0 \tag{5}$$

$$QU_2 + F.\nabla U_1 = \frac{\partial U_0}{\partial z} \tag{6}$$

$Y(z)$ is ergodic i.e. $\text{Null}(Q) = \text{Span}(\{1\})$. Thus Eq. (4) $\Longrightarrow U_0$ does not depend on y.

U_1 must satisfy

$$QU_1 = -F(y,x).\nabla U_0(z,x) \tag{7}$$

Q is not invertible, we know that $\text{Null}(Q) = \text{Span}(\{1\})$.
$\text{Null}(Q^*)$ has dimension 1 and is generated by the invariant probability $\mathbb{P}$.
By Fredholm alternative, the Poisson equation $QU = g$ has a solution U if g satisfies $g \perp \text{Null}(Q^*)$, i.e. $\int g d\mathbb{P} = 0$, i.e. $\mathbb{E}[g(Y(0))] = 0$.

Since the r.h.s. of Eq. (7) is centered, this equation has a solution U_1

$$U_1(z,y,x) = -Q^{-1}F(y,x).\nabla U_0(z,x)$$

$$U_1(z,y,x) = -Q^{-1}[F(y,x)].\nabla U_0(z,x) \tag{8}$$

up to an additive constant, where $-Q^{-1} = \int_0^\infty dz T_z$.

Substitute (8) into (6): $\frac{\partial U_0}{\partial z} = QU_2 + F.\nabla U_1$ and take the expectation w.r.t $\mathbb{P}$. We get that U_0 must satisfy

$$\frac{\partial U_0}{\partial z} = \mathbb{E}\left[F.\nabla(-Q^{-1}F.\nabla U_0)\right]$$

This is the solvability condition for (6) and this is the limit Kolmogorov equation for the process X^ε:

$$\frac{\partial U_0}{\partial z} = \mathcal{L}U_0$$

with the limit generator

$$\mathcal{L} = \int_0^\infty \mathbb{E}\left[F.\nabla(T_z F.\nabla)\right] dz$$

Using the probabilistic representation of the semi-group e^{zQ} we get

$$\mathcal{L} = \int_0^\infty \mathbb{E}[F(Y(0),x).\nabla F(Y(z),x).\nabla] dz$$

Rigorous proof: The generator

$$\mathcal{L}^\varepsilon = \frac{1}{\varepsilon^2} Q + \frac{1}{\varepsilon} F(y,x).\nabla$$

of $(X^\varepsilon(.), Y(\frac{.}{\varepsilon^2}))$ is such that

$$f(Y(\frac{z}{\varepsilon^2}), X^\varepsilon(z)) - f(Y(\frac{s}{\varepsilon^2}), X^\varepsilon(s)) - \int_s^z \mathcal{L}^\varepsilon f(Y(\frac{u}{\varepsilon^2}), X^\varepsilon(u)) du$$

is a martingale for any test function f.

$\Longrightarrow$ Convergence of martingale problems.

Convergence of martingale problems

Assume for a while: $\forall f \in \mathcal{C}_b^\infty$, *there exists* f^ε *such that:*

$$\sup_{x \in K, y \in S} |f^\varepsilon(y,x) - f(x)| \xrightarrow{\varepsilon \to 0} 0, \qquad \sup_{x \in K, y \in S} |\mathcal{L}^\varepsilon f^\varepsilon(y,x) - \mathcal{L} f(x)| \xrightarrow{\varepsilon \to 0} 0.$$

Assume tightness and extract $\varepsilon_p \to 0$ such that $X^{\varepsilon_p} \to X$.

Take $z_1 < ... < z_n < s < z$ and $h_1, ..., h_n \in \mathcal{C}_b^\infty$:

$$\mathbb{E}\Big[\Big(f^\varepsilon(Y(\frac{z}{\varepsilon^2}), X^\varepsilon(z)) - f^\varepsilon(Y(\frac{s}{\varepsilon^2}), X^\varepsilon(s)) - \int_s^z \mathcal{L}^\varepsilon f^\varepsilon(Y(\frac{u}{\varepsilon^2}), X^\varepsilon(u))du\Big) h_1(X^\varepsilon(z_1))...h_n(X^\varepsilon(z_n))\Big] = 0$$

Take $\varepsilon_p \to 0$ so that $X^{\varepsilon_p} \to X$:

$$\mathbb{E}\Big[\Big(f(X(z)) - f(X(s)) - \int_s^z \mathcal{L} f(X(u))du\Big) h_1(X(z_1))...h_n(X(z_n))\Big] = 0$$

X is solution of the martingale problem associated to $\mathcal{L}$.

