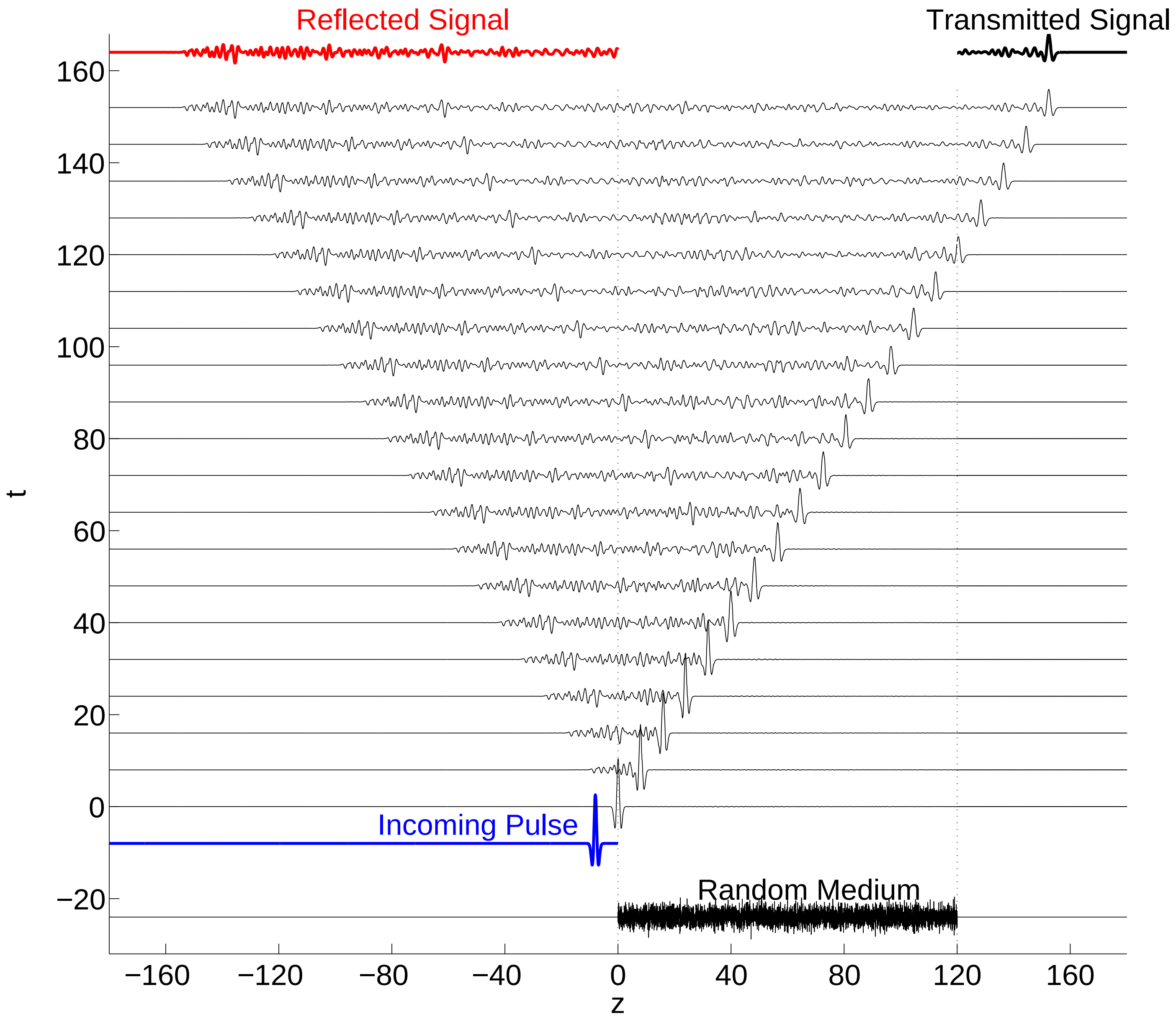
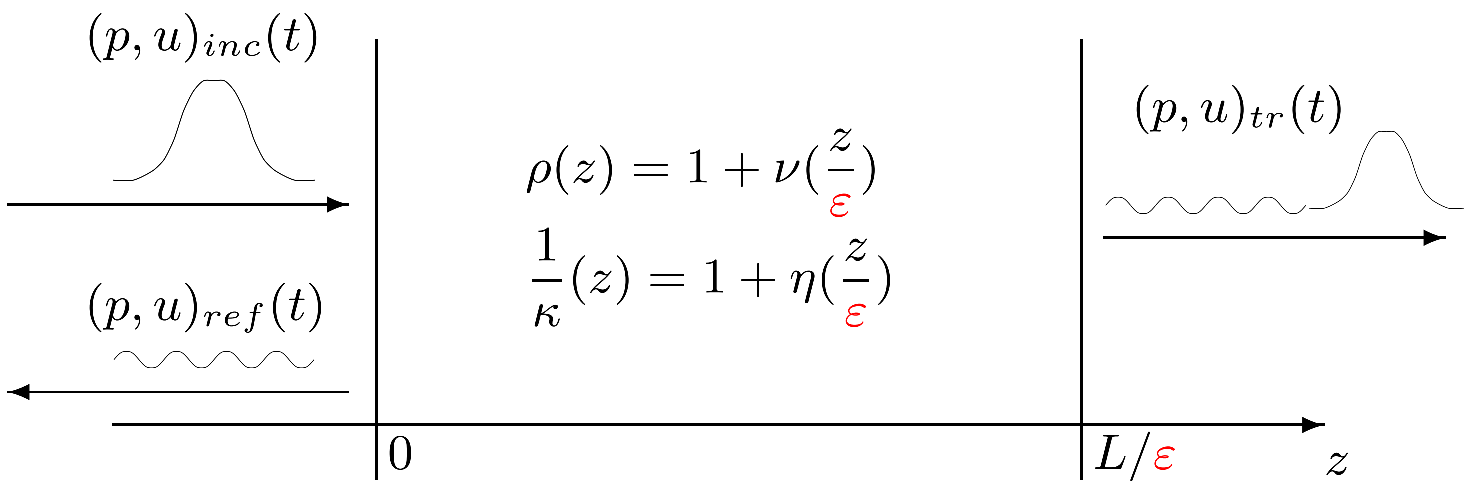
Long distance propagation



Long distance propagation $l_c \ll \lambda \ll L$

Acoustic equations for pressure p and velocity u:

$$\frac{\partial p}{\partial t} + \kappa(z)\frac{\partial u}{\partial z} = 0$$
$$\rho(z)\frac{\partial u}{\partial t} + \frac{\partial p}{\partial z} = 0$$



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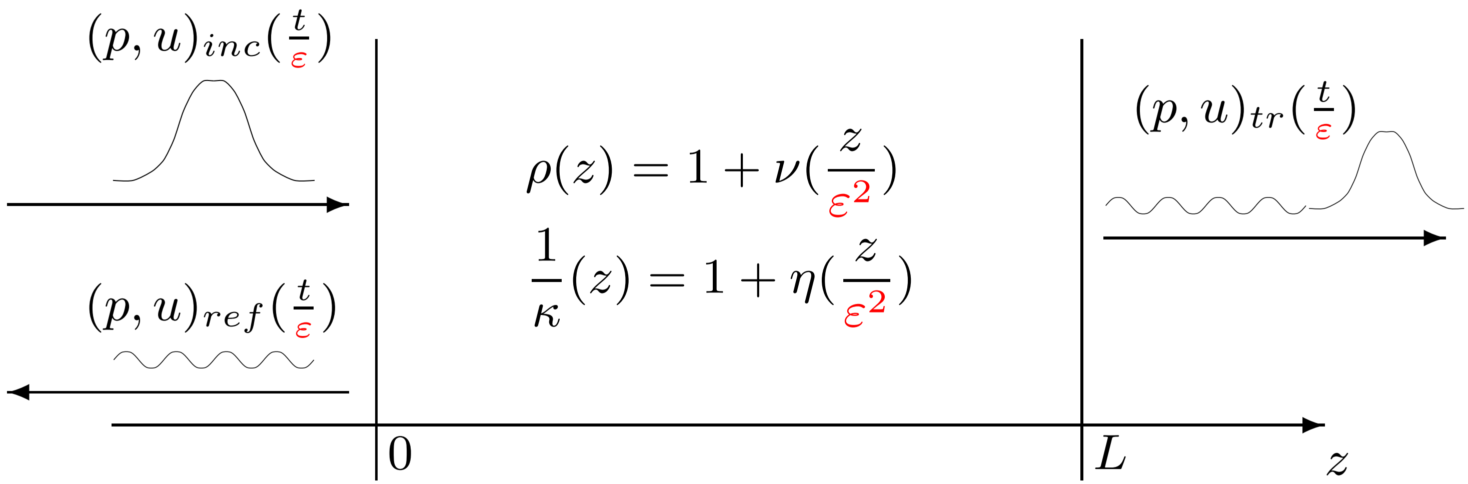

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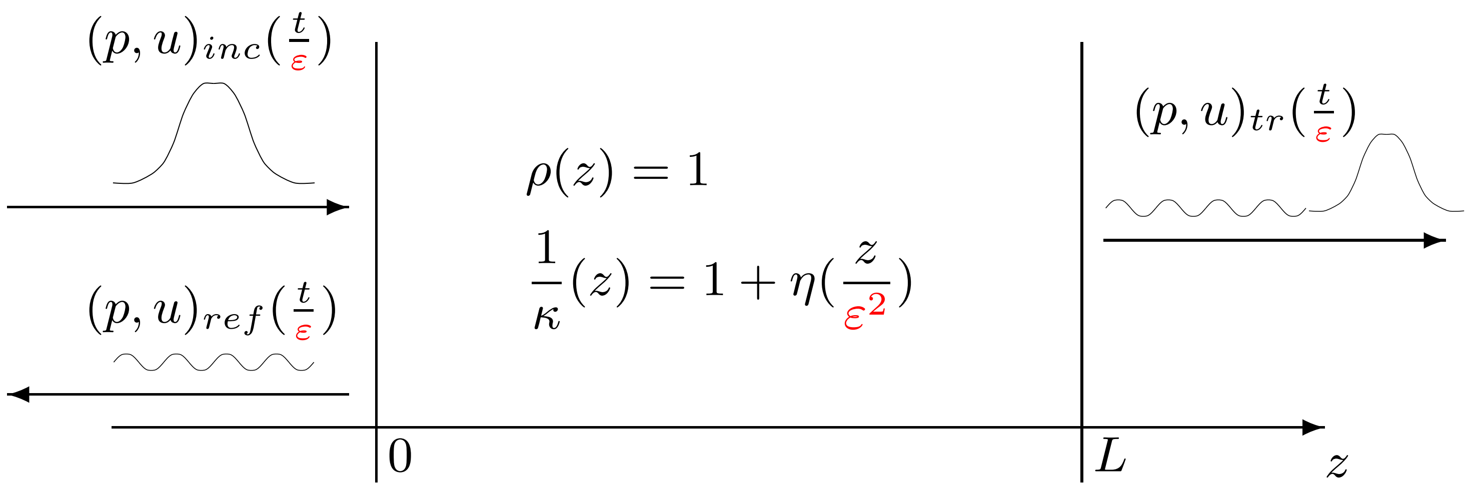


IC: right-going pulse incoming from the left homogeneous half-space $f(\frac{t}{\varepsilon})$.

