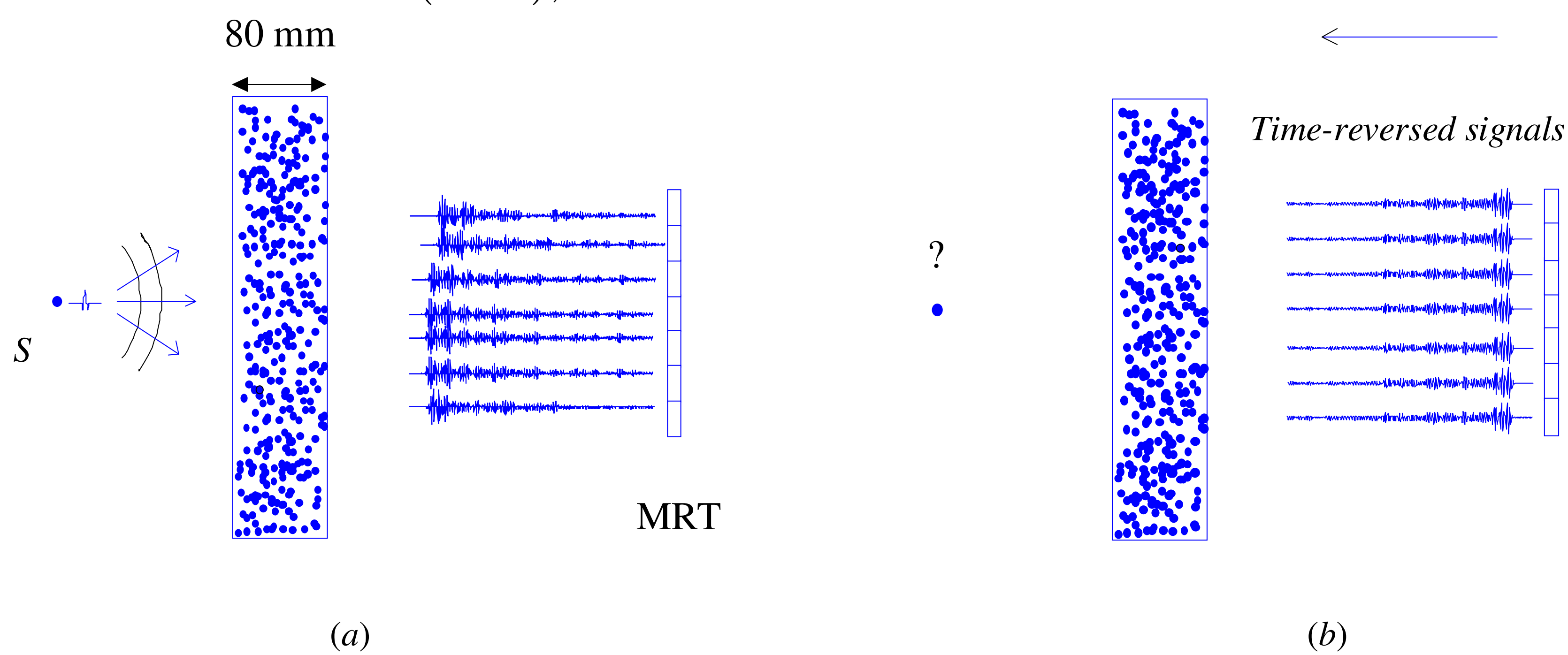
Ultrasound experiment by M. Fink

cf. A. Tourin, M. Fink, and A. Derode, Multiple scattering of sound, Waves Random Media **10** (2000), R31-R60.



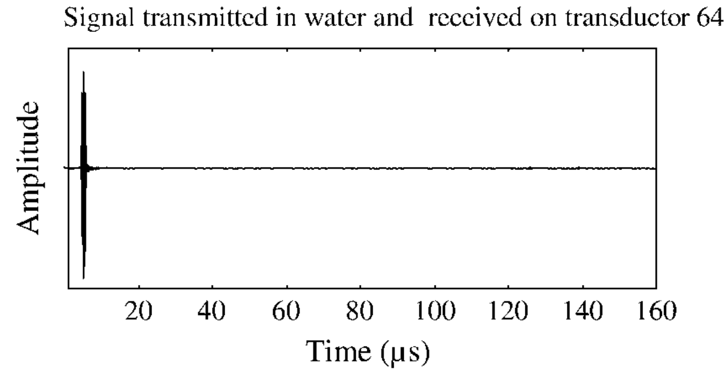
Experimental set-up for a time-reversal experiment through a multiple-scattering medium:

(a) first step, the source sends a pulse through the sample, the transmitted wave is recorded by the TRM.

(b) second step, the multiply scattered signals have been time-reverted, they are retransmitted by the TRM, and S records the reconstructed pressure field.

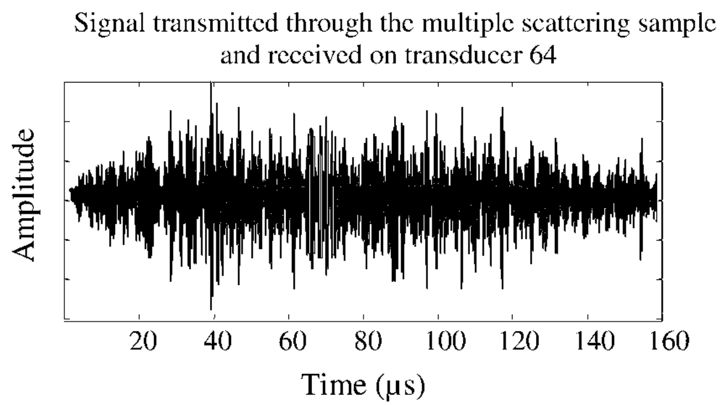
Experimental observations

(*a*)



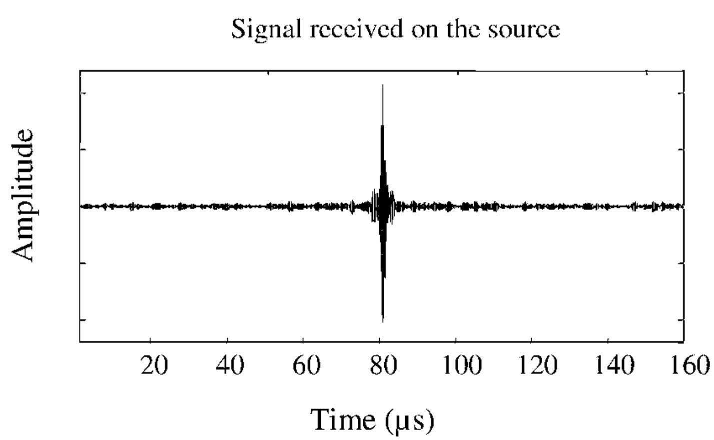
The source emits a short
1 μs pulse.

(*b*)



The TRM records a long
scattered signal.