Perturbed test function method

Proposition: $\forall f \in \mathcal{C}_b^\infty$, *there exists a family* f^ε *such that:*

$$\sup_{x\in K, y\in S} |f^\varepsilon(y,x) - f(x)| \stackrel{\varepsilon\to 0}{\longrightarrow} 0, \qquad \sup_{x\in K, y\in S} |\mathcal{L}^\varepsilon f^\varepsilon(y,x) - \mathcal{L}f(x)| \stackrel{\varepsilon\to 0}{\longrightarrow} 0.$$

Proof: Define $f^\varepsilon(y,x) = f(x) + \varepsilon f_1(y,x) + \varepsilon^2 f_2(y,x)$.
Applying $\mathcal{L}^\varepsilon = \frac{1}{\varepsilon^2} Q + \frac{1}{\varepsilon} F(y,x).\nabla$ to f^ε, one gets:

$$\mathcal{L}^\varepsilon f^\varepsilon = \frac{1}{\varepsilon}\left(Qf_1 + F(y,x).\nabla f(x)\right) + \left(Qf_2 + F.\nabla f_1(y,x)\right) + O(\varepsilon).$$

Define the corrections f_j as follows:

1. $$f_1(y,x) = -Q^{-1}\left(F(y,x).\nabla f(x)\right).$$

Q has an inverse on the subspace of centered functions.

$$f_1(y,x) = \int_0^\infty du \mathbb{E}[F(Y(u),x).\nabla f(x) | Y(0) = y].$$

2. $$f_2(y,x) = -Q^{-1}\left(F.\nabla f_1(y,x) - \mathbb{E}[F.\nabla f_1(y,x)]\right).$$

It remains: $\mathcal{L}^\varepsilon f^\varepsilon = \mathbb{E}[F.\nabla f_1(y,x)] + O(\varepsilon)$.

One-dimensional case

$$\frac{dX^\varepsilon}{dz} = \frac{1}{\varepsilon} F\left(Y(\frac{z}{\varepsilon^2}), X^\varepsilon(z)\right), \qquad X^\varepsilon(z=0) = x_0 \in \mathbb{R}$$

Then $X^\varepsilon \xrightarrow{\varepsilon \to 0} X$ where X is the diffusion process with generator

$$\mathcal{L} = a(x)\frac{\partial^2}{\partial x^2} + b(x)\frac{\partial}{\partial x}$$

with

$$\begin{aligned} a(x) &= \int_0^\infty du \mathbb{E}\left[F(Y(0),x)F(Y(u),x)\right] \\ b(x) &= \int_0^\infty du \mathbb{E}\left[F(Y(0),x)\partial_x F(Y(u),x)\right] \end{aligned}$$

The limit process can be identified as the solution of the **stochastic differential equation**

$$dX = b(X)dz + \sqrt{2a(X)}dW_z$$

where W is a Brownian motion.

Limit theorems - Random vs. periodic

$$\frac{dX^\varepsilon}{dz}(z) = \frac{1}{\varepsilon} F(Y(\frac{z}{\varepsilon^2}), X^\varepsilon(z), \frac{z}{\varepsilon^{2+c}}), \qquad X^\varepsilon(0) = x_0 \in \mathbb{R}^d.$$

$F(y, x, \phi)$ is periodic with respect to ϕ.

Case 1. Slow phase: $-2 < c < 0$ and $\mathbb{E}[F(Y(0), x, \phi)] = 0$.

Case 2. Fast phase: $c = 0$ and $\langle \mathbb{E}[F(Y(0), x, \phi)] \rangle_\phi = 0$.

Case 3. Ultra-fast phase: $c > 0$ and $\langle \mathbb{E}[F(Y(0), x, \phi)] \rangle_\phi = 0$.

The processes $(X^\varepsilon(z))_{z \geq 0}$ converge to X with generator $\mathcal{L}_j$:

$$\mathcal{L}_1 f(x) = \left\langle \int_0^\infty du \mathbb{E}\left[F(Y(0), x, .).\nabla \left(F(Y(u), x, .).\nabla f(x)\right)\right] \right\rangle_\phi,$$

$$\mathcal{L}_2 f(x) = \int_0^\infty du \left\langle \mathbb{E}\left[F(Y(0), x, .).\nabla \left(F(Y(u), x, . + u).\nabla f(x)\right)\right] \right\rangle_\phi,$$

$$\mathcal{L}_3 f(x) = \int_0^\infty du \mathbb{E}\left[\langle F(Y(0), x, .)\rangle_\phi .\nabla \left(\langle F(Y(u), x, .)\rangle_\phi .\nabla f(x)\right)\right].$$

The averaging theorem revisited

Consider the random differential equation

$$\frac{dX^\varepsilon}{dz} = F\left(Y\left(\frac{z}{\varepsilon}\right), X^\varepsilon(z)\right), \quad X^\varepsilon(0) = x_0$$

where we do not assume that $F(y, x)$ is centered. We denote its mean by

$$\bar{F}(x) = \mathbb{E}[F(Y(0), x)]$$

Then $(Y(z/\varepsilon), X^\varepsilon(z))$ is a Markov process with generator

$$\mathcal{L}^\varepsilon = \frac{1}{\varepsilon}Q + F(y, x) \cdot \nabla$$

Let $f(x)$ be a test function. Define $f^\varepsilon(y, x) = f(x) + \varepsilon f_1(y, x)$ where f_1 solves the Poisson equation

$$Qf_1(y, x) + \left[F(y, x) \cdot \nabla f(x) - \bar{F}(x) \cdot \nabla f(x)\right] = 0$$

We get $\mathcal{L}^\varepsilon f^\varepsilon(y, x) = \bar{F}(x) \cdot \nabla f(x) + O(\varepsilon)$. Therefore the processes $X^\varepsilon(z)$ converge to the solution of the martingale problem associated with the generator $\mathcal{L}f(x) = \bar{F}(x) \cdot \nabla f(x)$. The solution is the deterministic process $\bar{X}(z)$

$$\frac{d\bar{X}}{dz} = \bar{F}(\bar{X}(z)), \quad \bar{X}(0) = x_0.$$