Non destructive testing
using non linear vibroacoustic

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1 Introduction

Several recent experimental studies show that it is possible to detect defects in a structure by considering its vibro- acoustic response to an external actuation.

1.1 Some previous papers

On this topic there is a vast literature in applied physics. We recall some papers related to the use of the frequency response for non destructive testing; in particular generation of higher harmonics, cross-modulation of a high frequency by a low frequency:

- In Ekimov-Didenkulov-Kasakov (1999), [2], the authors report experiments with *torsional waves in a rod with a crack*: they use HF torsional wave (20kHz) and a LF flexural wave (12 Hz).
- In Zaitsev-Sas (1999), [12], the authors report experiments with *plate vibration* submitted to LF (20-60Hz) vibration by a shaker and HF (15-30 kHz) oscillations by a piezo-actuator. They notice that weak modulation side-lobes are observed for the undamaged sample but drastic increase in nonlinear vibro-acoustic of the damaged sample. Some theoretical explanations are provided.
- Other results may be found in Sedunov-Tsionsky-Donskoy(2002) [3],Sutin-Donskoy (1998), [1], Moussatov-Castagnede-Gusev(2002), [5] ...
- GDR 2501 (Etude de la propagation ultrasonore en milieux inhomognes en vue du controle non destructif)

In Vanderborck-Lagier-Groby (2003) [10], "a vibro-acoustic method, based on frequency modulation, is developed in order to detect defects on *aluminum and concrete beams*. Flexural waves are generated at two very separated frequencies by the way of two piezoelectric transducers. The low one corresponds to the first resonance f_m, the second one to a high non modal frequency f_p. The nonlinear response, due to the defects inside the structure, is detected by non-zero flexural waves at $f_p \pm n f_m$ frequencies.

see Vanderborck-Lagier(2004) $\longmapsto$ beam experimentation

Very recent experiments

- have been performed on a real bridge by G. Vanderborck with four prestressed cables: two undamaged cables, a damaged cable and a safe one but damaged at the anchor;
- these experiments have been performed in the frame of the European program “Promoting competitive and sustainable growth ”of 15/12/99.
- The cables are roughly 100 m long, 4 tones weight, 15cm in diameters.
- The experiments have proved the presence of the damaged cable but also the safe one damaged at the anchor.
- Routine experimental checking with the lower eigenfrequencies had **only** proved only the presence of the very damaged cable by comparison with data collected 15 years ago.
- See Vanderborck-Lagier(2004) [13] for a presentation of the results of the experiment with a new post processing graphic presentation of experimental results.

Difficulties of the experiments:

- non linearities of the shakers (including piezoelectric actuators)
- Natural non linearities: supports, links of complex multi structures as air planes, bridges etc

Orientation

We intend to present simple spring mass models, simple bar models with damage and use **asymptotic expansions and numerical methods** to try to get results which show some similarity with the experiments of [10]. Asymptotic expansions have been used for at least a century and for example has been used recently for numerical approximation of bifurcation of structures in PotierFerry-Cochelin and coworkers (1993) [4].

The key idea is to look at the solution in the frequency domain for the experiments and consequently for the numerics.

In a paper to be submitted (Lagier-Vandeborck) [7] several types of nonlinearities of defects are considered: contact elasticity, threshold contact model, nonlinear filling material. This last case will be considered for bar models: it may happen in case of corrosion: the voided crack is filled by a new dusty material: then the elastic crack response is related to the elastic properties of the filler. In this case it seems reasonable to consider a nonlinear elastic relation for the filler.

2 Background of Fourier transform

2.1 Basic formulas

2.1.1 Fourier transform

$$(\mathcal{F}f)(\nu) = \hat{f}(\nu) = \int_{\mathbf{R}} f(t)e^{-2\pi i\nu t}dt \tag{2.1}$$

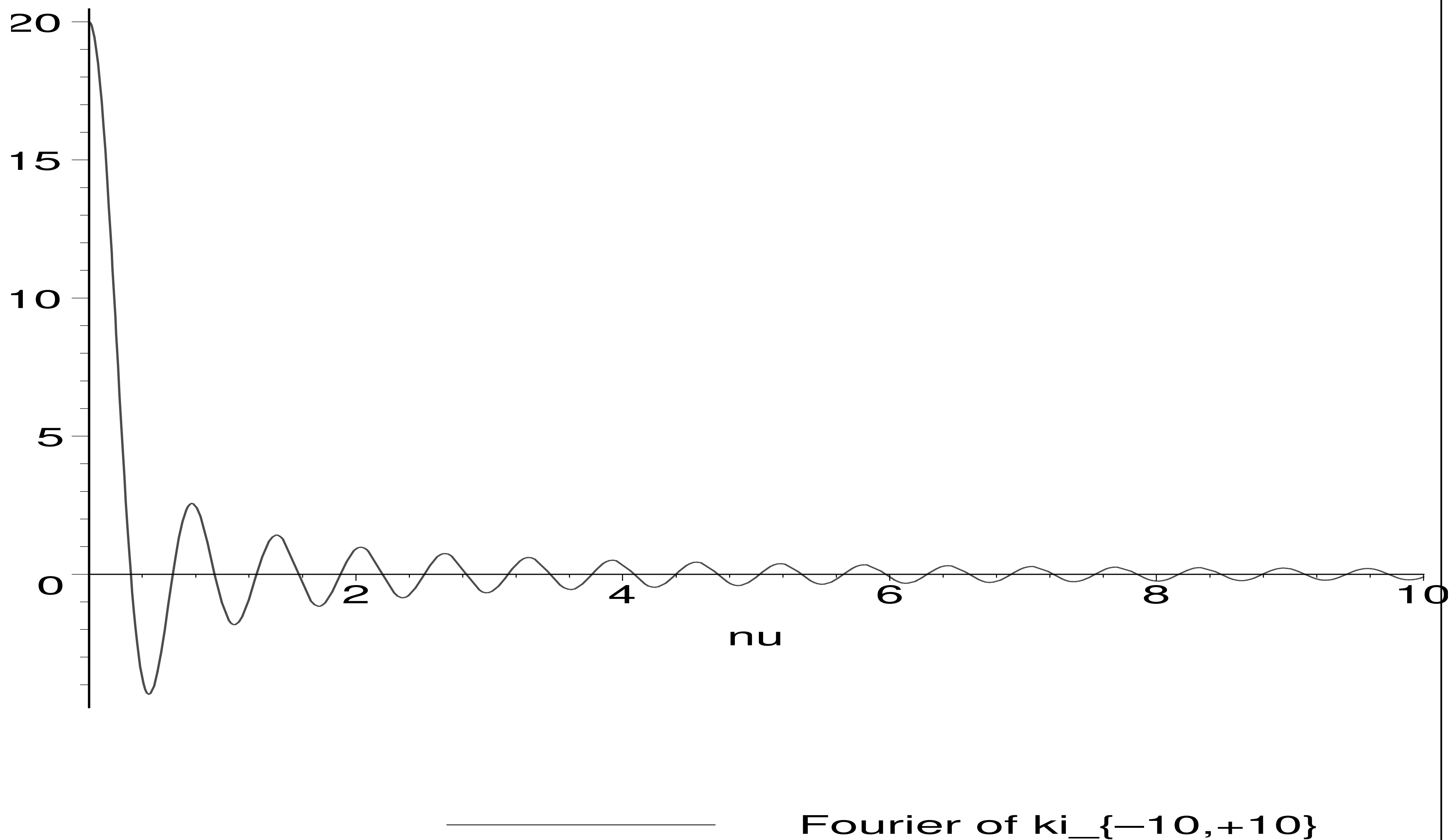
$$\tag{2.2}$$

$$\mathbf{sinc}(2\pi\nu A) \quad \widehat{\chi_{[0,A]}} = e^{-i\pi A\nu}\frac{sin(\pi A\nu)}{\pi\nu} = Ae^{-i\pi A\nu}\mathbf{sinc}(\pi A\nu) \tag{2.3}$$

$$\text{with } \textit{the sampling function (“sinus cardinal”)} \quad \boldsymbol{sinc}(t) = \frac{sin(t)}{t} \tag{2.4}$$

$$\mathcal{F}(e^{2\pi iat}) = \delta_a, \text{ and } \mathcal{F}(e^{2\pi iat}T) = \tau_{-a}\hat{T} = \delta_a * \hat{T} \tag{2.5}$$

$$\mathcal{F}(cos(2\pi iat)\chi_{[0,A]}) = \frac{A}{2}(\tau_a e^{-i\pi A\nu}\mathbf{sinc}(\pi\nu\mathbf{A}) + \tau_{-a}e^{-i\pi A\nu}\mathbf{sinc}(\mathbf{2}\pi\nu\mathbf{A})) \tag{2.6}$$