(*c*)



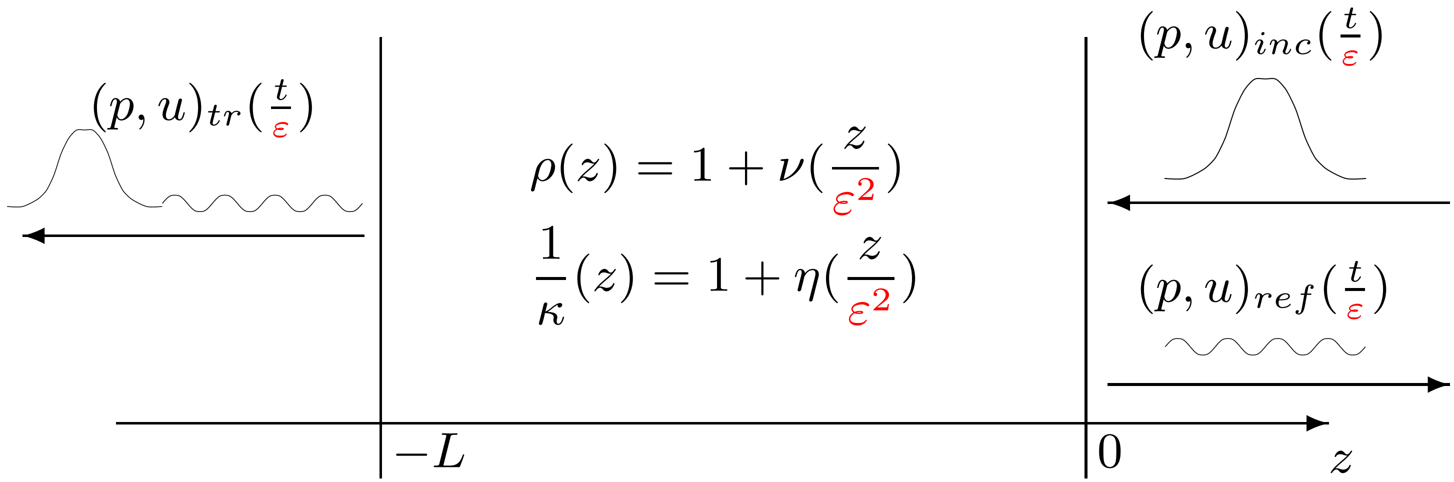
Recompression at the source
location after propagation
of the time-reversed wave.

Scattering of an acoustic pulse in random media

Acoustic equations for pressure p and speed u:

$$\frac{\partial p}{\partial t} + \kappa(z)\frac{\partial u}{\partial z} = 0$$

$$\rho(z)\frac{\partial u}{\partial t} + \frac{\partial p}{\partial z} = 0$$



IC: left-going pulse incoming from the right homogeneous half-space.

$$m(z) = \eta(z) + \nu(z), \qquad n(z) = \eta(z) - \nu(z)$$

Local velocity: $c(z) = \sqrt{\kappa(z)/\rho(z)}$.

Local impedance: $\zeta(z) = \rho(z)c(z)$.

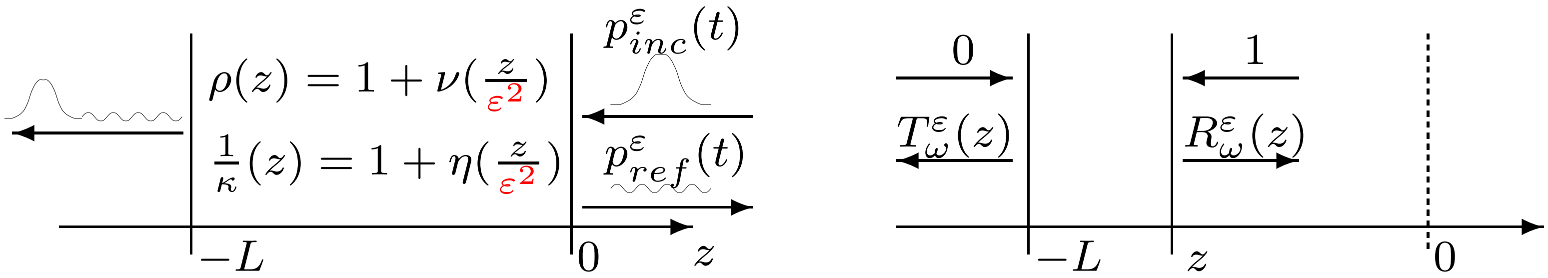
Integral representation of the reflected signal

Send a left-going pulse $f(\frac{t}{\varepsilon})$:

$$p^{\varepsilon}_{inc}(t, z = 0) = f(\frac{t}{\varepsilon}) = \frac{1}{2\pi} \int e^{-\frac{i\omega t}{\varepsilon}} \hat{f}(\omega) d\omega$$

Reflected signal:

$$p^{\varepsilon}_{ref}(t) = \frac{1}{2\pi} \int e^{-\frac{i\omega t}{\varepsilon}} \hat{f}(\omega) R^{\varepsilon}_{\omega})(0) d\omega$$



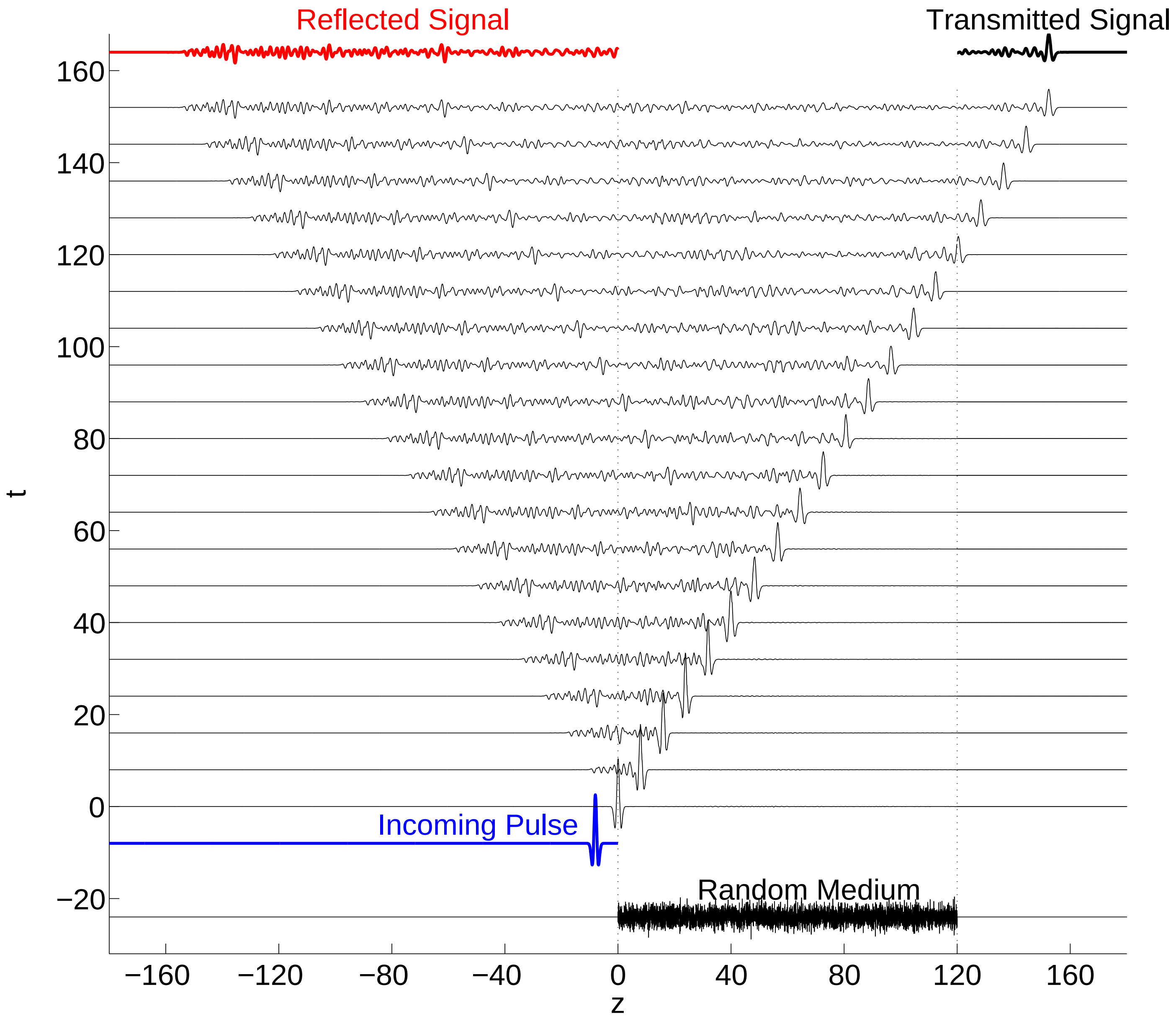
$R^{\varepsilon}_{\omega}(z)$ is the reflection coefficient for a random slab $[-L, z]$:

$$\frac{dR^{\varepsilon}_{\omega}}{dz} = -\frac{i\omega}{\varepsilon} m(\frac{z}{\varepsilon^2}) R^{\varepsilon}_{\omega} + \frac{i\omega}{2\varepsilon} n(\frac{z}{\varepsilon^2}) e^{-\frac{2i\omega z}{\varepsilon}} (R^{\varepsilon}_{\omega})^2 - \frac{i\omega}{2\varepsilon} n(\frac{z}{\varepsilon^2}) e^{\frac{2i\omega z}{\varepsilon}},$$

