Modelling junctions for class of Second–Order models of traffic flow

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SUMMING UP

Solutions to a network problem

... can be constructed by solving one (half)-Riemann problems for *each* incoming or outgoing road:

$$\partial_t \begin{pmatrix} \rho_i \\ \rho_i w_i \end{pmatrix} + \partial_x \begin{pmatrix} \rho_i v_i \\ \rho_i v_i w_i \end{pmatrix} = 0,\ U_i(x,0) = \begin{pmatrix} U_i^- & x < x_0 \\ U_i^+ & x > x_0 \end{pmatrix}. \qquad (1)$$

... have parts of their initial data unknown, i.e., right U_i^+ (left) state for incoming (outgoing) roads.

... are constructed such that arising waves in the solution travel with negative (incoming) or positive (outgoing) speed, only.

... are such that, cars passing through a junction conserve their own (Lagrangian) property (like a "color") or formally conserve the value $w = w_i(U_{i,0})$ $i \in \delta^-$.

THE CASE OF A $2 \mapsto 1$ INTERSECTION

Last property restated important observation in a microscopic view for a junction where two roads merge:

Cars from both incoming roads enter the outgoing road and we see a mixture on the outgoing road.

Next steps:

- Mathematical statement of the above observation in the microscopic situation (Follow–The–Leader model)
- Reinterpretation for the macroscopic setting (Homogenization limit)
- Translation from Lagrangian to Eulerian coordinate system
- Solving the (half-)Riemann Problems at the intersection

Finally, the $n \mapsto m$ junction is discussed

THE MICROSCOPIC SITUATION AT THE INTERSECTION

Returning to the situation on the outgoing and taking a discrete, microscopic view, i.e., considering the Follow–The–Leader Model

Assume cars entering from road one and two in an alternate way, then, on the *outgoing* road, the picture near the junction for constant initial data is as follows:

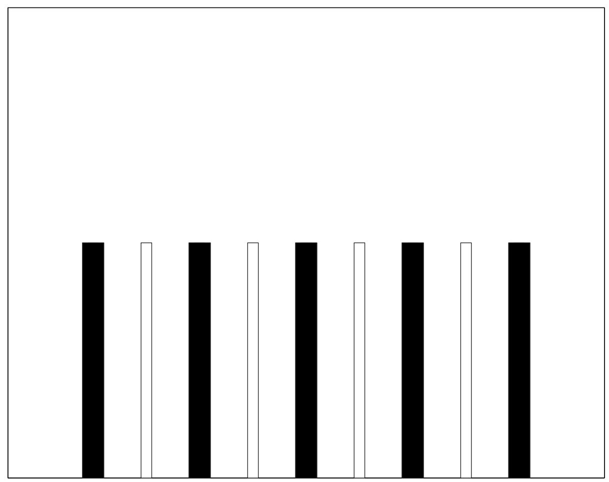

Figure 1: Each bar corresponds to a car. Cars from road one conserve their property (black) and so do cars from road two (white).

Now, imaging a zooming of the above situation: We eventually observe oscillations in w on the outgoing road.

OSCILLATING INITIAL DATA

Question: Given a sequence of oscillating initial data $w(X,0)$ and constant initial data $v(X,0)$, is there a solution to the AR–model (in Lagrangian coordinates)?

Answer: Yes, see Bagnerini, Rascle (2003): There exists a (homogenized) solution (τ^*, v^*, w^*) and a family of measures μ_X (associated with a sequence of approximate solutions of the Follow–The–Leader model), such that

$$\partial_t \tau^* - \partial_X v^* = 0 \quad (2a)$$

$$\partial_t w^* = 0 \quad (2b)$$

$$\tau^*(X,t) = \int P^{-1}(w^* - v^*) d\mu_X(w) \quad (2c)$$

and $w^*(X,0) = \int w d\mu_X(w)$ obtained as limit $\Delta X \to 0$ of the solutions to the semi–discretization (Follow–The–Leader model).

In the special case of initial data oscillating between two values w_1 (black) and w_2 (white): $\mu_X = \frac{1}{2}\left(\delta_{w_1} + \delta_{w_2}\right)$.

TRANSLATION TO EULERIAN COORDINATES

Up to now: On the outgoing road and near the junction we can obtain a macroscopic description of the situation by considering the homogenized solution (τ^*, w^*, v^*)

Next, we express this solution in Eulerian coordinates.

– We rewrite (2c) as $v = w - P_3^*(\tau)$ for the *fixed* (homogenized) value $w = \int w d\mu(w) = \frac{1}{2}(w_1(U_{1,0}) + w_2(U_{2,0})) =: \bar{w}_3$. I.e., for each fixed value of τ we define P_3^* so, that (v, τ) is a solution to (2c).