20
15
10
5
0
2
4
6
8
10
nu
Fourier of ki_{−10,+10}

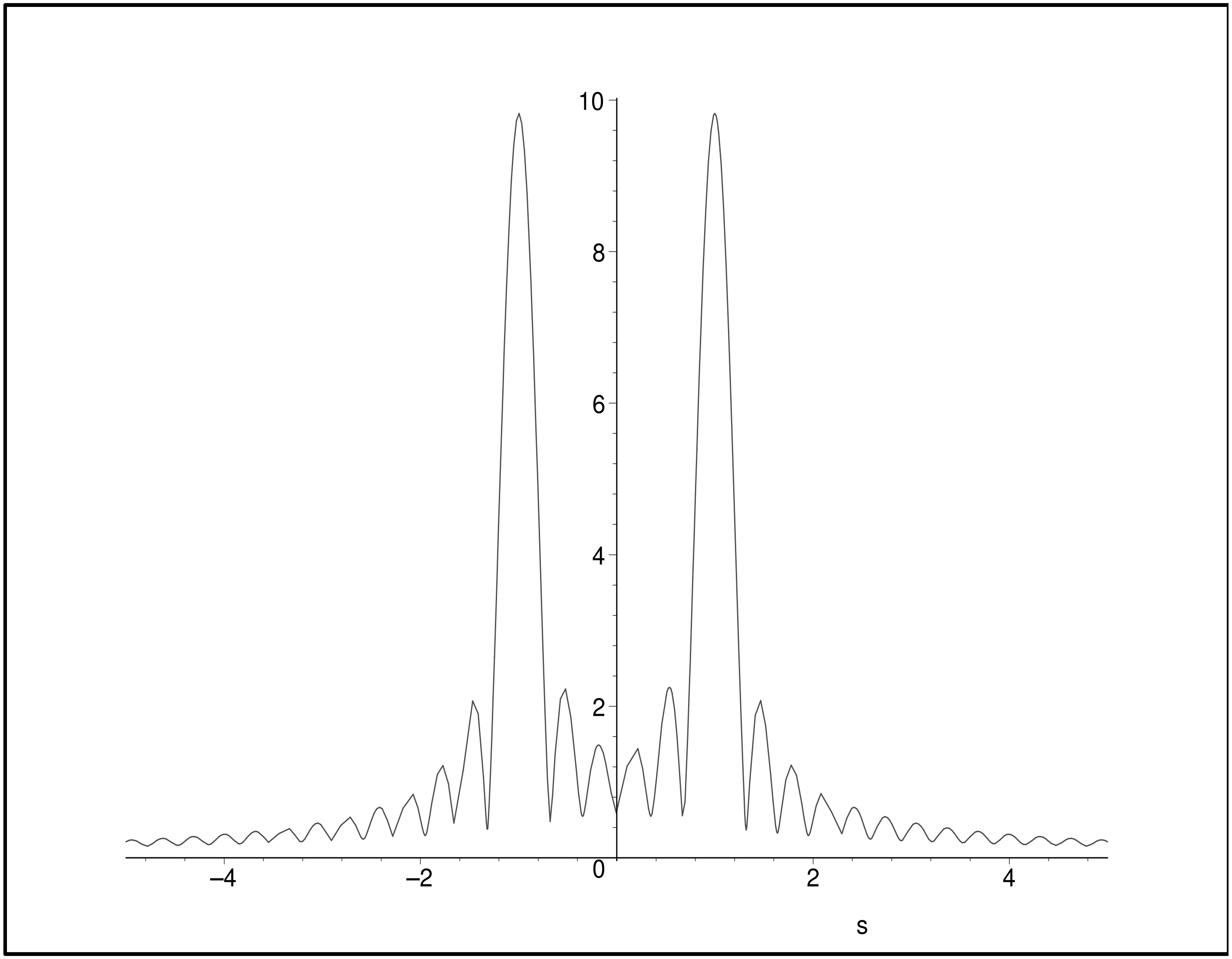


Figure 1: Fourier of $sin(t)\chi_{-10,10}(t)$

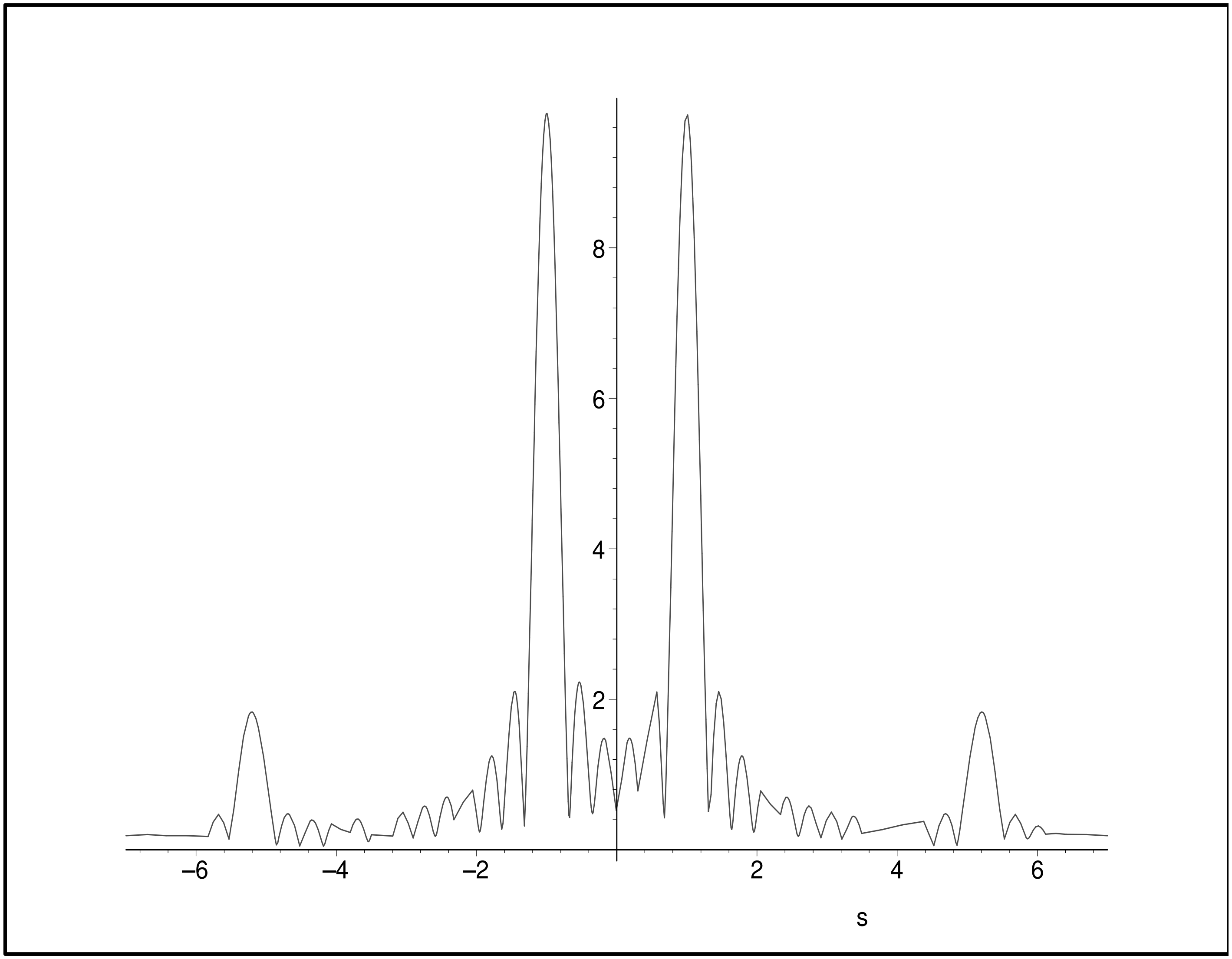


Figure 2: Fourier of $(sin(t) + 0.2sin(5.2t))\chi_{-10,10}(t)$

2.2 Numerical computation of Fourier transform

Sum of two sinus

`sin_p_sin.sci`

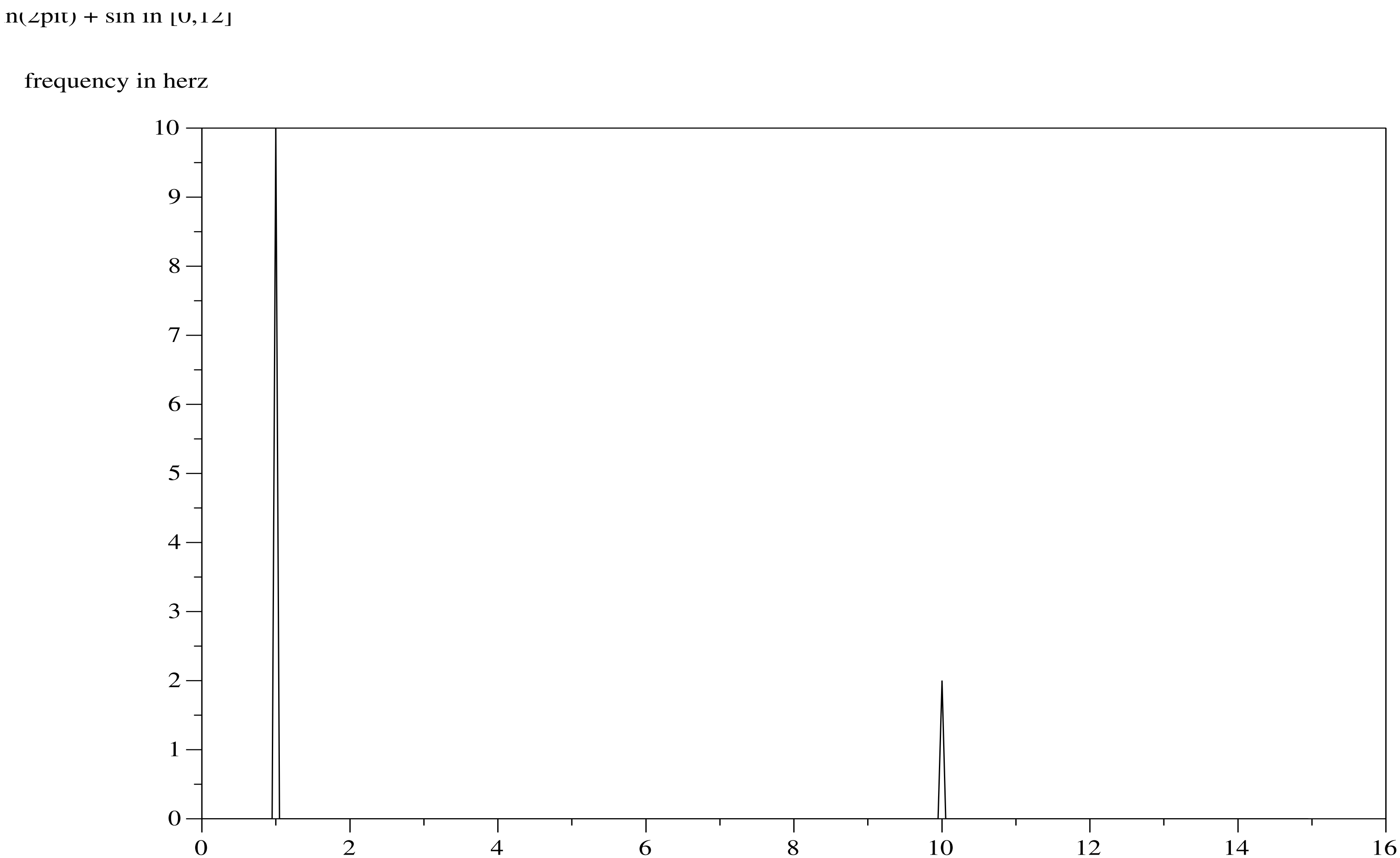


Figure 3: Norm of the Fourier transform of $sin(2\pi t) + .2sin(20\pi t)$ in $[0, 50]$ herz

3 Simplest mechanical example

in which we can exhibit intermodulations.

We consider a 1 d.o.f example of a spring mass system with a **non linear** spring.

In many situations in solid mechanics, it is common to assume **small load hypothesis** which is modeled with a **small parameter** ϵ; for a 1 dof:

$$my'' + k_1 y + k_3 y^3 = \tilde{\epsilon}\tilde{F} cos(\tilde{\alpha}\tilde{t}) \tag{3.1}$$

The solution is of order $O(\epsilon)$ so that we can perform the change of function: $Y = \frac{y}{\epsilon}$ which is solution of:

$$mY'' + k_1 Y + \tilde{\epsilon}^2 k_3 Y^3 = \tilde{F} cos(\tilde{\alpha}\tilde{t}) \tag{3.2}$$

$$\text{it can be put in dimensionless form introducing} \tag{3.3}$$

$$\text{a characteric time and lenth: } T^*, L^* \tag{3.4}$$

$$\text{and puting: } t = \frac{\tilde{t}}{T^*}, u = \frac{Y}{L^*} \tag{3.5}$$

$$\ddot{u} + \frac{k_1}{m}T^{*2}u + \tilde{\epsilon}^2\frac{k_3}{m}T^{*2}L^{*2}u^3 = \frac{\tilde{F}T^{*2}}{mL^*}cos(\tilde{\alpha}T^*t) \tag{3.6}$$

$$\text{possible choice: } T^* = \sqrt{\frac{m}{k_1}} \text{ and set } \boxed{\epsilon = \tilde{\epsilon}^2\frac{k_3}{m}T^{*2}L^{*2}}, \tag{3.7}$$

$$F = \frac{\tilde{F}T^{*2}}{mL^*},\ \alpha = \tilde{\alpha}T^*,\ \text{one obtains:} \tag{3.8}$$

$$\ddot{u} + u + \epsilon u^3 = F\cos(\alpha t) \tag{3.9}$$

4 Duffing equation

This model equation will be solved **numerically** on one side and with **asymptotic expansions** on the other side.

To study the three body problem, Lagrange introduced an averaging method; Poincaré used it also for celestial mechanics; introductory and theoretical background may be found in in Roseau ([11]). and Verhulst, Sanders-Verhulst ([14, 6]); many practical problems are considered in Nayfeh ([8, 9]).

$$\ddot{u} + u + \epsilon u^3 = F\cos(\alpha t) \qquad (4.1)$$

4.1 Solution by double scale expansion

"Naive expansion " *fails* to obtain bounded solutions, shift in fundamental and intermodulation frequency. Many methods: including averaging, double scale expansion ... Following notations of [8], we seek a double scale expansion of the solution, setting:

$$T_0 = t,\ T_1 = \epsilon t$$

so that

$$\frac{d}{dt} = D_0 + \epsilon D_1 + \ldots,\ \frac{d^2}{dt^2} = D_0^2 + 2\epsilon D_0 D_1 + \ldots$$

$$u = u_0(T_0, T_1) + \epsilon u_1(T_0, T_1) + \ldots$$

from which we get the first two equations:

$$\begin{cases} D_0^2 u_0 + u_0 = F\cos(\alpha T_0) \\ D_0^2 u_1 + u_1 = -2D_0 D_1 u_0 - u_0^3 \end{cases} \tag{4.2}$$

The solution u_0 may be expressed as:

$$u_0 = a(T_1)\cos(T_0 + \beta(T_1)) - \phi\cos(\alpha T_0) \quad \text{with } \phi = \frac{F}{-1+\alpha^2} \tag{4.3}$$

$$\text{or } u_0 = A(T_1)e^{iT_0} - \frac{\phi}{2}e^{i\alpha T_0} + \textbf{c.c.} \text{ with } A = \frac{1}{2}ae^{i\beta} \tag{4.4}$$

4.1.1 First order term

The first order term is solution of (after the computation of u_0^3):

$$D_0^2 u_1 + u_1 = -(2iA' + 3(A\bar{A} + \frac{\phi^2}{2})A)e^{iT_0} + \tag{4.5}$$

$$(6A\bar{A} + \frac{3}{4}\phi^2)\frac{\phi}{2}e^{i\alpha T_0} - A^3 e^{3iT_0} + \frac{\phi^3}{8}e^{3i\alpha T_0} + \tag{4.6}$$

$$\frac{3A^2\phi}{2}e^{i(2+\alpha)T_0} + \frac{3\bar{A}^2\phi}{2}e^{i(-2+\alpha)T_0} \tag{4.7}$$

$$-\frac{3A\phi^2}{4}e^{i(1+2\alpha)T_0} - \frac{3A\phi^2}{4}e^{i(1-2\alpha)T_0} + \textbf{c.c.} \tag{4.8}$$

In the right hand side the term in e^{iT_0} produces unbounded terms (the so called secular terms of celestial mechanics). We eliminate this term by imposing:

$$2iA' + 3A^2\bar{A} + \frac{3A\phi^2}{2} = 0 \quad \text{using } A = \frac{a}{2}e^{i\beta} \tag{4.9}$$

$$ia' - a\beta' + \frac{3a^3}{8} + \frac{3a\phi^2}{4} = 0 \tag{4.10}$$

$$a \text{ is constant and } \beta = \beta_1 T_1 + \beta_0 \quad \text{with } \beta_1 = \frac{3}{4}\left(\frac{a^2}{2} + \phi^2\right) \tag{4.11}$$

An approximate solution:

$$u \simeq u_0 + \epsilon u_1$$

The level of the computed lobes of $u = u_0 + \epsilon u_1$ is ($a = \phi$ when $u(0) = 0$ and $\dot{u}(0) = 0$): angular frequency $\mapsto$ lobe.

$$1 + \epsilon\beta_1 \mapsto \frac{T_{max}}{2} a \tag{4.12}$$

$$\alpha \mapsto \frac{T_{max}}{2} \left(-\phi + \epsilon \frac{3.\phi.(2.a^2 + \phi^2)}{4(1 - \alpha^2)} \right) \tag{4.13}$$

$$3.(1 + \epsilon.\beta_1) \mapsto \epsilon \frac{T_{max}}{2} \frac{a^3}{32} \tag{4.14}$$

$$(2 + \alpha + 2.\epsilon.\beta_1) \mapsto \epsilon \frac{T_{max}}{2} \frac{(3.a^2.\phi/4)}{(1 - (2 + \alpha)^2)} \tag{4.15}$$

$$(-2 + \alpha - 2.\epsilon.\beta_1) \mapsto \epsilon \frac{T_{max}}{2} \frac{(3.a^2.\phi/4)}{(1 - (-2 + \alpha)^2)} \tag{4.16}$$

$$(1 + 2.\alpha + \epsilon.\beta_1) \mapsto -\epsilon \frac{T_{max}}{2} \frac{(3.a.\phi^2)}{(4.(1 - (1 + 2.\alpha)^2))} \tag{4.17}$$

$$(-1 + 2.\alpha - \epsilon.\beta_1) \mapsto -\epsilon \frac{T_{max}}{2} \frac{(3.a.\phi^2)}{(4.(1 - (1 - 2.\alpha)^2))}] \tag{4.18}$$

$$3.\alpha \mapsto \epsilon \frac{T_{max}}{2} \frac{\phi^3}{(4.(1 - 9.\alpha^2))} \tag{4.19}$$

4.1.2 Behavior of intermodulation picks

$$(2+\alpha+2.\epsilon.\beta_1) \mapsto \epsilon \frac{T_{max}}{2} \frac{(3.a^2.\phi/4)}{(1-(2+\alpha)^2)} \tag{4.20}$$

Increase with ϕ and ϵ, decrease with α; for the mechanical non linear spring

$$\boxed{\epsilon = \tilde{\epsilon}^2 \frac{k_3}{m} T^{.2} L^{.2}}$$

ϵ increases with k_3, the damage!