Introduce the right-going mode $A = u + p$ and left-going mode $B = u - p$ that satisfy:

$$\frac{\partial}{\partial z}\begin{pmatrix} A \\ B \end{pmatrix} = \left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}\eta(\frac{z}{\varepsilon^2})\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\right)\frac{\partial}{\partial t}\begin{pmatrix} A \\ B \end{pmatrix}$$

Correlation radius $\sim \varepsilon^2 \ll$ wavelength $\sim \varepsilon \ll$ propagation distance ~ 1.

$$A(t,0) = f(\frac{t}{\varepsilon}), \qquad B(t,L) = 0$$

Observe the transmitted wave around the expected arrival time $t = L$:

$$A(L + \varepsilon s, L)_{s \in (-\infty,\infty)}$$

Define

$$a^\varepsilon(s,z) = A(z + \varepsilon s, z), \qquad b^\varepsilon(s,z) = B(-z + \varepsilon s, z)$$

Take a scaled Fourier transform ε:

$$\hat{a}^\varepsilon(\omega,z) = \int e^{i\omega s} a^\varepsilon(s,z) ds, \qquad \hat{b}^\varepsilon(\omega,z) = \int e^{i\omega s} b^\varepsilon(s,z) ds$$

In the frequency domain:

$$\frac{d}{dz}\begin{pmatrix} \hat{a}^\varepsilon \\ \hat{b}^\varepsilon \end{pmatrix} = \mathbf{H}^\varepsilon_\omega(z) \begin{pmatrix} \hat{a}^\varepsilon \\ \hat{b}^\varepsilon \end{pmatrix}, \qquad \mathbf{H}^\varepsilon_\omega(z) = \frac{i\omega}{2\varepsilon}\eta(\frac{z}{\varepsilon^2})\begin{pmatrix} 1 & -e^{-2i\omega\frac{z}{\varepsilon}} \\ e^{2i\omega\frac{z}{\varepsilon}} & -1 \end{pmatrix}$$

with the boundary conditions $\hat{a}^\varepsilon(\omega,0) = \hat{f}(\omega)$ and $\hat{b}^\varepsilon(\omega,L) = 0$.

Let $(\hat{\alpha}^\varepsilon, \hat{\beta}^\varepsilon)$ solution of:

$$\frac{d}{dz}\begin{pmatrix} \hat{\alpha}^\varepsilon \\ \hat{\beta}^\varepsilon \end{pmatrix} = \mathbf{H}_\omega^\varepsilon(z)\begin{pmatrix} \hat{\alpha}^\varepsilon \\ \hat{\beta}^\varepsilon \end{pmatrix},$$

with the initial conditions:

$$\hat{\alpha}^\varepsilon(\omega, 0) = 1, \qquad \hat{\beta}^\varepsilon(\omega, 0) = 0$$

By symmetry, $(\overline{\hat{\beta}^\varepsilon}, \overline{\hat{\alpha}^\varepsilon})$ is another solution, and therefore $\mathbf{P}_\omega^\varepsilon(z) = \begin{pmatrix} \hat{\alpha}^\varepsilon(\omega, z) & \overline{\hat{\beta}^\varepsilon}(\omega, z) \\ \hat{\beta}^\varepsilon(\omega, z) & \overline{\hat{\alpha}^\varepsilon}(\omega, z) \end{pmatrix}$ is the fundamental matrix (propagator) of the system:

$$\frac{d}{dz}\mathbf{P}_\omega^\varepsilon = \mathbf{H}_\omega^\varepsilon(z)\mathbf{P}_\omega^\varepsilon, \qquad \mathbf{P}_\omega^\varepsilon(0) = \mathbf{I}$$

By linearity:

$$\begin{pmatrix} \hat{a}^\varepsilon(\omega, L) \\ \hat{b}^\varepsilon(\omega, L) \end{pmatrix} = \mathbf{P}_\omega^\varepsilon(L)\begin{pmatrix} \hat{a}^\varepsilon(0, \omega) \\ \hat{b}^\varepsilon(0, \omega) \end{pmatrix}$$

Using $\hat{b}^\varepsilon(\omega, L) = 0$ and $\hat{a}^\varepsilon(\omega, 0) = \hat{f}(\omega)$:

$$\hat{b}^\varepsilon(\omega, 0) = R_\omega^\varepsilon(L)\hat{f}(\omega), \qquad \hat{a}^\varepsilon(\omega, L) = T_\omega^\varepsilon(L)\hat{f}(\omega)$$

$$R_\omega^\varepsilon(L) = -(\hat{\beta}^\varepsilon/\overline{\hat{\alpha}^\varepsilon})(\omega, L), \qquad T_\omega^\varepsilon(L) = (1/\overline{\hat{\alpha}^\varepsilon})(\omega, L),$$

Trace($\mathbf{H}_\omega^\varepsilon$) $= 0$, therefore $\det(\mathbf{P}_\omega^\varepsilon) = 1$ and $|\hat{\alpha}^\varepsilon|^2 - |\hat{\beta}^\varepsilon|^2 = 1$:

$$|R_\omega^\varepsilon(L)|^2 + |T_\omega^\varepsilon(L)|^2 = 1$$

Transmitted field:

$$A(L + \varepsilon s, L) = a^\varepsilon(s, L) = \frac{1}{2\pi} \int e^{-i\omega s} \hat{f}(\omega) T_\omega^\varepsilon(L) d\omega$$

The convergence of $(a^\varepsilon(s, L))_{s \in \mathbb{R}}$ requires:

- relative compacity,
- convergence of the finite-dimensional distributions.

The finite-dimensional distributions are characterized by the moments

$$\mathbb{E}[a^\varepsilon(s_1, L)^{p_1} \dots a^\varepsilon(s_k, L)^{p_k}]$$

for any real $s_1 < \dots < s_k$ and integer $p_1, \dots, p_k$.

Lemma. *The transmitted field* $((a^\varepsilon(s,L))_{-\infty<s<\infty})_{\varepsilon>0}$ *is relatively compact in* $\mathcal{C}_b(\mathbb{R},\mathbb{R})$*.*

Proof. We must show that, $\forall h>0$, there is a compact K in $\mathcal{C}_b(\mathbb{R},\mathbb{R})$ such that

$$\sup_{\varepsilon>0}\mathbb{P}(a^\varepsilon(\cdot,L)\in K)\geq 1-h$$

We have $a^\varepsilon(s,L)=\dfrac{1}{2\pi}\displaystyle\int e^{-i\omega s}\hat{f}(\omega)T^\varepsilon_\omega(L)d\omega$.
On the one hand $a^\varepsilon(s,L)$ is bounded by:

$$|a^\varepsilon(s,L)|\leq\frac{1}{2\pi}\int|\hat{f}(\omega)|d\omega$$

On the other hand the modulus of continuity

$$M^\varepsilon(\delta)=\sup_{|s_1-s_2|\leq\delta}|a^\varepsilon(s_1,L)-a^\varepsilon(s_2,L)|$$

is bounded by

$$M^\varepsilon(\delta)\leq\frac{1}{2\pi}\int\sup_{|s_1-s_2|\leq\delta}|1-\exp(i\omega(s_1-s_2))||\hat{f}(\omega)|d\omega$$

which goes to 0 as $\delta\to 0$.

First-order moment:

$$\mathbb{E}[a^\varepsilon(s,L)] = \frac{1}{2\pi}\int e^{-i\omega s}\hat{f}(\omega)\mathbb{E}[T^\varepsilon_\omega(L)]d\omega \quad \text{with } T^\varepsilon_\omega(L) = 1/\overline{\hat{\alpha}^\varepsilon}(\omega,L)$$

Let us fix ω and denote $X_1^\varepsilon(z) = \mathrm{Re}(\hat{\alpha}^\varepsilon(\omega,z))$, $X_2^\varepsilon(z) = \mathrm{Im}(\hat{\alpha}^\varepsilon(\omega,z))$, $X_3^\varepsilon = \mathrm{Re}(\hat{\beta}^\varepsilon(\omega,z))$ and $X_4^\varepsilon(z) = \mathrm{Im}(\hat{\beta}^\varepsilon(\omega,z))$. The process X^ε satisfies:

$$\frac{dX^\varepsilon(z)}{dz} = \frac{1}{\varepsilon}\mathbf{F}_\omega\left(\eta(\frac{z}{\varepsilon^2}), \frac{z}{\varepsilon}\right)X^\varepsilon(z),$$

with the initial conditions $X_1^\varepsilon(0) = 1$ and $X_{j'}^\varepsilon(0) = 0$ if $j' = 2,3,4$, where

$$\mathbf{F}_\omega(\eta,h) = \frac{\omega\eta}{2}\begin{pmatrix} 0 & -1 & -\sin(2\omega h) & \cos(2\omega h) \\ 1 & 0 & -\cos(2\omega h) & -\sin(2\omega h) \\ -\sin(2\omega h) & -\cos(2\omega h) & 0 & 1 \\ \cos(2\omega h) & -\sin(2\omega h) & -1 & 0 \end{pmatrix}$$

Diffusion-approximation: X^ε converges in distribution to X Markov with generator

$$\mathcal{L} = \sum_{i,j=1}^{4} a_{ij}(X)\frac{\partial^2}{\partial X_i \partial X_j}$$

$$a_{11} = \frac{\gamma\omega^2}{8}\left(X_2^2 + \frac{X_3^2+X_4^2}{2}\right), \quad a_{12} = \frac{\gamma\omega^2}{8}\left(-X_1X_2\right), \quad \gamma = 2\int_0^\infty \mathbb{E}[\eta(0)\eta(z)]dz$$

Compute $\mathcal{L}\big((X_1 - iX_2)^{-1}\big) = -\gamma\omega^2(X_1 - iX_2)^{-1}/4$.
The moment $\phi(z) = \mathbb{E}[(X_1(z) - iX_2(z))^{-1}] = \lim_{\varepsilon\to 0}\mathbb{E}[1/\overline{\hat{\alpha}^\varepsilon}(\omega, z)]$ satisfies

$$\frac{d\phi}{dz} = \mathbb{E}\big[\mathcal{L}(X_1 - iX_2)^{-1}\big] = -\frac{\gamma\omega^2}{4}\phi, \quad \phi(0) = 1.$$

where

$$\gamma = 2\int_0^\infty \mathbb{E}[\eta(0)\eta(z)]dz$$

Solution: $\phi(L) = \exp(-\alpha\gamma^2 L/4)$. The expectation $a^\varepsilon(s, L)$ converges:

$$\mathbb{E}[a^\varepsilon(s, L)] \xrightarrow{\varepsilon\to 0} \frac{1}{2\pi}\int e^{-i\omega s}\hat{f}(\omega)\exp(-\gamma\omega^2 L/4)d\omega$$

General moment:

$$\mathbb{E}\left[a^\varepsilon(s_1, L)^{p_1} ... a^\varepsilon(s_k, L)^{p_k}\right] =$$

$$= \frac{1}{(2\pi)^n}\int ... \int \prod_{\substack{1\le j\le k\\ 1\le l\le p_j}} \hat{f}(\omega_{j,l})e^{-i\omega_{j,l}s_j}\mathbb{E}\left[\prod_{\substack{1\le j\le k\\ 1\le l\le p_j}} T^\varepsilon_{\omega_{j,l}}(L)\right]\prod_{\substack{1\le j\le k\\ 1\le l\le p_j}} d\omega_{j,l}$$

One must compute the limit moments $\mathbb{E}[T^\varepsilon_{\omega_1}(L)\dots T^\varepsilon_{\omega_n}(L)]$ for n distinct frequencies $(\omega_i)_{i=1,\dots,n}$.

