Calculating maps between n-categories

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Introduction

This short note addresses the problem of how to calculate in a reasonable way the homotopy classes of maps between two n-categories (by which we mean Tamsamani's n-nerves [10]). The closed model structure of [9] gives an abstract way of calculating this but it isn't very concrete so we would like a more down-to-earth calculation. For the purposes of the present note we shall use the closed model structure of [9] to prove that our method gives the right answer. It might be possible to develop the theory entirely using the present calculation as the definition of "map", but that is left as an open problem.

Denote by $n-Cat$ the category of n-categories introduced by Tamsamani [10], and by $L(n-Cat)$ the Dwyer-Kan simplicial localization [2] where we divide out by the equivalences between n-categories (Tamsamani calls these "external equivalences"). Note that the 1-category obtained from $L(n-Cat)$ by applying π_0 to the simplicial Hom sets is just the Gabriel-Zisman [4] localization $Ho(n-Cat)$ which was introduced in [10].

We work with the model category nPC of n-precats introduced in [9], and we shall adopt the notations as from there. In particular, nPC is the category of presheaves over the category Θ^n.

In [9] the procedure for calculating the homotopy classes of maps from A to B consists of choosing a fibrant replacement $B \to B'$ and looking at maps from A to B'; two maps are homotopic if there is a homotopy $A \times \overline{I} \to B$ relating them (where $\overline{I}$ is the 1-category with two isomorphic objects). The problem is that the notion of fibrant replacement is not very explicit, depending on the addition of a wide class of pushouts by trivial cofibrations to B to get B'. Here we would like to pursue the alternative strategy of replacing A by a "free cofibrant" object $F \to A$, much as was done for morphisms of diagrams in Bousfield-Kan [1]. Then we look at maps from F to B. The advantage is that F can be made much more explicit, and we shall only need to assume that B is an n-category.

The interest of having such a description developed mainly out of conversations with A. Hirschowitz during and in the aftermath of [6].

The notion of free cofibration is a basic consequence of the shape of cells coming from Tamsamani's original "constancy conditions" and the corresponding quotient construction for Θ^n as a quotient of Δ^n.

At the first level in the multisimplicial approach to n-categories, one meets cells which look like k-simplices for any k. A k-simplex corresponds to k one-morphisms which can be composed, together with their compositions. On this level, the notion of free cofibration is easy to imagine: it is just the inclusion of the set of $k+1$ vertices (i.e. the objects), into the k-simplex. Our fundamental observation is that this definition extends into the other simplicial directions (corresponding to higher arrows) in a natural way.

Once one has noticed the notion of "free cofibration" of n-precats, the argument is a straightforward application of postmodern closed model category techniques as developped principally by Reedy, Dwyer, Kan, Hirschhorn [8] [2] [3] [5].

We end up in Corollary 9 with a very explicit description of how to calculate the set of homotopy classes of maps from A to B in $Ho(n-Cat)$.

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Free cofibrations

We define a class of morphisms in nPC called *free cofibrations*. These are analogues of the Bousfield-Kan cofibrations in the theory of diagrams [1]. This class is generated by the following *elementary free cofibrations*. Suppose $M = (m_1, \ldots, m_k)$ is an object in Θ^n. Then $h(M) \in nPC$, and we define the *elementary free cofibration with target* $h(M)$ denoted

$$\partial(M) \to h(M)$$

in the following way. For any other index object $N \in \Theta^n$, the elements of $\partial(M)_N$ are defined as those elements $a \in h(M)_N$ corresponding to maps $a : N \to M$ such that a factors through some map of the form

$$(m_1, \ldots, m_{k-1}, 0) \to (m_1, \ldots, m_k) = M.$$

In other words, $\partial(M)$ is the union of the images of the $m_k + 1$ morphisms $\hat{M} \to M$ where $\hat{M} := (m_1, \ldots, m_{k-1}, 0)$.

