Wave propagation and time reversal in the parabolic regime

Context: time-reversal experiments in ultrasound acoustics.

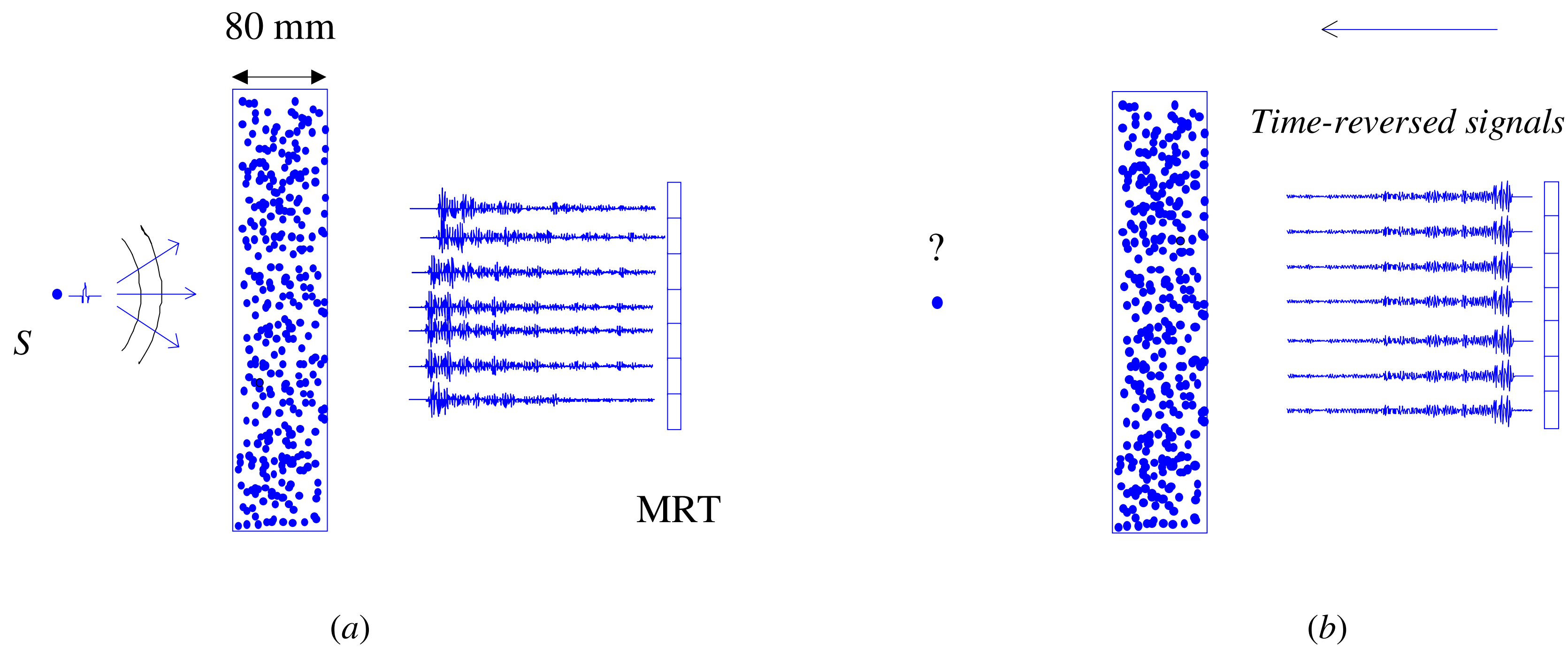
Experimental observations:
- robust spatial refocusing
- focal spot smaller in random medium than in homogeneous medium

Analysis of the mechanisms responsible for statistically stable time reversal.

The analysis is based on the semi-classical limit of the Schrödinger equation with a random potential.

