FORMALIZED PROOF, COMPUTATION, AND THE CONSTRUCTION PROBLEM IN ALGEBRAIC GEOMETRY

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It has become a classical technique to turn to theoretical computer science to provide computational tools for algebraic geometry. A more recent transformation is that now we also get *logical* tools, and these too should be useful in the study of algebraic varieties. The purpose of this note is to consider a very small part of this picture, and try to motivate the study of computer theorem-proving techniques by looking at how they might be relevant to a particular class of problems in algebraic geometry. This is only an informal discussion, based more on questions and possible research directions than on actual results.

This note amplifies the themes discussed in my talk at the “Arithmetic and Differential Galois Groups” conference (March 2004, Luminy), although many specific points in the discussion were only finished more recently.

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1. The construction problem

One of the basic problems we currently encounter is to give constructions of algebraic varieties along with computations of their topological or geometric properties. We summarize here some of the discussion in [100].

Hodge theory tells us much about what *cannot* happen. However, within the restrictions of Hodge theory, we know very little about natural examples of what *can* happen. While a certain array of techniques for constructing varieties is already known, these don’t yield sufficiently many examples of the complicated topological behavior we expect. And even for the known constructions, it is very difficult to calculate the properties of the constructed varieties.

This has many facets. Perhaps the easiest example to state is the question of what collections of Betti numbers (or Hodge numbers) can arise for an algebraic variety (say, smooth and maybe projective)? For the present discussion we pass directly on to questions about the fundamental group.

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What types of π_1 can arise? We know a somewhat diverse-sounding collection of examples: lattices, braid groups (in the quasiprojective case) [81], all kinds of virtually abelian groups, solvable groups [102], plenty of calculations for plane complements of line arrangements and other arrangements in low degrees [70] [32] [4], Kodaira surfaces, many examples of non-residually finite groups [113]. Which π_1's have nontrivial representations? Recall for example an old result:

Theorem *Any nonrigid representation of a Kähler group in $PSL(2,\mathbb{C})$ comes by pullback from a curve.*

Conversely, there exist nonrigid representations of rank > 2 which don't come by pullback from curves. However, in a more extended sense all of the known examples of representations come from rigid representations (which conjecturally are motivic) and from representations on curves, by constructions involving Grothendieck's "six operations" (cf [78]). In particular, the irreducible components of moduli varieties of flat connexions M_{DR} which are known, are all isomorphic to moduli varieties of representations on curves.

An early example of this phenomenon was Lawrence's construction of representations of the braid group [65]. For braid groups or generalized mapping class groups, Kontsevich has a conjecture dating from around 15 years ago, which would give an explicit description of what all representations should be in terms of higher direct images. (These two things should have been mentionned in [100]).

Nonetheless, over general quasiprojective varieties it seems likely that there are other "new" representations but that we don't know about them because it is difficult to master the computational complexity of looking for them.

An intermediate construction might be as follows: suppose we have a family $\{V_t\}$ of local systems on X, such that there is a closed locus $Z \subset M_{DR}(X)$ where $\dim H^i(X_y, V_t)$ jumps for $t \in Z$. Then the family $\{R^i\pi_*\}_{t\in Z}$ might be a component of the moduli space of local systems on Y. Thus the whole topic of variation of differential Galois groups could lead to some "semi-new" components in this way. Nonetheless, this doesn't go too far toward the basic question of finding cases where there are lots of representations for a general reason.

2. Logic and calculation

The construction problem results in a complex logical and computational situation, not directly amenable either to pure theoretical considerations, or to brute-force calculation. This could open up the road to a new type of approach, in a direction which was forseen by the INRIA group in Rocquencourt, when they baptised their research group "Logi-Cal". The idea behind this name was that it is becoming necessary to combine logic and calculation. The origins of this requirement lay in computer science, exemplified for example by the notions of "proof-carrying code" and verified and extracted programs. The "Logi-Cal" idea was very cogently explained by Benjamin Werner in an exposé in Nice a few years ago, in which he described its possible applications to pure mathematics using the example of the four-color theorem. He explained that it would be good to have a proof of the four-color theorem which combines computer verification of the theoretical details of the argument, with the computer computations which form the heart of the proof. He said that we could hope to have the whole thing contained in a single document verified by a single program. In a spectacular advance, this project has recently been completed by G. Gonthier, who gives a full computer-verified proof of the four-color theorem in Coq [45].