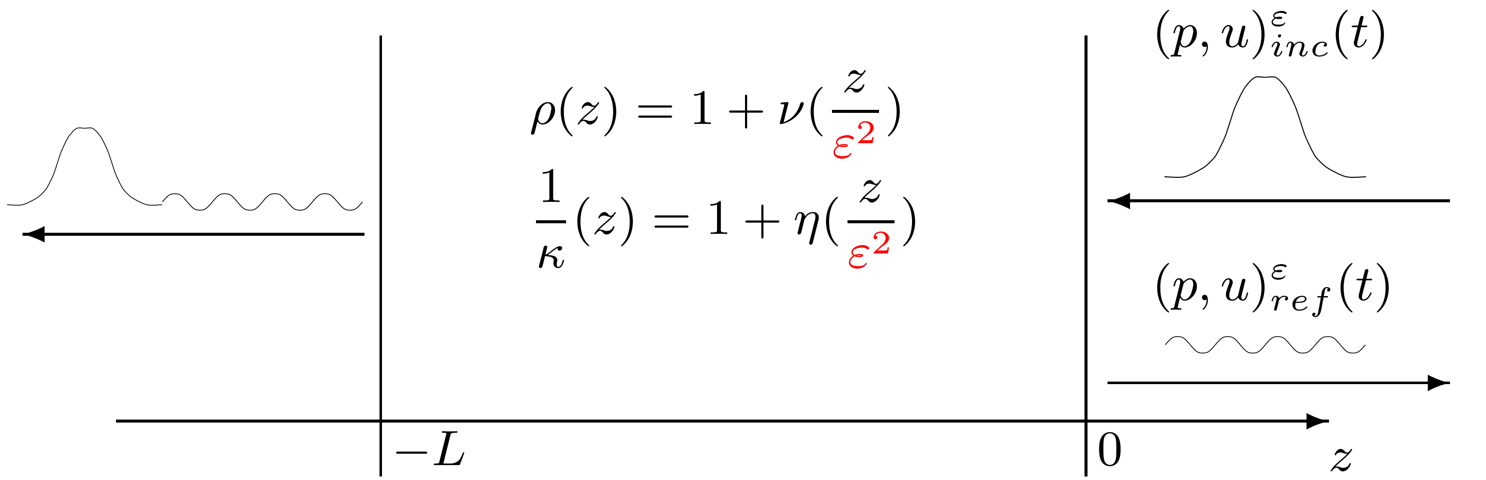
with the initial condition at $z = -L$: $R^{\varepsilon}_{\omega}(z = -L) = 0$.

Energy conservation $|R^{\varepsilon}_{\omega}|^2 + |T^{\varepsilon}_{\omega}|^2 = 1 \rightarrow$ uniform boundedness of R^{ε}_{ω}.

Numerical simulation



Time reversal in reflection (TR)

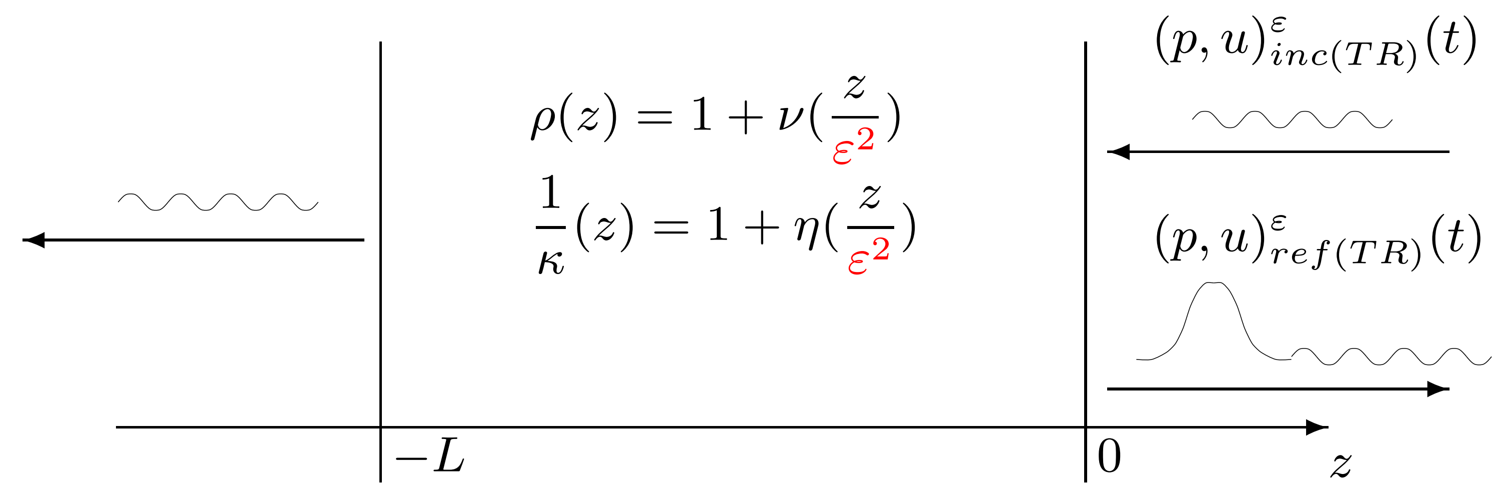


Denote $p^{\varepsilon}_{inc}(t) = f(\frac{t}{\varepsilon})$.

Record $p^{\varepsilon}_{ref}(t)$ up to time t_1.

Cut a piece $p^{\varepsilon}_{ref,cut}(t) = p^{\varepsilon}_{ref}(t)G_1(t)$, with $\text{supp}(G_1) \subset [0, t_1]$.

Time reverse and send back $p^{\varepsilon}_{inc(TR)}(t) = p^{\varepsilon}_{ref,cut}(t_1 - t)$.