Ultrasound experiment by M. Fink

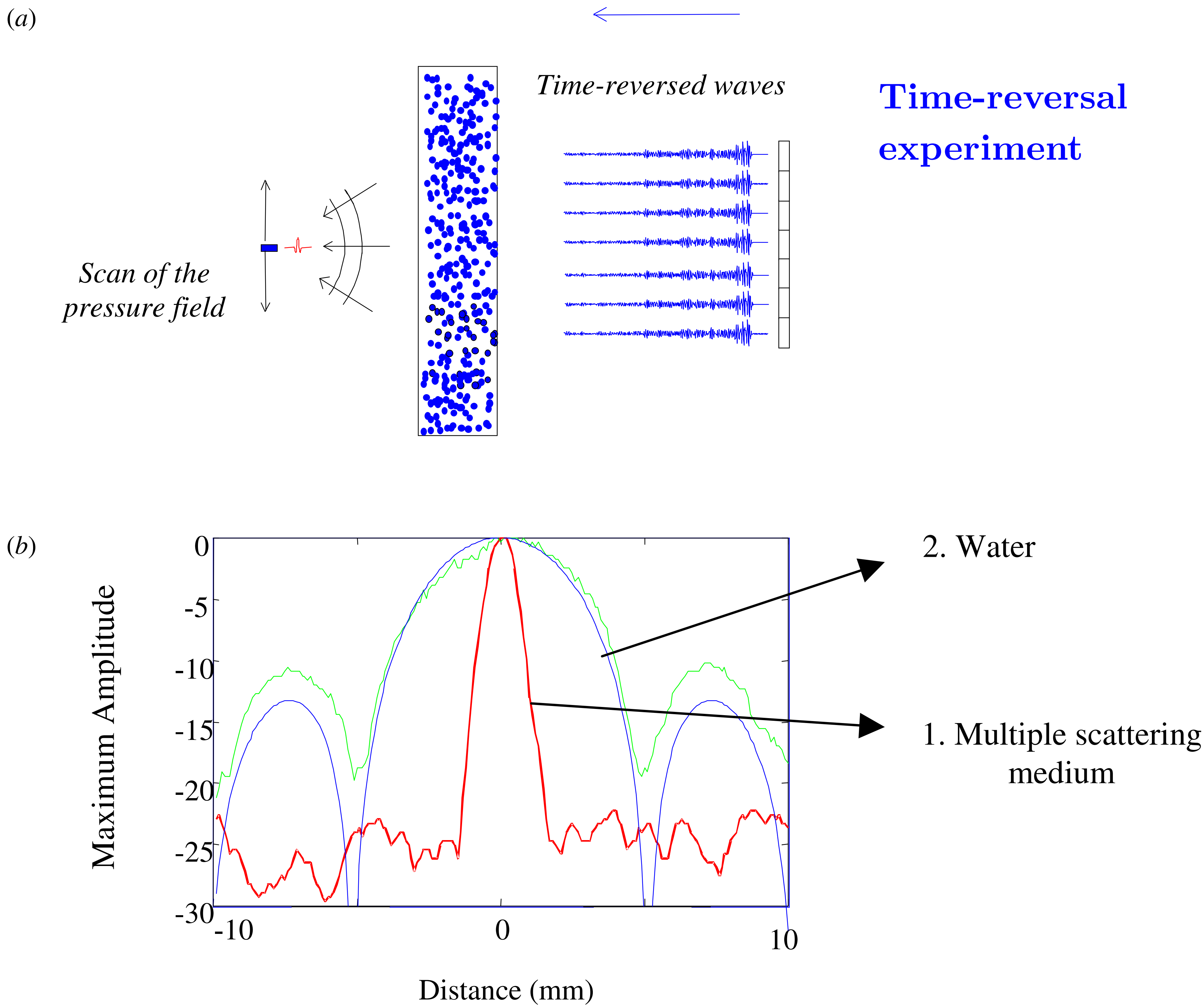
cf. A. Tourin, M. Fink, and A. Derode, Multiple scattering of sound, Waves Random Media **10** (2000), R31-R60.