Behavior of the rate of lobe at $2+\alpha+2\epsilon\beta_1$ over main lobe at α: see figure 24.

4.2 Admissible parameters

an obvious limitation: the angular frequencies should remain in the order they get for very small ϵ. After manipulations:

$$5(1 + \frac{9\epsilon\phi^2}{8}) < \alpha$$

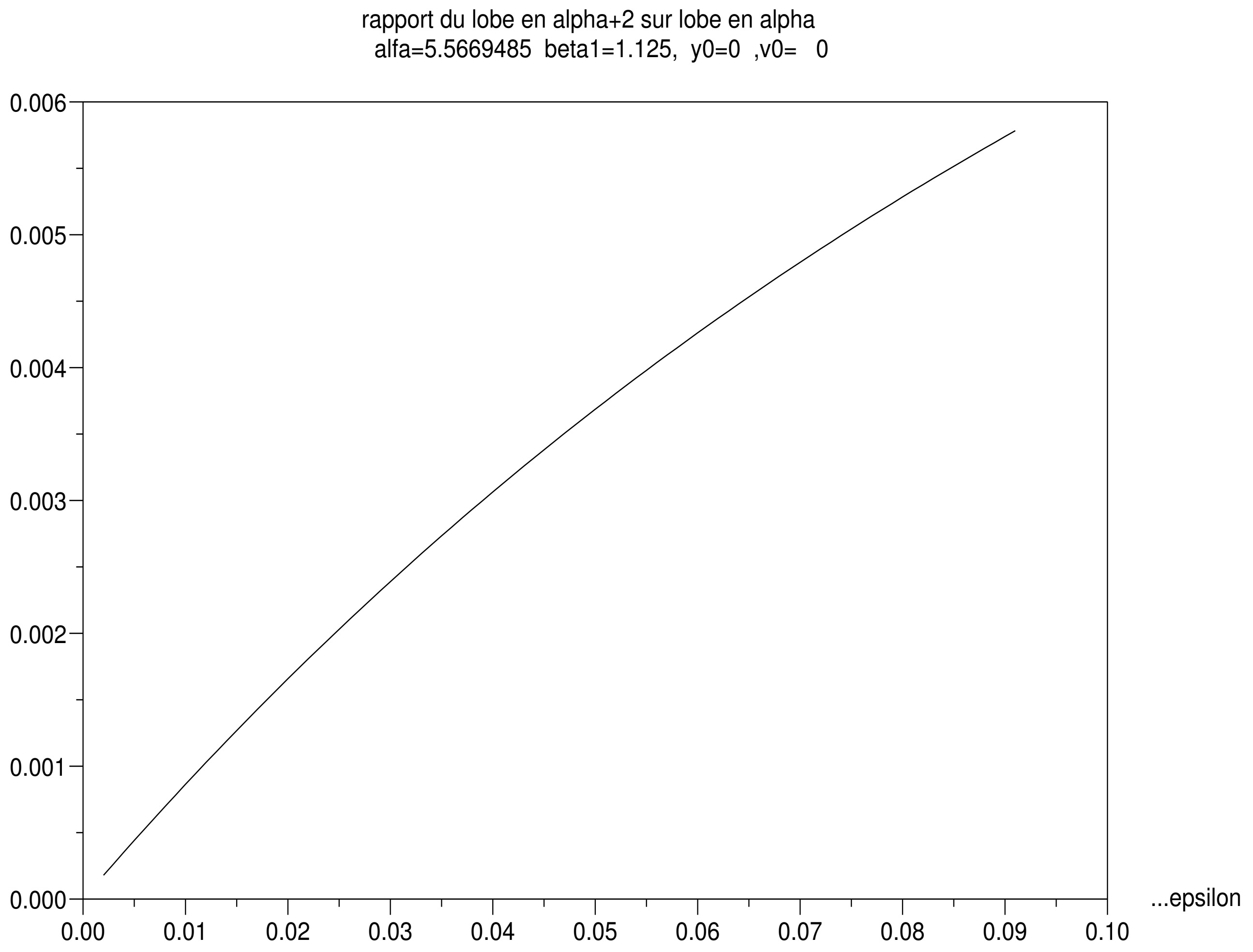


Figure 4: Rate of intermodulation to main lobe ($\nu_{\omega_0} = 1$)

4.3 Numerical issues

For $\alpha = 40\pi, \omega_0 = 2\pi, F = 100,\ \phi = .6348445087e - 2,\ \alpha\phi = .7977691380,$ (4.21)

$$v_1 = -.7578806812, \psi = -5.366518580,\ 3/4\psi^2\phi = -.1371241364 \quad (4.22)$$

$$\phi^3 \ll \psi^3 \quad (4.23)$$

General tendency:

- The pick of $3\omega_0$ is much larger than the pick in $\alpha \pm 2\omega_0$ which are the most natural picks in the experiments;
- it is delicate to find datas such that the secondary picks at $\alpha \pm 2\omega_0$ actually appear when the differential equation is solved numerically.
- Question: algorithm and software for detecting the secondary picks?
- then find (by optimization) datas such that the secondary picks are important: criteria for damage.