Thomas Hales' "Flyspeck project" [50] is another current example of a project in the direction of using computer proof techniques to combine theory with calculation, in that case for the proof of the Kepler conjecture.

It seems clear that this very nice idea should have repercussions for a much wider array of topics. The possibility of combining logic and computation will open up new routes in algebraic geometry. This is because there are questions such as those related to the construction problem above, which are susceptible neither to pure reasoning nor to pure computation. At this conference Andy Magid mentionned an interesting case: he had tried some time ago to compute examples of positive-dimensional representation varieties for finitely presented groups with more relations than generators (cf [6] [47] [30]). He reported that the computational complexity of the question (which depends on parameters like the number of generators, the number and length of the relations, and the value of n if we look for representations in $GL(n)$) became overwhelming even for very small parameters. This suggests that it would be impossible to envision a brute-force search for examples. In the algebraic-geometric case, we might want to take concrete varieties, compute presentations for their fundamental groups (using braid-group techniques for example) and then compute the representation spaces. Magid's remarks suggest that a brute-force approach to this computation will not be feasible. On the other hand, purely theoretical techniques are unlikely to answer the most interesting question in this regard, namely: are there new or exceptional examples which are not accounted for by known theoretical reasons? Thus the interest of looking for a mixed approach combining theory and computation. Implementation of such an approach could be significantly enhanced by computer-formalized proof techniques providing an interface between theory and calculation.

Another example seen in this conference was Ehud Hrushowski's talk about algorithmic solutions to the problem of computing differential Galois groups. While showing that in principle there were algorithms to make the computation, it appeared likely that the complexity of the algorithms would be too great to permit their direct implementation. It would be good to have precise information about the complexity of this kind of question. This undoubtedly would require substantial input from algorithmic complexity theory. Some things are known for related problems, see [115] for example.

The known bounds tend to be be high, so again one would like to envision a mixed approach in which theory provides shortcuts in determination of the answers. An interesting theoretical question is then to what extent there is a relation between proof complexity for the theory part [27], and algorithmic complexity for the calculational part.

Of course mixing between theory and computation has always taken place within mathematical work, a good example is [49]. There have also recently been advances in the use of algorithmic methods to attack problems such as the topology of real varieties [14] [15]. The editor points out [26] which constitutes a striking example (for the case of the Nullstellensatz) where mathematical theory can considerably improve computational bounds.

The relevance of computerized formulation of the theory part is that it might well permit the process to go much farther along, as it would make available the advances in computational power to both sides of the interaction. Currently we can benefit from advanced computational power on the calculation side, but this can outstrip the capacity of theory to keep up. This phenomenon was emphasized by Alain Connes in his talk (and subsequent comments) at the PQR conference in Brussels, June 2003. He pointed out that with computer algebra programs he could come up with new identities which took pages and pages just to print out; and that it would be good to have tools for interpreting this new information which often surpasses our classical human sensory capacities. It is possible that interface tools could be of some help, but likely in the end that we would want to connect these things directly to theoretical proof software—a step which might on some levels bypass human understanding altogether.

A related area in which it might be useful to have a mixture of theory and computation when looking for construction results is the Hodge conjecture. There are many concrete situations in which we expect to find certain algebraic cycles, but don't in general know that they exist. For example, the Lefschetz operators or Kunneth projectors are automatically Hodge cycles. It would be interesting to take explicit varieties and search for algebraic cycles representing these Hodge classes. As in the search for representations, a brute-force approach would probably run out of steam pretty fast, and it would be interesting to see what a mixed approach could attain. A related question is the search for constructions of varieties where the Lefschetz or Kunneth operators are topologically interesting, namely cases where the cohomology is not mostly concentrated in the middle dimension.