– Due to results of Klar, Rascle et. al. (2002) we can rewrite the Lagrangian in Eulerian coordinates (even for weak solutions) with $p_3^*(\rho) = P_3^*(1/\rho)$.

– In the (x,t)–plane the portion of road 3 concerned with this self-similar, homogenized flow is a triangle bounded by $x = a_3$ and by $x = a_3 + t\, v_{3,0}$ for initial data $v_{3,0}$ on the outgoing road $j = 3$. In this plane $w_3^*(U) = v + p_3^*(\rho)$ is constant and equal to the homogenized value $\frac{1}{2}(w_1(U_{1,0}) + w_2(U_{2,0}))$.

IMPLICATIONS OF THE PREVIOUS DISCUSSION (I/II)

Construction of the solution at a $2 \mapsto 1$ junction ($\delta^- = \{1,2\}, \delta^+ = 3$) with constant initial data $U_{i,0}$ and assuming fluxes entering in an alternating way.

- Compute the homogenized value $\bar{w}_3 := \frac{1}{2}(w_1(U_{1,0}) + w_2(U_{2,0}))$ and the homogenized function p_3^*.
- Solve a maximization problem at the interface to obtain unique flux at the intersection:

$$\max q_1 + q_2 \text{ subject to} \tag{3a}$$

$$0 \le q_i \le d_i\left(\rho_{i,0}; w_i(U) = v + p_i(\rho), w_1(U_{1,0})\right) \;\; i = 1,2 \tag{3b}$$

$$0 \le q_3 := q_1 + q_2 \le s\left(\rho^m; w_3^*(U) = v + p_3^*(\rho), \bar{w}_3\right), \tag{3c}$$

$$q_1 = q_2. \tag{3d}$$

where (ρ^m, v^m) is the point of intersection in the $\rho - \rho\, v$ plane of the level curve $\{w_3^* = \bar{w}_3\}$ and $\{v_3(U) = v_{3,0}\}$; (3b) guarantees waves of negative speed; (3c) guarantees waves of positive speed and incorporates the homogenized function w_3^* and value $\bar{w}$.

EXAMPLE FOR THE OUTGOING ROAD

Recall, the homogenized value $\bar{w}_3 := w_1(U_{1,0}) + w_2(U_{2,0})$ and the homogenized functions $\tau^* = (P_3^*)^{-1}(\bar{w}_3 - v^*)$ or equivalently $v + p_3^*(\rho) = \bar{w}_3$.