Introduce $X^\varepsilon_{4j+1}(z) = \mathrm{Re}(\hat{\alpha}^\varepsilon(\omega_j, z))$, $X^\varepsilon_{4j+2}(z) = \mathrm{Im}(\hat{\alpha}^\varepsilon(\omega_j, z))$, $X^\varepsilon_{4j+3}(z) = \mathrm{Re}(\hat{b}^\varepsilon(\omega_j, z))$ and $X^\varepsilon_{4j+4}(z) = \mathrm{Im}(\hat{b}^\varepsilon(\omega_j, z))$, $j = 1, \dots, n$. X^ε satisfies

$$\frac{dX^\varepsilon(z)}{dz} = \frac{1}{\varepsilon}\mathbf{F}\left(\eta(\frac{z}{\varepsilon^2}), \frac{z}{\varepsilon}\right) X^\varepsilon(z),$$

with $X^\varepsilon_{4j+j'}(0) = 1$ if $j' = 1$, $X^\varepsilon_{4j+j'}(0) = 0$ if $j' = 2, 3, 4$, where

$$\mathbf{F}(\eta, h) = \oplus_{j=1}^n \mathbf{F}_{\omega_j}(\eta, h)$$

Diffusion-approximation: $X^\varepsilon \to X$ Markov with generator $\mathcal{L}$.

If we denote $\phi^\varepsilon(z) = \mathbb{E}[T^\varepsilon_{\omega_1}(z) \dots T^\varepsilon_{\omega_n}(z)]$ then $\phi(z) = \lim_{\varepsilon\to 0} \phi^\varepsilon(z)$ satisfies:

$$\frac{d\phi(z)}{dz} = -\frac{2\gamma \sum_k \omega_k^2 + \gamma \sum_{k\neq l} \omega_k \omega_l}{8}\phi(z), \quad \phi(0) = 1.$$

This equation is satisfied by: $\tilde{\phi}(z) = \mathbb{E}\left[\prod_k \tilde{T}(\omega_k, z)\right]$ with:

$$\tilde{T}(\omega, z) = \exp(i\frac{\omega\sqrt{\gamma}}{2}W_z - \frac{\omega^2\gamma}{8}z)$$

where W_z is a Brownian motion (W_z is a random variable, with Gaussian density, mean 0 and variance z).

Thus $\phi(L) = \tilde{\phi}(L)$ and the limit in distribution of $a^\varepsilon(s,L)$ is:

$$\frac{1}{2\pi}\int e^{-i\omega s}\hat{f}(\omega)\exp(i\frac{\omega\sqrt{\gamma}}{2}W_L - \frac{\omega^2\gamma}{8}L)d\omega$$

Proposition. $(a^\varepsilon(s,L))_{s\in\mathbb{R}}$ *converge in distribution to* $(\bar{a}(s,L))_{s\in\mathbb{R}}$

$$\bar{a}(s,L) = K_{\mathrm{ODA}} * f\left(s - \frac{\sqrt{\gamma}}{2}W_L\right)$$

where

$$K_{\mathrm{ODA}}(t) = \frac{\sqrt{2}}{\sqrt{\pi\gamma L}}\exp\left(-\frac{2t^2}{\gamma L}\right)$$

The initial pulse f is modified in two ways:

- deterministic spreading (convolution with a Gaussian kernel).
- random time delay $\sim W_L$.

Effective convection-diffusion:

$$d\bar{a} = \frac{\sqrt{\gamma}}{2}\frac{\partial\bar{a}}{\partial s}\circ dW_z + \frac{\gamma}{8}\frac{\partial^2\bar{a}}{\partial s^2}dz$$

$$d\bar{a} = \frac{\sqrt{\gamma}}{2}\frac{\partial\bar{a}}{\partial s}dW_z + \frac{\gamma}{4}\frac{\partial^2\bar{a}}{\partial s^2}dz$$

$\bar{a}$ is the asymptotic pulse front. Its (deterministic) energy is:

$$\mathcal{E}_{coh} = \int \left| K_{\mathrm{ODA}} * f(s) \right|^2 ds$$

If $f(t)$ is a signal with carrier frequency ω_0:

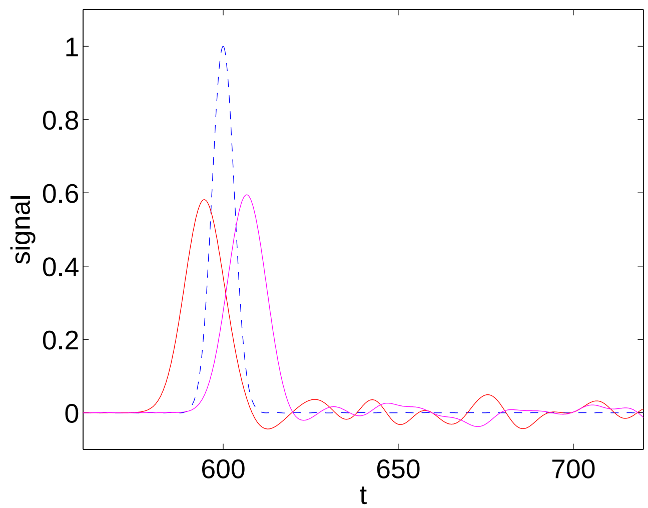
$$f(t) = \cos(\omega_0 t) \exp(-t^2 \delta\omega^2), \quad \delta\omega \ll \omega_0$$

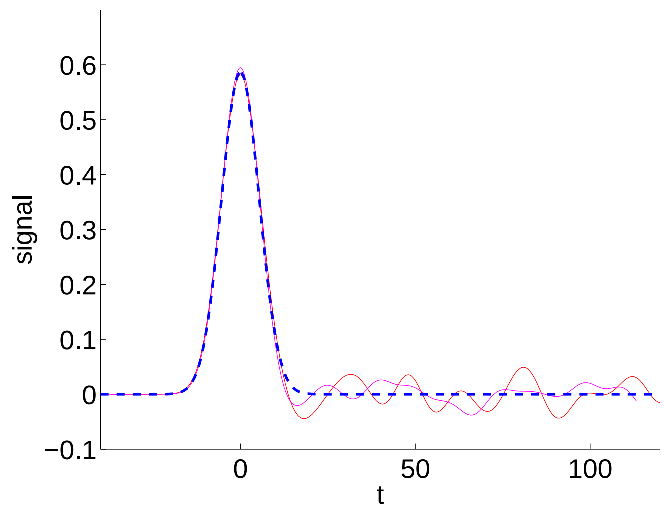
then

$$\begin{aligned} \mathcal{E}_{coh}(L) &= \mathcal{E}(0) \frac{1}{\sqrt{1+\gamma\delta\omega^2 L}} \exp\left(-\frac{\gamma\omega_0^2 L}{4(1+\gamma\delta\omega^2 L)} \right) \\ &\simeq \mathcal{E}(0) \exp\left(-\frac{\gamma\omega_0^2 L}{2} \right) \end{aligned}$$

$\hookrightarrow$ Exponential decay of the transmitted coherent energy.

Comparison theory - numerics





Time profiles of the transmitted wave at $z = L$

Without time shift

Red, purple: two numerical results obtained with two realizations of the random medium

Blue: original pulse $f(t)$

With time shift

Blue: theoretical (deterministic) transmitted pulse shape $K_{\mathrm{ODA}} * f(t)$

Remark on the mean field approach

Consider $f(z=0,t) = f_0(t) = \exp(-\frac{t^2}{2})$, and

$$f(z,t) = f_0(t - W_z)$$

where W_z is a Brownian motion, with pdf

$$p_z(w) = \frac{1}{\sqrt{2\pi z}} \exp\left(-\frac{w^2}{2z}\right)$$

For a realization: pulse shape preserved, random time delay.

For the mean field:

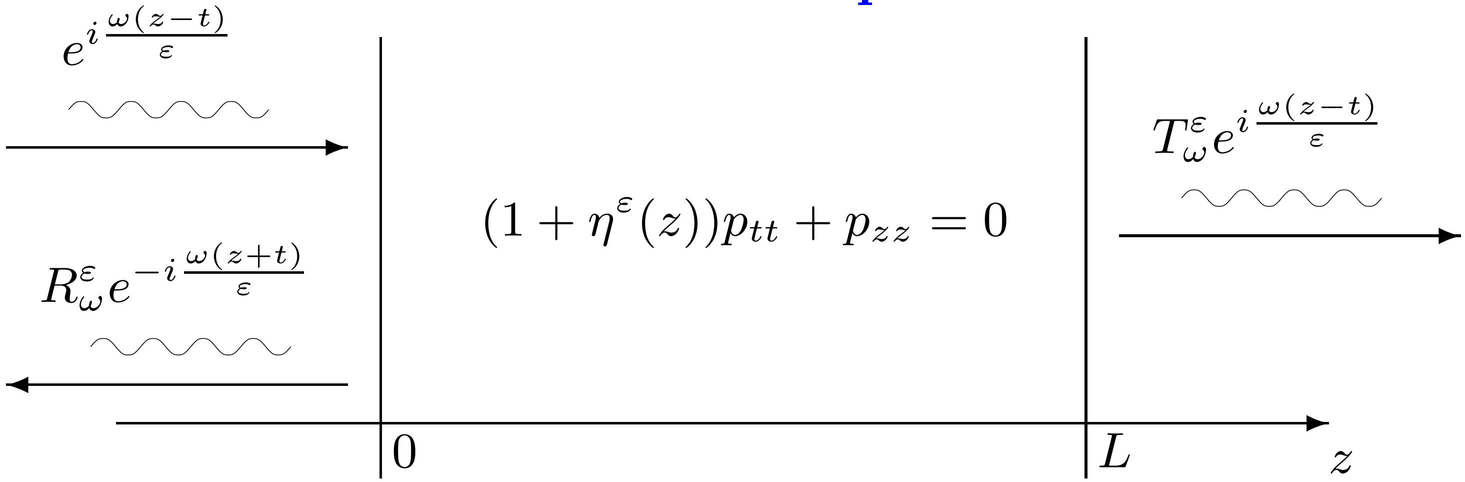
$$\begin{aligned}
\bar{f}(t,z) := \mathbb{E}[f(t,z)] \quad &= \quad \int f_0(t-w) p_z(w) dw \\
&= \quad \frac{1}{\sqrt{1+z}} \exp\left(-\frac{t^2}{2(1+z)}\right)
\end{aligned}$$

which means that $\bar{f}$ satisfies a diffusion equation:

$$\frac{\partial \bar{f}}{\partial z} = \frac{1}{2} \frac{\partial^2 \bar{f}}{\partial t^2}$$

$\hookrightarrow$ the mean field can be very different from the field.

