Pulse propagation and time reversal in random waveguides

Context: time-reversal experiments in underwater acoustics.

Experimental observations:
- robust spatial refocusing
- diffraction-limited focal spot

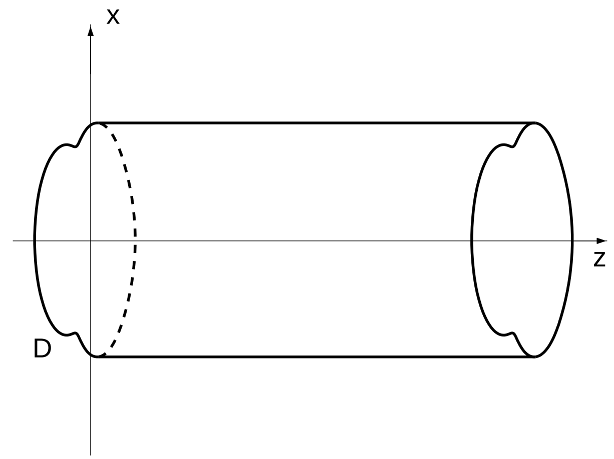
[1] W. A. Kuperman, W. S. Hodgkiss, H. C. Song, T. Akal, C. Ferla, and D. R. Jackson, Phase conjugation in the ocean, experimental demonstration of an acoustic time-reversal mirror, J. Acoust. Soc. Am. **103** (1998), 25-40.
[2] H. C. Song, W. A. Kuperman, and W. S. Hodgkiss, Iterative time reversal in the ocean, J. Acoust. Soc. Am. **105** (1999), 3176-3184.

Analysis of the mechanisms responsible for statistically stable time reversal.

Perfect acoustic waveguide

waveguide cross-section

$\mathcal{D} \subset \mathbb{R}^2$



$$\bar{\rho}\frac{\partial \mathbf{u}}{\partial t} + \nabla p = \mathbf{F}\,, \qquad \frac{1}{\bar{K}}\frac{\partial p}{\partial t} + \nabla \cdot \mathbf{u} = 0\,, \text{ for } \mathbf{x} \in \mathcal{D} \text{ and } z \in \mathbb{R}\,.$$

p is the acoustic pressure, $\mathbf{u}$ is the acoustic velocity.

$\bar{\rho}$ is the density of the medium, $\bar{K}$ is the bulk modulus.

The source is modeled by the forcing term $\mathbf{F}(t, \mathbf{r})$.

Wave equation with the sound speed $\bar{c} = \sqrt{\bar{K}/\bar{\rho}}$:

$$\Delta p - \frac{1}{\bar{c}^2}\frac{\partial^2 p}{\partial t^2} = \nabla.\mathbf{F} \qquad \text{for } \mathbf{x} \in \mathcal{D} \text{ and } z \in \mathbb{R}\,.$$

Dirichlet boundary conditions

$$p(t, \mathbf{x}, z) = 0 \qquad \text{for } \mathbf{x} \in \partial\mathcal{D} \text{ and } z \in \mathbb{R}\,.$$

Time harmonic wave equation $k = \omega/\bar{c}$

$$\partial_z^2 \hat{p}(\omega, \mathbf{x}, z) + \Delta_\perp \hat{p}(\omega, \mathbf{x}, z) + k^2(\omega)\hat{p}(\omega, \mathbf{x}, z) = 0$$

Spectrum of $\Delta_\perp$ with Dirichlet BC = infinite number of discrete eigenvalues

$$-\Delta_\perp \phi_j(\mathbf{x}) = \lambda_j \phi_j(\mathbf{x}),\ \mathbf{x} \in \mathcal{D}, \qquad \phi_j(\mathbf{x}) = 0,\ \mathbf{x} \in \partial\mathcal{D}\,,\ \text{for } j = 1, 2, \ldots$$

Number of propagating modes $N(\omega)$:

$$\lambda_{N(\omega)} \leq k(\omega) < \lambda_{N(\omega)+1}\,,$$

Propagating modes $1 \leq j \leq N(\omega)$:

$$\hat{p}_j(\omega, \mathbf{x}, z) = \phi_j(\mathbf{x}) e^{\pm i\beta_j(\omega) z}, \qquad \beta_j(\omega) = \sqrt{k^2(\omega) - \lambda_j}\,.$$

Evanescent modes $j > N(\omega)$:

$$\hat{q}_j(\omega, \mathbf{x}, z) = \phi_j(\mathbf{x}) e^{\pm \beta_j(\omega) z}, \qquad \beta_j(\omega) = \sqrt{\lambda_j - k^2(\omega)}\,.$$

Excitation Conditions for a Source

Source localized in the plane $z = 0$:

$$\mathbf{F}(t, \mathbf{x}, z) = f(t)\delta(\mathbf{x} - \mathbf{x}_0)\delta(z)\mathbf{e}_z \, .$$

$$\hat{p}(\omega, \mathbf{x}, z) = \left[\sum_{j=1}^{N} \frac{\hat{a}_j(\omega)}{\sqrt{\beta_j(\omega)}} e^{i\beta_j z} \phi_j(\mathbf{x}) + \sum_{j=N+1}^{\infty} \frac{\hat{c}_j(\omega)}{\sqrt{\beta_j(\omega)}} e^{-\beta_j z} \phi_j(\mathbf{x}) \right] \mathbf{1}_{(0,\infty)}(z)$$
$$+ \left[\sum_{j=1}^{N} \frac{\hat{b}_j(\omega)}{\sqrt{\beta_j(\omega)}} e^{-i\beta_j z} \phi_j(\mathbf{x}) + \sum_{j=N+1}^{\infty} \frac{\hat{d}_j(\omega)}{\sqrt{\beta_j(\omega)}} e^{\beta_j z} \phi_j(\mathbf{x}) \right] \mathbf{1}_{(-\infty,0)}(z) \, ,$$

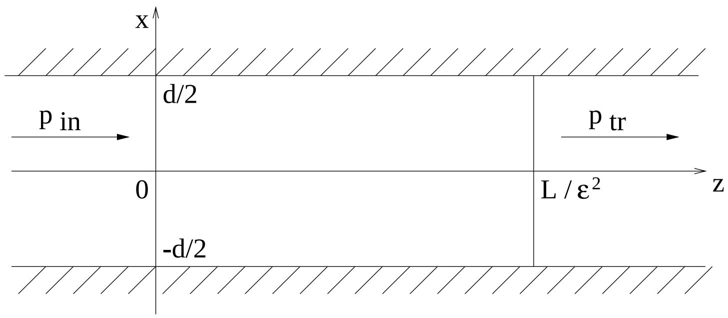
with

$$\hat{a}_j(\omega) = -\hat{b}_j(\omega) = \frac{\sqrt{\beta_j(\omega)}}{2} \hat{f}(\omega)\phi_j(\mathbf{x}_0) \, ,$$
$$\hat{c}_j(\omega) = -\hat{d}_j(\omega) = -\frac{\sqrt{\beta_j(\omega)}}{2} \hat{f}(\omega)\phi_j(\mathbf{x}_0) \, .$$

For $k(\omega)z \gg 1$:

$$\hat{p}(\omega, \mathbf{x}, z) = \sum_{j=1}^{N(\omega)} \frac{\hat{a}_j(\omega)}{\sqrt{\beta_j(\omega)}} \phi_j(\mathbf{x}) e^{i\beta_j(\omega) z}$$