An *elementary free cofibration* is any cofibration corresponding to some M as above. A *free cofibration* is any cofibration that is obtained as a sequential (possibly transfinitely sequential) pushout by elementary free cofibrations. In other words the class of free cofibrations is the class generated by the elementary free ones in the sense of [3].

Lemma 1 *For any M the "boundary" $\partial(M)$ is itself obtained from $\emptyset$ by a sequence of pushouts along elementary free cofibrations.*

At the top level, $\partial(M)$ is obtained by glueing together $m_k + 1$ copies of $h(\hat{M})$ along $\partial(\hat{M})$.

Proof: The statement about what happens at the top level follows directly from the shapes of Tamsamani's cells; or alternatively from the definition of the category Θ^n as a quotient of Δ^n. Indeed from the definition, $\partial(M)$ is clearly the image of the disjoint union U of $m_k + 1$ copies of $h(\hat{M})$. Two elements of U_N go to the same element of $\partial(M)_N$ if and only if they go to the same element of $h(M)_N$. Thus two elements of U_N represented by $a, a' : N \rightarrow \hat{M}$ (corresponding to different copies of $h(\hat{M})$ i.e. to different maps $0 \rightarrow m_k$) go to the same element, if and only if their compositions $N \rightarrow M$ are the same. In view of the definition of Θ^n, this holds if and only if a and a' factor through some map $N \rightarrow \hat{\hat{M}}$. This gives the last statement of the lemma.

The first statement follows by induction in the following way. Let $M^{[i]}$ denote the object $(m_1, \ldots, m_i, 0)$. Thus if M has length k then $\hat{M} = M^{[k-1]}$. By induction using the statement about what happens on the top level, we get that $\partial(M^{[i]})$ is obtained from $\partial(M^{[i-1]})$ by adding on $m_i + 1$ copies of $h(M^{[i-1]})$. Each of these copies is obtained by taking the coproduct with the cofibration

$$\partial(M^{[i-1]}) \rightarrow h(M^{[i-1]}).$$

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The main point behind the notion of free cofibrations is the following homotopy lifting property.

Proposition 2 *Suppose B is an n-category and $i : B \rightarrow B'$ is a fibrant replacement. Then any free cofibration satisfies the up-to-homotopy lifting property for i. More precisely, if $A \rightarrow C$ is a free cofibration and if*

$$\begin{array}{ccc} A & \rightarrow & B \\ \downarrow & & \downarrow \\ C & \rightarrow & B' \end{array}$$

is a commutative diagram, then there is a northeast diagonal map $g : C \to B$ which is a lifting up to homotopy, i.e. there exists a homotopy $C \times \overline{I} \to B'$ fixing A, starting at the original map $C \to B'$ and ending at a map which factors through $g : C \to B$.

Proof: It suffices to consider the case where the cofibration $A \to C$ is of the form $\partial(M) \to h(M)$. Let k be the length of M and denote by $m := m_k$. A map $h(M) \to B$ (resp. $h(M) \to B'$) is the same thing as an object of the set

$$B_M = B_{\hat{M},m} \quad (resp. B'_M = B'_{\hat{M},m})$$

Let E (resp. E') denote the $n+1-k$-category $B_{\hat{M}/}$ (resp. $B_{\hat{M}/}$). Then a map $f : h(M) \to B$ (resp. $f' : h(M) \to B'$) is the same thing as an object of E_m (resp. E'_m).

Fix a morphism $a : \partial(M) \to B$. This may be thought of as a collection of elements

$$\alpha_0, \ldots, \alpha_m \in B_{\hat{M}}$$

which agree over $\partial(\hat{M})$. In turn these correspond to objects of E. A map f or f' as above which restricts to a on $\partial(M)$ is just an element

$$f \in E_{m/}(\alpha_0, \ldots, \alpha_m)$$

or

$$f' \in E'_{m/}(\alpha_0, \ldots, \alpha_m).$$

The fact that $B \to B'$ is an equivalence of n-categories yields the statement that the map

$$i(a) : E_{m/}(\alpha_0, \ldots, \alpha_m) \to E'_{m/}(\alpha_0, \ldots, \alpha_m)$$

is an equivalence of $n-k$-categories [10]. In particular, given f' in the right side, there is an f on the left which maps to an object equivalent to f'. It is straightforward to turn this equivalence between $i(a)f$ and f' into a homotopy from f' to $i \circ f$, using the fibrant property of B' and traditional closed model category techniques. ///

The following lemma establishes a relationship between lifting for the elementary free cofibrations, and being an equivalence.