Experimental set-up for a time-reversal experiment through a multiple-scattering medium:

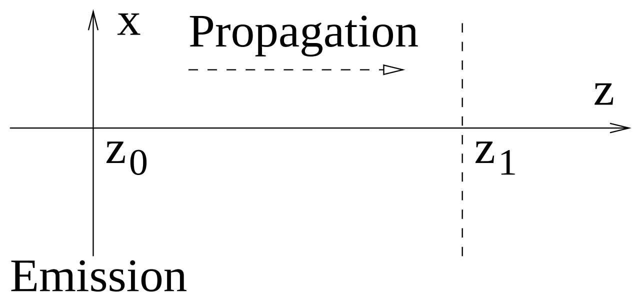
(a) first step, the source sends a pulse through the sample, the transmitted wave is recorded by the TRM.

(b) second step, the multiply scattered signals have been time-reverted, they are retransmitted by the TRM, and S records the reconstructed pressure field.



(a)
Time-reversed waves
Time-reversal experiment
Scan of the pressure field
(b)
Maximum Amplitude
0
-5
-10
-15
-20
-25
-30
-10
0
10
Distance (mm)
2. Water
1. Multiple scattering medium

Paraxial wave propagation



Propagation direction z. Initial condition located in the plane $z = z_0$.

Inhomogeneous medium with local celerity $c(r) = \dfrac{c_0}{n(r)}$, $r = (z, x)$.

The field $u(r, t)$, $t \in \mathbb{R}$, $r \in \mathbb{R}^3$, reads as:

$$u(r, t) = \int e^{-i\omega t} \hat{u}(r, \omega) d\omega$$

with $\hat{u}(r, \omega) = e^{ikz} \phi(r, \omega)$, where $k = \omega / c_0$ and ϕ is solution of:

$$2ik\phi_z + \Delta_\perp \phi + k^2 (n^2(r) - 1)\phi = 0, \qquad \phi(z = z_0, x, \omega) = \phi_0(x, \omega)$$

$\Delta_\perp$ =transverse Laplacian (w.r.t. x).

From now on we set $\mu = n^2 - 1$.

μ is a stationary, zero-mean, ergodic process.

Green function

Fundamental solution $G(z_0, z, x_0, x, \omega)$:

$$2ikG_z + \Delta_\perp G + k^2 \mu(z, x) G = 0, \qquad (k = \omega / c_0)$$

starting from $G(z_0, z = z_0, x_0, x, \omega) = \delta(x - x_0)$.

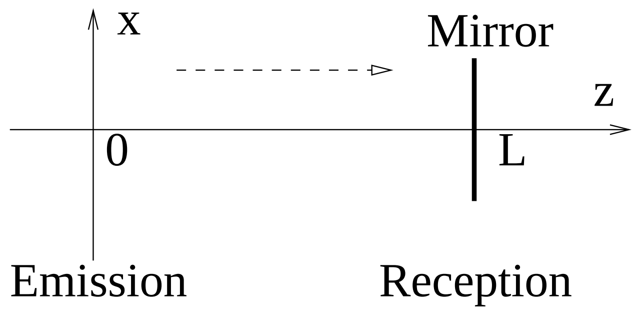
Solution ϕ starting from $\phi_0(x, \omega)$ at $z = 0$:

$$\phi(z, x, \omega) = \int G(0, z, x_0, x, \omega) \phi_0(x_0, \omega) dx_0$$

In homogeneous media $(\mu \equiv 0)$:

$$G(z_0, z, x_0, x, \omega) = \frac{\exp\left(\frac{ik|x - x_0|^2}{2|z - z_0|}\right)}{2i\pi \frac{|z - z_0|}{k}}$$

Time reversal setup



<u>1) Forward emission.</u>

We emit at $z = 0$: $u_0(x,t) = \int e^{-i\omega t} \hat{u}_0(x,\omega) d\omega$.

We receive and record at $z = L$, $y \in$mirror:

$$u_F(L,y,t) = \int e^{i\omega(\frac{L}{c_0}-t)} \phi_F(L,y,\omega) d\omega$$

where

$$\phi_F(L,y,\omega) = \int G(0,L,x,y,\omega) \hat{u}_0(x,\omega) dx$$

We shift the time origin:

$$\tilde{u}_F(L,y,t) := u_F(L,y,t-\frac{L}{c_0}) = \int e^{-i\omega t} \phi_F(L,y,\omega) d\omega$$

2) Time reversal.