Time harmonic problem



$$|R^\varepsilon_\omega(L)|^2 + |T^\varepsilon_\omega(L)|^2 = 1, \qquad |T^\varepsilon_\omega(L)|^2 = 1/|\hat{\alpha}^\varepsilon(\omega, L)|^2$$

The rescaled process $v^\varepsilon := \hat{\alpha}^\varepsilon(\omega, \varepsilon z)e^{-i\omega z} + \hat{\beta}^\varepsilon(\omega, \varepsilon z)e^{i\omega z}$ is solution of:

$$v^\varepsilon_{zz} + \omega^2 \left(1 + \eta(\frac{z}{\varepsilon})\right) v^\varepsilon = 0$$

starting from $v^\varepsilon(0) = 1$, $v^\varepsilon_z(0) = -i\omega$. Let $r^\varepsilon(\omega, z) := |v^\varepsilon|^2 + |v^\varepsilon_z|^2/\omega^2$.
If

$$\gamma^\varepsilon(\omega) = \lim_{z\to\infty} \frac{1}{z} \ln r^\varepsilon(\omega, z) > 0$$

then:

$$\lim_{L\to\infty} \frac{1}{L} \ln |T^\varepsilon_\omega(L)|^2 = -\frac{1}{\varepsilon}\gamma^\varepsilon(\omega)$$

because $r^\varepsilon(\omega, L) = 1 + 2|\hat{\alpha}^\varepsilon(\omega, \varepsilon L)|^2 = 1 + 2/|T^\varepsilon_\omega(\varepsilon L)|^2$.

Localization length

Result on the random harmonic oscillator:

$$v_{zz} + \omega^2 \left(1 + \eta(\frac{z}{\varepsilon})\right) v = 0, \qquad v(0) = 1, \qquad v_z(0) = -i\omega$$

The modulus $r(\omega, L) := |v|^2 + |v_z|^2/\omega^2$ satisfies

$$\lim_{L \to \infty} \frac{1}{L} \ln r(\omega, L) = G^\varepsilon = \frac{\gamma\omega^2}{4}\varepsilon + O(\varepsilon^2)$$

with $\gamma := 2 \int_0^\infty \mathbb{E}[\eta(0)\eta(z)]dz$.

Therefore:

$$\lim_{L \to \infty} \frac{1}{L} \ln |T_\omega^\varepsilon(L)|^2 = -\frac{1}{L_{loc}^\varepsilon} \text{ almost surely}$$

with

$$\frac{1}{L_{loc}^\varepsilon} = \frac{\gamma\omega^2}{4} + O(\varepsilon)$$

Power transmission

Proposition. $|T_\omega^\varepsilon(L)|^2$ *converge in distribution to the Markov process* $\mathcal{T}_\omega(L)$ *with generator:*

$$\mathcal{L}_\omega = \frac{1}{4}\gamma\omega^2\mathcal{T}^2(1-\mathcal{T})\frac{\partial^2}{\partial\mathcal{T}^2} - \frac{1}{4}\gamma\omega^2\mathcal{T}^2\frac{\partial}{\partial\mathcal{T}}.$$

Proof. We have $|T_\omega^\varepsilon|^2 = 1 - |\Gamma_\omega^\varepsilon|^2$ where $\Gamma_\omega^\varepsilon(L) = \hat{\beta}^\varepsilon/\hat{\alpha}^\varepsilon(\omega, L)$ is solution of:

$$\frac{d\Gamma_\omega^\varepsilon}{dL} = \frac{i\omega}{2\varepsilon}\eta(\frac{L}{\varepsilon^2})\left(e^{2i\omega\frac{L}{\varepsilon}} - 2\Gamma_\omega^\varepsilon + e^{-2i\omega\frac{L}{\varepsilon}}\Gamma_\omega^{\varepsilon\,2}\right), \qquad \Gamma_\omega^\varepsilon(0) = 0.$$

Introduce the process $X^\varepsilon := (r^\varepsilon, \psi^\varepsilon) := (|\Gamma_\omega^\varepsilon|^2, \arg(\Gamma_\omega^\varepsilon))$ solution of:

$$\frac{dX^\varepsilon}{dL}(L) = \frac{1}{\varepsilon}F\left(\eta(\frac{L}{\varepsilon^2}), X^\varepsilon(L), \frac{L}{\varepsilon}\right),$$

where F is given by:

$$F(\eta, r, \psi, l) = \frac{\omega\eta}{2}\begin{pmatrix} -2\sin(\psi - 2\omega l)(r^{3/2} - r^{1/2}) \\ -2 + \cos(\psi - 2\omega l)(r^{1/2} + r^{-1/2}) \end{pmatrix}$$

Apply the diffusion-approximation theorem.

In particular, we can compute $\mathbb{E}[\mathcal{T}_\omega(L)] = \lim_{\varepsilon\to 0} \mathbb{E}[|T_\omega^\varepsilon(L)|^2]$:

$$\begin{aligned}
\mathbb{E}[\mathcal{T}_\omega(L)] &= \frac{4}{\sqrt{\pi}} \exp\left(-\frac{\gamma\omega^2 L}{16}\right) \int_0^\infty dx \frac{x^2 e^{-x^2}}{\cosh(\sqrt{\gamma\omega^2 L/(2\sqrt{2})}x)} \\
&\underset{L\gg 1}{\simeq} \frac{c_1(\omega)}{L^{3/2}} \exp\left(-\frac{\gamma\omega^2 L}{16}\right).
\end{aligned}$$

Therefore

$$\lim_{L\to 0} \frac{1}{L} \ln \mathbb{E}[\mathcal{T}_\omega(L)] = -\frac{\gamma\omega^2}{16}$$

Ooups ! We have shown that, with probability 1:

$$\lim_{L\to\infty} \frac{1}{L} \ln |T_\omega^\varepsilon(L)|^2 = -\frac{1}{L_{loc}^\varepsilon} \text{ with } \frac{1}{L_{loc}^\varepsilon} = \frac{\gamma\omega^2}{4} + O(\varepsilon)$$

Is it because $\lim_{L\to\infty} \lim_{\varepsilon\to 0} \neq \lim_{\varepsilon\to 0} \lim_{L\to\infty}$?

No: we can show that, with probability 1:

$$\lim_{L\to\infty} \frac{1}{L} \ln \mathcal{T}_\omega(L) = -\frac{\gamma\omega^2}{4}$$

We also have:

$$\mathbb{E}[\mathcal{T}_\omega(L)^n] \underset{L\gg 1}{\simeq} \frac{c_n(\omega)}{L^{3/2}} \exp\left(-\frac{\gamma\omega^2 L}{16}\right).$$

The asymptotic power transmission coefficient $\mathcal{T}_\omega(L)$

$\mathcal{T}_\omega(L) = 2/(1+\rho(L))$ where ρ is solution of the SDE:

$$d\rho = \frac{\sqrt{\gamma}}{\sqrt{2}}\omega\sqrt{\rho^2-1}dW_L + \frac{\gamma}{2}\omega^2\rho dL, \quad \rho(0) = 1.$$

The behavior for large L is dominated by the drift $\Longrightarrow \rho \gg 1$:

$$d\rho \simeq \frac{\sqrt{\gamma}}{\sqrt{2}}\omega\rho dW_L + \frac{\gamma}{2}\omega^2\rho dL$$

$$\hookrightarrow \rho(L) \simeq \exp\left(\frac{\sqrt{\gamma}}{\sqrt{2}}\omega W_L + \frac{\gamma}{4}\omega^2 L\right).$$

Asymptotic $L \gg 1$ *:*

With probability ~ 1 we have $W_L \sim \sqrt{L} \ll L$, therefore

$\mathcal{T}_\omega(L) \sim \exp\left(-\gamma\omega^2 L/4\right) = \exp(-L/L_{loc})$.

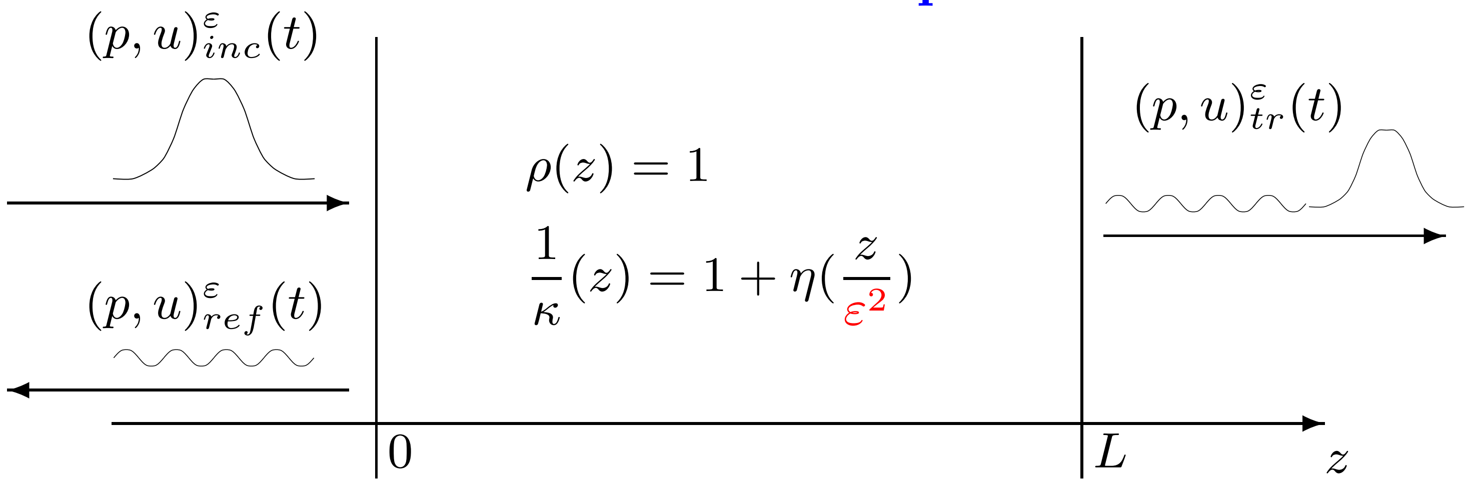
However, if $\frac{\sqrt{\gamma}}{\sqrt{2}}\omega W_L + \frac{\gamma}{4}\omega^2 L \le 0$, then $\rho \lesssim 1$ and $\mathcal{T}_\omega(L) \sim 1$!