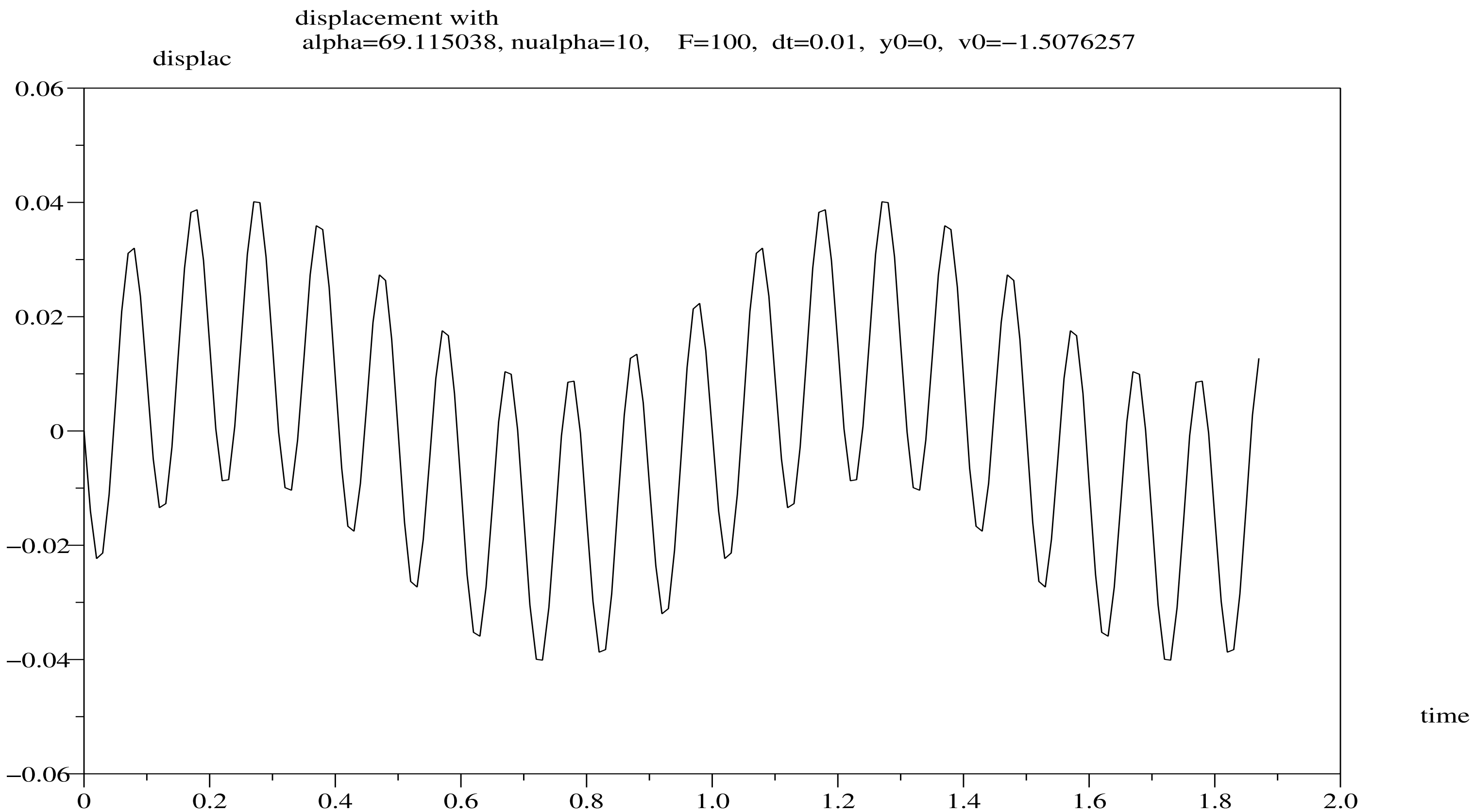


Figure 5: Linear response y, $\nu_{\omega_0} = 1$

y3

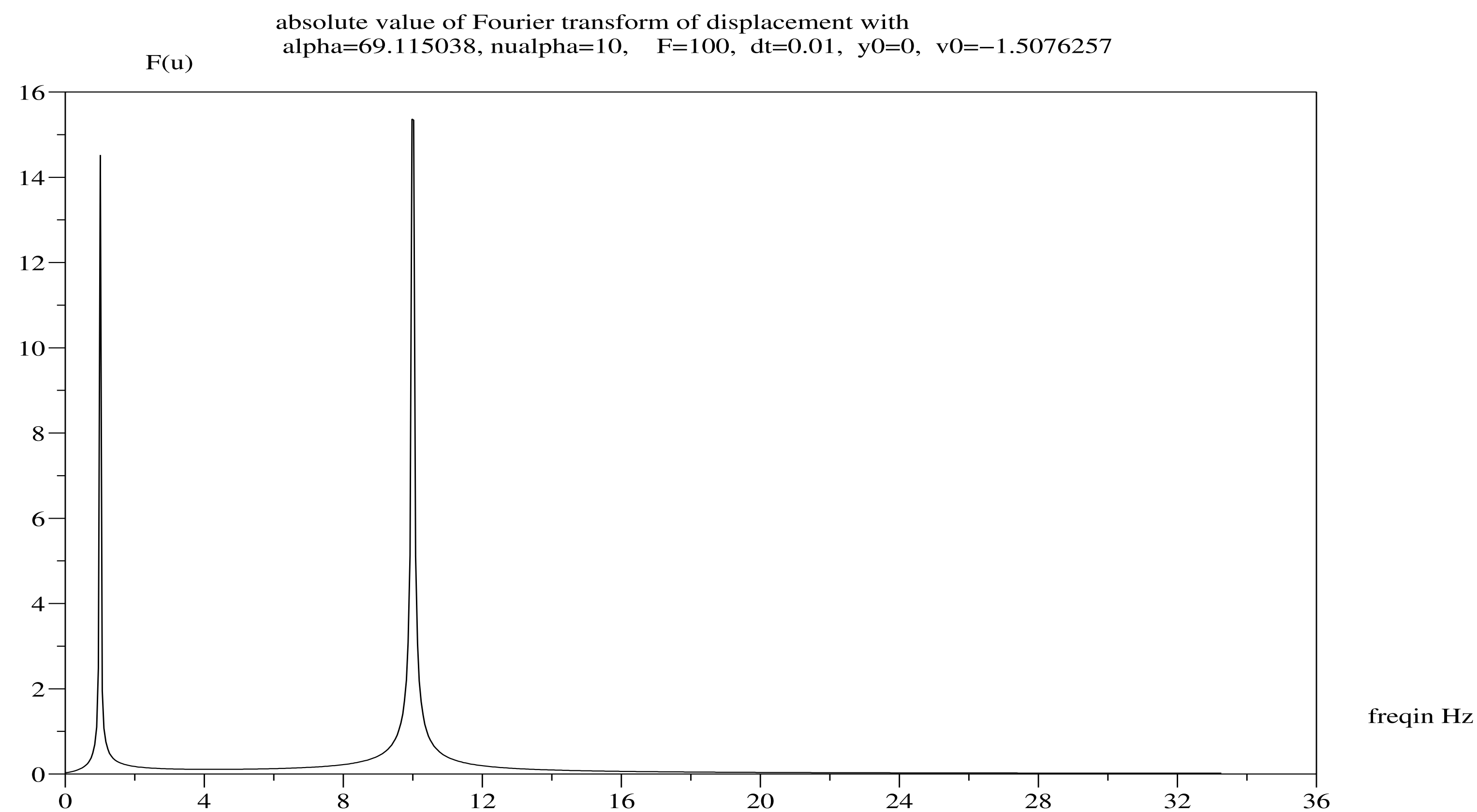


Figure 6: Fourier of linear response, $\nu_{\omega_0} = 1$

Fy3

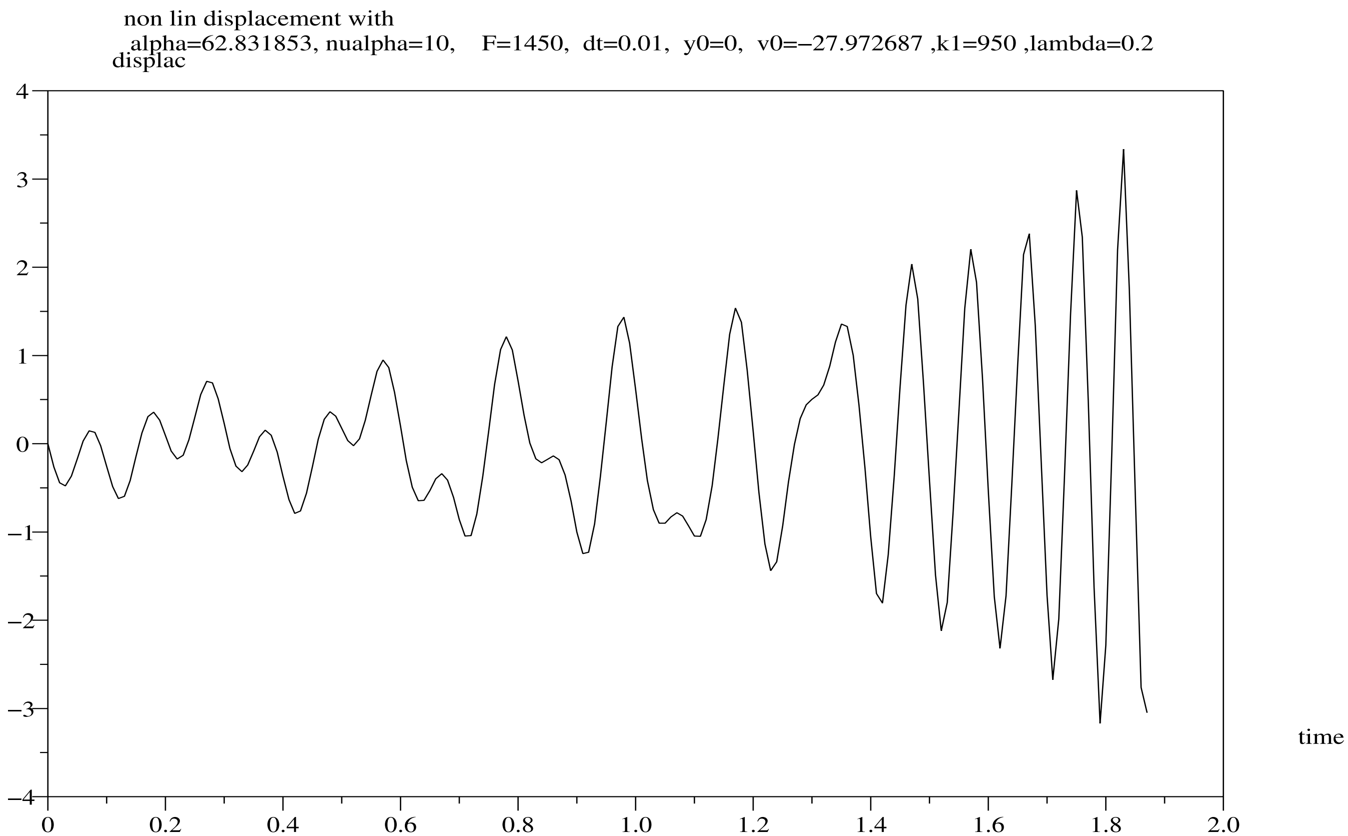


Figure 7: zoom of non linear response y, $\nu_{\omega_0} = 1$

y1

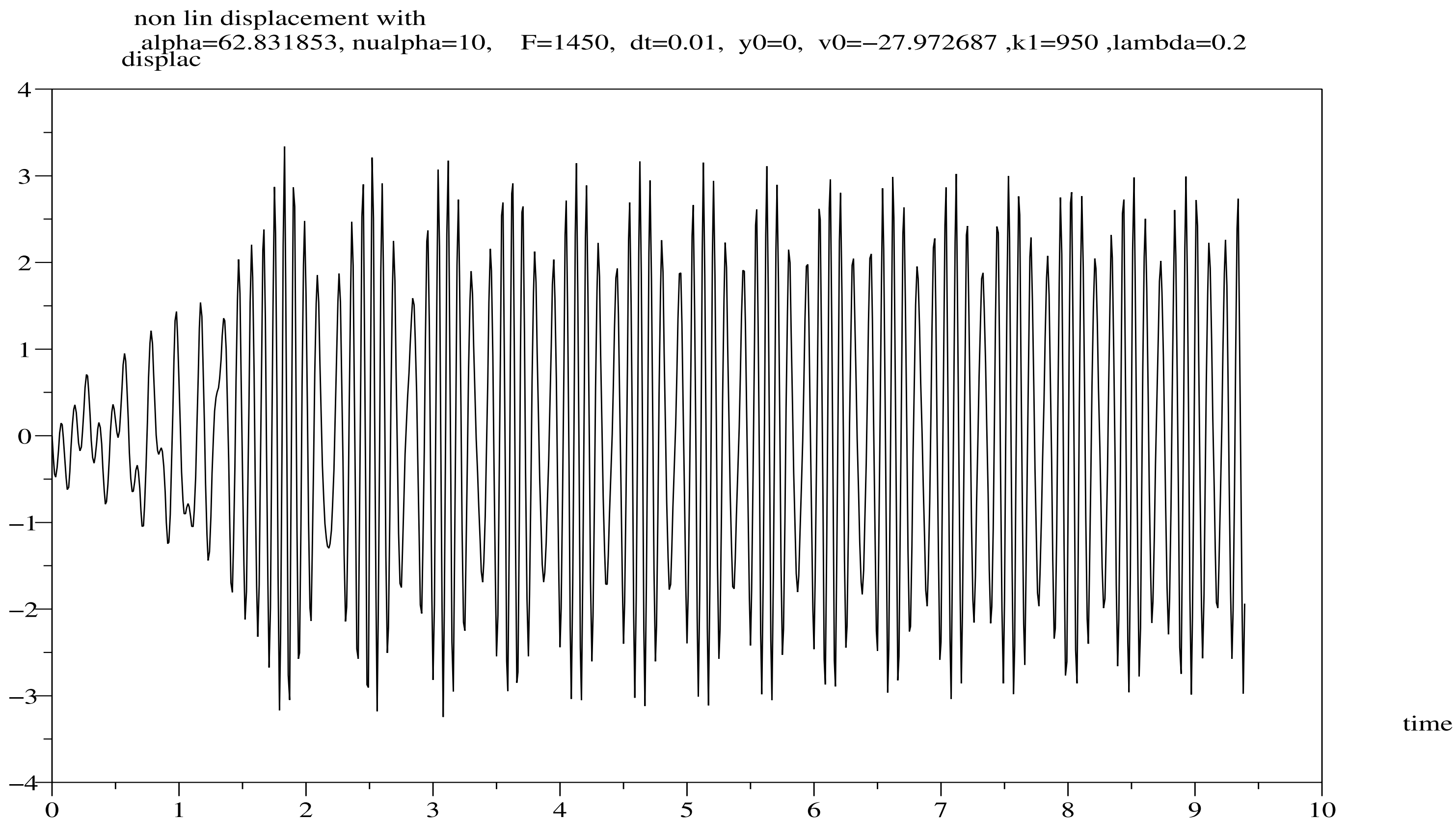


Figure 8: non linear response y, $\nu_{\omega_0} = 1$

y1lon

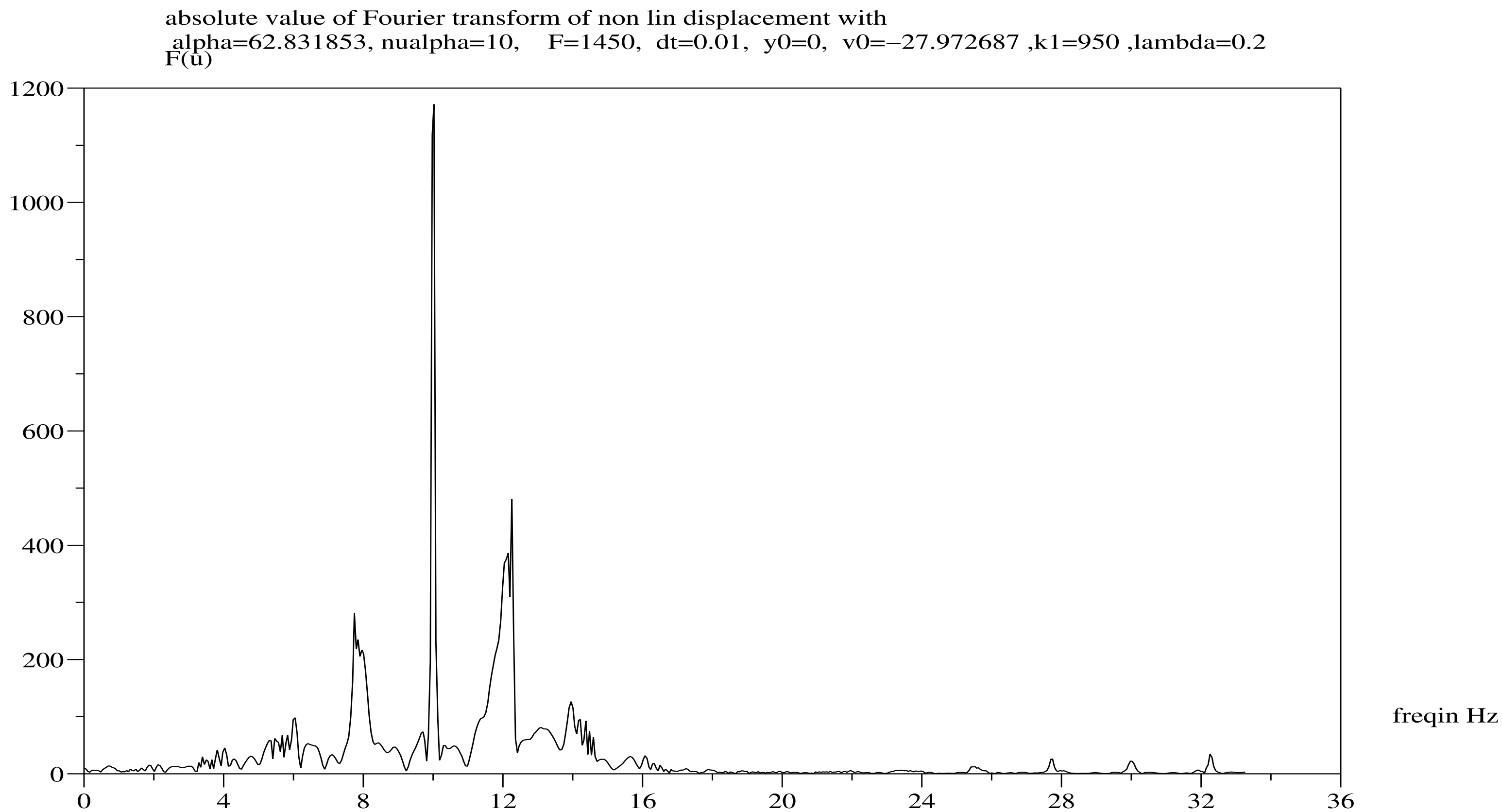


Figure 9: Fourier of non linear response y, $\nu_{\omega_0} = 1$

Fy1

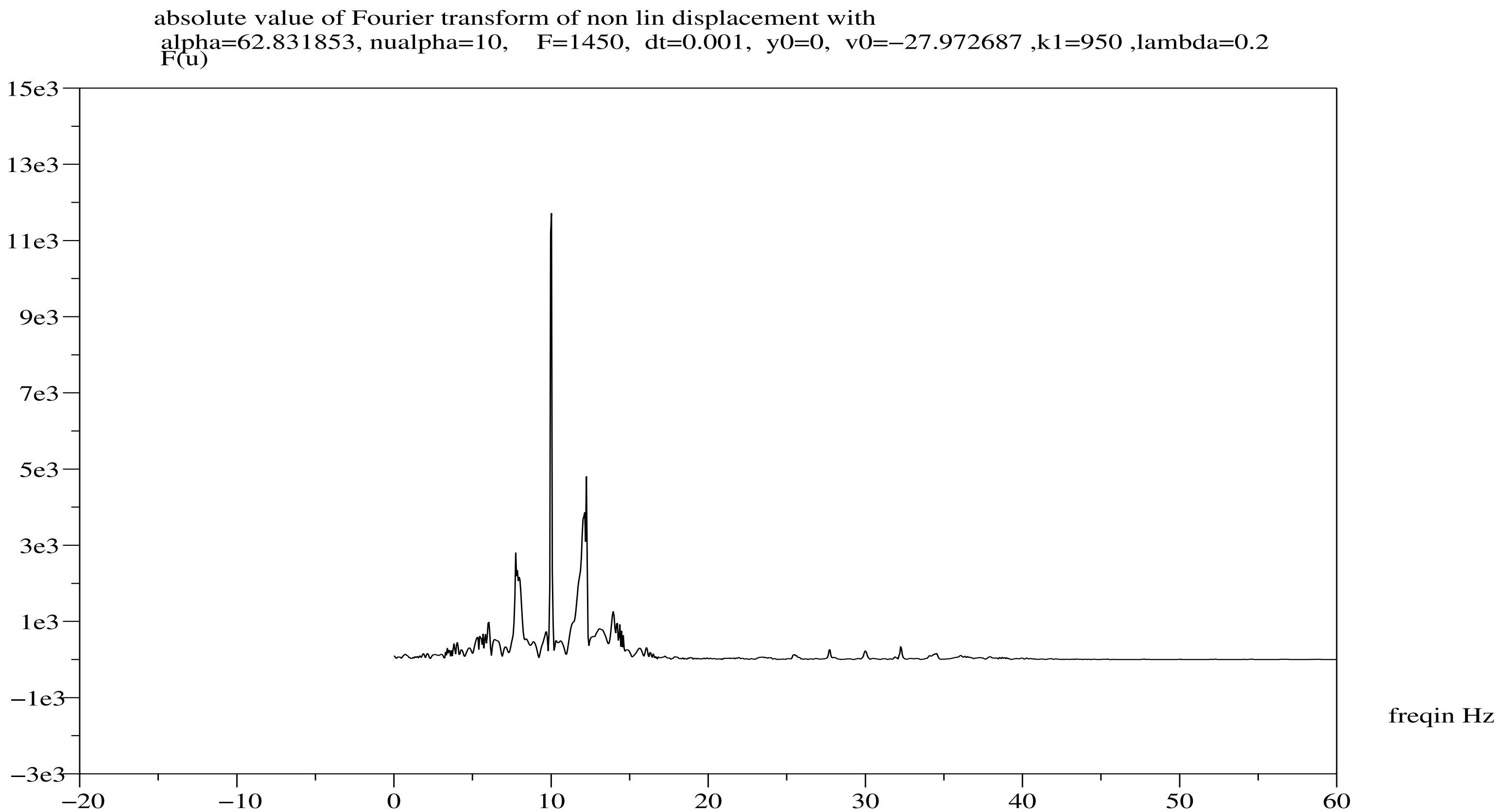
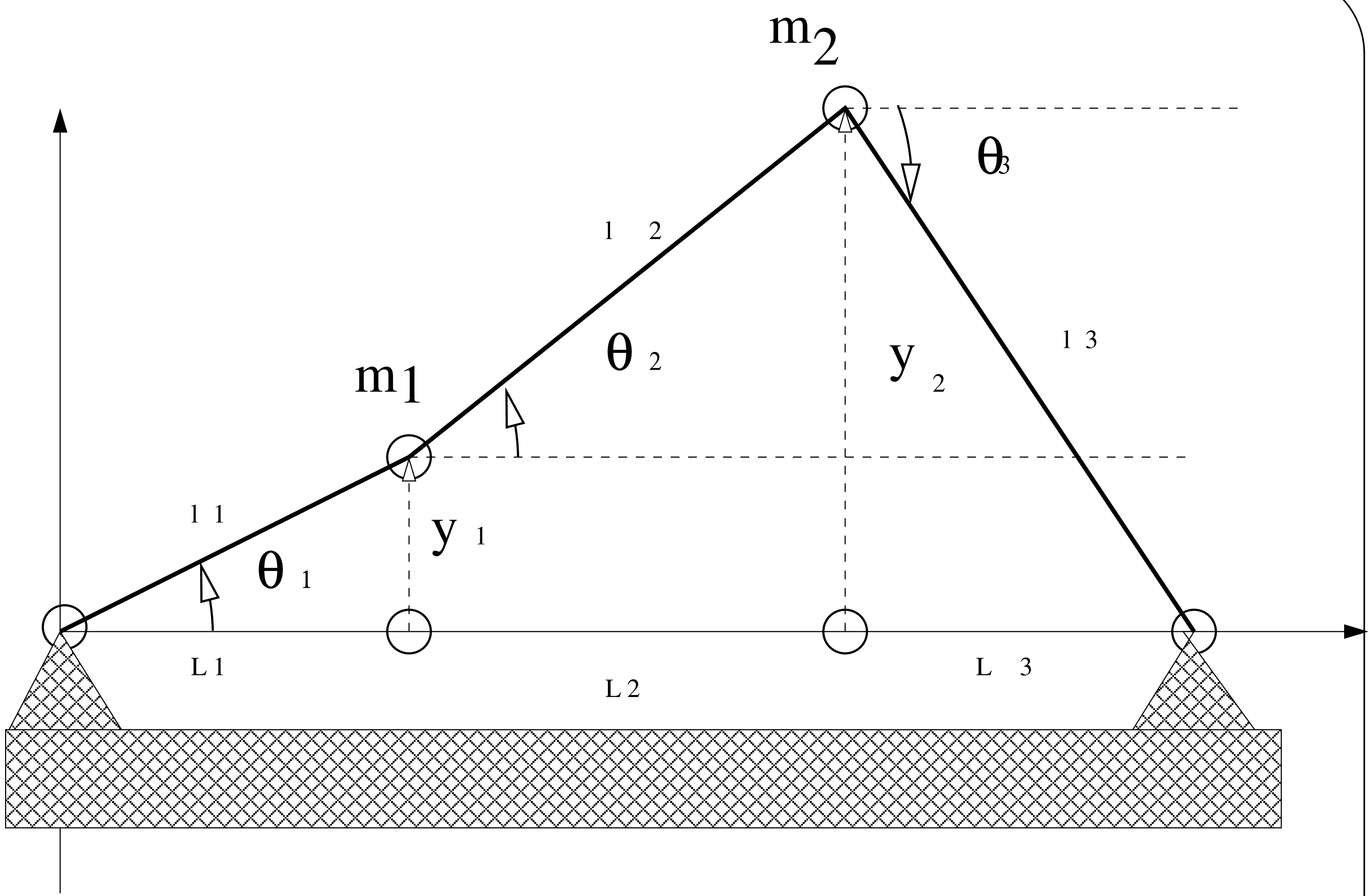


Figure 10: Fourier of non linear response y $\quad \nu_{\omega_0} = 1$

Fy1dte4



Two masses on stretched cables

5 Transverse vibrations: vibrating masses on streched cables in large displacement

Work performed with Theissen (doctoral student of U. Muenster); Erasmus students N. Goris and I. Altrogge worked on this topic during their stay in UNSA (2004-2005). We consider n masses attached to horizontal springs (or cables) which are in tension T_0, at rest ; the tension is positive when the cable is in traction which is assumed; at rest the mass m_i is submited to the force T the masses are moving (vertically) transversely to the springs; we denote by uper case letters quantities in the rest position and lower case in the current configuration.

5.1 Masses in vertical displacement

Here we assume that the masses can move only vertically.

- L_i lenth at rest; l_i lenth at time t; as the masses are moving vertically:

$l_i^2 = L_i^2 + (y_i - y_{i-1})^2$

- and **the change of tension of the linear elastic spring due to the change of of lenth**
  $T_i = T_0 + k_i[l_i(y) - L_i] = T_0 + k_i(\sqrt{L_i^2 + (y_i - y_{i-1})^2} - L_i)$. this tension is directed along the axis of the spring.

- Denote by θ_i, the angle of the spring with the horizontal axis, we have

- $y_1 = L_1 tan(\theta_1), \quad y_i - y_{i-1} = L_i tan(\theta_i) \quad\quad y_n - y_{n-1} = L_n tan(\theta_n).$

We enforce $y_n = 0$. See the picture with two masses and 3 cables.

The equation of the dynamics:

$$m_i y_i" = -T_i sin(\theta_i) + T_{i+1} sin(\theta_{i+1}) + u_i \quad i = 1 \ldots n \tag{5.1}$$

where $-T_i sin(\theta_i) + T_{i+1} sin(\theta_{i+1})$ is the vertical component of the force acting on mass i; we assume that there is no horizontal movement so the horizontal component of the force does not work. The applied load on mass i is denoted by u_i; it is the control to be determined.

with one degree of freedom section 5.2

Set

$$\zeta_i = \frac{(y_i - y_{i-1})}{L_i}, \quad \text{and note that } sin(arctan(\zeta_i)) = \frac{\zeta_i}{\sqrt{1+\zeta_i^2}} \text{ so that} \tag{5.2}$$