Perturbed waveguide: Time harmonic approach



$$\rho(\mathbf{r})\frac{\partial \mathbf{u}}{\partial t} + \nabla p = \mathbf{F}\,,$$
$$\frac{1}{K(\mathbf{r})}\frac{\partial p}{\partial t} + \nabla\cdot\mathbf{u} = 0\,,$$

$$\frac{1}{K(\mathbf{x},z)} = \begin{cases} \frac{1}{\bar{\bar{K}}}\left(1+\varepsilon\nu(\mathbf{x},z)\right) & \text{for} \quad \mathbf{x}\in\mathcal{D}, \quad z\in[0,L/\varepsilon^2] \\ \frac{1}{\bar{\bar{K}}} & \text{for} \quad \mathbf{x}\in\mathcal{D}, \quad z\in(-\infty,0)\cup(L/\varepsilon^2,\infty) \end{cases}$$
$$\rho(\mathbf{x},z) = \bar{\rho} \quad \text{for} \quad \mathbf{x}\in\mathcal{D}, \quad z\in(-\infty,\infty)$$

Perturbed wave equation with Dirichlet boundary conditions:

$$\Delta\hat{p}(\omega,\mathbf{x},z) + k^2(1+\varepsilon\nu(\mathbf{x},z))\hat{p}(\omega,\mathbf{x},z) = 0\,.$$

Wave mode expansions:

$$\hat{p}(\mathbf{x},z) = \sum_{j=1}^{N}\phi_j(\mathbf{x})\hat{p}_j(z) + \sum_{j=N+1}^{\infty}\phi_j(\mathbf{x})\hat{q}_j(z)$$

Right-going and left-going mode amplitudes $\hat{a}_j(z)$ and $\hat{b}_j(z)$:

$$\hat{p}_j = \frac{1}{\sqrt{\beta_j}}\left(\hat{a}_j e^{i\beta_j z} + \hat{b}_j e^{-i\beta_j z}\right)\,, \qquad \frac{d\hat{p}_j}{dz} = i\sqrt{\beta_j}\left(\hat{a}_j e^{i\beta_j z} - \hat{b}_j e^{-i\beta_j z}\right)\,, \qquad j\le N$$

Coupled mode equations

Neglect evanescent modes.

Coupled mode equations for $j \leq N$:

$$\frac{d\hat{a}_j}{dz} = \frac{i\varepsilon k^2}{2} \sum_{1\leq l\leq N} \frac{C_{jl}(z)}{\sqrt{\beta_j \beta_l}} \left(\hat{a}_l e^{i(\beta_l - \beta_j)z} + \hat{b}_l e^{-i(\beta_l + \beta_j)z} \right)$$

$$\frac{d\hat{b}_j}{dz} = -\frac{i\varepsilon k^2}{2} \sum_{1\leq l\leq N} \frac{C_{jl}(z)}{\sqrt{\beta_j \beta_l}} \left(\hat{a}_l e^{i(\beta_l + \beta_j)z} + \hat{b}_l e^{i(\beta_j - \beta_l)z} \right)$$

where

$$C_{jl}(z) = \int_{\mathcal{D}} \phi_j(\mathbf{x}) \phi_l(\mathbf{x}) \nu(\mathbf{x}, z) d\mathbf{x}$$

Boundary conditions:

$$\hat{a}_j(0) = \hat{a}_{j,0}, \qquad \hat{b}_j(\frac{L}{\varepsilon^2}) = 0$$

Rescaling:

$$\hat{a}_j^\varepsilon(z) = \hat{a}_j(\frac{z}{\varepsilon^2}), \qquad \hat{b}_j^\varepsilon(z) = \hat{b}_j(\frac{z}{\varepsilon^2})$$

$\hookrightarrow$ Diffusion approximation theorem.

The forward scattering approximation

Diffusion-approximation $\Longrightarrow$ multi-dimensional diffusion process.

Coupling coefficients between left and right-going modes:

$$\int_0^\infty \mathbb{E}[C_{jl}(0)C_{jl}(z)] \cos\left((\beta_j(\omega)+\beta_l(\omega))z\right) dz, \qquad j,l = 1,\cdots,N(\omega)\,.$$

Coupling coefficients between right-going modes:

$$\int_0^\infty \mathbb{E}[C_{jl}(0)C_{jl}(z)] \cos\left((\beta_j(\omega)-\beta_l(\omega))z\right) dz, \qquad j,l = 1,\cdots,N(\omega)\,.$$

We can neglect the left-going (backward) propagating modes if the first type of coefficients are negligible compared to the second ones.

$\longrightarrow$ reduced system:

$$\begin{aligned}
\frac{d\hat{a}^\varepsilon}{dz} &= \frac{1}{\varepsilon} M\left(\frac{z}{\varepsilon^2}\right)\hat{a}^\varepsilon(z) \\
M_{jl}(z) &= \frac{ik^2}{2\sqrt{\beta_j\beta_l}} C_{jl}(z) e^{i(\beta_l-\beta_j)z}
\end{aligned}$$

The mode amplitudes $(\hat{a}_j^{\varepsilon}(\omega, z))_{j=1,\cdots,N}$ converge in distribution as $\varepsilon \to 0$ to a diffusion process $(\hat{a}_j(\omega, z))_{j=1,\cdots,N}$ whose infinitesimal generator is

$$\begin{aligned}
\mathcal{L} &= \frac{1}{4}\sum_{j\neq l}\Gamma_{jl}^{(c)}(\omega)\left(A_{jl}\overline{A_{jl}} + \overline{A_{jl}}A_{jl}\right) + \frac{1}{2}\sum_{j,l}\Gamma_{jl}^{(1)}(\omega)A_{jj}\overline{A_{ll}} \\
&\quad + \frac{i}{4}\sum_{j\neq l}\Gamma_{jl}^{(s)}(\omega)(A_{ll} - A_{jj}),\\
A_{jl} &= \hat{a}_j\frac{\partial}{\partial \hat{a}_l} - \overline{\hat{a}_l}\frac{\partial}{\partial \overline{\hat{a}_j}} = -\overline{A_{lj}}.
\end{aligned}$$

$$\begin{aligned}
\Gamma_{jl}^{(c)}(\omega) &= \frac{\omega^4}{2\bar{c}^4\beta_j(\omega)\beta_l(\omega)}\int_0^{\infty}\cos\left((\beta_j(\omega) - \beta_l(\omega))z\right)\mathbb{E}[C_{jl}(0)C_{jl}(z)]dz \text{ if } j \neq l,\\
\Gamma_{jj}^{(c)}(\omega) &= -\sum_{n\neq j}\Gamma_{jn}^{(c)}(\omega),
\end{aligned}$$

Mean mode amplitudes

The expected values of the mode amplitudes $\mathbb{E}[\hat{a}_j^\varepsilon(\omega, z)]$ converge as $\varepsilon \to 0$ to $\mathbb{E}[\hat{a}_j(\omega, z)]$ given by

$$\mathbb{E}[\hat{a}_j(\omega, z)] = \exp\left(q_j(\omega) z\right) \hat{a}_{j0}(\omega)$$

$$\mathrm{Re}(q_j(\omega)) < 0$$

$\hookrightarrow$ exponential damping of the mean amplitudes.

Mean mode powers

The mode powers $(|\hat{a}_j^\varepsilon(\omega,z)|^2)_{j=1,\cdots,N}$ converge in distribution as $\varepsilon \to$ to $(P_j(\omega,z))_{j=1,\cdots,N}$ whose infinitesimal generator is

$$\mathcal{L}_P = \sum_{j \neq l} \Gamma_{jl}^{(c)}(\omega) \left[P_l P_j \left(\frac{\partial}{\partial P_j} - \frac{\partial}{\partial P_l} \right) \frac{\partial}{\partial P_j} + (P_l - P_j) \frac{\partial}{\partial P_j} \right]$$

$\hookrightarrow$ diffusion on $\mathcal{H}_N = \left\{ (P_j)_{j=1,\cdots,N},\, P_j \geq 0,\, \sum_{j=1}^N P_j = R_0^2 \right\}$

where $R_0^2 = \sum_{j=1}^{N} |\hat{a}_{j,0}|^2$

The mean mode powers $\mathbb{E}[|\hat{a}_j^\varepsilon(\omega,z)|^2]$ converge to $P_j^{(1)}(\omega,z)$

$$\frac{dP_j^{(1)}}{dz} = \sum_{n \neq j} \Gamma_{jn}^{(c)}(\omega) \left(P_n^{(1)} - P_j^{(1)} \right)$$

starting from $P_j^{(1)}(\omega, z=0) = |\hat{a}_{j,0}|^2$, $j = 1, \cdots, N$.