Event of probability:

$\mathbb{P}(\frac{\sqrt{\gamma}}{\sqrt{2}}\omega W_L < -\frac{\gamma}{4}\omega^2 L) = \mathbb{P}(W_1 < -\frac{\sqrt{\gamma\omega^2 L}}{2\sqrt{2}}) \sim \exp(-\gamma\omega^2 L/16)$.

This set of exceptional realizations imposes the values of the moments of $\mathcal{T}_\omega(L)$!

Transmission of a pulse



$$p^{\varepsilon}_{inc}(t) = f^{\varepsilon}(t), \qquad f^{\varepsilon}(t) = \frac{1}{2\pi}\int \hat{f}^{\varepsilon}(\omega)e^{-i\omega t}d\omega$$

$$\hat{f}^{\varepsilon}(\omega) = \sqrt{\varepsilon}\hat{f}(\varepsilon\omega) \Longleftrightarrow f^{\varepsilon}(t) = \frac{1}{\sqrt{\varepsilon}}f(\frac{t}{\varepsilon})$$

$$\mathcal{E}_{inc} := \frac{1}{2}\int |p^{\varepsilon}_{inc}(t)|^2 dt = \frac{1}{2\pi}\int |\hat{f}^{\varepsilon}(\omega)|^2 d\omega = \frac{1}{2\pi}\int |\hat{f}(\omega)|^2 d\omega$$

$$p^{\varepsilon}_{ref}(t,z) = -\frac{1}{2\pi\sqrt{\varepsilon}}\int \hat{f}(\omega)R^{\varepsilon}_{\omega}(L)e^{i\frac{\omega(-z-t)}{\varepsilon}}d\omega, \quad z \leq 0,$$

$$p^{\varepsilon}_{tr}(t,z) = \frac{1}{2\pi\sqrt{\varepsilon}}\int \hat{f}(\omega)T^{\varepsilon}_{\omega}(L)e^{i\frac{\omega(z-t)}{\varepsilon}}d\omega, \quad z \geq L,$$

Transmitted energy:

$$\mathcal{E}_T^\varepsilon(L) = \frac{1}{2}\int |p_{tr}^\varepsilon(t,L)|^2 dt = \frac{1}{2\pi}\int |\hat{f}(\omega)|^2 |T_\omega^\varepsilon(L)|^2 d\omega$$

We have

$$\begin{aligned}
\mathbb{E}\left[\mathcal{E}_T^\varepsilon(L)\right] \quad &= \quad \frac{1}{2\pi}\int |\hat{f}(\omega)|^2 \mathbb{E}[|T_\omega^\varepsilon(L)|^2] d\omega \\
&\xrightarrow{\varepsilon\to 0} \quad \frac{1}{2\pi}\int |\hat{f}(\omega)|^2 \mathbb{E}[\mathcal{T}_\omega(L)] d\omega
\end{aligned}$$

The computation of the high-order moments require the frequency correlation:

$$\mathbb{E}\left[\mathcal{T}^{\varepsilon}(L)^{2}\right] = \frac{1}{4\pi^2}\int\int |\hat{f}(\omega)|^2|\hat{f}(\omega')|^2\mathbb{E}\left[|T^{\varepsilon}_{\omega}(L)|^2|T^{\varepsilon}_{\omega'}(L)|^2\right] d\omega d\omega'$$

Let

$$\omega_1 = \omega - h\varepsilon^a/2 \text{ and } \omega_2 = \omega + h\varepsilon^a/2$$

1. If $a = 0$, then $(|T^{\varepsilon}_{\omega_1}(L)|^2, |T^{\varepsilon}_{\omega_2}(L)|^2)$ converge to $(\mathcal{T}_{\omega-h/2}(L), \mathcal{T}_{\omega+h/2}(L))$ where $\mathcal{T}_{\omega-h/2}$ and $\mathcal{T}_{\omega+h/2}$ are independent Markov processes with generators $\mathcal{L}_{\omega-h/2}$ and $\mathcal{L}_{\omega+h/2}$.

2. If $a = 1$, then $(|T^{\varepsilon}_{\omega_1}(L)|^2, |T^{\varepsilon}_{\omega_2}(L)|^2)$ converge to $(\mathcal{T}_1(L), \mathcal{T}_2(L))$ where $(\mathcal{T}_1(L), \mathcal{T}_2(L), \theta(L))$ is a diffusion process. In particular

$$\mathbb{E}\left[\mathcal{T}_1(L)\mathcal{T}_2(L)\right] - \mathbb{E}\left[\mathcal{T}_1(L)\right]\mathbb{E}\left[\mathcal{T}_2(L)\right] = f(h, L), \text{ with } f(h, L) \xrightarrow{h\to\infty} 0$$

$\hookrightarrow$ Decorrelation in frequency ! Frequency correlation radius $\sim \varepsilon$

Consider the second moment:

$$\begin{aligned}
\mathbb{E}\left[\mathcal{E}_T^\varepsilon(L)^2\right] &= \frac{1}{4\pi^2}\int\int|\hat{f}(\omega)|^2|\hat{f}(\omega')|^2\mathbb{E}\left[|T_\omega^\varepsilon(L)|^2|T_{\omega'}^\varepsilon(L)|^2\right]d\omega d\omega' \\
&\xrightarrow{\varepsilon\to 0} \frac{1}{4\pi^2}\int\int|\hat{f}(\omega)|^2|\hat{f}(\omega')|^2\mathbb{E}[\mathcal{T}_\omega(L)\mathcal{T}_{\omega'}(L)]d\omega d\omega' \\
&= \frac{1}{4\pi^2}\int\int|\hat{f}(\omega)|^2|\hat{f}(\omega')|^2\mathbb{E}[\mathcal{T}_\omega(L)]\mathbb{E}[\mathcal{T}_{\omega'}(L)]d\omega d\omega' \\
&= \left(\frac{1}{2\pi}\int|\hat{f}(\omega)|^2\mathbb{E}[\mathcal{T}_\omega(L)]d\omega\right)^2
\end{aligned}$$

Therefore:

$$\mathbb{E}\left[\left(\mathcal{E}_T^\varepsilon(L)-\mathbb{E}[\mathcal{E}_T^\varepsilon(L)]\right)^2\right]\xrightarrow{\varepsilon\to 0} 0$$

The process $\mathcal{E}_T^\varepsilon(L)$ converges in $L^2(\mathbb{P})$ to $\mathcal{E}_T(L)$:

$$\mathcal{E}_T(L)=\frac{1}{2\pi}\int|\hat{f}(\omega)|^2\mathbb{E}[\mathcal{T}_\omega(L)]d\omega$$

The process $\mathcal{E}_T^\varepsilon(L)$ converges in probability to $\mathcal{E}_T(L)$:

$$\text{For any } \delta>0,\ \mathbb{P}\left(|\mathcal{E}_T^\varepsilon(L)-\mathcal{E}_T(L)|>\delta\right)\leq\frac{\mathbb{E}\left[(\mathcal{E}_T^\varepsilon(L)-\mathcal{E}_T(L))^2\right]}{\delta^2}\xrightarrow{\varepsilon\to 0} 0$$

The decorrelation in frequency implies the self-averaging in time of the transmitted energy !

Application: $f(t) = \cos(\omega_0 t) \exp(-t^2 \delta\omega^2)$ with $\varepsilon \ll \delta\omega \ll \omega_0$.
The transmitted wave consists of
1) a coherent wave whose energy decays as $\exp(-\gamma\omega_0^2 L/4)$,
2) incoherent waves whose energy decays as $\exp(-\gamma\omega_0^2 L/16)$.

The incoherent waves dominate

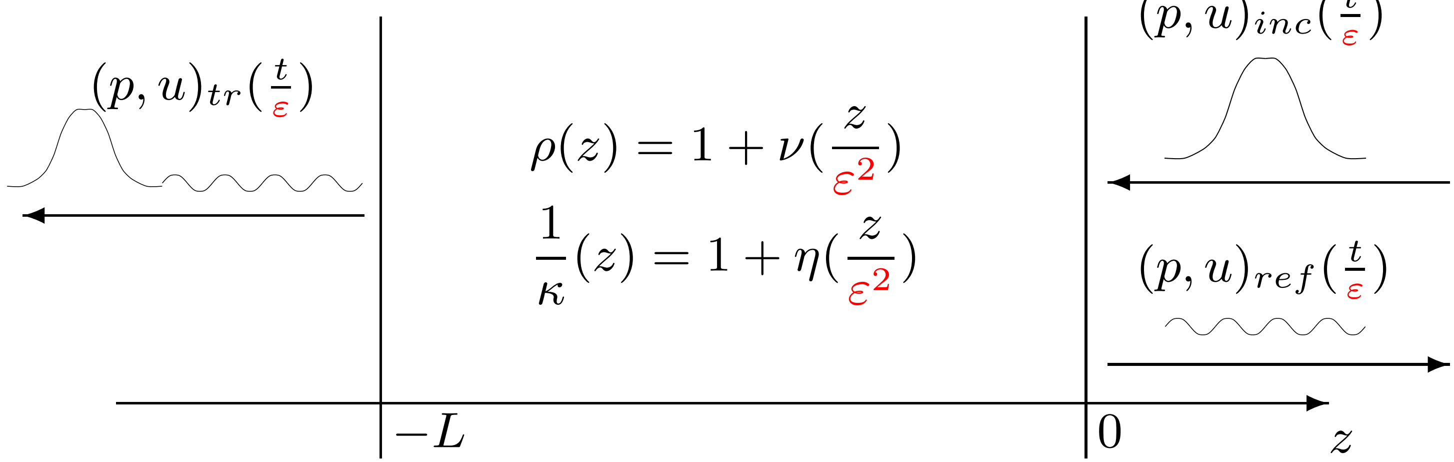
In the regime $\gamma\omega_0^2 L \geq 1$:
- The incoherent waves dominate
- The reflected waves dominate the transmitted waves.