We time-reverse the recorded signal $u_B(L,y,t) := \tilde{u}_F(L,y,-t)$:

$$u_B(L,y,t) = \int e^{i\omega t}\phi_F(L,y,\omega)d\omega \times \chi_M(y)$$

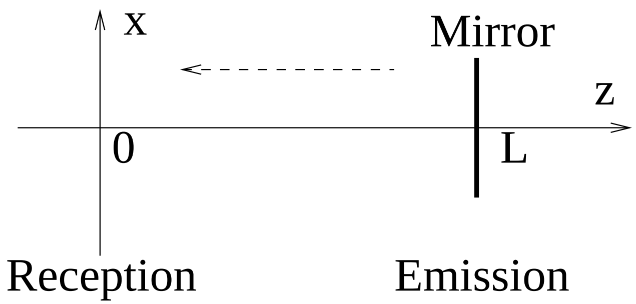
χ_M is the truncation function of the mirror.
If the mirror is square with side length d_M,
$\chi_M(y) = \mathbf{1}_{[-d_M/2,d_M/2]\times[-d_M/2,d_M/2]}(y)$.

Taking the complex conjugate (u_B is real-valued):

$$u_B(L,y,t) = \int e^{-i\omega t}\phi_B(L,y,\omega)d\omega$$

$$\begin{aligned} \text{with } \phi_B(L,y,\omega) &= \overline{\phi_F(L,y,\omega)}\chi_M(y) \\ &= \int \overline{G(0,L,x,y,\omega)\hat{u}_0(x,\omega)}dx\chi_M(y) \end{aligned}$$

We emit from the mirror $u_B(L,y,t)$.



3) Reception of the backward signal.

We receive at $z = 0$: $u_B(x,t) = \int e^{i\omega(\frac{L}{c_0} - t)} \phi_B(0,x,\omega) d\omega$

We shift the time origin: $\tilde{u}_B(x,t) := u_B(x, t - \frac{L}{c_0}) = \int e^{-i\omega t} \phi_B(0,x,\omega) d\omega$

where

$$
\begin{aligned}
\phi_B(0,x,\omega) &= \int G(L,0,y,x,\omega) \phi_B(L,y,\omega) dy \\
&= \int G(L,0,y,x,\omega) \overline{\phi_F(L,y,\omega)} \chi_M(y) dy \\
&= \int \int G(L,0,y,x,\omega) \overline{G(0,L,\eta,y,\omega) \hat{u}_0(\eta,\omega)} \chi_M(y) d\eta dy
\end{aligned}
$$

Furthermore $G(0,L,\eta,y,\omega) = G(L,0,y,\eta,\omega)$.

Finally, we receive the signal

$$\tilde{u}_B(x,t) = \int e^{-i\omega t} \phi_B(0,x,\omega) d\omega$$

with

$$\phi_B(0,\xi,\omega) = \int\int \Gamma(L,y,y,\eta,\xi,\omega) \overline{\hat{u}_0(\eta,\omega)} \chi_M(y) d\eta dy$$

where $\Gamma(L,x,y,\xi,\eta,\omega) := G(L,0,x,\xi,\omega)\overline{G(L,0,y,\eta,\omega)}$.

In homogeneous medium, Γ is explicitly known $\Rightarrow$ the double integral can be computed.

In random media: the statistical distribution of Γ is important.

Wigner of Γ

$$2ik\frac{\partial\Gamma}{\partial L} + (\Delta_x - \Delta_y)\Gamma + k^2(\mu(L,x) - \mu(L,y))\Gamma = 0$$

starting from $\Gamma(L=0,x,y,\xi,\eta,\omega) = \delta(x-\xi)\delta(y-\eta)$.