$$\tag{5.3}$$

possible approximations:

$$T_i sin(\theta_i) = T_0\zeta_i + (T_0 - k_i L_i)(-\frac{1}{2}{\zeta_i}^3 + \frac{3}{8}{\zeta_i}^5 + O\left({\zeta_i}^7\right)) \tag{5.4}$$

Same expansion for $T_{i+1} sin(\theta_{i+1})$ with $\zeta_{i+1} = \frac{(y_{i+1} - y_i)}{L_{i+1}}$ jump to one degree of freedom section 5.2

5.1.1 Linearized equation

$$m_i y_i'' = -T_0 \left(\frac{(y_i - y_{i-1})}{L_i} + \frac{(y_{i+1} - y_i)}{L_{i+1}} \right) + u_i$$

corrector equations may be obtained; details for 1 d.o.f below. **??**

5.2 Case with 1 d.o.f

5.2.1 Model with 1 d.o.f

In this case, with $y_0 = 0,\ y_2 = 0$ we have

$$m_1 y_1" = -T_1 sin(\theta_1) + T_2 sin(\theta_2) + u_1 \tag{5.5}$$

with $\theta_1 = atan(y_1/L_1), \quad \theta_2 = -atan(y_1/L_2)$

$$m_1 y_1" = -T_1 sin(atan(\frac{y_1}{L_1})) - T_2 sin(atan(\frac{y_1}{L_2})) + u_1 \tag{5.6}$$

The numerical solution of this model may be performed without stiff hypothesis with `scilab` routine `ode`; (sin(tan) is Lipshitz) but

it is not obvious to prescribe the right mechanical constants

to obtain clear intermodulation peaks;

also trouble of the experiments!

5.2.2 Approximation

Here set $\zeta_1 = \frac{y_1}{L_1}$, $\zeta_2 = -\frac{y_1}{L_2}$. Start from previous approximation

$$-T_1 sin(\theta_1) + T_2 sin(\theta_2) = \tag{5.7}$$

$$T_0(\zeta_2 - \zeta_1) - (T_0 - k_1 L_1)(-\frac{\zeta_1^3}{2} + \frac{3\zeta_1^5}{8}) + (T_0 - k_2 L_2)(-\frac{\zeta_2^3}{2} + \frac{3\zeta_2^5}{8}) + O(\zeta_1^7 + \zeta_2^7), \tag{5.8}$$

$$\text{expand } y_1 = \epsilon\eta_1 + \epsilon^2\eta_2 + \epsilon^3\eta_3 + O(\epsilon^4) \text{ to get} \tag{5.9}$$

$$-T_1 sin(\theta_1) + T_2 sin(\theta_2) = \tag{5.10}$$

$$-\epsilon T_0(\frac{1}{L_1} + \frac{1}{L_2})\eta_1 - \epsilon^2 T_0(\frac{1}{L_1} + \frac{1}{L_2})\eta_2 - \epsilon^3 T_0(\frac{1}{L_1} + \frac{1}{L_2})\eta_3 + \tag{5.11}$$

$$\frac{\epsilon^3}{2}\left(\frac{T_0 - k_1 L_1}{L_1^3} + \frac{T_0 - k_2 L_2}{L_2^3}\right)\eta_1^3 + O(\epsilon^4) \tag{5.12}$$

- The term in ϵ provides the linearised equation,

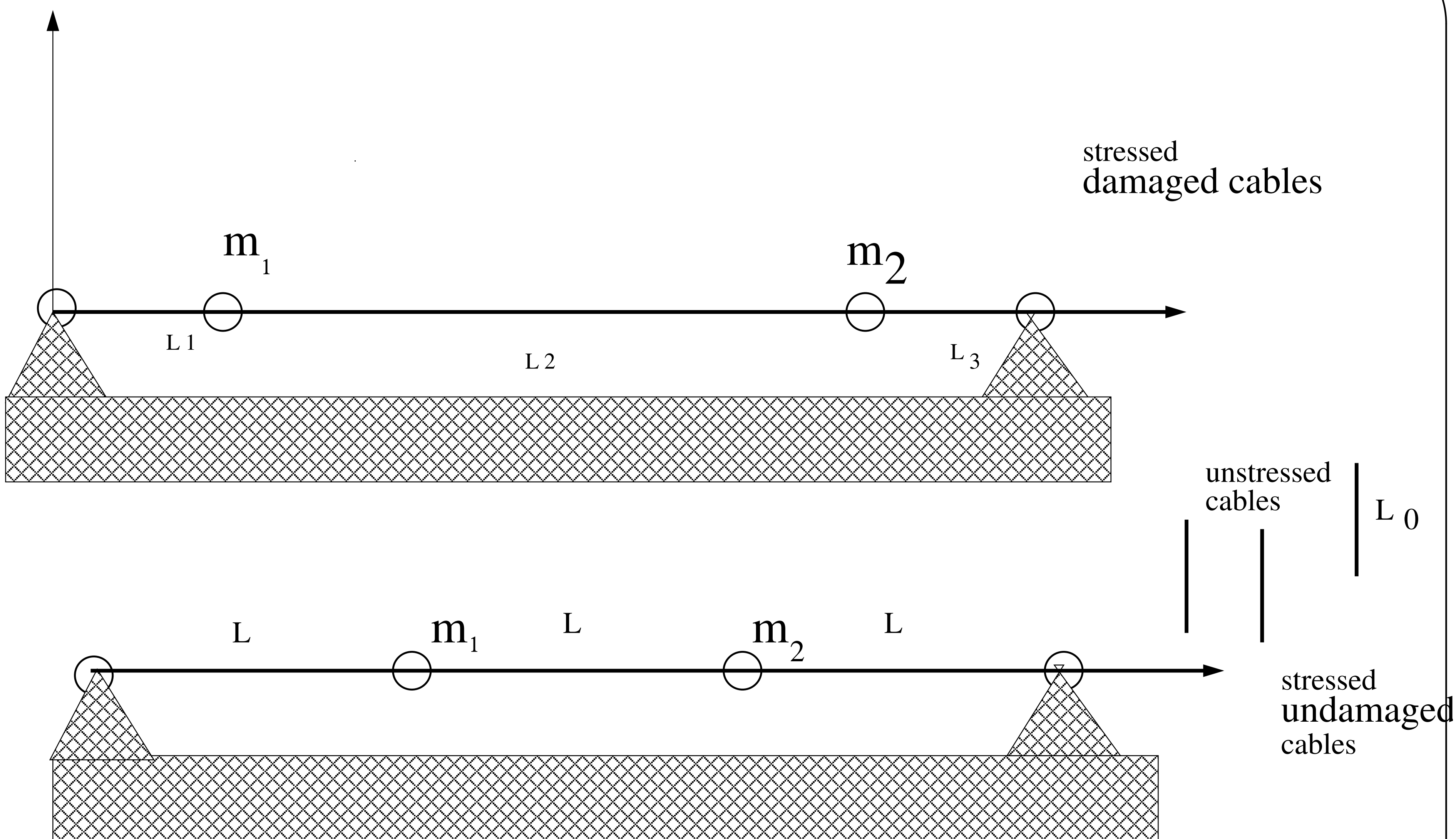
$$m_1\eta_1" = -T_0(\frac{1}{L_1}+\frac{1}{L_2})\eta_1 + u_1 \tag{5.13}$$

- the second equation provides $\eta_2 = 0$
- and the term in ϵ^3,

$$m\eta_3'' = T_0(\frac{1}{L_1}+\frac{1}{L_2})\eta_3 + \frac{1}{2}\left(\frac{T_0 - k_1 L_1}{L_1^3} + \frac{T_0 - k_2 L_2}{L_2^3}\right)\eta_1^3 \tag{5.14}$$

  equation similar to what is obtained for the simplest mechanical example!

jump to non linear string section 6



Two masses on stretched cables

5.2.3 A possible damage of a cable

is breakage of several fibers, this will cause decrease of rigidity k_1 say for cable 1.