This shows the asymptotic *equipartition* of mode energy:

$$\sup_{j=1,\cdots,N} \left| P_j^{(1)}(\omega,z) - \frac{1}{N} R_0^2 \right| \leq C \exp\left(-\frac{z}{L_{\text{equip}}} \right)$$

where $\frac{1}{L_{\text{equip}}}$ = second eigenvalue of $\Gamma^{(c)}$.

Fluctuations theory

$$P^{(2)}_{jl}(\omega,z) = \lim_{\varepsilon\to 0} \mathbb{E}\left[|\hat{a}^{\varepsilon}_j(\omega,z)|^2 |\hat{a}^{\varepsilon}_l(\omega,z)|^2\right] = \mathbb{E}[P_j(\omega,z)P_l(\omega,z)]\,.$$

Using the generator $\mathcal{L}_P$ we get a system of ordinary differential equations for limit forth moments $(P^{(2)}_{jl})_{j,l=1,\cdots,N}$ which has the form

$$\frac{dP^{(2)}_{jj}}{dz} = \sum_{n\neq j} \Gamma^{(c)}_{jn}\left(4P^{(2)}_{jn} - 2P^{(2)}_{jj}\right)\,,$$

$$\frac{dP^{(2)}_{jl}}{dz} = -2\Gamma^{(c)}_{jl}P^{(2)}_{jl} + \sum_{n}\Gamma^{(c)}_{ln}\left(P^{(2)}_{jn} - P^{(2)}_{jl}\right) + \sum_{n}\Gamma^{(c)}_{jn}\left(P^{(2)}_{ln} - P^{(2)}_{jl}\right), \qquad j\neq l\,.$$

The normalized correlation

$$\mathrm{Cor}(P_j,P_l)(z) := \frac{P^{(2)}_{jl}(z) - P^{(1)}_j(z)P^{(1)}_l(z)}{P^{(1)}_j(z)P^{(1)}_l(z)}\,,$$

has the following asymptotic form

$$\mathrm{Cor}(P_j,P_l)(z) \stackrel{z\to\infty}{\longrightarrow} \begin{cases} -\dfrac{1}{N+1} & \text{if } j\neq l\,, \\ \dfrac{N-1}{N+1} & \text{if } j = l\,. \end{cases}$$

When N is large: P_j uncorrelated, with exponential distribution.

Narrowband pulse propagation in a random waveguide

Point-like narrowband source term

$$\mathbf{F}^\varepsilon(t,\mathbf{x},z) = f^\varepsilon(t)\delta(\mathbf{x}-\mathbf{x}_0)\delta(z)\mathbf{e}_z$$
$$f^\varepsilon(t) = f(\varepsilon^2 t)e^{i\omega_0 t}, \qquad \hat{f}^\varepsilon(\omega) = \frac{1}{\varepsilon^2}\hat{f}\left(\frac{\omega-\omega_0}{\varepsilon^2}\right)$$

Transmitted pulse:

$$p_{tr}^\varepsilon(t,\mathbf{x},L) = p_{tr}\left(\frac{t}{\varepsilon^2},\mathbf{x},\frac{L}{\varepsilon^2}\right)$$

$$p_{tr}^\varepsilon(t,\mathbf{x},L) = \frac{1}{4\pi\varepsilon^2}\int\sum_{j,l=1}^N \frac{\sqrt{\beta_l}}{\sqrt{\beta_j}}\phi_j(\mathbf{x})\phi_l(\mathbf{x}_0)\hat{f}\left(\frac{\omega-\omega_0}{\varepsilon^2}\right)T_{jl}^\varepsilon(\omega)e^{i\frac{\beta_j(\omega)L-\omega t}{\varepsilon^2}}d\omega$$

where $T_{jl}^\varepsilon = \hat{a}_j^\varepsilon(L)$ when $\hat{a}_j^\varepsilon(0) = \delta_{jl}$ (T^ε: transfer matrix).

$$\begin{aligned} p_{tr}^\varepsilon(t,\mathbf{x},L) &= \frac{1}{4\pi}\int\sum_{j,l=1}^N \frac{\sqrt{\beta_l}}{\sqrt{\beta_j}}\phi_j(\mathbf{x})\phi_l(\mathbf{x}_0) \\ &\quad \times\hat{f}(h)T_{jl}^\varepsilon(\omega_0+\varepsilon^2 h)e^{i\frac{\beta_j(\omega_0)L-\omega_0 t}{\varepsilon^2}}e^{i[\beta_j'(\omega_0)L-t]h}dh \end{aligned}$$

Homogeneous waveguide

In a homogeneous waveguide we have $T^\varepsilon_{jl} = \delta_{jl}$ and

$$p^\varepsilon_{tr}(t, \mathbf{x}, L) = \frac{1}{2} \sum_{j=1}^{N} \phi_j(\mathbf{x}) \phi_j(\mathbf{x}_0) e^{i \frac{\beta_j(\omega_0) L - \omega_0 t}{\varepsilon^2}} f\left(t - \beta_j'(\omega_0) L\right)$$

Transmitted wave = superposition of modes.

Each mode propagates with its velocity $1/\beta_j'(\omega_0)$.

Modal dispersion grows like $\sim L$.

Random waveguide

Mean amplitude:

$$\mathbb{E}[p_{tr}^{\varepsilon}(t,\mathbf{x},L)] = \frac{1}{4\pi}\int \sum_{j,l=1}^{N} \frac{\sqrt{\beta_l}}{\sqrt{\beta_j}} \phi_j(\mathbf{x})\phi_l(\mathbf{x}_0) \times \hat{f}(h)\mathbb{E}[T_{jl}^{\varepsilon}(\omega_0+\varepsilon^2 h)] e^{i\frac{\beta_j(\omega_0)L-\omega_0 t}{\varepsilon^2}} e^{i[\beta_j'(\omega_0)L-t]h} dh$$

Mean intensity:

$$\mathbb{E}[p_{tr}^{\varepsilon}(t,\mathbf{x},L)^2] = \frac{1}{16\pi^2}\int \sum_{j,l,m,n=1}^{N} \frac{\sqrt{\beta_l}}{\sqrt{\beta_j}} \phi_j(\mathbf{x})\phi_l(\mathbf{x}_0)\frac{\sqrt{\beta_n}}{\sqrt{\beta_m}} \phi_m(\mathbf{x})\phi_n(\mathbf{x}_0) \times \hat{f}(h)\overline{\hat{f}(h')}\mathbb{E}[T_{jl}^{\varepsilon}(\omega_0+\varepsilon^2 h) T_{mn}^{\varepsilon}(\omega_0+\varepsilon^2 h')] e^{i\frac{[\beta_j(\omega_0)-\beta_m(\omega_0)]L}{\varepsilon^2}} e^{i[\beta_j'(\omega_0)L-t]h - i[\beta_m'(\omega_0)L-t]h'} dh dh'$$

Autocorrelation function of the transmission coefficients

For fixed indices m and n:

$$U^{\varepsilon}_{jl}(\omega,h,z)=T^{\varepsilon}_{jm}(\omega,z)\overline{T^{\varepsilon}_{ln}(\omega-\varepsilon^2 h,z)}$$

is the solution of

$$\begin{aligned}
\frac{dU^{\varepsilon}_{jl}}{dz} = \; & \frac{ik^2}{2\varepsilon}\left(\frac{C_{jj}(\frac{z}{\varepsilon^2})}{\beta_j(\omega)}-\frac{C_{ll}(\frac{z}{\varepsilon^2})}{\beta_l(\omega-\varepsilon^2 h)}\right)U^{\varepsilon}_{jl} \\
& +\frac{ik^2}{2\varepsilon}\sum_{j_1\neq j}\frac{C_{jj_1}(\frac{z}{\varepsilon^2})}{\sqrt{\beta_j\beta_{j_1}(\omega)}}e^{i(\beta_{j_1}-\beta_j)(\omega)\frac{z}{\varepsilon^2}}U^{\varepsilon}_{j_1 l} \\
& -\frac{ik^2}{2\varepsilon}\sum_{l_1\neq l}\frac{C_{ll_1}(\frac{z}{\varepsilon^2})}{\sqrt{\beta_l\beta_{l_1}(\omega-\varepsilon^2 h)}}e^{i(\beta_l-\beta_{l_1})(\omega-\varepsilon^2 h)\frac{z}{\varepsilon^2}}U^{\varepsilon}_{jl_1}\,,
\end{aligned}$$

with the initial conditions $U^{\varepsilon}_{jl}(\omega,h,z=0)=\delta_{mj}\delta_{nl}$.