Finally we mention a more vague direction. In the above examples we are looking for constructions with a certain desired topological or geometrical behavior. However, it may also be interesting to consider the question of what we get when we look at an arbitrary algebraic-geometric construction process or algorithm. This type of question is related to the field of dynamical systems, and has been popularized by S. Wolfram. There are probably many places to look for interesting processes in algebraic geometry. Insofar as a given process produces an infinite, combinatorially arranged collection of output, it opens up questions of asymptotic behavior, and more generally the arrangement of results with respect to measurable properties on the output, as well as dependence on the algorithm in question. For this type of research it would seem essential to have tools relating theoretical properties in algebraic geometry to algorithmic questions.

3. The Bogomolov-Gieseker inequality for filtered local systems

We go back to look more closely at the computational issues in constructing representations of algebraic fundamental groups. There are various different possible approaches:
–construct the representations directly on a presentation of π_1;
–construct directly the connections (E, ∇) or the Higgs bundles (E, θ);
–in the quasiprojective case, construct directly parabolic bundles, logarithmic connections, or "filtered local systems".

Most work up to now on the first approach has already had the flavor of mixing computation and theory [81] [88] [70] [46] [34] [25]. For the second and third approaches, there is a *Bogomolov-Gieseker inequality* lurking about. The basic example is the classical $3c_2 - c_1^2 \geq 0$ for surfaces of general type, with equality implying uniformization by the ball (and in particular the uniformization gives a representation $\pi_1 \to SU(2,1)$). This was used in Livné's construction [80]. Subsequent results, as is well-known, concern stable vector bundles and extensions to the cases of Higgs and parabolic structures [35] [114] [18] [66] [67] [69] [78] [82] [98] [105]. In all these cases, we only obtain representations in the case of equality, so it is hard to find numerical genericity conditions which imply existence. The Bogomolov-Gieseker inequalities come up in a fundamental way in the analysis of the quasiprojective case, where it seems to be a problem of finding special configurations, say of divisors in the plane, as well as special configurations of filtrations and weights to assign to the divisors, so that equality will hold in the BGI.

For parabolic vector bundles, this problem has been considered and solutions were found in D. Panov's thesis [87]. His investigation is deeper than the general remarks we make below. One interesting point, showing the need for computer verification of the interface between theory and computation, is that in order to successfully get at the problem of looking for solutions, Panov had to spend some time and energy correcting a computational or typographical error in Biswas' calculation of the parabolic Chern classes [19].

We will look at one facet of the problem—the case of filtered local systems—for which at least the basic definitions are elementary. By a *filtration* of a vector space we shall mean a filtration indexed by real numbers cf [98]. In particular gr_F^α is nonzero for only a finite number of reals α. A filtration can be multiplied by a positive real number λ: define $(\lambda\mathcal{F})_\alpha := \mathcal{F})_{\lambda^{-1}\alpha}$.

Fix a surface X with a divisor D which we shall assume (at first) to have normal crossings. A *filtered local system* is a local system L on $U := X - D$ together with a filtration $\mathcal{F}_i$ at the nearby fiber to each irreducible component of D. Recall that if D_i is a component and T_i a tubular neighborhood of D_i in U_i then the nearby fiber is a fiber of the local system at a point $P_i \in T_i$. We require that the filtration $\mathcal{F}_i$ be invariant under the monodromy over T_i. A parabolic version of the Riemann-Hilbert correspondence makes filtered local systems correspond to parabolic logarithmic connexions (this was pointed out for curves in [98] and presumably it works similarly in higher dimensions; also it was well-known in $\mathcal{D}$-module theory for the case of integer filtrations). We obtain the *Chern classes* of a filtered local system denoted $c_i(L, \mathcal{F})$ which could be defined as the parabolic Chern classes of the corresponding parabolic logarithmic connexion. We have the following formulae. The first Chern class is given (as a cycle on X) by

$$c_1(L, \mathcal{F}) = -\sum_{\alpha,i} \alpha \dim \left(gr^{\mathcal{F}_i}_{\alpha}(L_{P_i}) \right) \cdot D_i.$$