Scattering of an acoustic pulse in random media

Acoustic equations for pressure p and speed u:

$$\frac{\partial p}{\partial t} + \kappa(z)\frac{\partial u}{\partial z} = 0$$

$$\rho(z)\frac{\partial u}{\partial t} + \frac{\partial p}{\partial z} = 0$$



IC: left-going pulse incoming from the right homogeneous half-space.

$$m(z) = \eta(z) + \nu(z), \qquad n(z) = \eta(z) - \nu(z)$$

Local velocity: $c(z) = \sqrt{\kappa(z)/\rho(z)}$.

Local impedance: $\zeta(z) = \rho(z)c(z)$.

The propagator $\mathbf{P}^{\varepsilon}_{\omega}(z) = \begin{pmatrix} \hat{\alpha}^{\varepsilon}(\omega,z) & \overline{\hat{\beta}^{\varepsilon}}(\omega,z) \\ \hat{\beta}^{\varepsilon}(\omega,z) & \overline{\hat{\alpha}^{\varepsilon}}(\omega,z) \end{pmatrix}$ is the fundamental matrix of the system:

$$\frac{d}{dz}\mathbf{P}^{\varepsilon}_{\omega} = \mathbf{H}^{\varepsilon}_{\omega}(z)\mathbf{P}^{\varepsilon}_{\omega}, \qquad \mathbf{H}^{\varepsilon}_{\omega}(z) = \frac{i\omega}{2\varepsilon}\begin{pmatrix} -m(\frac{z}{\varepsilon^2}) & -n(\frac{z}{\varepsilon^2})e^{\frac{2i\omega z}{\varepsilon}} \\ n(\frac{z}{\varepsilon^2})e^{-\frac{2i\omega z}{\varepsilon}} & m(\frac{z}{\varepsilon^2}) \end{pmatrix}$$

starting from $\mathbf{P}^{\varepsilon}_{\omega}(-L) = \mathbf{I}$. By linearity:

$$\mathbf{P}^{\varepsilon}_{\omega}(z)\begin{pmatrix} 0 \\ T^{\varepsilon}_{\omega}(z) \end{pmatrix} = \begin{pmatrix} R^{\varepsilon}_{\omega}(z) \\ 1 \end{pmatrix}$$

The reflection and transmission coefficients for the slab $[-L,z]$ are:

$$R^{\varepsilon}_{\omega}(z) = \frac{\overline{\hat{\beta}^{\varepsilon}(\omega,z)}}{\overline{\hat{\alpha}^{\varepsilon}(\omega,z)}}, \qquad T^{\varepsilon}_{\omega}(z) = \frac{1}{\overline{\hat{\alpha}^{\varepsilon}(\omega,z)}}.$$

$$\frac{dR^{\varepsilon}_{\omega}}{dz} = \overline{\frac{1}{\hat{\alpha}^{\varepsilon}}\frac{d\hat{\beta}^{\varepsilon}}{dz} - \frac{\hat{\beta}^{\varepsilon}}{(\hat{\alpha}^{\varepsilon})^2}\frac{d\hat{\alpha}^{\varepsilon}}{dz}}, \qquad \frac{dT^{\varepsilon}_{\omega}}{dz} = -\overline{\frac{1}{(\hat{\alpha}^{\varepsilon})^2}\frac{d\hat{\alpha}^{\varepsilon}}{dz}}$$

From the equations satisfied by $(\hat{\alpha}^{\varepsilon}, \hat{\beta}^{\varepsilon})$ we get

$$\frac{\partial R^{\varepsilon}_{\omega}}{\partial z} = \frac{i\omega}{\varepsilon}m(\frac{z}{\varepsilon^2})R^{\varepsilon}_{\omega} - \frac{i\omega}{2\varepsilon}n(\frac{z}{\varepsilon^2})e^{-\frac{2i\omega z}{\varepsilon}}(R^{\varepsilon}_{\omega})^2 - \frac{i\omega}{2\varepsilon}n(\frac{z}{\varepsilon^2})e^{\frac{2i\omega z}{\varepsilon}}$$
$$\frac{\partial T^{\varepsilon}_{\omega}}{\partial z} = -\frac{i\omega}{2\varepsilon}n(\frac{z}{\varepsilon^2})R^{\varepsilon}_{\omega}e^{-\frac{2i\omega z}{\varepsilon}}T^{\varepsilon}_{\omega} + \frac{i\omega}{2\varepsilon}m(\frac{z}{\varepsilon^2})T^{\varepsilon}_{\omega}$$

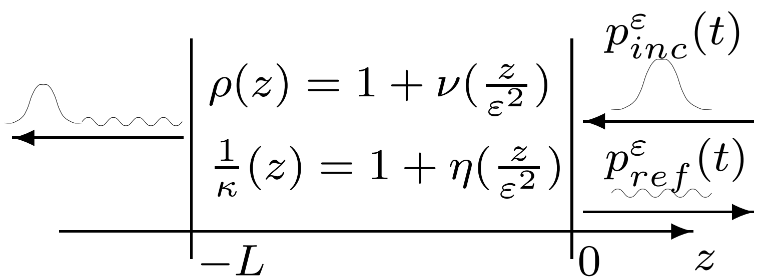
Integral representation of the reflected signal

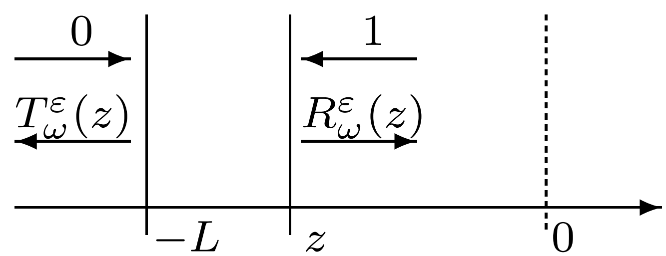
Send a left-going pulse $f(\frac{t}{\varepsilon})$:

$$p^{\varepsilon}_{inc}(t) = f(\frac{t}{\varepsilon}) = \frac{1}{2\pi}\int e^{-\frac{i\omega t}{\varepsilon}}\hat{f}(\omega)d\omega$$

Reflected signal:

$$p^{\varepsilon}_{ref}(t) = \frac{1}{2\pi}\int e^{-\frac{i\omega t}{\varepsilon}}\hat{f}(\omega)R^{\varepsilon}_{\omega}(0)d\omega$$





$R^{\varepsilon}_{\omega}(z)$ is the reflection coefficient for a random slab $[-L, z]$:

$$\frac{\partial R^{\varepsilon}_{\omega}}{\partial z} = \frac{i\omega}{\varepsilon}m(\frac{z}{\varepsilon^2})R^{\varepsilon}_{\omega} - \frac{i\omega}{2\varepsilon}n(\frac{z}{\varepsilon^2})e^{-\frac{2i\omega z}{\varepsilon}}(R^{\varepsilon}_{\omega})^2 - \frac{i\omega}{2\varepsilon}n(\frac{z}{\varepsilon^2})e^{\frac{2i\omega z}{\varepsilon}},$$

with the initial condition at $z = -L$: $R^{\varepsilon}_{\omega}(z = -L) = 0$.

Energy conservation $|R^{\varepsilon}_{\omega}|^2 + |T^{\varepsilon}_{\omega}|^2 = 1 \rightarrow$ uniform boundedness of R^{ε}_{ω}.

O'Doherty-Anstey theory

The front $b^\varepsilon(s,z) := B(z+\varepsilon s, z)$, converges as $\varepsilon \to 0$ to

$$\bar{b}(t,z) = K_{\mathrm{ODA}} * f\left(s - \frac{\sqrt{\gamma_m}}{2} W_z\right)$$

$$K_{\mathrm{ODA}}(t) = \frac{\sqrt{2}}{\sqrt{\pi \gamma_n z}} \exp\left(-\frac{2t^2}{\gamma_n z}\right)$$

$$\gamma_m = 2\int_0^\infty \mathbb{E}[m(0)m(z)]dz, \qquad \gamma_n = 2\int_0^\infty \mathbb{E}[n(0)n(z)]dz$$

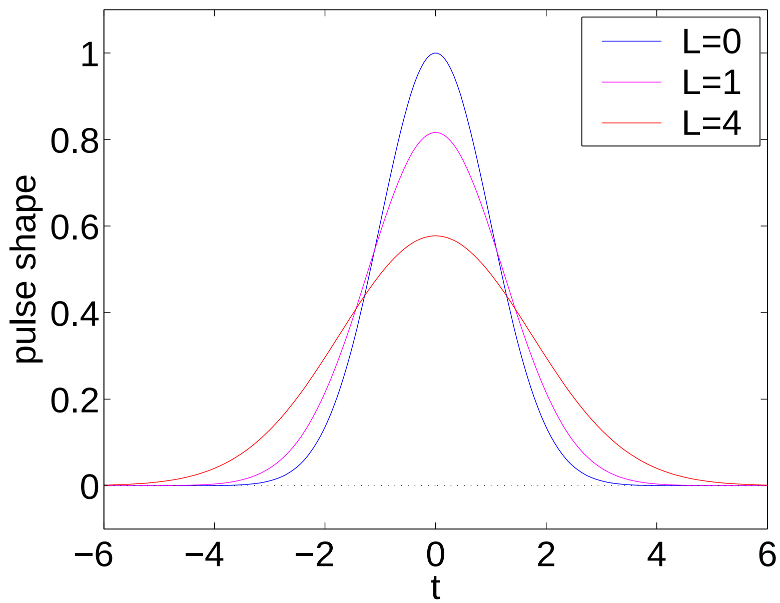
$\gamma_m \leftrightarrow$ time delay $\qquad \gamma_n \leftrightarrow$ reshaping.