- Let us start with undamaged cables of same rigidity k. If we note L_0, the common length of the unstressed cables, and L their common stressed lenth, their tension is $T_0 = k(L - L_0)$;
- now, after damage, $k_1 < k = k_2$, cable 1 becomes longer and cable 2 shorter, $L_1 > L_2$, the tension goes down to $T_{00} = k_1(L_1 - L_0) = k_2(L_2 - L_0)$;
- note the limit case of cable 1 broken is $k_1 = 0$ so that the cable 2 gets lenth L_0 but the system is no longer working properly!
- Before such a breakdown, if the change of tension is substantial, this causes a substantial change of the fundamental frequency; indeed, this is the routine monitoring of cable bridges!
- The nonlinear vibroacoustic testing aims at monitoring the cables before such a substantial change.

5.2.4 Datas

- L_0 unstressed length,
- L half of the lenth of the span, or lenth of each of the stressed undamaged cables.
- k undamaged spring constant,
- from which "undamaged" tension $T_0 = k(L - L_0)$,
- L_1 (with $L_0 < L_1 < L$) increased lenth of the damaged cable,
- from which, $L_2 = 2L - L_1$ decreased lenth of the undamged cable,
- from which "damaged"tension $T_{0d} = k(L_2 - L_0)$,
- from which spring constant of the damaged cable $k_1 = \frac{T_{od}}{L_1 - L_0}$

6 A non linear string model

A model of non linear string has been introduced first by Kirchoff in 1877 and rederived by Carrier in 1945.

$$y_{tt} - T(\int_0^l y_x^2) y_{xx} = f \tag{6.1}$$

For the classical linear string model, T is the tension of the string, **assumed to be constant**; in a next step, a natural asumption is:

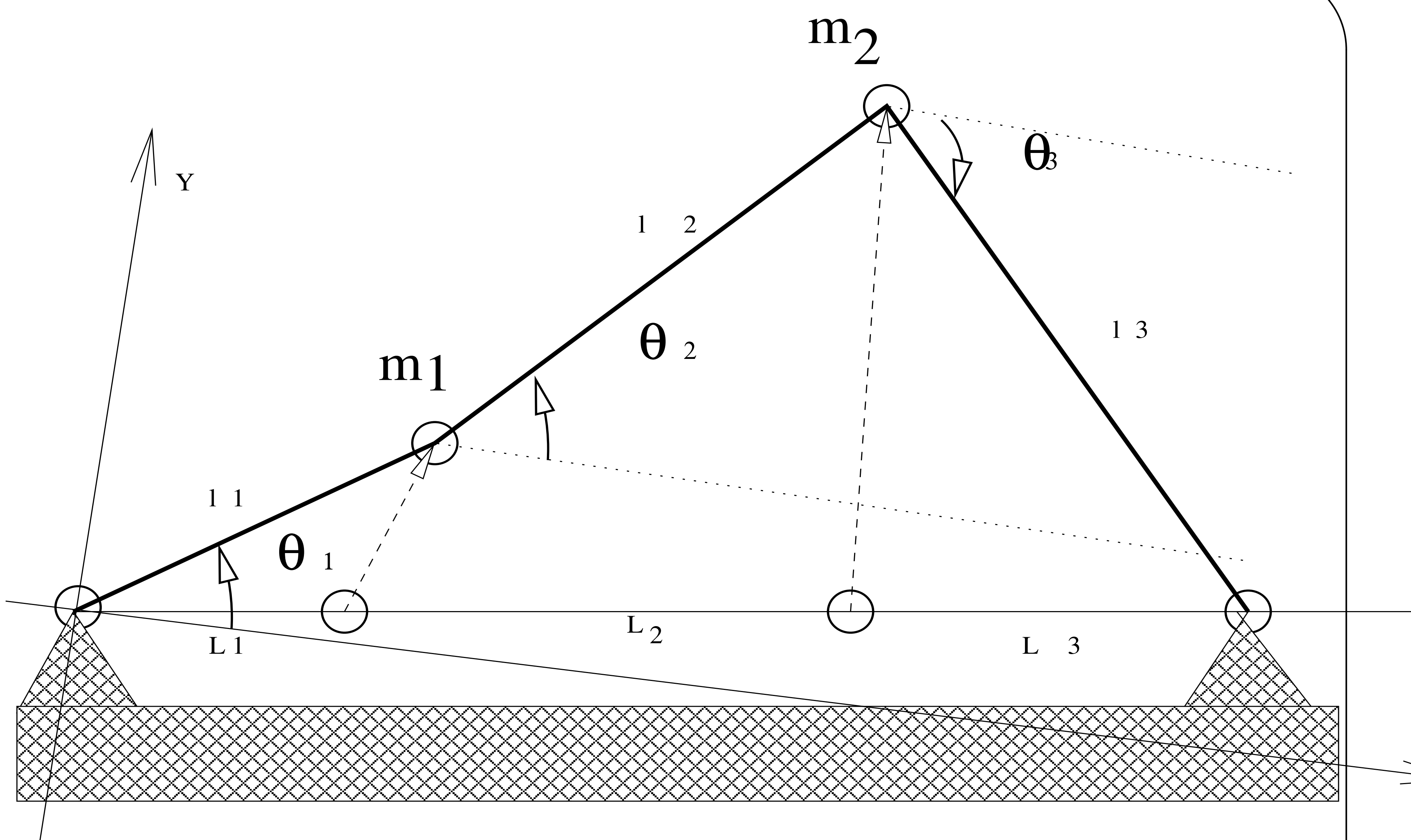
$$T = T_0 + k \int_0^l y_x^2$$

it involves the linearized change of lenth as the length of the deformed string is:

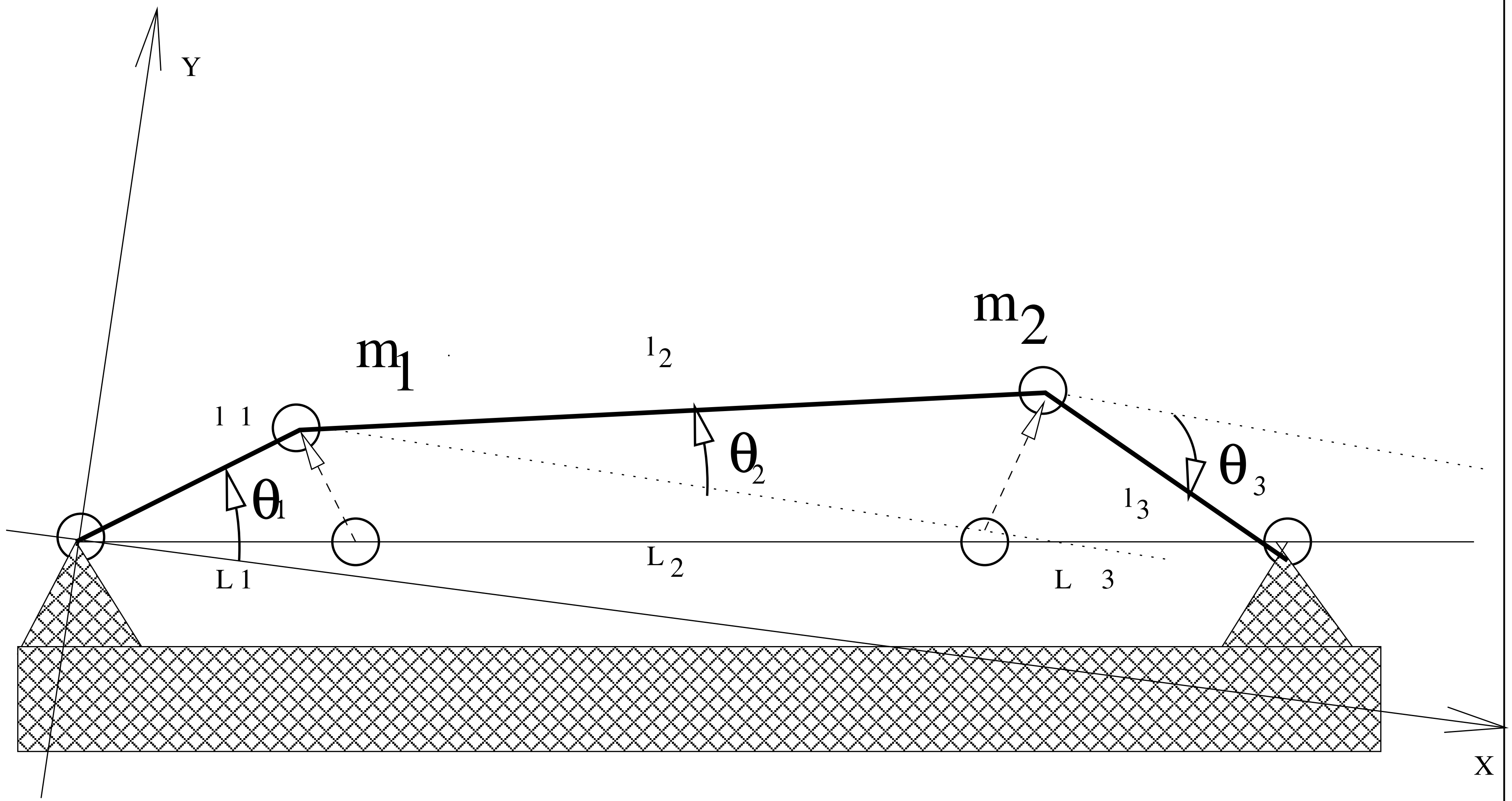
$$l(y) = \int_0^l \sqrt{1 + y_x^2}$$

- Several mathematical studies of this type of equations have been performed recently (Medeiros(1994), Clark- Lima (1997).
- Following the lines of the discrete model, we intend to **investigate** a string made of two materials (*safe and dameged*).
- For a damaged string, k will be small on a small portion of the string:

$$T = T_0 + k \int_0^{d-\epsilon} y_x^2 + k_d \int_{d-\epsilon}^{d+\epsilon} y_x^2 + k \int_{d+\epsilon}^{l} y_x^2$$



Two masses on stretched cables moving freely



Two masses on stretched cables (cable 2 damaged) moving freely

7 Masses moving freely in a plane

7.1 Model

Here, we assume that the masses can move freely; we denote:

$$\begin{pmatrix} X_i \\ Y_i \end{pmatrix} \text{ the position at rest, } \begin{pmatrix} x_i \\ y_i \end{pmatrix} \text{ the curent position} \tag{7.1}$$

- L_i lenth at rest; l_i lenth at time t; as the masses are moving freely:

$$l_i(x,y)^2 = \left((x_i - x_{i-1}) + (y_i - y_{i-1})\right)^2$$

- and the change of tension of the linear elastic spring due to the change of of lenth
  $T_i = T_{0,i} + k_i[l_i(x,y) - L_i] =$. this tension is directed along the axis of the spring:

$$\vec{T_i} = T_i \vec{\tau_i}$$

- Denote by θ_i, the angle of the spring with the horizontal axis, we have,

$$\vec{\tau}_i = \begin{pmatrix} cos\theta_i \\ sin\theta_i \end{pmatrix}$$

- $$y_i - y_{i-1} = l_i(x, y) sin\theta_i, \qquad x_i - x_{i-1} = l_i(x, y) cos\theta_i$$