Lemma 3 *Suppose that $f : A \to B$ is a morphism of n-precats, such that B is an n-category and such that f satisfies the on-the-nose lifting property for every elementary free cofibration. Then A is an n-category and f is an equivalence.*

Proof: We prove this by induction on n. If $m \in \Delta$ and $N \in \Theta^{n-1}$ then a diagram

$$\begin{array}{ccc} \partial(m,N) & \to & h(m,N) \\ \downarrow & & \downarrow \\ A & \to & B \end{array} .$$

is the same thing as a choice of $m+1$ objects $x_0, \ldots, x_m \in A_0$ together with a diagram

$$\begin{array}{ccc} \partial(N) & \to & h(N) \\ \downarrow & & \downarrow \\ A_{m/}(x_0, \ldots, x_m) & \to & B_{m/}(f(x_0), \ldots, f(x_m)). \end{array}$$

This can be seen by using the description given by Lemma 1.

Similarly, a lifting $h(m,N) \to A$ is the same thing as a lifting $h(N) \to A_{m/}(x_0, \ldots, x_m)$.

In particular if f satisfies the lifting property then so do the morphisms

$$f_{m/}(x) : A_{m/}(x_0, \ldots, x_m) \to B_{m/}(fx_0, \ldots, fx_m).$$

By induction we conclude that the $A_{m/}(x_0, \ldots, x_m)$ are $n-1$-categories and that these morphisms $f_{m/}(x)$ are equivalences. In turn, the fact that B is an n-category and that these morphisms are equivalences, implies that A is an n-category and that f is fully faithful. Essential surjectivity of f follows from surjectivity on objects which is a limiting case of the lifting property. ///

Remark: It would probably be a good idea to replace the notion of “easy equivalence” which was used at the start of [9], by the notion of morphism satisfying the lifting property of this lemma.

Good resolutions

Next we apply the strategy of Reedy-Dwyer-Kan for calculating the space of maps from an n-category A to another one B in the localized simplicial category $L(n-Cat)$, by using a cosimplicial resolution of A. The trick is to use a cosimplicial resolution which is Reedy-cofibrant in terms of free cofibrations in nPC.

Recall that a cosimplicial object $F^\cdot$ in a model category M is *Reedy-cofibrant* if for every k the morphism

$$Latch^k(F) \to F^k$$

is a cofibration in M. In our situation, we are not sure whether there exists a closed model structure on nPC which has free cofibrations as its fibrations (there probably does...). However, it still makes sense to ask that a cosimplicial object $F^\cdot$ be Reedy-cofibrant with respect to the class of free cofibrations: this just means that we ask for every k for the morphism

$$Latch^k(F) \to F^k$$

to be a free cofibration. We call such a resolution, *Reedy-free-cofibrant.*

The cornerstone of our calculation is the construction of a Reedy-free-cofibrant resolution of any n-category A.

Proposition 4 *Suppose A is an n-category. Then there is a natural functorial cosimplicial object $F^\cdot$ in nPC together with a morphism to the constant cosimplicial object associated to A (we denote this by $F^\cdot \to A$) such that each stage $F^k \to A$ is an equivalence of n-categories, and such that $F^\cdot$ is Reedy-cofibrant with respect to the class of free cofibrations in nPC.*

Proof: There are two canonical ways to complete any morphism of n-precats $U \to V$ to a diagram

$$U \to W \to V$$

where the first morphism is a free cofibration and the second morphism satisfies the lifting property with respect to the elementary free cofibrations.