Expand $\beta_\cdot(\omega - \varepsilon^2 h)$ with respect to ε.

Introduce the Fourier transform

$$V_{jl}^\varepsilon(\omega, \tau, z) = \frac{1}{2\pi} \int e^{-ih(\tau - \beta_l'(\omega) z)} U_{jl}^\varepsilon(\omega, h, z) dh\,,$$

solution of

$$\begin{aligned}
\frac{\partial V_{jl}^\varepsilon}{\partial z} + \beta_l'(\omega) \frac{\partial V_{jl}^\varepsilon}{\partial \tau} \quad &= \quad \frac{ik^2}{2\varepsilon} \left(\frac{C_{jj}(\frac{z}{\varepsilon^2})}{\beta_j(\omega)} - \frac{C_{ll}(\frac{z}{\varepsilon^2})}{\beta_l(\omega)} \right) V_{jl}^\varepsilon \\
&+ \frac{ik^2}{2\varepsilon} \sum_{j_1 \neq j} \frac{C_{jj_1}(\frac{z}{\varepsilon^2})}{\sqrt{\beta_j \beta_{j_1}(\omega)}} e^{i(\beta_{j_1} - \beta_j)(\omega) \frac{z}{\varepsilon^2}} V_{j_1 l}^\varepsilon \\
&- \frac{ik^2}{2\varepsilon} \sum_{l_1 \neq l} \frac{C_{ll_1}(\frac{z}{\varepsilon^2})}{\sqrt{\beta_l \beta_{l_1}(\omega)}} e^{i(\beta_l - \beta_{l_1})(\omega) \frac{z}{\varepsilon^2}} V_{jl_1}^\varepsilon\,,
\end{aligned}$$

with the initial conditions $V_{jl}^\varepsilon(\omega, h, z = 0) = \delta_{mj} \delta_{nl} \delta(\tau)$.

$\hookrightarrow$ diffusion approximation theorem.

Autocorrelation function of the transmission coefficients

The autocorrelation function of the transmission coefficients at two nearby frequencies admits a limit as $\varepsilon \to 0$.

$$\begin{aligned}
\mathbb{E}[T^{\varepsilon}_{jj}(\omega,L)\overline{T^{\varepsilon}_{ll}(\omega-\varepsilon^2 h,L)}] &\stackrel{\varepsilon\to 0}{\longrightarrow} e^{Q_{jl}(\omega)L} \text{ if } j\neq l\,,\\
\mathbb{E}[T^{\varepsilon}_{jl}(\omega,L)\overline{T^{\varepsilon}_{jl}(\omega-\varepsilon^2 h,L)}] &\stackrel{\varepsilon\to 0}{\longrightarrow} e^{-i\beta'_j(\omega)hL}\int \mathcal{W}^{(l)}_j(\omega,\tau,L)e^{ih\tau}d\tau\,,\\
\mathbb{E}[T^{\varepsilon}_{jm}(\omega,L)\overline{T^{\varepsilon}_{ln}(\omega-\varepsilon^2 h,L)}] &\stackrel{\varepsilon\to 0}{\longrightarrow} 0 \text{ in the other cases }\,,
\end{aligned}$$

where $(\mathcal{W}^{(l)}_j(\omega,\tau,z))_{j=1,\cdots,N(\omega)}$ is the solution of the system of transport equations

$$\frac{\partial \mathcal{W}^{(l)}_j}{\partial z}+\beta'_j(\omega)\frac{\partial \mathcal{W}^{(l)}_j}{\partial \tau}=\sum_{n\neq j}\Gamma^{(c)}_{jn}(\omega)\left(\mathcal{W}^{(l)}_n-\mathcal{W}^{(l)}_j\right)$$

starting from $\mathcal{W}^{(l)}_j(\omega,\tau,z=0)=\delta(\tau)\delta_{jl}$.

The damping coefficients $Q_{jl}(\omega)<0 \Longrightarrow$ the coherent field decays exponentially.

$$\Gamma^{(c)}_{jn}=\frac{\omega^4}{4\bar{c}^4\beta_j(\omega)\beta_n(\omega)}\int_{-\infty}^{\infty}\cos\left((\beta_j(\omega)-\beta_n(\omega))z\right)\mathbb{E}[C_{jn}(0)C_{jn}(z)]dz\ >0$$

Probabilistic representation

Let us define the jump Markov process J_z whose state space is $\{1, \cdots, N(\omega)\}$ and whose infinitesimal generator is

$$\mathcal{L}\phi(j) = \sum_{l \neq j} \Gamma_{jl}^{(c)}(\omega) \left(\phi(l) - \phi(j)\right) .$$

We also define the process $\mathcal{B}_z$ by

$$\mathcal{B}_z = \int_0^z \beta'_{J_s} ds \, , z \geq 0 \, .$$

Kolmogorov equation:

$$\int_{\tau_0}^{\tau_1} \mathcal{W}_j^{(n)}(\omega, \tau, L) d\tau = \mathbb{P}\left(J_L = j \ , \ \mathcal{B}_L \in [\tau_0, \tau_1] \mid J_0 = n\right) .$$

(J_z) is an ergodic Markov chain with uniform stationary distribution.
From the ergodic theorem we have

$$\frac{\mathcal{B}_z}{z} \xrightarrow{z \to \infty} \overline{\beta'(\omega)} \, , \text{ where } \overline{\beta'(\omega)} = \frac{1}{N(\omega)} \sum_{j=1}^{N(\omega)} \beta'_j(\omega)$$

Exponential convergence rate $= 1/L_{\text{equip}}$

By applying a central limit theorem for functionals of ergodic Markov processes,

$$\frac{\mathcal{B}_z - \overline{\beta'(\omega)} z}{\sqrt{z}} \stackrel{z\to\infty}{\longrightarrow} \mathcal{N}(0, \sigma^2_{\beta'(\omega)})$$

where

$$\sigma^2_{\beta'(\omega)} = 2 \int_0^\infty \mathbb{E}_e \left[(\beta'_{J_0}(\omega) - \overline{\beta'(\omega)})(\beta'_{J_s}(\omega) - \overline{\beta'(\omega)}) \right] ds$$

$$\mathcal{W}_j^{(n)}(\omega, \tau, L) \stackrel{L \gg L_{\mathrm{equip}}}{\simeq} \frac{1}{N(\omega)} \frac{1}{\sqrt{2\pi\sigma^2_{\beta'(\omega)} L}} \exp\left(- \frac{(\tau - \overline{\beta'(\omega)} L)^2}{2\sigma^2_{\beta'(\omega)} L} \right)$$

Diffusion approximation for the system of transport equations $\mathcal{W}_j^{(n)}(\omega, \tau, L) \stackrel{L \gg L_{\mathrm{equip}}}{\simeq} \mathcal{W}(\omega, \tau, L)$ with

$$\frac{\partial \mathcal{W}}{\partial z} + \overline{\beta'(\omega)} \frac{\partial \mathcal{W}}{\partial \tau} = \frac{1}{2} \sigma^2_{\beta'(\omega)} \frac{\partial^2 \mathcal{W}}{\partial \tau^2}$$