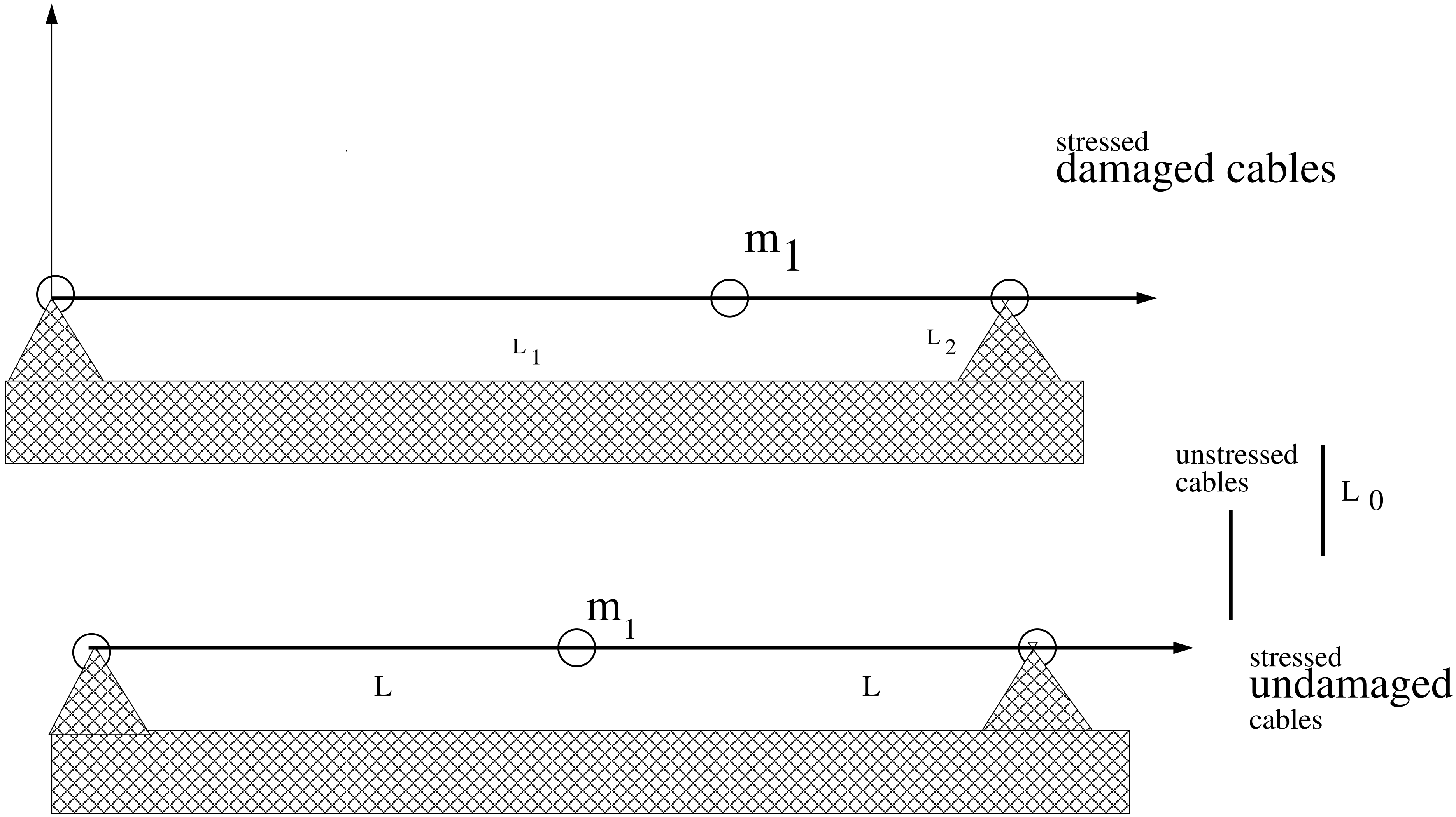
Equation of the dynamics

$$\begin{cases} m_i x_i" = -T_i cos(\theta_i) + T_{i+1} cos(\theta_{i+1}) + f_i \quad i = 1 \ldots n \\ m_i y_i" = -T_i sin(\theta_i) + T_{i+1} sin(\theta_{i+1}) + g_i \quad i = 1 \ldots n \end{cases} \tag{7.2}$$

We can express θ_i with respect to x_i, y_i, to obtain:

$$\begin{cases} m_i x_i" = -T_i(x,y)\frac{x_i - x_{i-1}}{l_i(x,y)} + T_{i+1}(x,y)\frac{x_{i+1} - x_i}{l_{i+1}(x,y)} + f_i \quad i = 1 \ldots n \\ m_i y_i" = -T_i(x,y)\frac{y_i - y_{i-1}}{l_i(x,y)} + T_{i+1}(x,y)\frac{y_{i+1} - y_i}{l_{i+1}(x,y)} + g_i \quad i = 1 \ldots n \end{cases} \tag{7.3}$$

7.2 A possible damaged model



One mass on stretched cables

7.2.1 possible damaged cable

is breakage of several fibers, this will cause decrease of rigidity k_1 say for cable 1.

- Let us start with undamaged cables of same rigidity k. If we note L_0, the common length of the unstressed cables, and L their common stressed lenth, their tension is $T_0 = k(L - L_0)$;
- now, after damage, $k_1 < k = k_2$, cable 1 becomes longer and cable 2 shorter, $L_1 > L_2$, the tension goes down to $T_{0dam} = k_1(L_1 - L_0) = k_2(L_2 - L_0)$;
- note the limit case of cable 1 broken is $k_1 = 0$ so that the cable 2 gets lenth L_0 but the system is no longer working properly!
- Before such a breakdown, if the change of tension is substantial, this causes a substantial change of the fundamental frequency; indeed, this is the routine monitoring of cable bridges!

The aim of non linear vibroacoustic testing :
monotoring the cables before
such a substantial change!

7.2.2 Datas

- L_0 unstressed length,
- L half of the lenth of the span, or lenth of each of the stressed undamaged cables.
- k undamaged spring constant, k_1 damaged spring constant,
- from which "undamaged" tension $T_0 = k(L - L_0)$,
- $T_{0dam} = k_1(L_1 - L_0) = k((2L - L_1) - L_0)$
- $L_1 = \frac{k-k_1}{k+k_1}(L - L_0) + L$ increased lenth of the damaged cable,
- from which, $L_2 = 2L - L_1$ decreased lenth of the undamged cable,

7.2.3 Damage and symmetry breaking

Lenthy computations by expansion:

$$x = L_1 + \epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 + \ldots \quad (7.4)$$

$$y = \epsilon v_1 + \epsilon^2 v_2 + \epsilon^3 v_3 + \ldots \quad (7.5)$$

show that the symetry breaking due to the lenth increase of the damaged cable causes the apparition of non linear terms which cancel for an undamaged system;

a substantial increase of the intermodulation lobes should appear!

Jump to bar model section 9

8 Actively controled system, non destructive testing

- The case of an actively controled system is prospective; real experiments are not yet performed.
- **Idea:** to detect damage in real time taking adavantage of the data processed by the real time actuators used for the optimal control; real time control, research group: “Echtzeit Optimierung grosser Systeme”in Germany.
- Example of the vibrating masses: the forces u_i are now the control we consider the simple case of a quadratic functional:

$$F(u) = \int_0^{t_f} \left(\sum_i u_i^2(t) \right) dt$$

with final time conditions:

$$y_i(t_f) = 0, \; y_i'(t_f) = 0$$

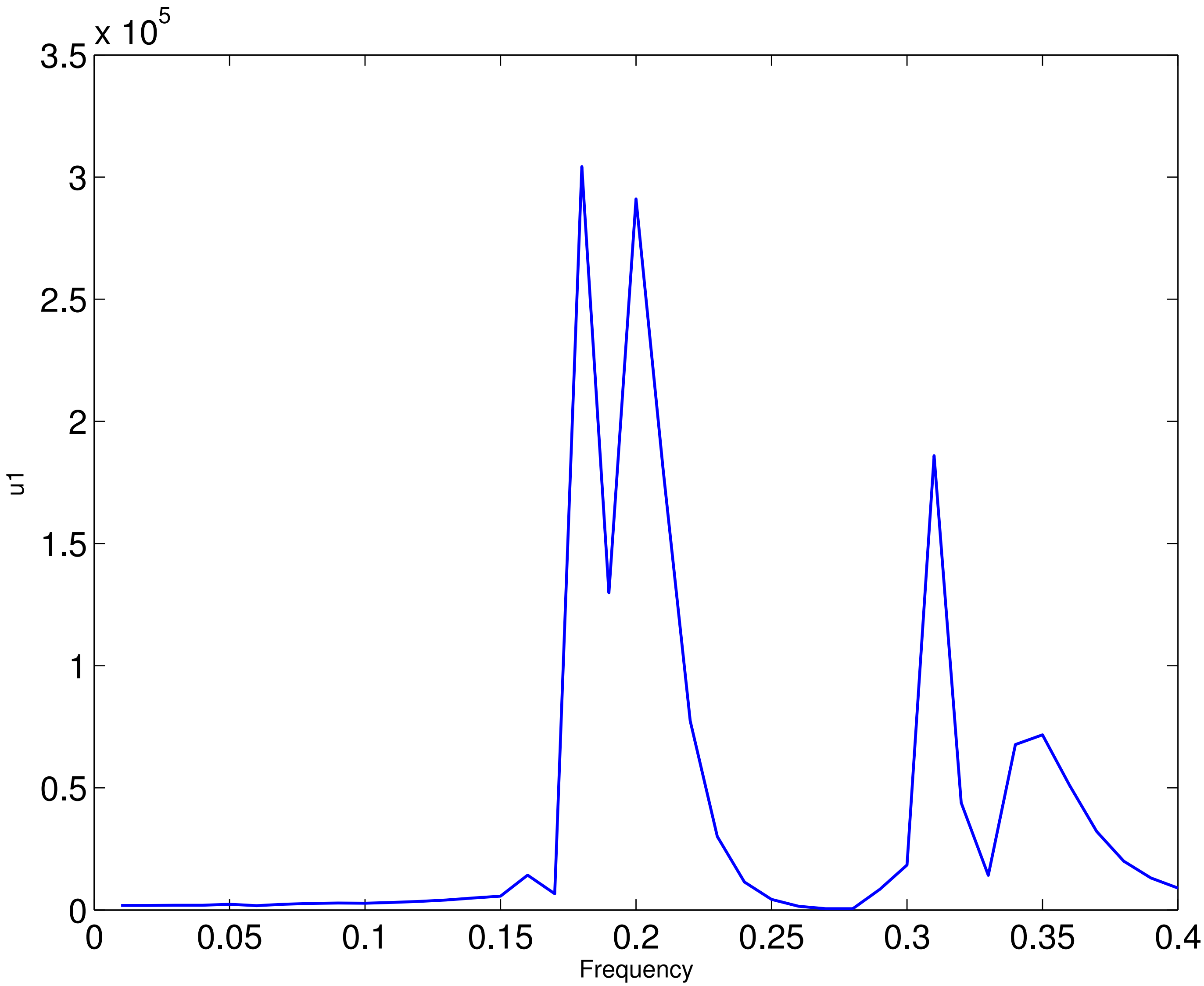
- The initial conditions may be seen as a perturbation of the system, the active control brings to rest the system;

- this process is supposed to be performed regularly during the lifetime of the system; in practice y_i is measured by sensors and the control u_i is a force performed by actuators; both devices transform electric energy in mechanical energy.
- the communication between both devices goes trough some computer
- If we are able to distinguish the response of a damaged system from an undamaged one, this opens the path of monitoring controled systems in real time as a dayly routine during their life.
- Numerical approach: to solve damaged and undamaged system and compare
- Perturbation approach, introduce a small parameter ϵ and expand the solution with respect to it; theoretical basis: the controled system should satisfy second order sufficient conditions (Malanowski, Maurer ...)

Datas for an example of controlled 2 masses worked out by by K. Theissen (U. Muenster)

$T_{0,1} = T_{0,2} = T_{0,3} =$	1
$k_1 = k_2 = k_3 =$	5
$m_1 = m_2 =$	1
$L_1 = L_2 = L_3 =$	1
t_f	100