The second Chern class combines a sum over intersection points Q of the divisors, plus self-intersection contributions of the components and the square of c_1. For each intersection point choose an ordering of the two associated indices and note them by j_Q, k_Q. Let Q' denote a point nearby to Q (in the intersection of the tubular neighborhoods T_{j_Q} and T_{k_Q}). Define the *local contribution*

$$c_2(L, \mathcal{F})_Q := -\sum_{\alpha,\beta} \alpha\beta \dim \left(gr^{\mathcal{F}_{j_Q}}_{\alpha} gr^{\mathcal{F}_{k_Q}}_{\beta}(L_{Q'}) \right),$$

then

$$c_2(L, \mathcal{F}) = \frac{1}{2} c_1(L, \mathcal{F})^2 - \frac{1}{2} \sum_{\alpha,i} \dim \left(gr^{\mathcal{F}_i}_{\alpha}(L_{P_i}) \right) \cdot \alpha^2 (D_i.D_i) - \sum_Q c_2(L, \mathcal{F})_Q \cdot Q.$$

The Chern classes allow us to define *stability and semistability* in the usual way by comparing the slope with slopes of subobjects. These conditions should be equivalent on filtered local systems and parabolic logarithmic connexions. Finally, there should be a *harmonic theory* comparing these objects with parabolic logarithmic Higgs bundles—where T. Mochizuki's work [77] [78] comes in. We won't say anything about that here[1] except to say that it should lead to a *Bogomolov-Gieseker inequality* (BGI) which we describe in a conjectural way. Here I would like to thank M. S. Narasimhan for pointing out recently that it would be good to investigate the BGI for logarithmic objects. He had in mind the logarithmic Higgs bundle case, but it seems likely that all three cases would be interesting and the simplest to explain and think about is filtered local systems.

[1] A glance at his papers should convince the average reader of the value of having the help of a computer to digest the argument.

The BGI would say that if $(L, \mathcal{F})$ is a filtered local system which is semistable with $c_1(L, \mathcal{F}) = 0$ then $c_2(L, \mathcal{F}) \geq 0$ and in case of equality we get some kind of pluriharmonic metric. The pluriharmonic metric should allow us to make a correspondence with parabolic Higgs bundles and to use the transformations discussed in [99] to obtain other different representations of $\pi_1(U)$.

The first case to look at is when L is a trivial local system of rank r which we denote by $\mathbb{C}^r$. It is easiest to understand the filtrations in this case, and also in this way we don't presuppose having any representations of $\pi_1(U)$. Even in this case, if equality could be obtained in the BGI then the transformations of [99] would yield nontrivial representations. By tensoring with a rank one filtered local system, we can assume that the filtrations are *balanced*:

$$\sum_{\alpha} \alpha \dim(gr^{\mathcal{F}}_{\alpha}(\mathbb{C}^r)) = 0.$$

This guarantees that the first Chern class will vanish. Now define the *product of two filtrations* by

$$\langle \mathcal{F}, \mathcal{G} \rangle := \sum_{\alpha, \beta} \alpha\beta \dim \left(gr^{\mathcal{F}}_{\alpha} gr^{\mathcal{G}}_{\beta}(\mathbb{C}^r) \right).$$

In this case the second Chern class (as a number) becomes

$$c_2(\mathbb{C}^r, \mathcal{F}) = -\frac{1}{2} \sum_{i,j} \langle \mathcal{F}_i, \mathcal{F}_j \rangle D_i . D_j .$$

The stability condition is that if $V \subset \mathbb{C}^r$ is any proper subspace, then

$$\sum_{\alpha, i} \alpha \dim(gr^{\mathcal{F}_i}_{\alpha}(V)) \deg(D_i) < 0.$$

The BGI can be stated as a theorem in this case:

Theorem *If $\{\mathcal{F}_i\}$ is a collection of filtrations satisfying the stability condition, then $c_2(\mathbb{C}^r, \mathcal{F}) \geq 0$, and if equality holds then there are irreducible representations of $\pi_1(X - D)$.*

The theorem in this case is a consequence of what is known for parabolic vector bundles. Indeed, the collection of filtrations also provides a parabolic structure for the trivial vector bundle (with the same Chern classes). If we use small multiples $\{\epsilon \mathcal{F}_i\}$, then the stability condition as described above implies stability of the parabolic bundle, so the Bogomolov-Gieseker inequality (plus representations in case of equality) for parabolic bundles [66] [67] [19] [87] gives the statement of the theorem.