Change of variables: $X = (x+y)/2$ and $Y = y - x$.

$$2ik\frac{\partial\Gamma}{\partial L} - 2\nabla_X \cdot \nabla_Y \Gamma + k^2\left(\mu(L, X - \frac{Y}{2}) - \mu(L, X + \frac{Y}{2})\right)\Gamma = 0$$

We introduce:

$$W(L,X,P,\xi,\eta,\omega) = \frac{1}{(2\pi)^2}\int e^{iP.Y}\Gamma(L,X,Y,\xi,\eta,\omega)dY$$

Γ is obtained by inverse transform:

$$\Gamma(L,y,y,\xi,\eta,\omega) = \int W(L,y,P,\xi,\eta,\omega)dP$$

W is solution of

$$k\frac{\partial W}{\partial L} + P.\nabla_X W = \mathcal{L}W$$

$$\mathcal{L}W = \frac{ik^2}{2}\int e^{-iQ.X}\hat{\mu}(L,Q)\left[W(L,X,P+\frac{Q}{2})\right.$$
$$\left. - W(L,X,P-\frac{Q}{2})\right] dQ$$

starting from

$$W(L=0,X,P) = \frac{1}{(2\pi)^2}e^{-iP.(\xi-\eta)}\delta\left(X - \frac{\xi+\eta}{2}\right)$$

where $\hat{\mu}$ is the Fourier transform of μ w.r.t. X:

$$\hat{\mu}(L,Q) = \frac{1}{(2\pi)^2}\int e^{iQ.X}\mu(L,X)dX$$

Homogeneous medium $\mu \equiv 0$

Solution: $W(L,X,P) = \frac{1}{(2\pi)^2} e^{-iP.(\xi-\eta)} \delta\left(X - \frac{LP}{k} - \frac{\xi+\eta}{2}\right)$.

Thus $\Gamma(L,y,y,\xi,\eta,k) = \frac{k^2}{(2\pi L)^2} e^{-i\frac{k}{L}(y-\frac{\xi+\eta}{2}).(\xi-\eta)}$.

If $u_0(t,x) = f(t)\delta_0(x)$, then $\phi_B(x,\omega) = \overline{\hat{f}(\omega)} e^{i\frac{\omega x^2}{2Lc_0}} \hat{\chi}_M\left(-\frac{\omega x}{c_0 L}\right)$.

In case of a square mirror:

$$\phi_B(x,\omega) = \overline{\hat{f}(\omega)} e^{i\frac{\omega x^2}{2Lc_0}} \operatorname{sinc}\left(\frac{x_1}{r_c}\right) \operatorname{sinc}\left(\frac{x_2}{r_c}\right)$$

where $r_c = \frac{2\pi L c_0}{\omega d_M}$.

In the time domain:

if $u_0(x,t) = \cos(\omega_0 t) v(t) \delta_0(x)$ with bandwidth$(v) \ll \omega_0$, then:

$$u_B(x,t) = \cos(\omega_0 t) v\left(-t - \frac{x^2}{2Lc_0}\right) \operatorname{sinc}\left(\frac{x_1}{r_c}\right) \operatorname{sinc}\left(\frac{x_2}{r_c}\right)$$

where $r_c = \frac{2\pi L c_0}{\omega_0 d_M} = \frac{\lambda_0 L}{d_M}$.

Standard results on radiative transport

$$i\frac{\partial \phi}{\partial t} + \frac{1}{2}\Delta\phi - V(x)\phi = 0, \quad x \in \mathbb{R}^d,\ t > 0$$

Usually $d = 3$. Start from $\phi(t = 0, x) = \phi_0(x)$ (smooth).

Wigner: $W(x,p) = \frac{1}{(2\pi)^d}\int e^{ip.y}\phi(x - \frac{1}{2}y)\overline{\phi(x + \frac{1}{2}y)}dy$

Property: $\int W(x,p)dp = |\phi(x)|^2$

W is solution of:

$$\frac{\partial W}{\partial t} + p.\nabla_x W + \mathcal{L}_x W = 0$$

$$\mathcal{L}_x W = i\int e^{-iq.x}\hat{V}(q)\left(W(x, p + \frac{q}{2}) - W(x, p - \frac{q}{2})\right)dq$$