Figure 11: Frequences of u_1



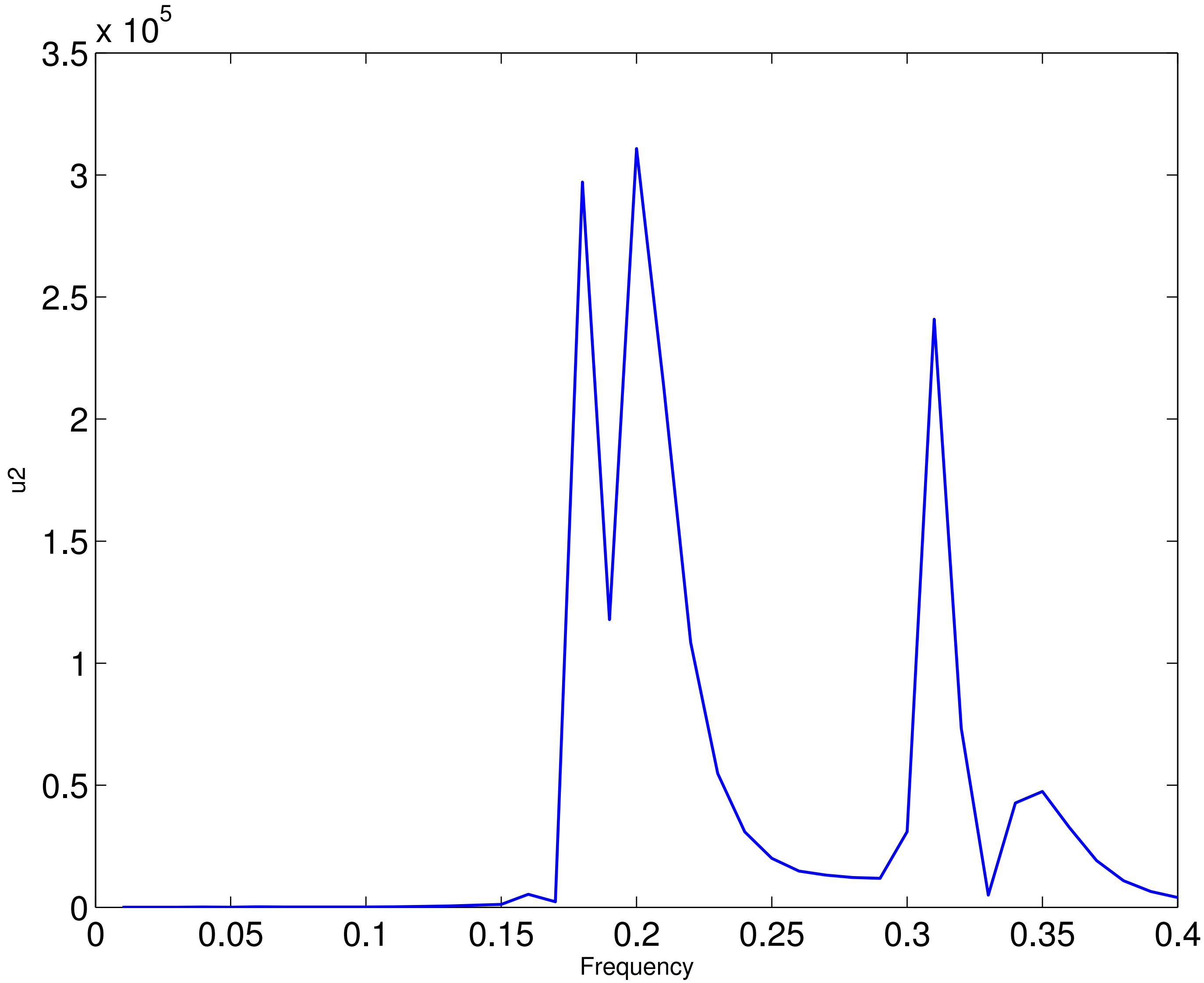
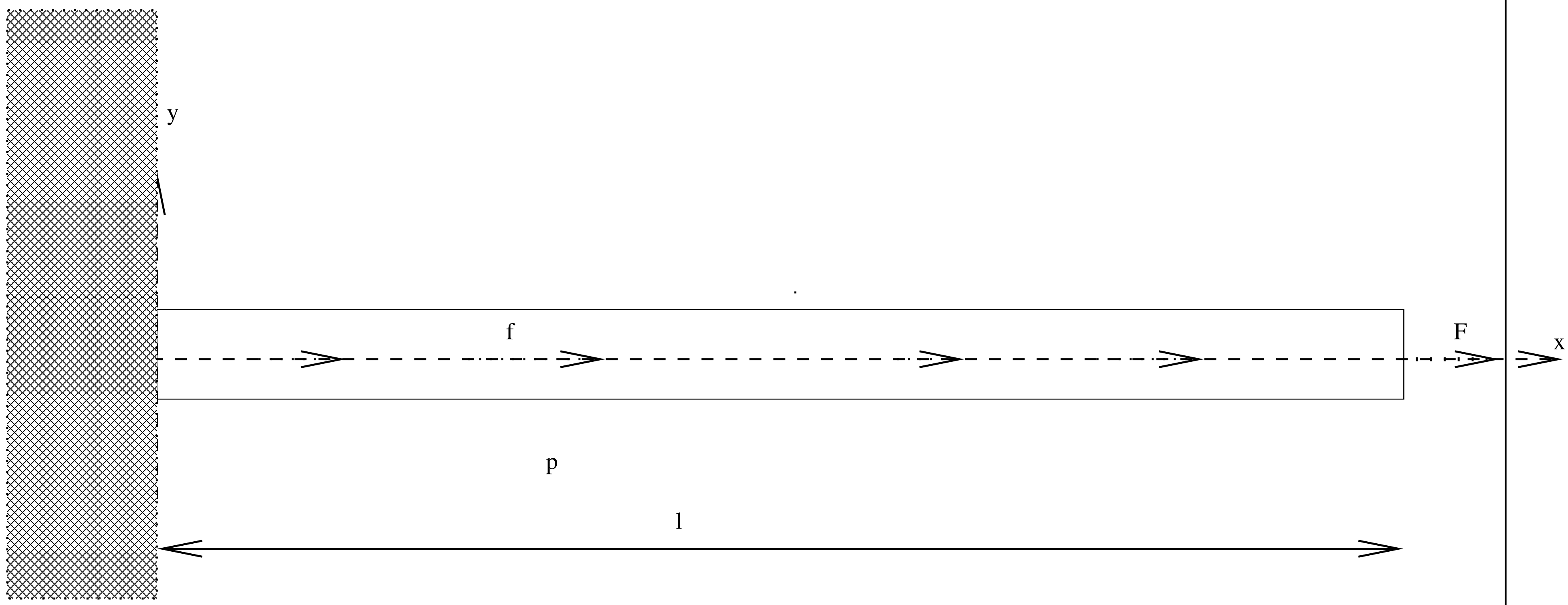


Figure 12: u_2


y
f
F
x
p
l

9 Bar models with defects

Bar models with longitudinal waves (dynamical traction and compression) are considered.

$$\rho \frac{\partial^2 u}{\partial t^2} - \frac{\partial n}{\partial x} = f(x,t) \tag{9.1}$$

With a non linear stress-strain law:

$$n = E(A\frac{\partial u}{\partial x} + \epsilon \chi_{[a,b]}(\frac{\partial u}{\partial x})^3) \tag{9.2}$$

Also a linear law is considered with a modified equation::

$$n = EA\frac{\partial u}{\partial x} \tag{9.3}$$

it may correspond to the action of a non linear spring acting on part of the bar :

$$\rho \frac{\partial^2 u}{\partial t^2} - \frac{\partial n}{\partial x} + \epsilon \chi_{[a,b]} u^3 = f(x,t) \tag{9.4}$$

We could as well assume that the applied load is of order epsilon without any assumption on the nonlinearity. Assuming ϵ to be small an approximate solution is searched for with the following "ansatz":

$$u = u_0 + \epsilon u_1 + \ldots \qquad \text{d'où} \tag{9.5}$$

$$u^3 = u_0^3 + 3\epsilon u_0^2 u_1 + \ldots \tag{9.6}$$

$$\frac{\partial u}{\partial x}^3 = \frac{\partial u_0}{\partial x}^3 + 3\epsilon \frac{\partial u_0}{\partial x}^2 \frac{\partial u_1}{\partial x} + \ldots \tag{9.7}$$

$$\tag{9.8}$$

From which we get for the non linear law:

$$n = E(A\frac{\partial u_0}{\partial x} + \epsilon\left(A\frac{\partial u_1}{\partial x} + \chi_{[a,b]}(\frac{\partial u_0}{\partial x})^3\right) + \ldots \tag{9.9}$$

and for the linear law:

$$n = EA(\frac{\partial u_0}{\partial x} + \epsilon\frac{\partial u_1}{\partial x}) + \ldots \tag{9.10}$$

Using these expansions, with the non linear law, the following system is obtained:

$$\begin{cases} \rho \frac{\partial^2 u_0}{\partial t^2} - EA \frac{\partial^2 u_0}{\partial x^2} = f(x,t) \\ \rho \frac{\partial^2 u_1}{\partial t^2} - EA \frac{\partial^2 u_1}{\partial x^2} = -E \frac{\partial}{\partial x} (\frac{\partial u_0}{\partial x})^3 \chi_{a,b} \end{cases} \tag{9.11}$$

For the modified equation the same equation for u_0 is found but for u_1:

$$\rho \frac{\partial^2 u_1}{\partial t^2} - EA \frac{\partial^2 u_1}{\partial x^2} = -(u_0)^3 \chi_{[a,b]} \tag{9.12}$$

Jump to the conclusion 10

Theoretical justification of the expansions:

Non liner law The situation is complex in full generality: non linear hyperbolic equations exhibit a singularity after a finite time! But: the experiments are performed during a short time interval and the Fourier transforms are computed on these time intervals! Following a suggestion of Guy Metivier we are addressing the problem during a small initial time interval in which the solution is smooth: plan to use an approximation of the equation with a fixed point method proposed in Majda. In any case we should smooth the characteristic function (the material is changing smoothly)!

Modified equation The situation is simpler; we can use a priori inequalities for this type of equation.

Jump to the conclusion 10

9.1 Explicit Solution

Coefficients are assumed to be constant and we consider:

Clamped at both ends: $u(x,0) = 0 = u(x,l)$; Eigenfunctions are introduced :

$$EA\frac{\partial^2 \phi}{\partial x^2} = -\lambda \rho \phi \tag{9.13}$$

$$\phi(0) = 0 = \phi(l) \tag{9.14}$$

we find $\lambda_k = \frac{k^2\pi^2}{l^2}\frac{EA}{\rho}$, on pose $\omega_k = \sqrt{\lambda_k}$ and the normalised eigenfunction: $\phi_k = \sqrt{\frac{2}{l}} sin(\frac{k\pi}{l}x)$.

9.1.1 Computation of u_0

let us consider | a force of frequency $\frac{\alpha}{2\pi}$ |

$$f(x,t) = Fcos(\alpha t)sin(\frac{k\pi}{l}x) \tag{9.15}$$

with initial velocity: $\frac{\partial u}{\partial t}(x,0) = 0$ The solution

$$u_0 = \frac{Fcos(\alpha t)}{\rho(-\alpha^2 + \lambda_k)} sin(\frac{k\pi}{l}x) \tag{9.16}$$

corresponding to an initial condition

$$u_0(x,0) = \frac{F}{\rho(-\alpha^2 + \lambda_k)} sin(\frac{k\pi}{l}x) \frac{\partial u_0}{\partial t}(x,0) \tag{9.17}$$

For the initial condition:

$$u_0(x,0) = a_0 sin(\frac{k\pi}{l}x) \tag{9.18}$$

the solution is:

$$u_0(x,0) = \left[\frac{F}{\rho(-\alpha^2+\lambda_k)}(cos(\alpha t) - cos(\omega_k t)) + a_0 cos(\omega_k t)\right] sin(\frac{k\pi}{l}x) \tag{9.19}$$

9.1.2 Computation of u_1

Considering the first solution with a global non linearity, we get:

$$u_0^3 = \frac{cos(\alpha t)^3}{\rho^3(-\alpha^2+\lambda_k)^3} sin^3(\frac{k\pi}{l}x) = \quad (9.20)$$

$$\frac{cos(\alpha t)^3}{\rho(-\alpha^2+\lambda_k)^3}\Big[cos(3\alpha t)sin(3\frac{k\pi x}{l}) - 3cos(3\alpha t)sin(\frac{k\pi x}{l}) \quad (9.21)$$

$$+3cos(\alpha t)sin(\frac{3k\pi x}{l}) - 9cos(\alpha t)sin(\frac{k\pi x}{l})\Big] \quad (9.22)$$

$$\frac{\partial u_0}{\partial x}^3 = \frac{k^3\pi^3}{l^3}u_0^3; \qquad \frac{\partial}{\partial x}\frac{\partial u_0}{\partial x}^3 = \frac{k^3\pi^3}{l^3}\frac{\partial u_0^3}{\partial x} = \quad (9.23)$$

$$(9.24)$$

solution u_1 with frŕfrequency $\frac{3\alpha}{2\pi}$

or $\frac{2\alpha}{2\pi}$ for a quadratic non linearity.

$$\frac{cos(\alpha t)^3}{\rho^3(-\alpha^2+\lambda_k)^3}\frac{k^4\pi^4}{l^4}\Big[3cos(3\alpha t)cos(3\frac{k\pi x}{l})+3cos(3\alpha t)cos(\frac{k\pi x}{l}) \quad (9.25)$$

$$+9cos(\alpha t)cos(\frac{3k\pi x}{l})-9cos(\alpha t)cos(\frac{k\pi x}{l})\Big] \quad (9.26)$$

Second case

for the second pair of boundary conditions, we set:

$$c = \frac{F}{\rho(-\alpha^2 + \lambda_k)} \quad d = \left(-\frac{F}{\rho(-\alpha^2 + \lambda_k)} + a_0 \right) \tag{9.27}$$

Now we have:

$$u_0 = (c\ cos(\alpha t) + d\ cos(\omega_k t))\ sin(\frac{k\pi x}{l}) \tag{9.28}$$

$$(u_0)^3 = \Big[\frac{c^3}{4}cos(3\alpha t) + \frac{3c}{2}(\frac{c^2}{2} + d^2)cos(\alpha t) +$$
$$\frac{3c^2d}{4}(cos((\omega_k + 2\alpha)t) + cos((\omega_k - 2\alpha)t)) +$$
$$\frac{3cd^2}{4}(cos((2\omega_k + \alpha)t) + cos((2\omega_k - \alpha)t) +)$$
$$+\frac{3d}{2}(\frac{d^2}{2} + c^2)cos(\omega_k t)\frac{d^3}{4}cos(3\omega_k t)\Big]$$
$$\frac{1}{4}\left(3sin(\frac{k\pi x}{l}) - sin(\frac{3k\pi x}{l})\right) \tag{9.29}$$

$$\tag{9.30}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u_0}{\partial x}\right)^3 = \frac{k^3\pi^3}{l^3}\frac{\partial u_0^3}{\partial x} = \tag{9.31}$$

$$\frac{3k^4\pi^4}{4l^4}\Big[\frac{c^3}{4}cos(3\alpha t) + \frac{3c}{2}(\frac{c^2}{2} + d^2)cos(\alpha t) +$$
$$\frac{3c^2d}{4}(cos((\omega_k + 2\alpha)t) + cos((\omega_k - 2\alpha)t)) +$$
$$\frac{3cd^2}{4}(cos((2\omega_k + \alpha)t) + cos((2\omega_k - \alpha)t) +)$$
$$+\frac{3d}{2}(\frac{d^2}{2} + c^2)cos(\omega_k t)\frac{d^3}{4}cos(3\omega_k t)\Big]$$
$$\left(cos(\frac{k\pi x}{l}) - cos(\frac{3k\pi x}{l})\right) \tag{9.32}$$

We notice clearly terms of frequency $\frac{\alpha}{2\pi}$ and $\frac{3\alpha}{2\pi}$ but also cross-modulations: $\frac{\omega_k+2\alpha}{2\pi}$ et $\frac{2\omega_k+\alpha}{2\pi}$ and frequencies $\frac{3\omega_k}{2\pi}$ $\frac{\omega_k}{2\pi}$. This last term provides secular terms for the corrector term u_1; they ought to be eliminated for example by using some renormalization technique:

$$t = s(1 + \epsilon\omega_1 + \dots) \tag{9.33}$$

We notice that the perturbation is larger if α is close to ω_k. this fact is used in practice: the applied load uses two frequencies with the low one at the first resonance in [10]. Here the low frequency is excited by the initial conditions.

10 Conclusion

- Some simple models governed by ODE or PDE show intermodulations;
- But the relative level of secondary peaks for a given set of datas deserves investigations: indeed it is also the difficulty of the real experiments
- The use of explicit expansions is necessary to understand the behavior of secondary peaks!
- Need to include other behaviors: shocks, friction
- Need of more precise models: non linear beams including tractional, flexural, torsional effects; plates, shells, smart materials (piezoelectric...).
- Mixture of local models for the defect and global models for the undamaged structure to obtain precise results at low computational cost.

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