Expression of the refocused pulse

The incoming signal $f(\frac{t}{\varepsilon})$

$$p_{inc}(t) = \frac{1}{2\pi} \int e^{-\frac{i\omega t}{\varepsilon}} \hat{f}(\omega) d\omega$$

propagates into the medium and generates the reflected signal:

$$p^{\varepsilon}_{ref}(t) = \frac{1}{2\pi} \int e^{-\frac{i\omega t}{\varepsilon}} \hat{f}(\omega) R^{\varepsilon}_{\omega}(0) d\omega$$

Record up to time t_1 and cut a piece of the recorded signal (i.e. $G_1(t) = \mathbf{1}_{[0,t_1]}(t)$)

$$p^{\varepsilon}_{ref,cut}(t) = p^{\varepsilon}_{ref}(t, z=0) G_1(t)$$

Time-reverse and send back into the medium:

$$\begin{aligned} p^{\varepsilon}_{inc(TR)}(t) &= p^{\varepsilon}_{ref}(t_1 - t) G_1(t_1 - t) \\ &= \frac{1}{2\pi\varepsilon} \int\int e^{-\frac{i\omega(t_1 - t)}{\varepsilon}} \hat{p}^{\varepsilon}_{ref}(\omega') \hat{G}_1(\frac{\omega - \omega'}{\varepsilon}) d\omega' d\omega \end{aligned}$$

The signal is real-valued:

$$p^{\varepsilon}_{inc(TR)}(t) = \frac{1}{2\pi\varepsilon} \int\int e^{\frac{i\omega(t_1 - t)}{\varepsilon}} \overline{\hat{p}^{\varepsilon}_{ref}}(\omega') \overline{\hat{G}_1}(\frac{\omega - \omega'}{\varepsilon}) d\omega' d\omega$$

The new signal propagates into the same medium and generates a new reflected signal observed at time $t_2 + \varepsilon s$:

$$p^{\varepsilon}_{ref(TR)}(t_2 + \varepsilon s) = \frac{1}{2\pi} \int \hat{p}^{\varepsilon}_{inc(TR)}(\omega) R^{\varepsilon}_{\omega}(0) e^{-\frac{i\omega t_1}{\varepsilon} - i\omega s} d\omega$$

Subsituting the expression of $\hat{p}^{\varepsilon}_{inc(TR)}$ into this equation:

$$\begin{aligned} p^{\varepsilon}_{ref(TR)}(t_2 + \varepsilon s) &= \frac{1}{(2\pi)^2 \varepsilon} \int \int e^{-i\omega s} e^{\frac{i\omega(t_1 - t_2)}{\varepsilon}} \overline{\hat{f}}(\omega') \overline{\hat{G}_1}(\frac{\omega - \omega'}{\varepsilon}) \\ &\times R^{\varepsilon}_{\omega}(0) \overline{R^{\varepsilon}_{\omega'}}(0) d\omega' d\omega. \end{aligned}$$

Change of variables $\omega' = \omega - \varepsilon h$:

$$\begin{aligned} p^{\varepsilon}_{ref(TR)}(t_2 + \varepsilon s) &= \frac{1}{(2\pi)^2} \int \int e^{-i\omega s} e^{\frac{i\omega(t_1 - t_2)}{\varepsilon}} \overline{\hat{f}}(\omega - \varepsilon h) \overline{\hat{G}_1}(h) \\ &\times R^{\varepsilon}_{\omega}(0) \overline{R^{\varepsilon}_{\omega - \varepsilon h}(0)} dh \, d\omega. \end{aligned}$$

The autocorrelation function $\overline{R^{\varepsilon}_{\omega'}(0)} R^{\varepsilon}_{\omega}(0)$ plays a primary role.

Refocusing at $t_2 = t_1$.

We have obtained:

$$\mathbb{E}\left[R^{\varepsilon}_{\omega+\frac{\varepsilon h}{2}}(0)\overline{R^{\varepsilon}_{\omega-\frac{\varepsilon h}{2}}(0)}\right] \xrightarrow{\varepsilon\to 0} \int \mathcal{W}_1(0,\omega,\tau)e^{ih\tau}d\tau$$

In the time domain:

$$\begin{aligned}
\mathbb{E}\left[p^{\varepsilon}_{ref(TR)}(t_1+\varepsilon s)\right] \quad & \xrightarrow{\varepsilon\to 0} \quad \frac{1}{(2\pi)^2}\int\int e^{-i\omega s}\overline{\hat{f}}(\omega)\overline{\hat{G}_1}(h) \\
& \qquad \times\left[\int \mathcal{W}_1(0,\omega,\tau)e^{ih\tau}d\tau\right]dh\, d\omega \\
& = \quad \frac{1}{2\pi}\int\int e^{-i\omega s}\overline{\hat{f}}(\omega)G_1(\tau)\mathcal{W}_1(0,\omega,\tau)d\tau d\omega \\
& = \quad \frac{1}{2\pi}\int e^{-i\omega s}\overline{\hat{f}}(\omega)\hat{K}_{TR}(\omega)d\omega \\
& = \quad \left(f(-\cdot) * K_{TR}\right)(s)
\end{aligned}$$

where

$$\hat{K}_{TR}(\omega) = \int G_1(\tau)\mathcal{W}_1(0,\omega,\tau)d\tau$$

This results only holds true in average (averaging over all possible realizations of the medium) !

Frequency correlation of R^ε_ω

Let us consider the forth-order moment at 4 different frequencies of R^ε_ω

$$\begin{aligned}
&\Big| \mathbb{E}\left[R^\varepsilon_{\omega_1+\frac{\varepsilon h_1}{2}} \overline{R^\varepsilon_{\omega_1-\frac{\varepsilon h_1}{2}}} R^\varepsilon_{\omega_2+\frac{\varepsilon h_2}{2}} \overline{R^\varepsilon_{\omega_2-\frac{\varepsilon h_2}{2}}} \right] \\
&-\mathbb{E}\left[R^\varepsilon_{\omega_1+\frac{\varepsilon h_1}{2}} \overline{R^\varepsilon_{\omega_1-\frac{\varepsilon h_1}{2}}} \right] \mathbb{E}\left[R^\varepsilon_{\omega_2+\frac{\varepsilon h_2}{2}} \overline{R^\varepsilon_{\omega_2-\frac{\varepsilon h_2}{2}}} \right] \Big| \\
&\quad \stackrel{\varepsilon\to 0}{\longrightarrow} 0
\end{aligned}$$

if $\omega_1 \neq \omega_2$.

$$p^{\varepsilon}_{ref(TR)}(t_1+\varepsilon s) = \int\int e^{-i\omega s} g(\omega,h) R^{\varepsilon}_{\omega+\frac{\varepsilon h}{2}} \overline{R^{\varepsilon}_{\omega-\frac{\varepsilon h}{2}}} d\omega dh$$

$$\begin{aligned}
\mathbb{E}\left[p^{\varepsilon}_{ref(TR)}(t_1+\varepsilon s)^2\right] &= \int..\int d\omega_1 d\omega_2 dh_1 dh_2 e^{-i(\omega_1+\omega_2)s} g(\omega_1,h_1) g(\omega_2,h_2) \\
&\quad \times \mathbb{E}\left[R^{\varepsilon}_{\omega_1+\frac{\varepsilon h_1}{2}} \overline{R^{\varepsilon}_{\omega_1-\frac{\varepsilon h_1}{2}}} R^{\varepsilon}_{\omega_2+\frac{\varepsilon h_2}{2}} \overline{R^{\varepsilon}_{\omega_2-\frac{\varepsilon h_2}{2}}}\right] \\
&\overset{\varepsilon\to 0}{\simeq} \int..\int d\omega_1 d\omega_2 dh_1 dh_2 e^{-i(\omega_1+\omega_2)s} g(\omega_1,h_1) g(\omega_2,h_2) \\
&\quad \times \mathbb{E}\left[R^{\varepsilon}_{\omega_1+\frac{\varepsilon h_1}{2}} \overline{R^{\varepsilon}_{\omega_1-\frac{\varepsilon h_1}{2}}}\right] \mathbb{E}\left[R^{\varepsilon}_{\omega_2+\frac{\varepsilon h_2}{2}} \overline{R^{\varepsilon}_{\omega_2-\frac{\varepsilon h_2}{2}}}\right] \\
&\overset{\varepsilon\to 0}{\simeq} \mathbb{E}\left[p^{\varepsilon}_{ref(TR)}(t_1+\varepsilon s)\right]^2
\end{aligned}$$

$\mathrm{Var}\left(p^{\varepsilon}_{ref(TR)}(t_1+\varepsilon s)\right) \overset{\varepsilon\to 0}{\longrightarrow} 0 \Longrightarrow$ Convergence in L^2 and in probability of $p^{\varepsilon}_{ref(TR)}(t_1+\varepsilon s)$.