The "small" way is to complete the diagram by adding in lifts of all elementary free cofibrations which don't already have lifts. This is of course to be prefered in calculations, but it is not functorial.

The "big" way to complete the above diagram is to add in lifts of all elementary free cofibrations (without looking to see whether there is already a lift), and iterate over the first infinite ordinal ω. The big way is functorial, so we adopt it for the theory.

If V is an n-category then by Lemma 3, W is also an n-category and the morphism $W \to V$ is an equivalence.

Using this construction, the standard construction of the Reedy-cofibrant cosimplicial resolution (adding in each stage by enforcing the lifting property) works to give the desired $F^{\cdot}$.

In order to deal with the degeneracies (although it isn't clear to me that we really need them) one needs the following observation: if $U \to V \leftarrow W$ is a diagram of n-categories such that both maps satisfy the lifting property of Lemma 3, then the fiber product $U \times_V W \to V$ also satisfies the lifting property and in particular it is again an n-category equivalent to the other ones.

We explicitly describe the first couple of steps in the standard construction. First choose a free cofibrant F^0 with map $F^0 \to A$ satisfying the lifting property of Lemma 3. Then choose F^1 fitting into the diagram

$$F^0 \sqcup F^0 \to F^1 \to F^0$$

so that the first map is a free cofibration and the second map (the "degeneracy") satisfies the lifting property. Let C be the pushout

$$C := (F^1 \cup^{F^0} F^1) \cup^{F^0 \sqcup F^0} F^1,$$

in other words it is three copies of F^1 glued together at three copies of F^0 in the form of a triangle. It is the latching object: $C = Latch^2(F)$. We have a map

$$C \to F^1 \times_{F^0} F^1$$

where the first projection is the identity in the first and last components of C and the degeneracy on the middle component; and the second projection is the identity on the second and third components and the degeneracy on the first component (for the author at least it was easier to write down the dual map for simplicial objects). From the previous paragraph the target $F^1 \times_{F^0} F^1$ is again an n-category mapping via an equivalence to F^0. Choose F^2 to fit into a diagram

$$C \to F^2 \to F^1 \times_{F^0} F^1$$

where the first map is a free cofibration and the second map satisfies the lifting property of Lemma 3.

For the rest of the cosimplicial object, continue in the same way.

The whole cosimplicial object projects to F^0 by the degeneracy maps, and this in turn maps to A. We obtain the map $F^{\cdot} \rightarrow A$ which on every stage is an equivalence of n-categories $F^k \cong A$. ///

Applying the up-to-homotopy lifting property of Proposition 2 to the resolutions provided by Proposition 4 will give the calculation we are looking for. This is resumed in the following proposition.

Proposition 5 *Suppose $F^{\cdot} \rightarrow A$ is a Reedy-free-cofibrant cosimplicial resolution as in Proposition 4. Suppose $i : B \rightarrow B'$ is a fibrant replacement of an n-category B. Then the morphism of simplicial sets*

$$Hom_{nPC}(F^{\cdot}, B) \rightarrow Hom_{nPC}(F^{\cdot}, B')$$

is a weak equivalence.

Proof: This general property of maps having the up-to-homotopy lifting property with respect to a class of "cofibrations" probably goes back to Reedy [8]. We give an argument for completeness.

Suppose K is a finite simplicial set. Recall that by adjunction we can define an operation

$$K \mapsto K \otimes F^{\cdot} \in nPC$$

such that a map

$$K \rightarrow Hom_{nPC}(F^{\cdot}, B)$$

is the same thing as a map $K \otimes F^{\cdot} \rightarrow B$. We claim that if $K \rightarrow L$ is a cofibration of finite simplicial sets then

$$K \otimes F^{\cdot} \rightarrow L \otimes F^{\cdot}$$

is a free cofibration in nPC. Indeed, for the generating cofibrations of simplicial sets $\partial \Delta^k \rightarrow \Delta^k$ we just get back the original free cofibrations $Latch^k(F) \rightarrow F^k$ in the definition of Reedy-free-cofibrant resolution.