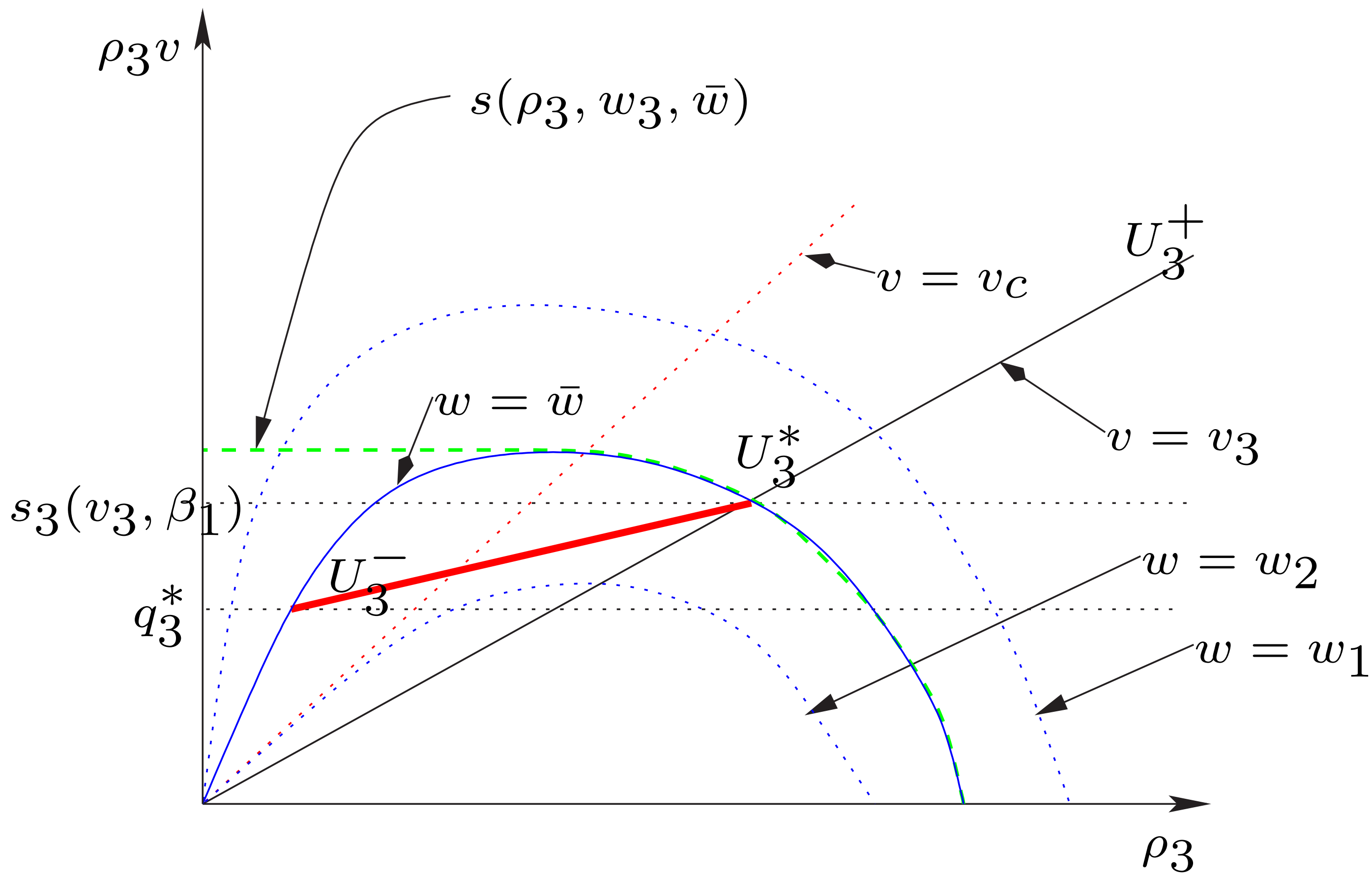

Figure 2: Supply s_3 corresponds to the curve $w(U) := v + p_3^*(\rho) = \bar{w}_3$ and to the unique point $U_3^m \equiv U_3^*$ on this curve with velocity $v_{3,0}$, with $w_1 \equiv w_1(U_{1,0})$, $w_2 \equiv w_2(U_{2,0})$.

IMPLICATIONS OF THE PREVIOUS DISCUSSION (II/II)

– For each $q_i, i = 1,2,3$ find the corresponding states $\bar{U}_i = (\bar{\rho}_i, \bar{\rho}_i\, v_i =: q_i)$ and solve the (half-)Riemann problems

$$\partial_t \begin{pmatrix} \rho_i \\ \rho_i w_i \end{pmatrix} + \partial_x \begin{pmatrix} \rho_i v_i \\ \rho_i v_i w_i \end{pmatrix} = 0,\ U_i(x,0) = \begin{pmatrix} U_i^- & x < x_0 \\ U_i^+ & x > x_0 \end{pmatrix} \tag{4}$$

where $i \in \delta^- : U_i^- = U_{i,0}, U_i^+ = \bar{U}_i$ and for $i = 3 \in \delta^+ :\ U_i^- = \bar{U}_i, U_i^+ = U_{i,0}$.

– By construction the solution U_i conserves the mass $\rho_1 v_1(x_0-,t) + \rho_2 v_2(x_0-,t) = \rho_3 v_3(x_0+,t)$, c.f. (3c).

– By construction the solution U_i conserves the (pseudo-)mass:

$$w_3 \rho_3 v_3(x_0+,t) \tag{5a}$$

$$= \bar{w}\ \rho_3 v_3(x_0+,t) = \frac{1}{2}\left(w_1(x_0-,t) + w_2(x_0-,t)\right) \rho_3 v_3(x_0+,t) \tag{5b}$$

$$= w_1 \rho_1 v_1(x_0-,t) + w_2 \rho_2 v_2(x_0-,t). \tag{5c}$$

SHORT SUMMARY

Consider three roads $i = 1, 2, 3$ with $a_1 = a_2 = -\infty, b_1 = b_2 = a_3$ and $b_3 = \infty$ and constant initial data $U_{i,0} = (\rho_{i,0}\rho_{i,0}v_{i,0}), i = 1, 2, 3$.

Then there exists a unique solution $U_i(x, t), i = 1, 2, 3$ of the (half-)Riemann problems at the junction with the following properties.

- $U_i(x, t)$ is a weak solution of the network problem, where $p_i^* \equiv p_i$ for the incoming roads $i = 1, 2$.