Decorrelation in frequency of $R^{\varepsilon}_{\omega} \Longrightarrow$ Self-averaging in time of $p^{\varepsilon}_{ref(TR)}$.

Convergence of the refocused pulse

The refocused signal $\left(p^{\varepsilon}_{ref(TR)}(t_1+\varepsilon s)\right)_{s\in(-\infty,\infty)}$ converges in probability as $\varepsilon \longrightarrow 0$ to

$$P_{ref(TR)}(s) = \left(f(-\,\cdot) * K_{TR}(\cdot)\right)(s)$$

$$\hat{K}_{TR}(\omega) = \int G_1(\tau)\mathcal{W}_1(0,\omega,\tau)d\tau$$

where $\mathcal{W}_1(0,\omega,\tau)$ is the **deterministic** density given by the system of transport equations.

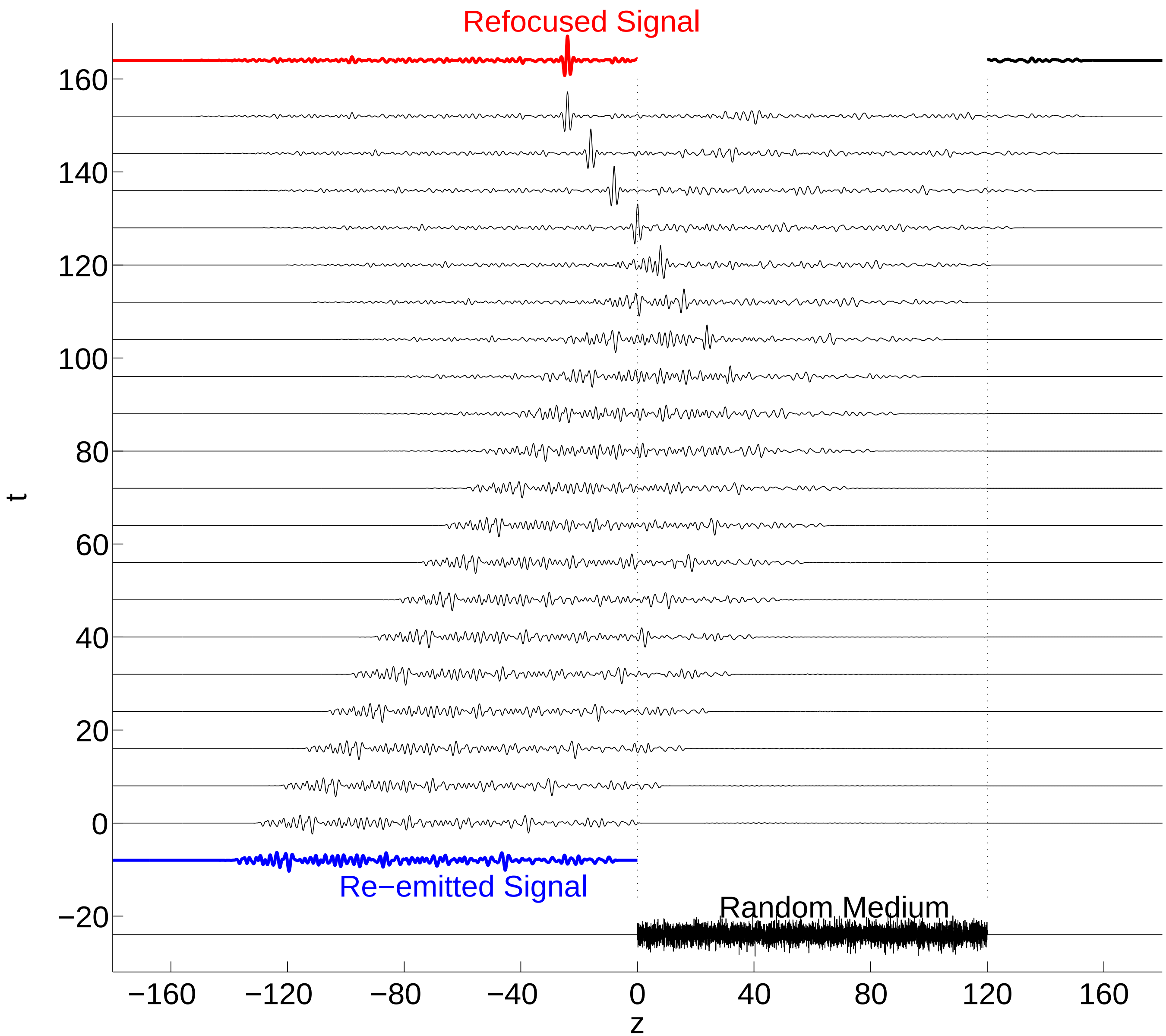
In particular, if $L \to \infty$:

$$\mathcal{W}_1(0,\omega,\tau) = \frac{8\gamma_n\omega^2}{\left(8+\gamma_n\omega^2\tau\right)^2}$$

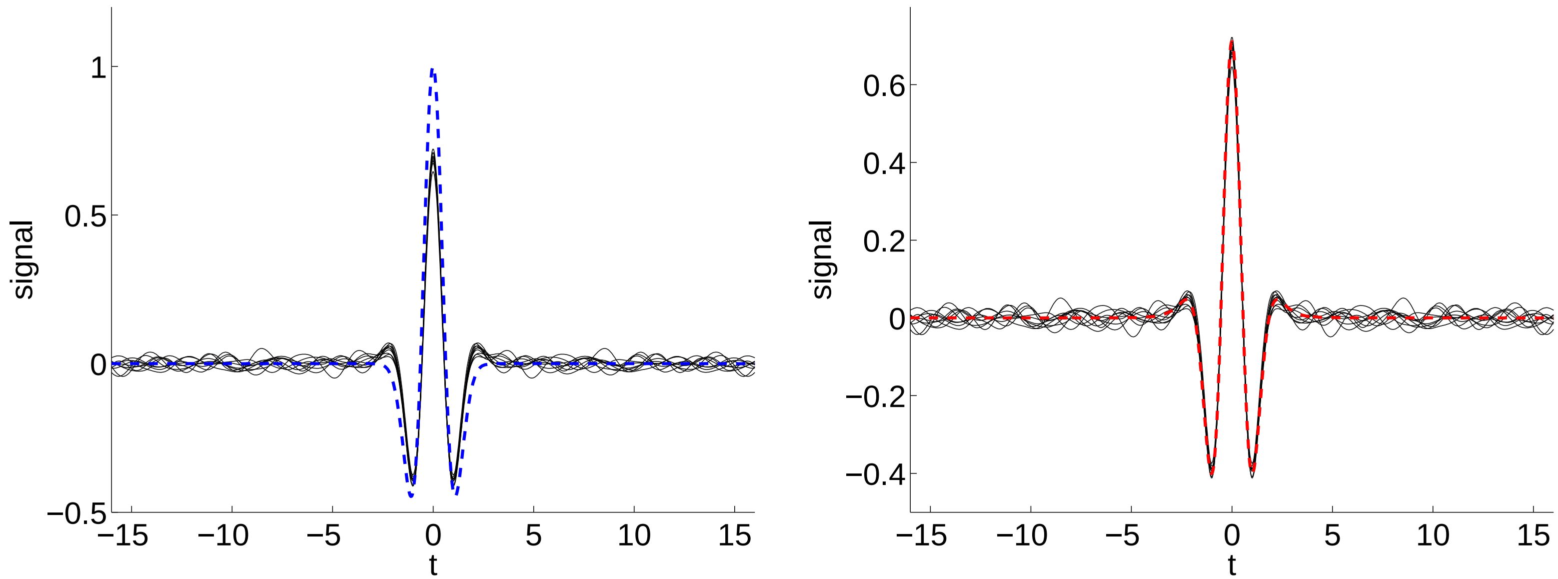
$\hookrightarrow$ The refocused pulse has deterministic center and shape.