The mean transmitted intensity

$$\mathbb{E}\left[\left|p_{tr}^{\varepsilon}(t,\mathbf{x},L)\right|^2\right] = I_1^{\varepsilon}(t,\mathbf{x},L) + I_2^{\varepsilon}(t,\mathbf{x},L)$$

The first term is the contribution of the coherent wave:

$$\begin{aligned} I_1^{\varepsilon}(t,\mathbf{x},L) \quad &\overset{\varepsilon\to 0}{\simeq} \quad \frac{1}{4}\sum_{j\neq m=1}^{N} \phi_j(\mathbf{x})\phi_j(\mathbf{x}_0)\phi_m(\mathbf{x})\phi_m(\mathbf{x}_0) e^{i\frac{[\beta_j(\omega_0)-\beta_m(\omega_0)]L}{\varepsilon^2}} \\ &\qquad \times e^{Q_{jm}(\omega_0)L} f(t-\beta_j'(\omega_0)L) f(t-\beta_m'(\omega_0)L)\,. \end{aligned}$$

We see that it decays exponentially with the propagation distance because of the damping factors $\exp(Q_{jm}(\omega_0)L)$.

The second term is the contribution of the incoherent waves:

$$I_2^{\varepsilon}(t,\mathbf{x},L) \overset{\varepsilon\to 0}{\simeq} \frac{1}{4}\sum_{j,l=1}^{N} \frac{\beta_l}{\beta_j}\phi_j^2(\mathbf{x})\phi_l^2(\mathbf{x}_0)\int \mathcal{W}_j^{(l)}(\omega_0,\tau,L) f(t-\tau)^2 d\tau\,.$$

The mean transmitted intensity

For $L \gg L_{\mathrm{equip}}$:

$$\lim_{\varepsilon \to 0} \mathbb{E}\left[p_{tr}^{\varepsilon}(t, \mathbf{x}, L)\right] \stackrel{L \gg L_{\mathrm{equip}}}{\simeq} 0$$

$$\lim_{\varepsilon \to 0} \mathbb{E}\left[\left|p_{tr}^{\varepsilon}(t, \mathbf{x}, L)\right|^2\right] \stackrel{L \gg L_{\mathrm{equip}}}{\simeq} H_{\omega_0, \mathbf{x}_0}(\mathbf{x}) \times [K_{\omega_0, L} * (f^2)](t),$$

where the spatial profile $H_{\omega_0, \mathbf{x}_0}$ and the time convolution kernel $K_{\omega_0, L}$ are

$$H_{\omega_0, \mathbf{x}_0}(\mathbf{x}) = \frac{1}{4N(\omega_0)} \sum_{j=1}^{N(\omega_0)} \frac{\phi_j^2(\mathbf{x})}{\beta_j(\omega_0)} \times \sum_{l=1}^{N(\omega_0)} \phi_l^2(\mathbf{x}_0) \beta_l(\omega_0),$$

$$K_{\omega_0, L}(t) = \frac{1}{\sqrt{2\pi \sigma_{\beta'(\omega_0)}^2 L}} \exp\left(-\frac{(t - \overline{\beta'(\omega_0)} L)^2}{2\sigma_{\beta'(\omega_0)}^2 L}\right).$$

Universal spatial profile (independent of the statistics of the perturbations).

Time profile:

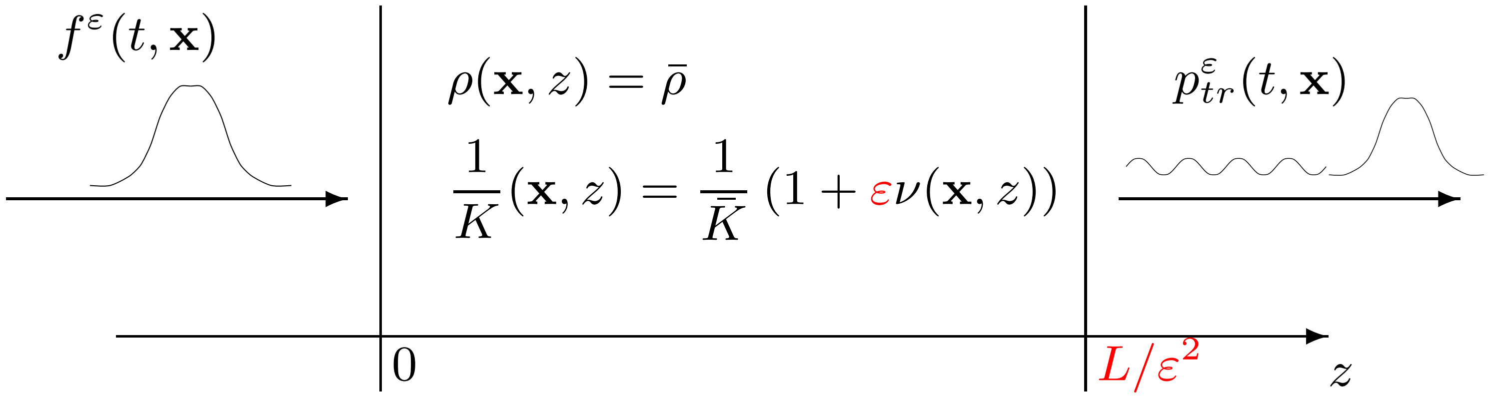
$1/\overline{\beta'(\omega_0)}$: group velocity = harmonic average of the mode velocity.

$\sigma_{\beta'(\omega_0)}^2$: group velocity dispersion.

Modal dispersion $\sim \sqrt{L}$.

Strong random coupling $\Longrightarrow$ reduction of modal dispersion.

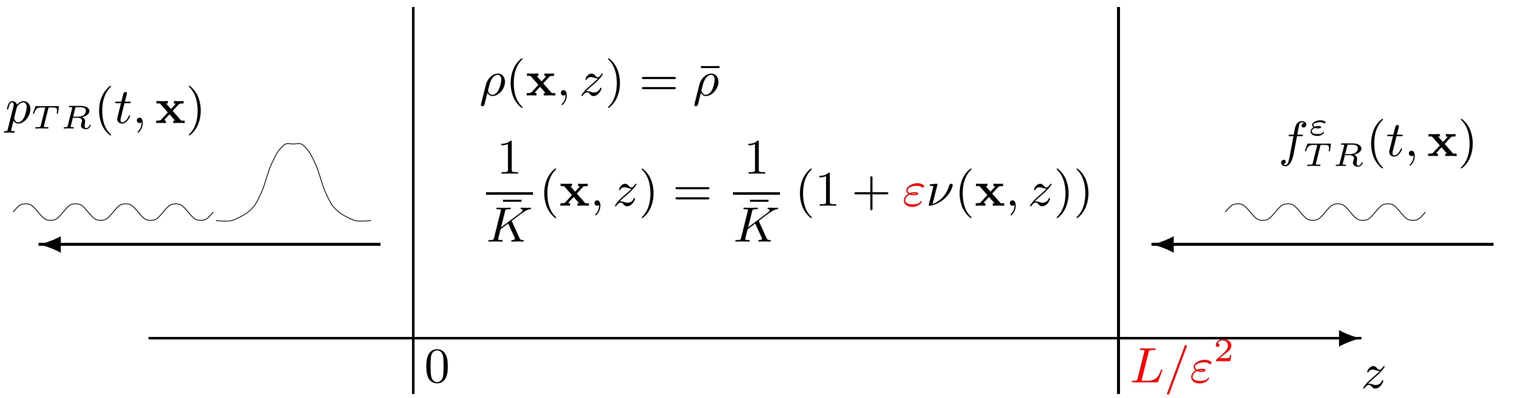
Time reversal setup



Record $p^\varepsilon_{tr}(t,\mathbf{x},L)$ up to time t_1 on the mirror $\mathbf{x}\in\mathcal{D}_M\subset\mathcal{D}$.