It may be interesting to think of the minimum of $c_2(\mathbb{C}^r, \mathcal{F})$ as some kind of measure of how far we are from having representations of $\pi_1(U)$. We need to be more precise because scaling the filtrations by a positive real number doesn't affect stability and it scales the second Chern class by the square.

Put

$$\|\mathcal{F}\|^2 := \sum_{i,\alpha} \|\alpha\|^2 \dim(gr_\alpha^{\mathcal{F}_i}) \cdot deg(D_i),$$

and

$$\Upsilon(X, D, r) := \min_{\{\mathcal{F}_i\}} \frac{c_2(\mathbb{C}^r, \mathcal{F})}{\|\mathcal{F}\|^2}$$

where the minimum is taken over collections of filtrations which give a nontrivial stable filtered structure with $c_1(\mathbb{C}^r, \mathcal{F}) = 0$ on the constant local system $\mathbb{C}^r$. The BGI says that $\Upsilon(X, D, r) \geq 0$ and in case of equality, there should[2] exist nontrivial representations of $\pi_1(X - D)$.

The above considerations lead to the question of how $\Upsilon(X, D, r)$ behaves for actual normal crossings configurations on surfaces X. For simplicity, (X, D) might come from a plane configuration after blowing up (for example, a plane configuration with only multiple intersections, where we blow up once at each intersection point). The first problem is computing $\Upsilon(X, D, r)$ and in particular calculating the local contributions to the second Chern class at points which are not normal crossings (discussed in [66]). Computation of $\Upsilon(X, D, r)$ involves searching through the possible configurations of filtrations. Most importantly, we would like to create configurations of divisors D_i in the plane which are interesting with respect to the invariant Υ.

This might be algorithmic: given some process for generating plane configurations, what are the distribution, asymptotic behavior and other properties of the resulting numbers Υ? But even before we get to infinite families of configurations, the simple problem of thoroughly analyzing what happens for specific configurations is a nontrivial computational problem. Calculation of algebro-geometric and specially topological properties of plane configurations goes back to Zariski and Hirzebruch, and much work in this direction continues (see Teicher *et al* [81] [94] ...). One of the main characteristics of these computations is that they require significant amounts of reasoning. Similarly, the computation of Donaldson invariants has required a significant amount of theoretical work [37] [86]. The problem we are proposing above, consideration of the behavior of the BGI and the minima Υ in the setting of a configuration, will quite likely fit into the same mold.

Back to the theoretical level, it might be interesting to look at whether we could have a Gromovian phenomenon [48] of simply connected varieties which look approximately non-simply-connected, which is to say that their "isoperimetric inequalities" are very bad, with relatively small loops being the boundaries only of very large homotopies. Also whether Bogomolov-Gieseker quantities such as $\Upsilon(X, D, r)$ being small (but nonzero) might detect it. And again, we would like to have information about the distribution of this phenomenon in combinatorial families of varieties.

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[2]This would require proving that the minimum is attained.

4. The foundations of category theory

Unfortunately, the visions sketched above contrast with the rather limited state of progress on the problem of computer formalization of theoretical mathematics such as algebraic geometry. The space between what we would like to do, and what we can concretely do right now is still much too big, but we have to start somewhere.

It is of course necessary to give a thorough overview of the many projects working in this direction all over the globe; but this has been or is in the process of being treated in other documents. In this note I will rather just describe the current state of my own progress on this matter.

There are two Coq develoments attached to the source file of the `arxiv` version of the present preprint.[3] One is a short self-contained file `fmachine.v` which is a little demonstration of how pure computer-programming can be done entirely within the Coq environment (we don't even need Coq's program-extraction mechanism). The example which is treated is a forward-reasoning program for a miniature style of first-order logic (compare [93]). Programs such as this one itself may or may not be useful for proof-checking in the future. The main point of interest is that we can write a program entirely within Coq; this might point the way for how to treat the programming side of things when we want to integrate computation with mathematical theory. The notion of Coq as a programming language was mentionned by S. Karrmann on the Coq-club mailing list [59].

The other development continues with the environment described in [101] where we axiomatized a very classical-looking ZFC within the type-theoretical environment of Coq, maintaining access to the type-theory side of things via the realization parameter R. This is based on a small set of axioms which purport to correspond to how types are implemented as sets, following Werner's paper [120] —we don't give any argument other than refering to [120] for why these axioms should be consistent (and from [76] it appears that giving a full proof of this would not be entirely straightforward).