$\hookrightarrow$ There is statistical stability.

Numerical simulation



Comparisons simulations - theory



Profiles of the refocused pulse at $z = 0$

Comparison with the input pulse $f(t)$

Comparison with the asymptotic formula $K_{\mathrm{TR}} * f(-t)$

Application to imagery

Goal: extract the information about the large-scale properties of the medium (ρ_0, κ_0)

$$\rho(z) = \rho_0(z)\left(1 + \nu(\frac{z}{\varepsilon^2})\right)$$
$$\frac{1}{\kappa(z)} = \frac{1}{\kappa_0(z)}\left(1 + \eta(\frac{z}{\varepsilon^2})\right)$$

This information is contained in $\mathcal{W}_1(0, \omega, \tau)$. The problem is to find a statistically stable estimator of $\mathcal{W}_1$.

Method:

1) Compare the input pulse $f(t)$ and the refocused pulse $K_{\rm TR} * f(-t)$.
$\hookrightarrow$ Extract $(\hat{K}_{\rm TR}(\omega))_\omega$.

$$\hat{K}_{\rm TR}(\omega) = \int G_1(\tau)\mathcal{W}_1(0, \omega, \tau) d\tau$$

2) Use different truncation functions G_1 to get $(\mathcal{W}_1(0, \omega, \tau))_{\omega,\tau}$.
$\hookrightarrow$ large-scale variations of the medium.

Statistically stable method, no local average is needed.

Application to communications

How to send a message $f(t)$ from E to R in a highly scattering medium ?

1) R emits a short, broadband pulse pulse $f_0(t)$.

2) E receives and records a noisy signal $G(t)$.

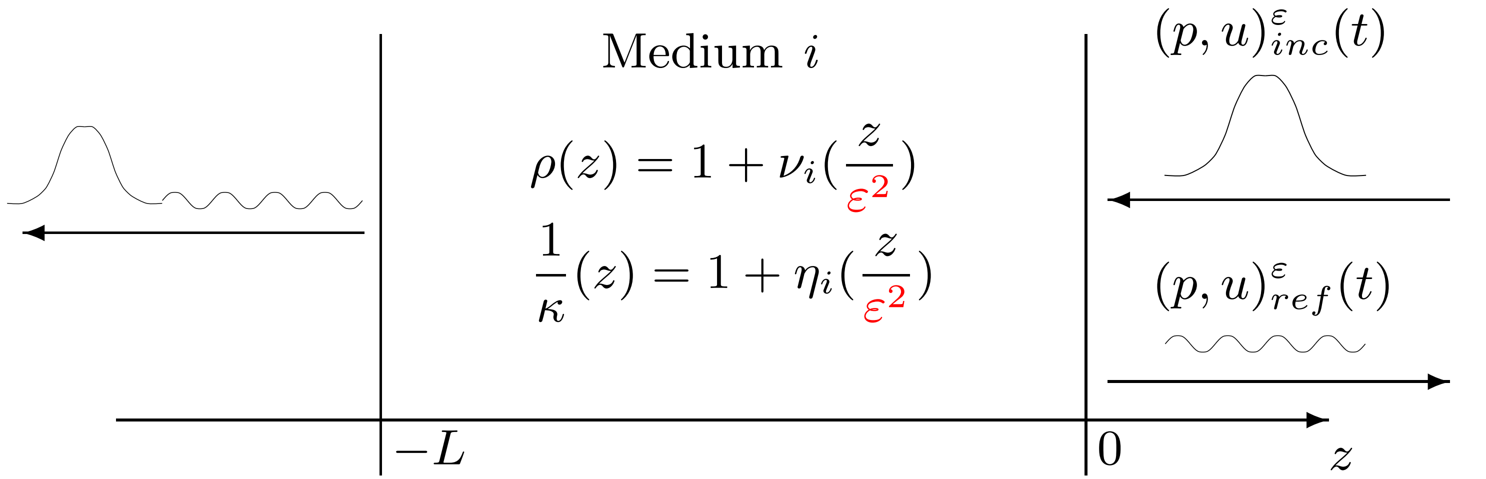
3) E emits $[f * G(-\cdot)](t)$.

4) R receives $[K_{\rm TR} * f_0(-\cdot) * f](t)$.

$\hookrightarrow$ R can extract f.

(OK in ocean acoustics, difficult in electromagnetics).

Time reversal in changing media

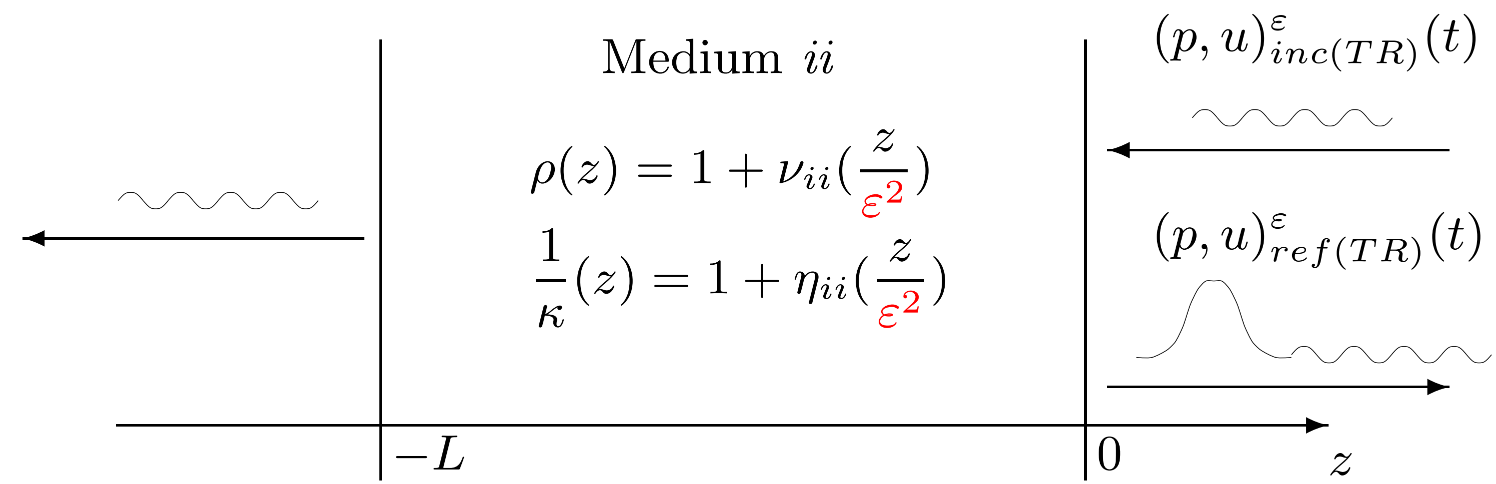


Denote $p^{\varepsilon}_{inc}(t) = f(\frac{t}{\varepsilon})$.

Record $p^{\varepsilon}_{ref}(t)$ up to time t_1.

Cut a piece $p^{\varepsilon}_{ref,cut}(t) = p^{\varepsilon}_{ref}(t)G_1(t)$, with $\text{supp}(G_1) \subset [0, t_1]$.