  For the outgoing road $i = 3$, we obtain two different expressions for p_i^* : In the $x - t$ plane, in a triangle near the junction, we consider the *homogenized solution* p_3^* defined as previously introduced. The triangle is bounded at any fixed time $t > 0$ by $x = a_3$ and $x = a_3 + tv_{3,0}$. In the remaining part of the outgoing road we have $p_3^* \equiv p_3$.

- In particular $U_3(a_3^+, t)$ satisfies $w_3^*(U_3(a_3^+, t)) := v_3(a_3^+, t) + p_3^*(\rho_3(a_3^+, t)) = \frac{1}{2}\left(w_1(U_{1,0}) + w_2(U_{2,0}),\right)$.

- The two incoming fluxes are equal, and the total flux $2(\rho_1 v_1)(b_1^-, t) = 2(\rho_2 v_2)(b_2^-, t) = (\rho_3 v_3)(a_3^+, t)$ is maximal subject to the other conditions.

For notation introduce initially *unknown* quantities

q_{ji} is the initially unknown flux going from road i to j

$q_j = \sum_{i \in \delta^-} q_{ji}$ is the total outgoing flux on road j at the intersection

$q_i = \sum_{j \in \delta^+} q_{ji}$ is the total incoming flux on road i

The proportion

$$\alpha_{ji} := \frac{q_{ji}}{q_i} \tag{6}$$

is the percentage of flux going from road i to road j. This controls the *distribution* of incoming flow.

The proportion

$$\beta_{ji} := \frac{q_{ji}}{q_j} \tag{7}$$

is the percentage of flux arriving on road j and coming from i. This controls the *mixture* on each outgoing road.

ASSUMPTIONS FOR THE GENERAL CASE

We collect the assertions of the previous discussions:

H1. We assume the proportions $\alpha_{ji} = q_{ji}/q_i$ of fluxes going from road $i \in \delta^-$ to $j \in \delta^+$, i.e., $A = (\alpha_{ji})_{(j,i)\in(\delta^+,\delta^-)}$, to be *known*.

H2. We assume the cars mix according to the proportion $\beta_{ji} = q_{ji}/q_j$, i.e., the homogenized value $\bar{w}_j$ on each outgoing road j fulfills

$$\bar{w}_j = \sum_{i\in\delta^-} \beta_{ji}\, w_i(U_{i,0}).$$

H3. We assume the ratios β_{ji} are *known*. This is enforced e.g. by assuming that the total incoming fluxes $(q_i)_{i\in\delta^-}$ are proportional to $(1,\ldots,1)$:

$$q_i = r\,1 \implies \beta_{ji} = \frac{\alpha_{ji}}{\sum_{j\in\delta^+}\alpha_{ji}}.$$

MAIN STATEMENT FOR THE GENERAL CASE

Consider a junction with m incoming and n outgoing roads, with constant initial data $U_{i,0} = (\rho_{i,0}, \rho_{i,0} v_{i,0})$ for all $i \in \delta^- \cup \delta^+$ under the assumptions (H1) to (H3).

Then there exists a unique solution $\{U_i(x,t)\}_i$ at the intersection which satisfies the following properties.

1 $\{U_i(x,t)\}_i$ is a weak entropy solution of the network problem and for $i \in \delta^-$: $p_i^\dagger \equiv p_i$.

For the outgoing roads $j \in \delta^+$ we obtain two different expressions for $p_j^\dagger$, depending on the region. In the $x - t$-plane in a triangle near the junction, we consider the homogenized solution and hence $p_j^\dagger(\cdot) = p_j^*(\cdot)$. This triangle is defined by $\{(x,t) : a_j \leq x \leq t v_{j,0}\}$ for any fixed time $t > 0$. Beyond this triangle we have $p_j^\dagger(\cdot) \equiv p_j(\cdot)$.

Furthermore, mass and (pseudo)-momentum are conserved through the junction by the solution $\{U_i\}_i$.

2 The incoming fluxes $(U_i(b_i-, t))_{i \in \delta^-}$ are proportional to $(1, \dots, 1)$ and distributed according to α_{ji}. Moreover, they are maximal subject to the other conditions.

SUMMARY & OUTLOOK

– We presented a solution for an arbitrary junction in the network conserving mass and (pseudo-)momentum

– Modelling of the coupling conditions is motivated by the microscopic interpretation of the AR–model

– Additional posed assumption in this talk: Mixture rule of the incoming fluxes, *but* other conditions are possible (c.f. recent work with S. Moutari)

– Up to now: Constant initial data on all roads

– Work in progress on numerical results