Re-scale: $V \to \sqrt{\varepsilon} V$, $t \to t/\varepsilon$ and $x \to x/\varepsilon$

$$i\varepsilon \frac{\partial \phi^\varepsilon}{\partial t} + \frac{\varepsilon^2}{2} \Delta \phi^\varepsilon - \sqrt{\varepsilon} V(\frac{x}{\varepsilon}) \phi^\varepsilon = 0, \quad \phi^\varepsilon(t,x) = \phi(\frac{t}{\varepsilon}, \frac{x}{\varepsilon})$$

Wigner: $W^\varepsilon(x,p) = \dfrac{1}{(2\pi)^d} \displaystyle\int e^{ip.y} \phi^\varepsilon(x - \frac{\varepsilon}{2} y) \overline{\phi^\varepsilon(x + \frac{\varepsilon}{2} y)} dy$

W^ε is solution of:

$$\frac{\partial W^\varepsilon}{\partial t} + p.\nabla_x W^\varepsilon + \frac{1}{\sqrt{\varepsilon}} \mathcal{L}_{\frac{x}{\varepsilon}} W^\varepsilon = 0$$

Result (Erdös-Yau;...): $\mathbb{E}\left[W^\varepsilon\right] \to \bar{W}$ solution of

$$\frac{\partial \bar{W}}{\partial t} + p.\nabla_x \bar{W} = \bar{\mathcal{L}} \bar{W}$$

$$\bar{\mathcal{L}} \bar{W} = 4\pi \int \hat{R}(q-p) \delta(q^2 - p^2) \left(\bar{W}(x,q) - \bar{W}(x,p) \right) dq$$

where $\hat{R}(q) = \dfrac{1}{(2\pi)^d} \displaystyle\int \mathbb{E}\left[V(x)V(x+y)\right] e^{iq.y} dy$

Back to W = Wigner of Γ

Weakly perturbed medium $\mu \to \sqrt{\varepsilon}\mu$.

Re-scale $L \to L/\varepsilon$ and $x \to x/\varepsilon$

W^ε is solution of

$$k\frac{\partial W^\varepsilon}{\partial L} + P.\nabla_X W^\varepsilon = \mathcal{L}^\varepsilon W^\varepsilon$$

$$\mathcal{L}^\varepsilon W^\varepsilon = \frac{ik^2}{2\sqrt{\varepsilon}} \int e^{-iQ.\frac{X}{\varepsilon}} \hat{\mu}(\frac{L}{\varepsilon}, Q) \left[W^\varepsilon(L, X, P + \frac{Q}{2}) - W^\varepsilon(L, X, P - \frac{Q}{2}) \right] dQ$$

Comparison with the "standard" configuration:

- L plays the role of t,
- 1+2D instead of 1+3D ,
- μ depends explicitly on L.

Result: $\mathbb{E}\left[W^\varepsilon\right] \to \bar{W}(L,X,P)$ solution of

$$k\frac{\partial \bar{W}}{\partial L} + P.\nabla_X \bar{W} = \bar{\mathcal{L}}\bar{W}$$

$$\bar{\mathcal{L}}\bar{W} = \frac{\pi k^3}{4}\int \hat{R}\Big(\frac{P^2-Q^2}{2k}, P-Q\Big)\big(\bar{W}(Q)-\bar{W}(P)\big)\,dQ$$

where $\hat{R}(\kappa,Q) = \frac{1}{(2\pi)^3}\int R(z,X)e^{iQ.X+i\kappa z}dX\,dz,$

$R(z,X) = \mathbb{E}\left[\mu(0,0)\mu(z,X)\right].$

Note: $\hat{R}$ is nonnegative-valued.

The proof is "easy" because of the explicit mixing w.r.t. L.

G. Bal, G. Papanicolaou and L. Ryzhik, Radiative transport limit for the random Schrodinger equation, Nonlinearity, 15, 2002, 513-529.

F. Poupaud and A. Vasseur, Classical and quantum transport in random media, Jour. Math. Pure et Appl., 82, 2003, 711-748.