Time reverse and send it back $p^{\varepsilon}_{inc(TR)}(t) = p^{\varepsilon}_{ref,cut}(t_1 - t)$.



Changing media

The density and compressibility fluctuations (ν_i, η_i) and (ν_{ii}, η_{ii}) are identically distributed.

Integrated correlation functions:

$$\gamma_n = 2\int_0^\infty \mathbb{E}[n(0)n(z)]dz, \qquad \gamma_m = 2\int_0^\infty \mathbb{E}[m(0)m(z)]dz$$

Degree of correlation $\delta \in [-1,1]$:

$$\delta_m = \frac{\int_0^\infty \mathbb{E}[m_i(0)m_{ii}(z)]dz}{\int_0^\infty \mathbb{E}[m_i(0)m_i(z)]dz}, \qquad \delta_n = \frac{\int_0^\infty \mathbb{E}[n_i(0)n_{ii}(z)]dz}{\int_0^\infty \mathbb{E}[n_i(0)n_i(z)]dz}$$

$\delta = 1 \leftrightarrow$ complete correlation. $\delta = 0 \leftrightarrow$ complete decorrelation.

Refocused pulse:

$$p^\varepsilon_{ref(TR)}(t_1 + \varepsilon s) = \frac{1}{(2\pi)^2}\int\int e^{-i\omega s}\overline{\hat{f}}(\omega - \varepsilon h)\overline{\hat{G}}_{t_1}(h) \times R^{\varepsilon,ii}_\omega(0)\overline{R^{\varepsilon,i}_{\omega-\varepsilon h}(0)}dh\, d\omega$$

where the reflection coefficient $R^{\varepsilon,i}_\omega$ satisfies the Ricatti equation:

$$\frac{dR^{\varepsilon,i}_\omega}{dz} = -i\frac{\omega}{\varepsilon}m_i(\frac{z}{\varepsilon^2})R^{\varepsilon,i}_\omega + \frac{i\omega}{2\varepsilon}n_i(\frac{z}{\varepsilon^2})e^{-\frac{2i\omega z}{\varepsilon}}(R^{\varepsilon,i}_\omega)^2 - \frac{i\omega}{2\varepsilon}n_i(\frac{z}{\varepsilon^2})e^{\frac{2i\omega z}{\varepsilon}}$$

Asymptotics of the refocused pulse

Expectation of the autocorrelation function $\overline{R^{\varepsilon,i}_{\omega}} R^{\varepsilon,ii}_{\omega}$:

$$\mathbb{E}\left[R^{\varepsilon,ii}_{\omega+\frac{\varepsilon h}{2}}(0) \overline{R^{\varepsilon,i}_{\omega-\frac{\varepsilon h}{2}}(0)} \right] \stackrel{\varepsilon\to 0}{\longrightarrow} \int v_1(0,\omega,\tau) e^{ih\tau} d\tau$$

where v_1 is given by a system of transport equations for $(v_p)_{p\in\mathbb{N}}$:

$$\begin{aligned} \frac{\partial v_p}{\partial z} + 2p \frac{\partial v_p}{\partial \tau} &= \frac{1}{4} \delta_n \gamma_n \omega^2 p^2 \left(v_{p+1} + v_{p-1} - 2 v_p \right) \\ &\quad - \frac{1-\delta_n}{2} \gamma_n \omega^2 p^2 v_p - (1-\delta_m) \gamma_m \omega^2 p^2 v_p \\ v_p(z=-L,\omega,\tau) &= \delta_0(\tau) \mathbf{1}_0(p) \end{aligned}$$

$\Rightarrow$ Convergence of the expectation of the refocused pulse:

$$\mathbb{E}[p^{\varepsilon}_{ref(TR)}(t_1+\varepsilon s)] \stackrel{\varepsilon\to 0}{\longrightarrow} \frac{1}{2\pi} \int\int e^{-i\omega s} \overline{\hat{f}}(\omega) G_1(\tau) v_1(0,\omega,\tau) d\tau d\omega$$

But:

$$\mathbb{E}[p^{\varepsilon}_{ref(TR)}(t_1+\varepsilon s)^2] \stackrel{\varepsilon\to 0}{\not\longrightarrow} \left[\frac{1}{2\pi} \int\int e^{-i\omega s} \overline{\hat{f}}(\omega) G_1(\tau) v_1(0,\omega,\tau) d\tau d\omega \right]^2$$

Equivalently $v_p(z,\omega,\tau,z) = \mathbf{E}[\mathcal{W}_p(z,\omega,\tau)]$ where $(\mathcal{W}_p)_{p\in\mathbb{N}}$ is solution of

$$\begin{aligned} d\mathcal{W}_p + 2p\frac{\partial \mathcal{W}_p}{\partial \tau}dz &= ip\omega\sqrt{2(1-\delta_m)\gamma_m}\mathcal{W}_p \circ dW_z \\ &\quad + \frac{1}{4}\delta_n\gamma_n\omega^2p^2\left(\mathcal{W}_{p+1} + \mathcal{W}_{p-1} - 2\mathcal{W}_p\right) - \frac{1-\delta_n}{2}\gamma_n\omega^2p^2\mathcal{W}_p \\ \mathcal{W}_p(z=-L,\omega,\tau) &= \delta_0(\tau)\mathbf{1}_0(p) \end{aligned}$$

W_z is a standard Brownian motion.

Thus

$$\mathbb{E}[p^{\varepsilon}_{ref(TR)}(t_1+\varepsilon s)] \stackrel{\varepsilon\to 0}{\longrightarrow} \frac{1}{2\pi}\int\int e^{-i\omega s}\overline{\hat{f}}(\omega)G_1(\tau)\mathbf{E}\left[\mathcal{W}_1(0,\omega,\tau)\right]d\tau d\omega$$

- Convergence of the finite-dimensional distributions:

$$\mathbb{E}\left[p^{\varepsilon}_{ref(TR)}(t_1+\varepsilon s_1)^{p_1}\dots p^{\varepsilon}_{ref(TR)}(t_1+\varepsilon s_k)^{p_k}\right] \stackrel{\varepsilon\to 0}{\longrightarrow}$$
$$\mathbf{E}\left[\prod_{1\le j\le k}\left(\tfrac{1}{2\pi}\int\int \mathcal{W}_1(0,\omega,\tau)\overline{\hat{f}}(\omega)e^{-i\omega s_j}G_1(\tau)d\omega d\tau\right)^{p_j}\right]$$

- Tightness of $(p^{\varepsilon}_{ref(TR)}(t_1+\varepsilon s))_{s\in(-\infty,\infty)}$ in the space of continuous functions.