Cut a piece $f^\varepsilon_{TR}(t,\mathbf{x})=p^\varepsilon_{tr}(t,\mathbf{x},L)G_1(t)G_2(\mathbf{x})$, with $\mathrm{supp}(G_1)\subset[0,t_1]$ and $\mathrm{supp}(G_2)\subset\mathcal{D}_M$.

Time reverse and re-emit $\mathbf{F}^\varepsilon_{TR}(t,\mathbf{x},z)=f^\varepsilon_{TR}(t,\mathbf{x})\mathbf{e}_z$.



Mirror coupling coefficients:

$$M_{jl} = \int_{\mathcal{D}} \phi_j(\mathbf{x}) G_2(\mathbf{x}) \phi_l(\mathbf{x}) d\mathbf{x}$$

- If the mirror spans the complete cross section $\mathcal{D}$ of the waveguide, then we have $G_2(\mathbf{x}) = 1$ and $M_{jl} = 1$ if $j = l$ and 0 otherwise.
- If the mirror is point-like at $\mathbf{x} = \mathbf{x}_1$, i.e. $G_2(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}_1)$, then $M_{jl} = \phi_j(\mathbf{x}_1)\phi_l(\mathbf{x}_1)$.

Intuition for a "Good" mirror: M almost diagonal.

Refocused pulse after TR:

$$\begin{aligned} p_{\mathrm{TR}}\left(\frac{t_{\mathrm{obs}}}{\varepsilon^2}, \mathbf{x}, 0\right) \quad = \quad & \frac{1}{8\pi^2} \sum_{j,l,m,n=1}^{N} \frac{\sqrt{\beta_l \beta_m}}{\sqrt{\beta_j \beta_n}} M_{mj} \phi_n(\mathbf{x}) \phi_l(\mathbf{x}_0) \\ & \times e^{i[\beta_m(\omega_0) - \beta_j(\omega_0)]\frac{L}{\varepsilon^2} + i\omega_0 \frac{t_1 - t_{\mathrm{obs}}}{\varepsilon^2}} \\ & \times \int\int \overline{T_{jl}^{\varepsilon}(\omega_0 + \varepsilon^2 h')} T_{mn}^{\varepsilon}(\omega_0 + \varepsilon^2 h) \\ & \times \overline{\hat{f}(h') \hat{G}_1(h - h')} e^{i[\beta_m'(\omega_0)h' - \beta_j'(\omega_0)h]L + ih(t_1 - t_{\mathrm{obs}})} dh dh' \end{aligned}$$

Simplification: $G_1 \equiv 1$ (record everything in time).

Homogeneous waveguide with a full mirror

Transfer matrix $T^{\varepsilon}_{jl} = \delta_{jl}$. Mirror coupling $M_{jl} = \delta_{jl}$.

$$p_{\mathrm{TR}}\left(\frac{t_{\mathrm{obs}}}{\varepsilon^2}, \mathbf{x}, 0\right) = \frac{1}{2} e^{i\omega_0 \frac{t_1 - t_{\mathrm{obs}}}{\varepsilon^2}} f\left(t_1 - t_{\mathrm{obs}}\right) \sum_{j=1}^{N} \phi_j(\mathbf{x})\phi_j(\mathbf{x}_0)$$

Planar waveguide with diameter d:
$$\left\{ \begin{array}{l} \phi_j(x) = \sqrt{2/d}\,\sin(\pi j x/d) \\ \lambda_j = \pi j/d \\ \beta_j = \sqrt{\omega^2/c^2 - \pi^2 j^2/d^2} \\ N(\omega) = [(\omega d)/(\pi \bar{c})] = [2d/\lambda_0] \end{array} \right. .$$

In the continuum limit $N \gg 1$ (i.e. $d \gg \lambda_0$) we have

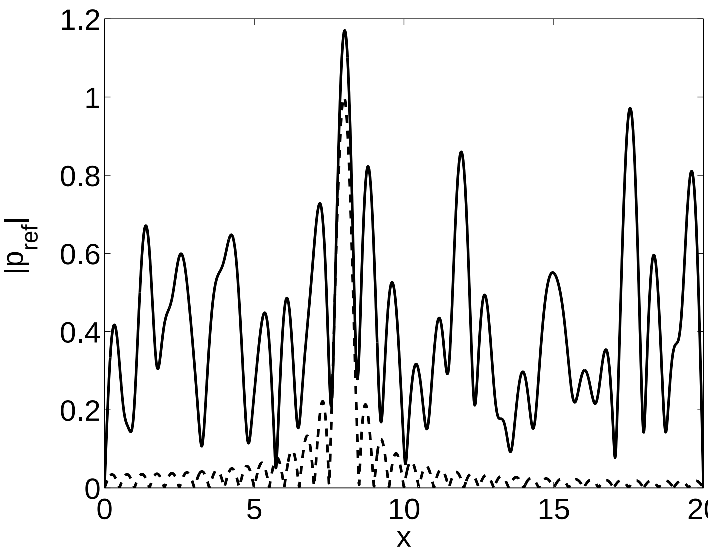
$$\frac{1}{2}\sum_{j=1}^{N} \phi_j(x)\phi_j(x_0) \overset{N \gg 1}{\simeq} \frac{1}{\lambda_0}\mathrm{sinc}\left(2\pi \frac{x - x_0}{\lambda_0}\right)$$

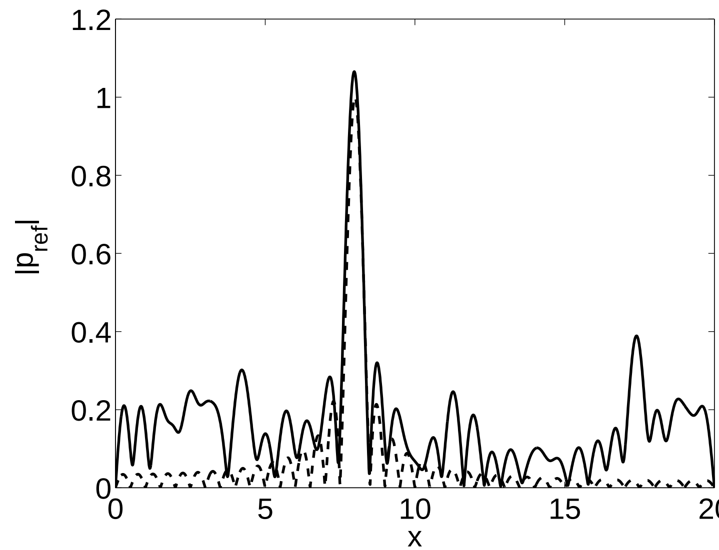
$\rightarrow$ diffraction-limited spot size, with focal radius $= \lambda_0/2$.

Homogeneous waveguide with a partial mirror

Transfer matrix $T^{\varepsilon}_{jl} = \delta_{jl}$.

$$p_{\mathrm{TR}}\left(\frac{t_{\mathrm{obs}}}{\varepsilon^2}, \mathbf{x}, 0\right) = \frac{1}{2} e^{i\omega_0 \frac{t_1 - t_{\mathrm{obs}}}{\varepsilon^2}} \sum_{j,m=1}^{N} e^{i[\beta_m - \beta_j](\omega_0)\frac{L}{\varepsilon^2}}$$
$$\times M_{mj} \phi_m(\mathbf{x}) \phi_j(\mathbf{x}_0) f\left([\beta'_m - \beta'_j](\omega_0) L + t_1 - t_{\mathrm{obs}}\right)$$





Mirror size $a = 2.5$ Mirror size $a = 10$

Transverse profile of the refocused field in a homogeneous waveguide with diameter $d = 20$ and length $L = 200$. Here $\lambda_0 = 1$, so there are 40 modes. Original source location is $x_0 = 8$.