Simplification of the effective transport equation

Approximation: $R(z,X) = m(z/z_c)r(X/l_c)$ with $l_c \gg z_c$. Then $\hat{R}(\kappa,Q) = z_c\hat{m}(\kappa z_c)l_c^2\hat{r}(Ql_c) \simeq \sigma^2 l_c^2 \hat{r}(Ql_c)$ where $\sigma^2 = z_c\hat{m}(0)$.

$$\int \hat{R}\Big(\frac{P^2-Q^2}{2k}, P-Q\Big)\big(\bar{W}(Q)-\bar{W}(P)\big)\,dQ$$

$$= \int \sigma^2 l_c^2 \hat{r}(l_c(P-Q))\big(\bar{W}(Q)-\bar{W}(P)\big)\,dQ$$

$$= \sigma^2 \int \hat{r}(Q')\big(\bar{W}(P+Q'/l_c)-\bar{W}(P)\big)\,dQ'$$

$$= \sigma^2 \int \hat{r}(Q')\Big(\nabla\bar{W}(P).Q'/l_c + \frac{1}{2}Q'.\nabla\nabla\bar{W}(P)Q'/l_c^2 + ...\Big)\,dQ'$$

The first term is zero ($\hat{r}$ is even). The second term is:

$$\int \hat{R}\Big(\frac{P^2-Q^2}{2k}, P-Q\Big)\big(\bar{W}(Q)-\bar{W}(P)\big)\,dQ = \frac{1}{2}D\Delta_P\bar{W}$$

with $D = \sigma^2 l_c^{-2}\int Q'^2\hat{r}(Q')dQ' = -\int \Delta_X R(z,0)dz \geq 0$

Resolution of the effective transport equation

$$\frac{\partial \bar{W}}{\partial L} = -\frac{P}{k}.\nabla_X \bar{W} + \frac{\pi k^2 D}{8}\Delta_P \bar{W}$$

starting from $W(L=0,X,P) = \frac{e^{-iP.(\xi-\eta)}}{(2\pi)^2}\delta\left(X - \frac{\xi+\eta}{2}\right)$

Probabilistic interpretation:

$$\begin{cases} dP(L) = \frac{\sqrt{\pi k^2 D}}{2} dB_L, & P(0) = P \\ dX(L) = -\frac{P(L)}{k} dL, & X(0) = X \end{cases}$$

and then $W(L,X,P) = \mathbb{E}\left[\frac{e^{-iP(L).(\xi-\eta)}}{(2\pi)^2}\delta\left(X(L) - \frac{\xi+\eta}{2}\right)\right]$

Thus $\Gamma(L,y,y,\xi,\eta) = \underbrace{\frac{k^2}{(2\pi L)^2} e^{-i\frac{k}{L}(y-\frac{\xi+\eta}{2}).(\xi-\eta)}}_{\text{homogeneous case}} e^{-\frac{\pi k^2 DL}{4}(\xi-\eta)^2}.$

With: $u_0(x,t) = \delta_0(x) f(t)$, $\hat{u}_0(\omega, x) = \delta_0(x)\hat{f}(\omega)$,

$$\mathbb{E}\left[\phi_B(x,\omega)\right] = \overline{\hat{f}(\omega)} e^{i\frac{\omega x^2}{2Lc_0}} \operatorname{sinc}\left(\frac{x_1}{r_c}\right) \operatorname{sinc}\left(\frac{x_2}{r_c}\right) \exp\left(-\frac{x^2}{r_a^2}\right)$$

where $r_c = \dfrac{2\pi L c_0}{\omega d_M}$ and $r_a = \dfrac{2c_0}{\omega\sqrt{\pi D L}}$.

In the time domain:

$u_0(x,t) = \cos(\omega_0 t) v(t) \delta_0(x)$ with bandwidth$(v) \ll \omega_0$:

$$\mathbb{E}\left[u_B(x,t)\right] = \cos(\omega_0 t) v\left(-t - \frac{x^2}{2Lc_0}\right) \operatorname{sinc}\left(\frac{x_1}{r_c}\right) \operatorname{sinc}\left(\frac{x_2}{r_c}\right) \boxed{\exp\left(-\frac{x^2}{r_a^2}\right)}$$

where $r_c = \dfrac{2\pi L c_0}{\omega_0 d_M} = \dfrac{\lambda_0 L}{d_M}$ and $r_a = \dfrac{2c_0}{\omega_0\sqrt{\pi D L}}$.