- Conclusion: $p^{\varepsilon}_{ref(TR)}(t_1+\varepsilon s)$ converges in distribution to $\frac{1}{2\pi}\int\int \mathcal{W}_1(0,\omega,\tau)\overline{\hat{f}}(\omega)e^{-i\omega s}G_1(\tau)d\omega d\tau$

Probabilistic representation of the transport equations

Consider our familiar jump (Markov) process $(N_z)_{z\geq -L}$ with state space $\mathbb{N}$ and generator

$$\mathcal{L}\phi(N) = \frac{1}{4}\gamma_n \omega^2 N^2 \left(\phi(N+1) + \phi(N-1) - 2\phi(N)\right)$$

Therefore (Feynman-Kac) :

$$\begin{aligned}\int_{\tau_0}^{\tau_1} \mathcal{W}_1(0,\omega,\tau)d\tau \quad = \quad & \mathbb{E}\Big[\exp\Big(i\sqrt{2\gamma_m(1-\delta_m)}\omega\int_{-L}^{0} N_{-L-s}dW_s\Big) \\ & \times \exp\Big(-\frac{1-\delta_n}{2}\gamma_n\omega^2\int_{-L}^{0} N^2_{-L-s}ds\Big) \\ & \times \mathbf{1}_0(N_0)\mathbf{1}_{[\tau_0,\tau_1]}\Big(\int_{-L}^{0} 2N_s ds\Big) \;\Big|\; N_{-L}=1\Big]\end{aligned}$$

where $\mathbb{E}$ is the expectation w.r.t the distribution of $(N_z)_{z\geq -L}$

Convergence of the refocused pulse

The refocused signal $\left(p^{\varepsilon}_{ref(TR)}(t_1 + \varepsilon s)\right)_{s \in (-\infty,\infty)}$ converges in distribution as $\varepsilon \longrightarrow 0$ to

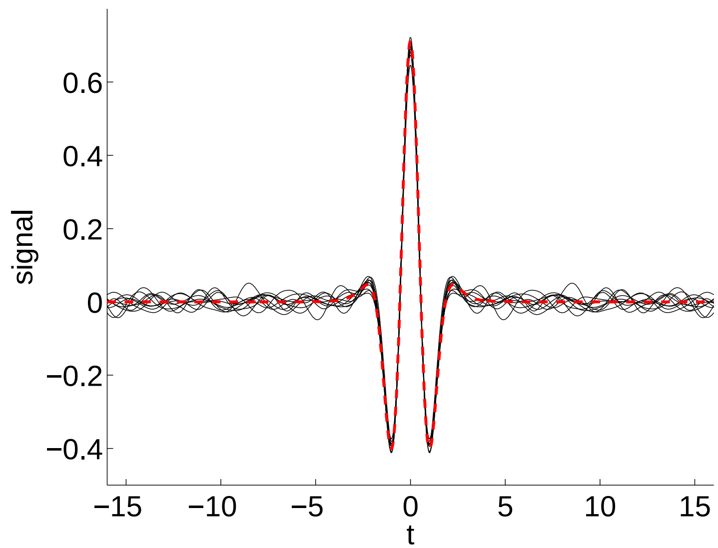
$$P_{ref(TR)}(s) = \left(f(-\,\cdot) * K_{TR}(\cdot)\right)(s)$$

$$\hat{K}_{TR}(\omega) = \int G_1(\tau)\mathcal{W}_1(0,\omega,\tau)d\tau$$

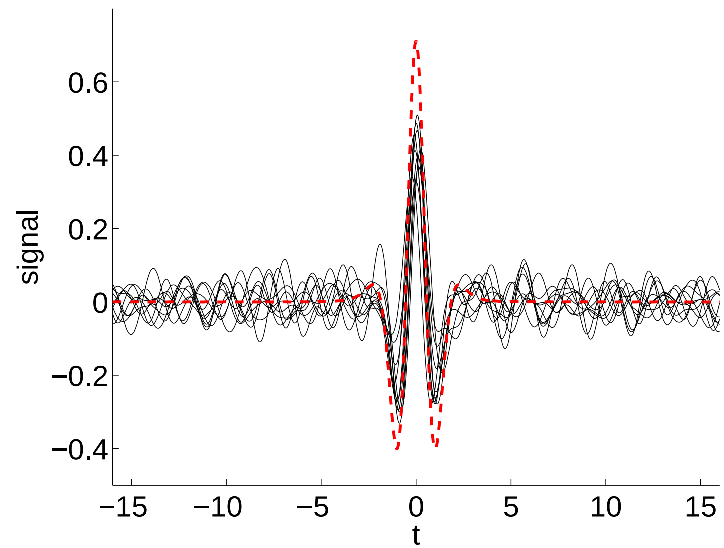
where $\mathcal{W}_1(0,\omega,\tau)$ is the **random** density given by the system of transport equations driven by the Brownian motion W_z.

$\hookrightarrow$ The refocused pulse has random center and shape.

$\hookrightarrow$ There is no statistical stability.



Fixed medium
$\delta_m = \delta_n = 1$



Changing medium
$\delta_m = \delta_n = 0.75$

Refocused pulses

Comparison of the refocused pulses from the numerical experiments and the expected refocused pulse shape obtained by the asymptotic theory.
Only the density is random $\kappa_i = \kappa_{ii} \equiv \kappa_0 = 1 \Rightarrow \gamma_m = \gamma_n$.

Mean refocused shape

$$\mathbb{E}[P_{ref(TR)}(s)] = \left(f(-\,\cdot) * \mathbb{E}[K_{TR}](\cdot)\right)(s)$$

$$\mathbb{E}[\hat{K}_{TR}](\omega) = \int G_1(\tau)\mathbb{E}[\mathcal{W}_1(0,\omega,\tau)]d\tau$$

For large slab $L \to \infty$:

$$\mathbb{E}[\mathcal{W}_1(0,\omega,\tau)] = \frac{\gamma_0\omega^2\delta_0}{8}\frac{1-\tanh^2\left(\frac{\sqrt{1-\delta_0^2}\gamma_0\omega^2\tau}{8}\right)}{\left[1+\frac{1}{\sqrt{1-\delta_0^2}}\tanh\left(\frac{\sqrt{1-\delta_0^2}\gamma_0\omega^2\tau}{8}\right)\right]^2}$$

where

$$\gamma_0 = \gamma_n + 2(1-\delta_m)\gamma_m \qquad \delta_0 = \frac{\delta_n\gamma_n}{\gamma_n + 2(1-\delta_m)\gamma_m}$$

If $G_1(t) = \mathbf{1}_{[0,t_1]}(t)$,

$$\mathbb{E}[\hat{K}_{TR}](\omega) = \delta_0\frac{\tanh\left(\frac{\sqrt{1-\delta_0^2}\gamma_0\omega^2 t_1}{8}\right)}{\sqrt{1-\delta_0^2}+\tanh\left(\frac{\sqrt{1-\delta_0^2}\gamma_0\omega^2 t_1}{8}\right)}$$

Pulse stabilization for a particular class

If $\underline{\delta_m = 1}$, then the signal $\left(p_{ref(TR)}(t_1 + \varepsilon s)\right)_{t \in (-\infty, \infty)}$ converges in probability to $P_{ref(TR)}(s)$ as $\varepsilon \to 0$ where $P_{ref(TR)}$ is the **deterministic** pulse shape:

$$P_{ref(TR)}(s) = \left(f(- \cdot) * K_{TR}(\cdot)\right)(s)$$

$$\hat{K}_{TR}(\omega) = \int G_1(\tau) \mathcal{W}_1(0, \omega, \tau) d\tau$$

For large slab $L \to \infty$:

$$\mathcal{W}_1(0, \omega, \tau) = \frac{\delta_n \gamma_n \omega^2}{8} \frac{1 - \tanh^2\left(\frac{\sqrt{1-\delta_n^2}\gamma_n \omega^2 \tau}{8}\right)}{\left[1 + \frac{1}{\sqrt{1-\delta_n^2}} \tanh\left(\frac{\sqrt{1-\delta_n^2}\gamma_n \omega^2 \tau}{8}\right)\right]^2}$$

where δ_n is the correlation degree.

Sufficient condition for $\delta_m = 1$: the local velocity is not changed by the time-perturbation $m_i \equiv m_{ii}$.

True in particular for Goupillaud medium.

Conclusion

Statistical stability of the refocused pulse depends on the statistical properties of the reflection coefficient R.

Asymptotic framework $\varepsilon \longrightarrow 0$:
- Frequency decorrelation of R.
- Moments of R satisfy a system of transport equations.
- Representation in terms of a canonical jump Markov process on the nonnegative integers.

No statistical stability if the medium is changing (except in special cases).