The mean refocused pulse in a random waveguide

$$\mathbb{E}\left[p_{\rm TR}\left(\frac{t_{\rm obs}}{\varepsilon^2},\mathbf{x},0\right)\right]=p_1^\varepsilon+p_2^\varepsilon$$

p_1^ε is the contribution of the coherent waves:

$$p_1^\varepsilon \overset{\varepsilon\to 0}{\simeq} \frac{1}{2}\sum_{j\neq m=1}^{N} M_{mj}\phi_m(\mathbf{x})\phi_j(\mathbf{x}_0)e^{i[\beta_m-\beta_j](\omega_0)\frac{L}{\varepsilon^2}+i\omega_0\frac{t_1-t_{\rm obs}}{\varepsilon^2}}$$
$$\times e^{Q_{jm}(\omega_0)L} f\left([\beta'_m-\beta'_j](\omega_0)L+t_1-t_{\rm obs}\right),$$

p_2^ε is the contribution of the refocused incoherent waves:

$$p_2^\varepsilon \overset{\varepsilon\to 0}{\simeq} \frac{1}{2}e^{i\omega_0\frac{t_1-t_{\rm obs}}{\varepsilon^2}} f\left(t_1-t_{\rm obs}\right)\sum_{j,l=1}^{N} M_{jj}\phi_l(\mathbf{x})\phi_l(\mathbf{x}_0)\int \mathcal{W}_j^{(l)}(\omega_0,L,d\tau)$$

In the equipartition regime:

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[p_{\rm TR}\left(\frac{t_{\rm obs}}{\varepsilon^2},\mathbf{x},0\right)\right] \overset{L\gg L_{\rm equip}}{\simeq} e^{i\omega_0\frac{t_1-t_{\rm obs}}{\varepsilon^2}} f\left(t_1-t_{\rm obs}\right)$$
$$\times\frac{1}{N(\omega_0)}\sum_{j}^{N(\omega_0)} M_{jj}\times\frac{1}{2}\sum_{l=1}^{N(\omega_0)}\phi_l(\mathbf{x})\phi_l(\mathbf{x}_0)$$

In the equipartition regime:

$$\lim_{\varepsilon\to 0}\left|\mathbb{E}\left[p_{\mathrm{TR}}\left(\frac{t_{\mathrm{obs}}}{\varepsilon^2},\mathbf{x},0\right)\right]\right| \quad \overset{L\gg L_{\mathrm{equip}}}{\sim} \quad \frac{1}{N(\omega_0)}\sum_{j}^{N(\omega_0)} M_{jj}\times\frac{1}{2}\sum_{l=1}^{N(\omega_0)}\phi_l(\mathbf{x})\phi_l(\mathbf{x}_0)$$

Off-diagonal terms $j\neq m$ are killed by the expectation.

$$\text{Planar waveguide: }\begin{cases}\phi_j(x)=\sqrt{2/d}\sin(\pi jx/d)\\ \beta_j=\sqrt{\omega^2/c^2-\pi^2j^2/d^2}\\ \lambda_j=\pi j/d, \qquad N(\omega)=[(\omega d)/(\pi\bar{c})]=[2d/\lambda_0]\end{cases}.$$

In the continuum limit $N\gg 1$ we have

$$\frac{1}{2}\sum_{l=1}^{N}\phi_l(x)\phi_l(x_0)\overset{N\gg 1}{\simeq}\frac{1}{\lambda_0}\operatorname{sinc}\left(2\pi\frac{x-x_0}{\lambda_0}\right)$$

→ diffraction-limited spot size.

→ holds true only in average.

Statistical stability

$$S^2 := \lim_{\varepsilon \to 0} \frac{\mathbb{E}\left[\left|p_{\rm TR}\left(\frac{t_{\rm obs}}{\varepsilon^2}, \mathbf{x}_0, 0\right)\right|^2\right] - \left|\mathbb{E}\left[p_{\rm TR}\left(\frac{t_{\rm obs}}{\varepsilon^2}, \mathbf{x}_0, 0\right)\right]\right|^2}{\left|\mathbb{E}\left[p_{\rm TR}\left(\frac{t_{\rm obs}}{\varepsilon^2}, \mathbf{x}_0, 0\right)\right]\right|^2}$$

We have statistical stability when S is small. In the equipartition regime :

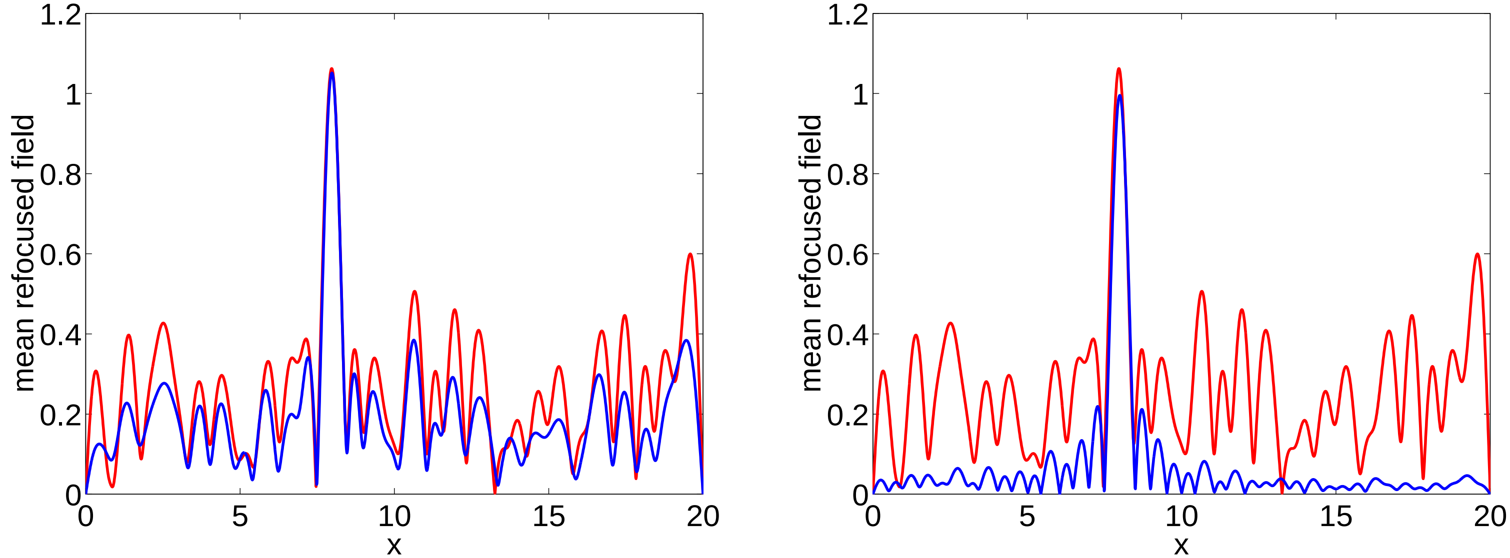
$$S^2 \stackrel{L \gg L_{\rm equip}}{\simeq} -\frac{1}{N+1} + \frac{N}{N+1}\frac{1}{Q_{\rm mirror}}, \qquad Q_{\rm mirror} = \frac{\sum_{j,l} M_{jj} M_{ll}}{\sum_{j,l} M_{jl}^2}$$

The quality factor $Q_{\rm mirror}$ depends only on the time reversal mirror.

Two extreme cases:

- If the time reversal mirror spans the waveguide cross-section, then $M_{jl} = \delta_{jl}$, $Q_{\rm mirror} = N$ and $S^2 = 0$, which is optimal.
- If the time reversal mirror is point-like at $\mathbf{x}_1$, then $M_{jl} = \phi_j(\mathbf{x}_1)\phi_l(\mathbf{x}_1)$, $Q_{\rm mirror} = 1$, and $S^2 = (N-1)/(N+1)$.