Coherent front pulse

Gaussian initial pulse

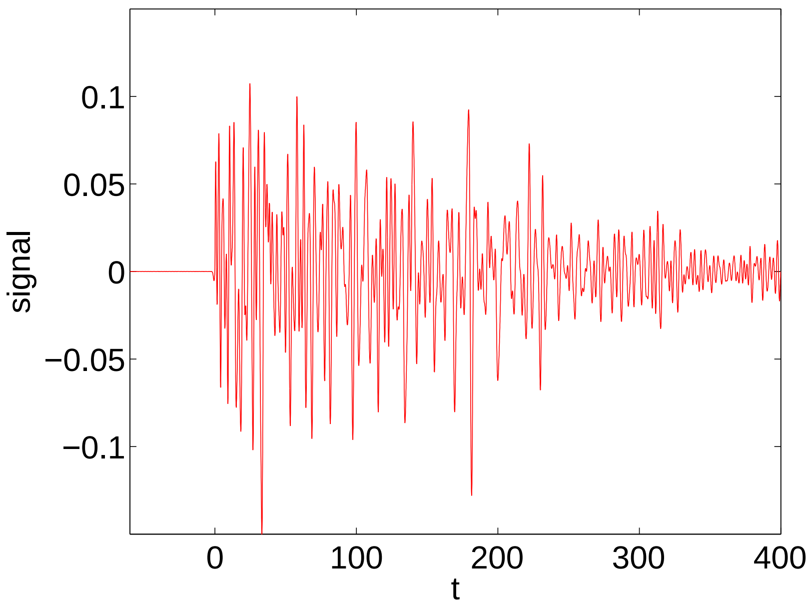
$$f(t) = \exp\left(-\frac{t^2}{2}\right)$$

$\gamma_n = 1/2$, $\gamma_m = 0$



The reflected wave

$$p_{ref}(t) = \frac{1}{2\pi} \int e^{-\frac{i\omega t}{\varepsilon}} \hat{f}(\omega) R^{\varepsilon}_{\omega}(0) d\omega$$



We have $\mathbb{E}[p_{ref}(t)] = 0$ (no coherent signal).

Reflected intensity:

$$\begin{aligned} p^2_{ref}(t) &= \frac{1}{4\pi^2} \int\int e^{\frac{i(\omega'-\omega)t}{\varepsilon}} R^{\varepsilon}_{\omega}(0) \overline{R^{\varepsilon}_{\omega'}(0)} \hat{f}(\omega) \overline{\hat{f}(\omega')} d\omega d\omega' \\ &= \frac{\varepsilon}{4\pi^2} \int\int e^{-iht} R^{\varepsilon}_{\omega+\frac{\varepsilon h}{2}}(0) \overline{R^{\varepsilon}_{\omega-\frac{\varepsilon h}{2}}(0)} \hat{f}(\omega + \frac{\varepsilon h}{2}) \overline{\hat{f}(\omega - \frac{\varepsilon h}{2})} d\omega dh \end{aligned}$$

The autocorrelation function of the reflection coefficient

Let us fix ω.

$$U_{1,1}^{\varepsilon}(z,\omega,h) = R^{\varepsilon}_{\omega+\frac{\varepsilon h}{2}}(z)\overline{R^{\varepsilon}_{\omega-\frac{\varepsilon h}{2}}(z)}$$

We look for $\lim_{\varepsilon\to 0}\mathbb{E}[U_{1,1}^{\varepsilon}(0,\omega,h)]$.

For $p,q\in\mathbb{N}$, $z\in[-L,0]$ we introduce

$$U_{p,q}^{\varepsilon}(z,\omega,h) = \left(R^{\varepsilon}_{\omega+\frac{\varepsilon h}{2}}(z)\right)^{p}\left(\overline{R^{\varepsilon}_{\omega-\frac{\varepsilon h}{2}}(z)}\right)^{q}$$

From the Ricatti equation satisfied by R^{ε}_{ω}:

$$\begin{aligned}
\frac{\partial U_{p,q}^{\varepsilon}}{\partial z} \quad = \quad & \frac{i\omega}{\varepsilon}m(\frac{z}{\varepsilon^2})(p-q)U_{p,q}^{\varepsilon} \\
& -\frac{i\omega}{2\varepsilon}n(\frac{z}{\varepsilon^2})e^{\frac{2i\omega z}{\varepsilon}}\left(pe^{ihz}U_{p-1,q}^{\varepsilon}-qe^{-ihz}U_{p,q+1}^{\varepsilon}\right) \\
& +\frac{i\omega}{2\varepsilon}n(\frac{z}{\varepsilon^2})e^{-\frac{2i\omega z}{\varepsilon}}\left(qe^{ihz}U_{p,q-1}^{\varepsilon}-pe^{-ihz}U_{p+1,q}^{\varepsilon}\right)
\end{aligned}$$

with $U_{p,q}^{\varepsilon}(z=-L,\omega,h)=\mathbf{1}_0(p)\mathbf{1}_0(q)$.
Take a Fourier transform with respect to h:

$$V_{p,q}^{\varepsilon}(z,\omega,\tau) = \frac{1}{2\pi}\int e^{-ih(\tau-(p+q)z)}U_{p,q}^{\varepsilon}(z,\omega,h)dh$$

$$\begin{aligned}\frac{\partial V^\varepsilon_{p,q}}{\partial z} &= -(p+q)\frac{\partial V^\varepsilon_{p,q}}{\partial \tau} + \frac{i\omega}{\varepsilon} m(\frac{z}{\varepsilon^2})(p-q)V^\varepsilon_{p,q} \\ & -\frac{i\omega}{2\varepsilon} n(\frac{z}{\varepsilon^2}) e^{\frac{2i\omega z}{\varepsilon}} \left(pV^\varepsilon_{p-1,q} - qV^\varepsilon_{p,q+1}\right) \\ & +\frac{i\omega}{2\varepsilon} n(\frac{z}{\varepsilon^2}) e^{-\frac{2i\omega z}{\varepsilon}} \left(qV^\varepsilon_{p,q-1} - pV^\varepsilon_{p+1,q}\right)\end{aligned}$$

starting from $V^\varepsilon_{p,q}(z=-L,\omega,\tau) = \delta(\tau)\mathbf{1}_0(p)\mathbf{1}_0(q)$.

Approximation-diffusion $\Longrightarrow$ $V^\varepsilon_{p,q}$ converges as $\varepsilon \to 0$ to a diffusion Markov process.

In particular $\mathbb{E}[V^\varepsilon_{p,p}(z,\omega,\tau)]$, $p \in \mathbb{N}$, converges to $\mathcal{W}_p(z,\omega,\tau)$:

$$\frac{\partial \mathcal{W}_p}{\partial z} + 2p\frac{\partial \mathcal{W}_p}{\partial \tau} = \frac{1}{4}\gamma_n \omega^2 p^2 \left(\mathcal{W}_{p+1} + \mathcal{W}_{p-1} - 2\mathcal{W}_p\right)$$
$$\mathcal{W}_p(z=-L,\omega,\tau) = \delta(\tau)\mathbf{1}_0(p)$$

where $\gamma_n = 2\int_0^\infty \mathbb{E}[n(0)n(z)]dz$.

We thus get the limit of the expectation of R^ε_ω:

$$\mathbb{E}\left[R^\varepsilon_{\omega+\frac{\varepsilon h}{2}}(0)\overline{R^\varepsilon_{\omega-\frac{\varepsilon h}{2}}(0)}\right] \stackrel{\varepsilon\to 0}{\longrightarrow} \int \mathcal{W}_1(0,\omega,\tau)e^{ih\tau}d\tau$$

Analysis of the transport equations

Let us introduce the jump Markov process $(N_z)_{z\geq -L}$ with state space $\mathbb{N}$ and infinitesimal generator

$$\mathcal{L}\phi(N) = \frac{1}{4}\gamma_n \omega^2 N^2 \left(\phi(N+1) + \phi(N-1) - 2\phi(N)\right)$$

We also define the process $\dfrac{\partial S_z}{\partial z} = -2N_z$.

Then (N_z, S_z) is Markovian with generator: $\mathcal{L} - 2N\frac{\partial}{\partial S}$.

The solution to

$$\frac{\partial \phi}{\partial z} + 2N\frac{\partial \phi}{\partial S} = \mathcal{L}\phi, \qquad \phi(z=-L, N, S) = \phi_0(N, S)$$

can be written as

$$\phi(z, N, S) = \mathbb{E}\left[\phi_0\left(N_z, S - 2\int_{-L}^{z} N_s ds\right) \mid N_{-L} = N\right]$$

Taking (formally) $\phi_0(N,\tau) = \mathbf{1}_0(N)\delta(\tau)$, we find $\phi(0,1,\tau) = \mathcal{W}_1(0,\omega,\tau)$

$$\int_{\tau_0}^{\tau_1} \mathcal{W}_1(0,\omega,\tau)d\tau = \mathbb{P}\left(N_0 = 0 \bigcap \int_{-L}^{0} 2N_s ds \in [\tau_0, \tau_1] \,|\, N_{-L} = 1\right)$$

Application 1: By taking $\tau_0 = 0$ and $\tau_1 = \infty$

$$\mathbb{E}\left[|R_\omega^\varepsilon|^2(0)\right] \xrightarrow{\varepsilon\to 0} \mathbb{P}(N_0 = 0 \,|\, N_{-L} = 1).$$

Duality formula: Let us introduce the diffusion process $(\theta_z)_{z\geq -L}$:

$$d\theta_z = \frac{\sqrt{\gamma_n}}{\sqrt{2}}\omega dW_z + \frac{\gamma_n}{4}\omega^2 \coth(\theta_z)dz.$$

We have

$$\mathbb{E}\left[\xi^{N_z} \mid N_{-L} = p_0\right] = \mathbb{E}\left[\tanh(\frac{\theta_z}{2})^{2p_0} \mid \theta_{-L} = 2 \,\operatorname{argtanh}(\sqrt{\xi})\right],$$

and in particular

$$\mathbb{E}\left[|R_\omega^\varepsilon|^2(0)\right] \xrightarrow{\varepsilon\to 0} \mathbb{P}(N_0 = 0 \,|\, N_{-L} = 1) = \mathbb{E}\left[\tanh(\frac{\theta_0}{2})^2 \mid \theta_{-L} = 0\right].$$

The pdf of θ_0 can be computed:

$$\mathbb{E}\left[|R_\omega^\varepsilon|^2(0)\right] \xrightarrow{\varepsilon\to 0} 1 - \exp\left(-\frac{L}{4L_{\mathrm{loc}}(\omega)}\right)\int_0^\infty \frac{2\pi x \sinh(\pi x)}{\cosh^2(\pi x)} e^{-x^2 L/L_{\mathrm{loc}}} dx$$

where $L_{\mathrm{loc}}(\omega) = \dfrac{4}{\gamma_n \omega^2}$.

Application 2: $L \to \infty$.

We study the limit distribution of the Markov (N_0, S_0) starting from $(N_{-L} = N, S_{-L} = 0)$ when $L \to \infty$.