Here we build on this by adding basic category theory. Newer—slightly updated–versions of the files from [101] are included with the present development (in particular one has to use the versions included here and not the older ones).[4] We treat the notions of category, functor and natural transformation. We construct the category of functors between two given categories. Then we treat limits and colimits, and give examples of categories.

Most of what we have done here—and more—has already been done some time ago in different contexts: Huet and Saibi, in Coq, in the context of

[3] Go to the `arxiv` preprint's abstract page, then to "other formats" and download the "Source" format. The result is a tar archive containing the tex source file for the preprint but also the `*.v` files in question. Compiles with `v8.0` of the Coq proof assistant.

[4] With this method of making public a continuing mathematical theory development project, the files bundled with a given preprint do not all represent new material: some are copies of previous ones possibly with slight modifications, while others are new but even the new ones will themselves be recopied in the future.

"setoids" [53]; several articles in Mizar [12]; and also[5] [13] [28] [29] [40] [42] [75] [79] [97] [117]. We don't actually claim that our present treatment has any particular advantages over the other ones; the reason for doing it is that we hope it will furnish a solid foundation for future attempts to treat a wider range of mathematical theories.

We use the following approach to defining the notion of category. A category is an uplet (with entries named over strings using the file `notation.v` as was explained in [101]) consisting of the set of objects, the set of morphisms, the composition function, the identity function, and a fifth place called the "structure" which is a hook allowing us to add in additional structure in the future if called for (e.g. monoidal categories will have the tensor product operation encoded here; closed model categories will have the fibration, cofibration and equivalence sets encoded here; sites will have the Grothendieck topology encoded here etc.). The elements of the set of morphisms are themselves assumed to be "arrows" which are triplets having a "source", a "target" and an "arrow" (to take care of the information about the morphism). In particular, the functions `source` and `target` don't depend on which category we are in. Functors and natural transformations are themselves arrows, so the functions `source` and `target` do a lot of work.

We treat limits in detail, and colimits by dualizing limits. The main technical work is directed toward the formalized proof of the following standard theorem.

Theorem *If* `a`, `b`, *and* `c` *are categories such that* `b` *admits limits over* `c`, *then* `functor_cat a b` *also admits limits over* `c`.

The proof is done in the file `fc_limits.v`. Intricacy comes from the need to use the universal property of the pointwise limits in order to construct the structural morphisms for the limiting functor, and then further work is needed to show that the functor constructed in this way is actually a limit. The corresponding result for colimits is obtained almost immediately by dualizing—the only subtlety being that `opp (functor_cat a b)` is not equal but only isomorphic to `functor_cat (opp a) (opp b)`. Because of this we need to make a preliminary study of the invariance of limiting properties under isomorphisms of categories. This discussion will have to be amplified in the future when we are able to treat equivalences of categories.

[5]There is an extensive discussion of references about mechanizing category theory in a thread of the QED mailing list, circa 1996, in response to a question posted by Clemens Ballarin. David Rydeheard mentions work in the systems Alf, LEGO and Coq, and work by Dyckhoff, Goguen, Hagino, Aczel, Cockett, Carmody and Walters, Fleming, Gunther, Rosebrugh, Gray, Watjen and Struckmann, Hasegawa, and Gehrke. Masami Hagiya mentions work of his student Takahisa Mohri. Ingo Dahn mentions a number of Mizar articles by Bylínski, Trybulec, Muzalewski, Bancerek, Darmochwal. Roger B. Jones mentions some work of his own. Pratt mentions work by Bruckland and Walters, and tools for computation with finite categories by Rosebrugh. And Amokrane Saibi mentions his work with Huet in Coq. Evidently this list would have considerably to be expanded for work up to the present day.

The importance of this theorem is its corollary that presheaf categories admit limits and colimits. This will (in the future) be essential to theories of sheaves and hence topoi; and theories of closed model categories, because many useful closed model categories take presheaf categories as their underlying categories, and one of the main conditions for a closed model category is that it should admit (at least finite) limits and colimits.