If $r_c < r_a$: Same result as in homogeneous medium

If $r_c > r_a$: $\mathbb{E}\left[u_B\right]$ spot thiner than in homogeneous medium !

But: the result holds true in average (averaging over all possible realizations of the medium).

Frequency correlation of W

We have shown that ($\varepsilon \to 0$):

$$\mathbb{E}\left[\phi_B(x,\omega)\right] = \int\int\int \bar{W}(L,y,P,x,\eta,\omega)\overline{\hat{u}_0(\eta,\omega)}\chi_M(y)dyd\eta dP$$

where $\bar{W} = \lim_{\varepsilon\to 0}\mathbb{E}\left[W^\varepsilon\right]$.

The frequency autocorrelation function of W^ε:

$$\mathbb{E}\left[W^\varepsilon(L,...,\omega+\varepsilon h/2)W^\varepsilon(L,...,\omega-\varepsilon h/2)\right] \stackrel{\varepsilon\to 0}{\longrightarrow} \gamma(L,...,\omega,h)$$

with $\gamma(L,...,\omega,h) \stackrel{h\to\infty}{\longrightarrow} 0$.

Thus

$$\mathbb{E}\left[W^\varepsilon(L,...,\omega_1)W^\varepsilon(L,...,\omega_2)\right] \simeq \bar{W}(L,...,\omega_1)\bar{W}(L,...,\omega_2)$$

as soon as $|\omega_1-\omega_2| \gg \varepsilon$.

Similarly

$$\mathbb{E}\left[\phi_B(x,\omega_1)\phi_B(x,\omega_2)\right] \simeq \mathbb{E}\left[\phi_B(x,\omega_1)\right]\mathbb{E}\left[\phi_B(x,\omega_2)\right]$$

as soon as $|\omega_1-\omega_2| \gg \varepsilon$.

Self-averaging in time

In the time domain: $\mathbb{E}\left[u_B(x,t)\right] = \int e^{-i\omega t}\mathbb{E}\left[\phi_B(x,\omega)\right] d\omega$,

$$\mathbb{E}\left[u_B(x,t)^2\right] = \mathbb{E}\left[\left(\int e^{-i\omega t}\phi_B(x,\omega)d\omega\right)^2\right]$$

$$\begin{aligned}
\mathbb{E}\left[u_B(x,t)^2\right] &= \int\int e^{-i(\omega_1+\omega_2)t}\mathbb{E}\left[\phi_B(x,\omega_1)\phi_B(x,\omega_2)\right] d\omega_1 d\omega_2 \\
&\overset{\varepsilon\to 0}{\simeq} \int\int e^{-i(\omega_1+\omega_2)t}\mathbb{E}\left[\phi_B(x,\omega_1)\right]\mathbb{E}\left[\phi_B(x,\omega_2)\right] d\omega_1 d\omega_2 \\
&= \mathbb{E}\left[u_B(x,t)\right]^2
\end{aligned}$$

Thus $\mathrm{Var}(u_B(x,t)) := \mathbb{E}\left[\left(u_B(x,t) - \mathbb{E}\left[u_B(x,t)\right]\right)^2\right] \overset{\varepsilon\to 0}{\longrightarrow} 0$.

This implies that, for any $\delta > 0$:

$$\mathbb{P}\Big(\left|u_B(x,t) - \mathbb{E}\left[u_B(x,t)\right]\right| > \delta\Big) \leq \frac{\mathrm{Var}\left(u_B(x,t)\right)}{\delta^2} \overset{\varepsilon\to 0}{\longrightarrow} 0$$

The frequency decorrelation implies the self-averaging in time.

Conclusion

Spatial refocusing enhanced by randomness.

Statistical stability of the refocused focal spot ensured by the frequency decorrelation of W.