0 is an absorbing state of $(N_z)_{z \geq -L}$ and $(N_0, 2\int_{-L}^{0} N_z dz)$ converges to $(0, \mu)$ where μ is a random variable with density P_N (pdf of μ starting from $N_0 = N$). It satisfies

$$\frac{\partial P_N}{\partial \tau} = \frac{1}{8}\gamma_n \omega^2 N \left(P_{N+1} - 2P_N + P_{N-1}\right), \qquad P_0(\tau) = \delta(\tau)$$

After some algebra:

$$P_N(\omega, \tau) = \frac{\partial}{\partial \tau}\left[\left(\frac{\gamma_n \omega^2 \tau}{8 + \gamma_n \omega^2 \tau}\right)^N \mathbf{1}_{[0,\infty)}(\tau)\right]$$

Finally

$$\mathbb{E}\left[R^{\varepsilon}_{\omega+\frac{\varepsilon h}{2}}(0)\overline{R^{\varepsilon}_{\omega-\frac{\varepsilon h}{2}}(0)}\right] \xrightarrow{\varepsilon \to 0} \int P_1(\omega, \tau) e^{ih\tau} d\tau$$

where

$$P_1(\omega, \tau) = \frac{8\gamma_n \omega^2}{(8 + \gamma_n \omega^2 \tau)^2} \mathbf{1}_{[0,\infty)}(\tau)$$

The reflected wave

Mean reflected intensity:

$$
\begin{aligned}
\frac{1}{\varepsilon}\mathbb{E}[p_{ref}^2(t)] &= \frac{1}{(2\pi)^2}\int\int e^{-iht}\mathbb{E}[R^{\varepsilon}_{\omega+\frac{\varepsilon h}{2}}(0)\overline{R^{\varepsilon}_{\omega-\frac{\varepsilon h}{2}}(0)}] \\
&\qquad \times \hat{f}(\omega+\frac{\varepsilon h}{2})\overline{\hat{f}}(\omega-\frac{\varepsilon h}{2})d\omega dh \\
&\xrightarrow{\varepsilon\to 0} \frac{1}{(2\pi)^2}\int\int e^{-iht}\int \mathcal{W}_1(0,\omega,\tau)e^{ih\tau}d\tau|\hat{f}(\omega)|^2 d\omega dh \\
&= \frac{1}{2\pi}\int \mathcal{W}_1(0,\omega,t)|\hat{f}(\omega)|^2 d\omega
\end{aligned}
$$

Autocorrelation function:

$$
\frac{1}{\varepsilon}\mathbb{E}[p_{ref}(t)p_{ref}(t+\varepsilon\tau)] \xrightarrow{\varepsilon\to 0} \frac{1}{2\pi}\int \mathcal{W}_1(0,\omega,t)|\hat{f}(\omega)|^2 e^{i\omega\tau}d\omega \qquad (1)
$$

Compute all moments:

a) For a fixed $t>0$, $\Big(\varepsilon^{-1/2}p_{ref}(t+\varepsilon\tau)\Big)_\tau$ converges to a zero-mean stationary Gaussian process with autocorrelation function (1).

b) For $0<t_1<\cdots<t_n$, the processes $\Big(\varepsilon^{-1/2}p_{ref}(t_j+\varepsilon\tau)\Big)_\tau$, $j=1,\cdots,n$, become independent as $\varepsilon\to 0$.

Explicit expressions for a random half-plane (1/2)

Case $L \to \infty$.

Mean reflected intensity:

$$\mathbb{E}[p_{ref}(t)^2] \xrightarrow{\varepsilon \to 0} I^\infty(t) := \frac{1}{2\pi} \int \frac{2/L_{\rm loc}(\omega)}{(2 + t/L_{\rm loc}(\omega))^2} |\hat{f}(\omega)|^2 d\omega$$

where $L_{\rm loc}(\omega) = 4/(\gamma\omega^2)$.

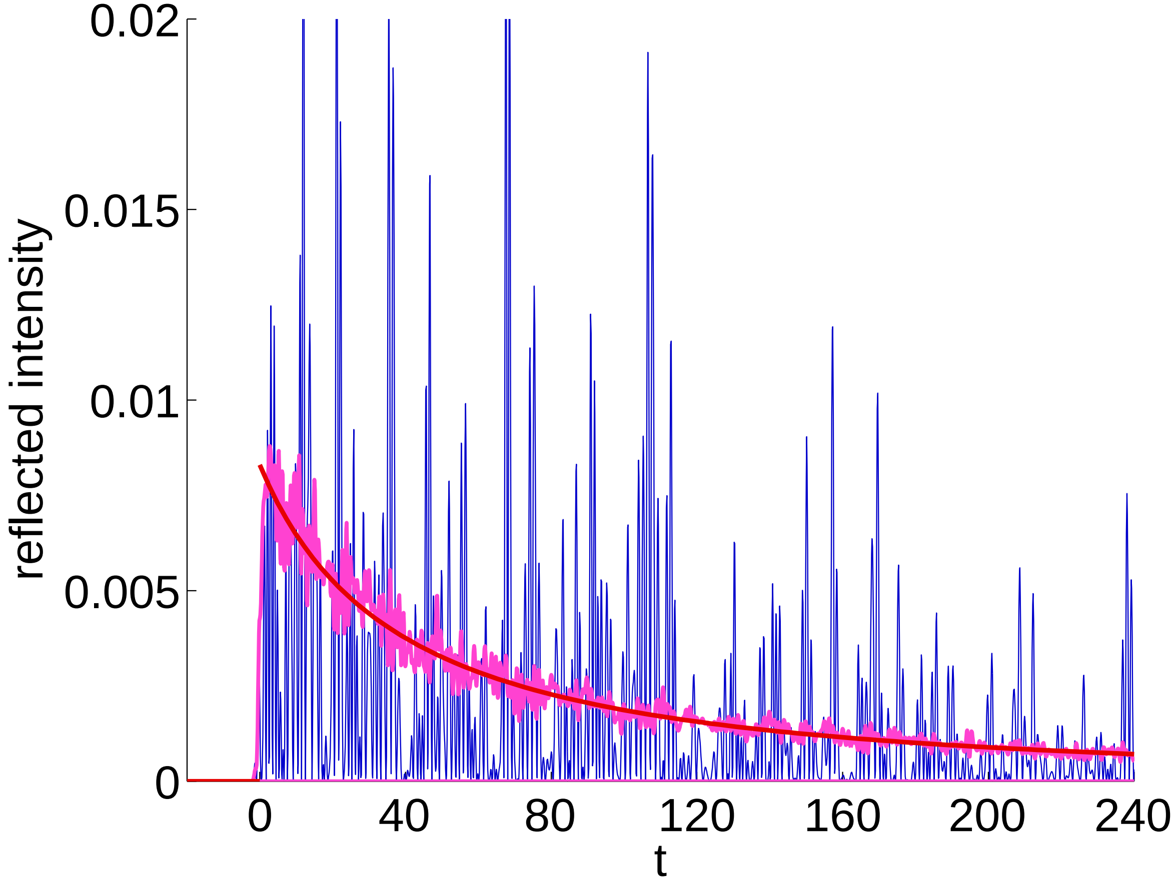
If the incident signal is a narrowband pulse with carrier frequency ω_0 and energy E_0, then

$$I^\infty(t) = \frac{2E_0/L_{\rm loc}(\omega_0)}{(2 + t/L_{\rm loc}(\omega_0))^2} = \frac{E_0/t_0}{(1 + t/t_0)^2}$$

where $t_0 = 2L_{\rm loc}(\omega_0)$.

$\hookrightarrow$ Very long power delay spread.

Half the reflected energy is captured in the time interval $[0, t_0]$ (time for a round trip until depth $L_{\rm loc}(\omega_0)$).



reflected intensity for one realization of the random medium (blue), mean intensity averaged over 10^3 realizations (magenta), theoretical expected reflected intensity (red).

Explicit expressions for a random half-plane (2/2)

$$\frac{1}{\varepsilon}\mathbb{E}[p_{ref}(t)p_{ref}(t+\varepsilon\tau)] \xrightarrow{\varepsilon\to 0} \frac{1}{2\pi}\int \mathcal{W}_1^\infty(\omega,t,0)|\hat{f}(\omega)|^2 e^{i\omega s}d\omega$$

$$\mathcal{W}_1^\infty(\omega,t,0) = \frac{2/L_{\rm loc}(\omega)}{(2+t/L_{\rm loc}(\omega))^2}\mathbf{1}_{[0,\infty)}(t) = \frac{8\gamma_n\omega^2}{(8+\gamma_n\omega^2 t)^2}\mathbf{1}_{[0,\infty)}(t)$$

The power spectral density of the reflected signal at time t is maximal at $\omega^*(t)$, where

$$t = \frac{2}{L_{\rm loc}(\omega^*(t))}\,,$$

or

$$\omega^*(t) = \sqrt{\frac{2t}{\gamma_n}}\,.$$

Applications to imaging

Assume the medium presents small-scale random fluctuations and large-scale deterministic variation that we want to image:

$$\frac{1}{\kappa(z)} = \frac{1}{\kappa_0(z)}\left(1+\eta(\frac{z}{\varepsilon^2})\right)$$

$$\rho(z) = \rho_0(z)\left(1+\nu(\frac{z}{\varepsilon^2})\right)$$

Idea: Perform a series of experiments where you probe the medium with a source f (or a set of sources).

Goal: extract information about the medium from the reflected waves.

Property: a Gaussian process is characterized by its autocorrelation function.
Consequence: All the information is in the autocorrelation function.

$$\mathbb{E}[p_{ref}(t+\varepsilon\tau)p_{ref}(t)] = 2\pi\varepsilon\int \mathcal{W}_1(0,\omega,t)|\hat{f}(\omega)|^2 e^{i\omega\tau}d\omega$$

Fortunately, $\mathcal{W}_1$ contains the information about the large-scale features of the medium.

Thus: if we get $\mathbb{E}[p_{ref}(t+\varepsilon\tau)p_{ref}(t)]$, then we get $\mathcal{W}_1$, and we "know" how to solve the inverse problem

$$\text{“}(\mathcal{W}_1(0,\omega,\tau))_{\omega,\tau} \mapsto \text{large-scale variations (such as } c_0(z))\text{”}$$

$$\frac{\partial \mathcal{W}_p}{\partial z} + \frac{2p}{c_0(z)}\frac{\partial \mathcal{W}_p}{\partial \tau} = \frac{1}{4}\gamma_n \omega^2 p^2 \left(\mathcal{W}_{p+1} + \mathcal{W}_{p-1} - 2\mathcal{W}_p\right)$$

$$V_p(z=-L,\omega,\tau) = \delta(\tau)\mathbf{1}_0(p)$$

Problem: we only have a single realization of the medium !

How to estimate $\mathbb{E}[p_{ref}(t+\varepsilon\tau)p_{ref}(t)]$?

Answer: Local average,

$$\frac{1}{\Delta t}\int_{t-\Delta t/2}^{t-\Delta t/2} p_{ref}(s+\varepsilon\tau)p_{ref}(s)ds$$

with $\varepsilon \ll \Delta t \ll 1$ (optimal choice not easy).

Or: time-windowed Fourier transform, wavelet decomposition, ...

Not very efficient...