Suppose $(*)$

$$\begin{array}{ccc} K & \rightarrow & Hom_{nPC}(F^{\cdot}, B) \\ \downarrow & & \downarrow \\ L & \rightarrow & Hom_{nPC}(F^{\cdot}, B') \end{array}$$

is a diagram. It corresponds to a diagram

$$\begin{array}{ccc} K \otimes F^{\cdot} & \rightarrow & B \\ \downarrow & & \downarrow \\ L \otimes F^{\cdot} & \rightarrow & B' \end{array}$$

and since we saw above that the left vertical arrow is a free cofibration, Proposition 2 gives a homotopy lifting. This means that there is a lifting in the original diagram such that the two resulting morphisms

$$L \rightrightarrows Hom_{nPC}(F^{\cdot}, B')$$

come from maps

$$L \otimes F^{\cdot} \rightrightarrows B'$$

which are related by a homotopy

$$L \otimes F^{\cdot} \times \overline{I} \rightarrow B'$$

or equivalently

$$L \otimes F^{\cdot} \rightarrow \underline{Hom}(\overline{I}, B').$$

Thus the two maps $L \rightrightarrows Hom_{nPC}(F^{\cdot}, B')$ are related by a "homotopy" which is a map

$$L \rightarrow Hom_{nPC}(F^{\cdot}, \underline{Hom}(\overline{I}, B')).$$

The operation $Hom_{nPC}(F^{\cdot}, -)$ is homotopy-invariant (and takes fibrations to fibrations) when the argument is a fibrant object of nPC. Thus it takes a path-space object in nPC in the sense of Quillen, such as $\underline{Hom}(\overline{I}, B')$, to a path-space object in the Kan closed model category of simplicial sets. In particular, if two maps

$$L \rightrightarrows Hom_{nPC}(F^{\cdot}, B')$$

are related by a "homotopy" in the above sense, then they are related by a Quillen homotopy in the closed model category of simplicial sets. We have now shown that the morphism of simplicial sets

$$Hom_{nPC}(F^{\cdot}, B) \rightarrow Hom_{nPC}(F^{\cdot}, B')$$

satisfies the up-to-homotopy lifting property with respect to any cofibration $K \rightarrow L$ of finite simplicial sets. This implies that it is a weak equivalence. ///

Note in the situation of the above proposition, the F^k and B are both n-categories. Recall that $n-Cat$ is a full subcategory of nPC so the morphism which is an equivalence by the proposition may be written as

$$Hom_{n-Cat}(F^{\cdot}, B) \rightarrow Hom_{n-Cat}(F^{\cdot}, B').$$

Corollary 6 *Suppose $F^{\cdot} \to A$ is a free-cofibrant cosimplicial resolution as in Proposition 4, and suppose B is an n-category. Then the simplicial set $Hom_{n-Cat}(F^{\cdot}, B)$ is naturally equivalent to the simplicial set of morphisms from A to B in the simplicial category $L(n-Cat)$.*

Proof: In [2] (see also [3]) it is shown that the simplicial set of morphisms in the Dwyer-Kan localization is naturally equivalent to the simplicial set obtained by using a Reedy-cofibrant cosimplicial resolution of the domain object and assuming that the range-object is fibrant. The resolution $F^{\cdot} \to A$ is Reedy-cofibrant for the usual notion of cofibration in the closed model structure of [9], because free cofibrations are in particular cofibrations. Thus there is a natural equivalence of simplicial sets

$$Hom_{L(nPC),\cdot}(A, B) \cong Hom_{nPC}(F^{\cdot}, B').$$

On the other hand, $L(n-Cat)$ is sandwiched in between $L(nPC_f)$ and $L(nPC)$ but the latter two are equivalent [2] so

$$Hom_{L(n-Cat),\cdot}(A, B) \cong Hom_{L(nPC),\cdot}(A, B).$$

Finally, by the previous proposition and composing, we obtain the equivalence

$$Hom_{L(n-Cat),\cdot}(A, B) \cong Hom_{nPC}(F^{\cdot}, B) = Hom_{n-Cat}(F^{\cdot}, B).$$