Numerical illustration: planar waveguide



$\sigma = 0.005$ $\sigma = 0.015$

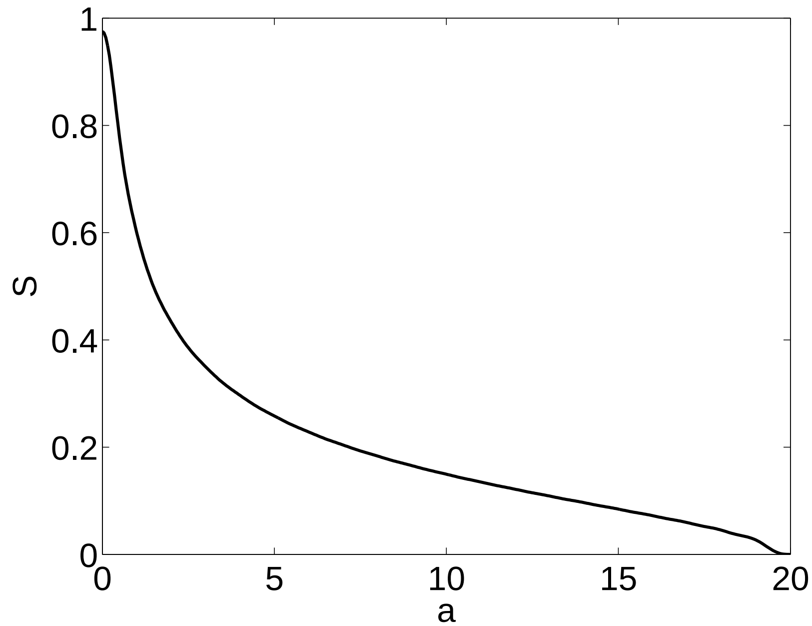
Transverse profile of the mean refocused field in a random waveguide with diameter $d = 20$, length $L = 200$, $\lambda_0 = 1$, mirror size $a = 5$, original source location $x_0 = 8$.

Random medium: correlation length $l_c = 0.25$ and standard deviation σ.

Red lines: spatial profile obtained in homogeneous medium $\sigma = 0$.

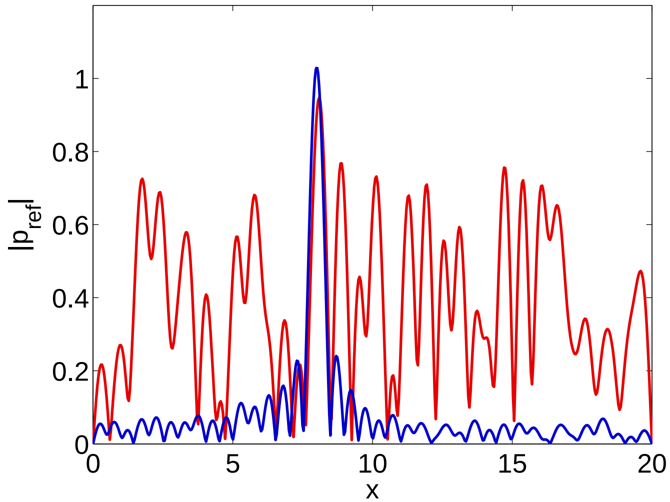
Blue lines: mean profile in a random waveguide.

Right figure is very close to the equipartition regime.

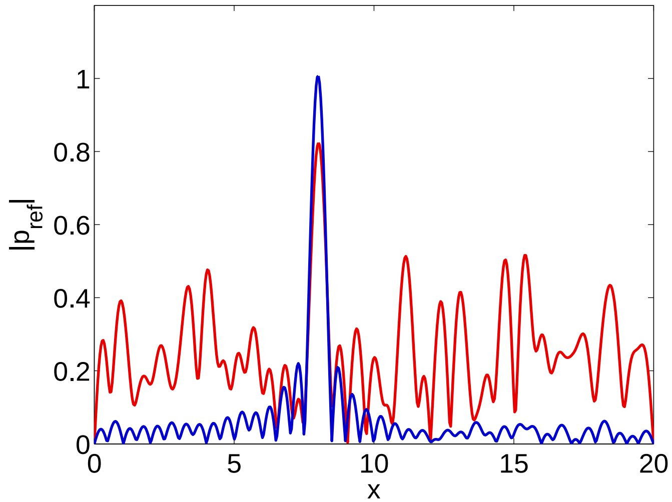


The relative standard deviation S of the refocused field in the equipartition regime as a function of the mirror size a. Here $d = 20$ and $\lambda_0 = 1$.

Numerical simulations: planar waveguide



$a = 2.5$



$a = 5$

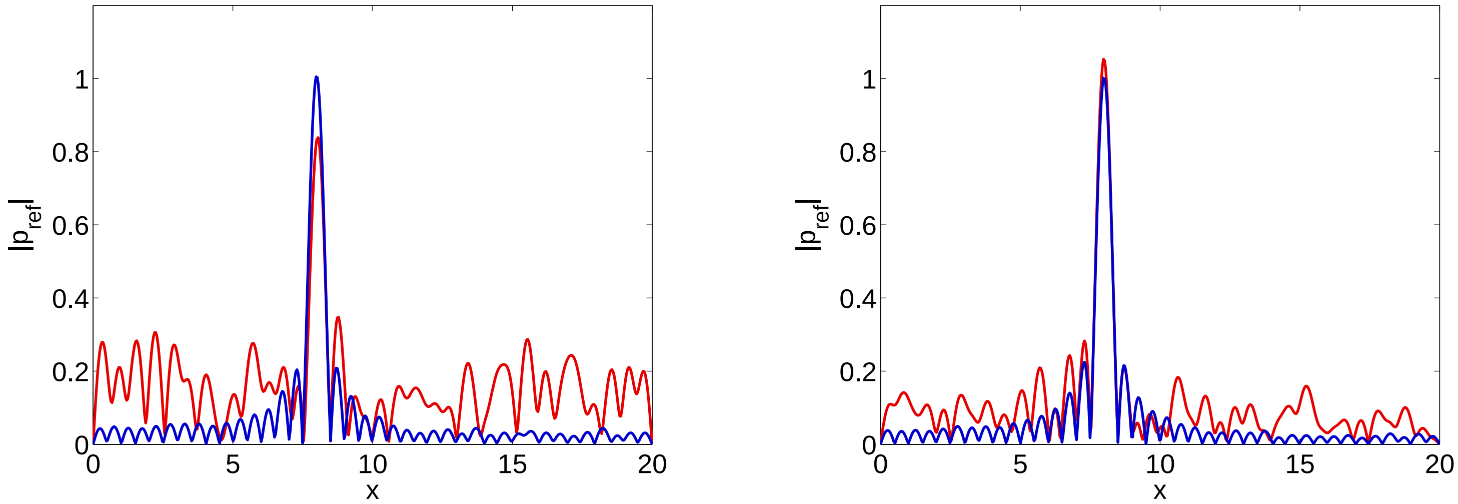
Transverse profile of the **refocused field** in a random waveguide with diameter $d = 20$, length $L = 200$, $\lambda_0 = 1$, mirror diameter a, original source location $x_0 = 8$.

Red lines: spatial profile obtained for a particular realization of the random medium.

Blue lines: mean profiles averaged over 100 realizations.

Parameters close to the equipartition regime.

Numerical simulations: planar waveguide



$a = 10$ $\qquad$ $a = 15$

Transverse profile of the **refocused field** in a random waveguide with diameter $d = 20$, length $L = 200$, $\lambda_0 = 1$, mirror diameter a, original source location $x_0 = 8$.

Red lines: spatial profile obtained for a particular realization of the random medium.

Blue lines: mean profiles averaged over 100 realizations.

Parameters close to the equipartition regime.

Conclusions

Mechanisms responsible for statistical stability in time-reversal:

1) broadband pulse

large number of uncorrelated frequency components

$\Longrightarrow$ self-averaging in time.

2) large mirror

large number of uncorrelated spatial modes

$\Longrightarrow$ self-averaging even for time-harmonic waves.