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Corollary 7 *Suppose $F^{\cdot} \to A$ is a free-cofibrant cosimplicial resolution as in Proposition 4, and suppose B is an n-category. Then the set of morphisms from A to B in $Ho(n-Cat)$ is calculated as*

$$Hom_{Ho(n-Cat)}(A, B) = \pi_0 Hom_{n-Cat}(F^{\cdot}, B).$$

Proof: This follows immediately from the fact that $Ho(n-Cat)$ is just π_0 of $L(n-Cat)$ (cf [2] for this fundamental property of the Dwyer-Kan localisation $L(-)$). ///

Further remarks

In order to calculate the π_0 (i.e. the set of morphisms in $Ho(n-Cat)$) we only need the first stage of the Reedy-free-cofibrant resolution:

$$F^0 \rightrightarrows F^1 \to A.$$

This yields a diagram of sets

$$Hom(F^1, B) \rightrightarrows Hom(F^0, B)$$

and $Hom_{Ho(n-Cat)}(A, B)$ is the set-theoretic coequalizer of these two morphisms. In particular, from this description two maps $F^0 \to B$ could represent homotopic maps from A to B, but only be related by a chain of two or more maps $F^1 \to B$. We shall now show that actually there is no need to refer to chains of equivalences.

Lemma 8 *Suppose F^0 is free-cofibrant, and*

$$F^0 \sqcup F^0 \to F^1$$

is a free cofibration, fitting into a diagram

$$F^0 \rightrightarrows F^1 \to A$$

where all maps to A are equivalences of n-categories. Then for any n-category B, two maps $F^0 \to B$ are homotopic if and only if their disjoint union extends to a map $F^1 \to B$.

Proof: Fix the diagram $F^0 \rightrightarrows F^1 \to A$ in question. We call a *basic homotopy* between maps $F^0 \to B$, any one which is obtained by a morphism $F^1 \to B$. To prove the lemma, it suffices to show that the composition of two basic homotopies is again homotopic to a basic homotopy. For this choose an extension of the resolution to level 2, i.e. with $F^2 \to A$ restricting to three copies of the map $F^1 \to A$. Let D denote the union of the first two copies of F^1 (it is in nPC but not an n-category). Then $D \to F^2$ is a weak equivalence of n-precats. It is also a free cofibration. Two homotopies result in a map $D \to B$; compose this to obtain a map $D \to B'$ which extends to $h' : F^2 \to B'$ since B' is fibrant. The fact that $D \to F^2$ is a free cofibration allows us to apply Proposition 2 to move h' back to a map $h : F^2 \to B$, which gives (by restricting to the third copy of F^1) a basic homotopy composing the first two given ones. ///

Corollary 9 *Suppose A is an n-category. Choose a diagram*

$$F^0 \rightrightarrows F^1 \to A$$

with F^0 a free cofibrant object and $F^0 \sqcup F^0 \to F^1$ a free cofibration, and all objects being n-categories mapping by equivalences to A. Then for any n-category B the set of homotopy classes of maps from A to B in $Ho(n-Cat)$ may be described as the set of maps $F^0 \to B$ modulo the relation that two such maps are equivalent if and only if there exists a map $F^1 \to B$ restricting to their disjoint union on $F^0 \sqcup F^0$.

Proof: This is what we just showed in the previous lemma. ///

Exercise 1: *Let C denote the 1-category with three objects $0, 1, 2$ and morphisms $f : 0 \to 1$ and $g : 1 \to 2$ composing to $gf : 0 \to 2$. Consider C as a 2-category, and calculate a free-cofibrant 2-category F with an equivalence $F \to C$.*

Exercise 2: *If F is a free cofibrant object then the set of maps from F to an n-category B has an easy inductive description in terms of the cells which were used to construct F. Write this out in detail.*

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