One task which is worth mentionning is that we construct examples of categories by various different methods, in the file `cat_examples.v`. The methods include subcategories of other categories; defining a category by its object set together with the set of arrows between each pair of objects; and function categories which come in two flavors, depending on whether we look at functions between the objects as sets themselves or functions between their underlying sets (denoted `U x`).

A different approach is called for when we want to construct and manipulate finite categories—important for example in relating classical limit constructions such as equalizers and fiber products, to the notion of limit as defined in general (done for (co)equalizers in `equalizer.v` and (co)fiber-products in `fiprod.v`). It doesn't seem efficient to manipulate finite sets by directly constructing them, but instead to build them with Coq's inductive type construction and then bring them into play using the realization parameter. This allows us to list the elements of a finite type by name, and then to manipulate them with the `match` construction. To bridge from here to the notion of category, we need to discuss the construction of categories (also functors and natural transformations) starting from type-theoretic data: these constructions `catyd`, `funtyd` and `nttyd` occupy a large place in `little_cat.v`.

We finish by pointing out how a theoretical category-theory development such as presented here, is relevant for some of the more long-range projects discussed in the beginning. This discussion is very related to L. Chicli's thesis [31] in which he used Huet-Saibi's category theory as the basis for the definition and construction of affine schemes. The basic point is that to manipulate the fundamental objects of modern (algebraic, analytic or even differential) geometry, we need to know what a ringed space is, and better yet a ringed site or ringed topos. Thus we need a theory of sheaves, and in particular a well-developed category theory, with functor categories, limits and colimits, etc. The next items which need to be treated in the present development are equivalences of categories, adjoint functors (and even fancier things like Kan extensions), over-categories, monomorphisms and epimorphisms, then sheaves and topoi.

If we want to access more recent developments in geometry, it will be essential to have good theories of (possibly monoidal) closed model categories starting with the small-object argument. On a somewhat different plane, it is clear that to manipulate many of the geometric questions discussed above, we will need to have a good development of linear algebra. This presents a number of categoric aspects, for example in the notions of additive and abelian categories (again possibly with tensor structures).

There remain some thorny notational dilemmas still to be worked out before we can do all of this. One example is that the right notion of "presheaf" is probably slightly different from that of a functor: we probably don't want to include the data of the target category. This is because in general the target category will be a big category for the universe we want to work in, whereas we would also like our presheaves to be elements of the universe, and indeed we don't necessarily want to specify which universe it is for a given presheaf. So we will probably have to define a presheaf as being a modified version of a functor where the `target` element of the arrow triple is set by default to `emptyset`. This is the kind of thing which is easy to say in a few phrases, but which in practice requires writing a whole new file containing material similar (but not identical) to what is in `functor.v`.

It seems likely that once the definitional work is finished, subsequent geometrical manipulations of these objects should be fairly easy to take care of, compared with the amount of foundational work necessary just to give the definitions. Unfortunately, as best as I know nobody has gotten far enough to test this out.

5. Finite categories

The work on formalization of category theory, *a priori* a waypoint along the path to formalizing algebraic geometry, also suggests its own research directions. When we are forced to look very closely at the foundational details of a subject, there stand out certain questions which would otherwise be overlooked in the usual rush to get on with the abstract theory. An example, strongly representative of the general problem of relating theory and computation, is the classification of finite categories [111]. For a given finite integer N, how many categories are there with N morphisms? What do diagrams or other standard categorical constructions (functors, natural transformations, limits, adjoints, Kan extensions ...) look like in these categories, perhaps in terms of asymptotic behavior with respect to N but also maybe just for small fixed values? What additional structures can these categories have?

The question of classification of finite categories has been treated in [111] [106] [109] [56] [61] from a universal-algebra point of view. Their idea is to define notions of *variety* or *pseudovariety* which are collections of objects closed under direct product and subquotient,[6] and from these references we know a lot about the structure and classification of pseudovarieties of finite categories. For example, Tilson proves a classification theorem for locally trivial categories, those being the ones with only identity endomorphisms of each object: the answer is that they are subquotients of products of

[6]These notions might be modifiable so as to be relevant to the problem of classifying representations of algebraic fundamental groups.

the two-arrow category whose limits are equalizers [111]. Related are [1], [5] [9] [57] [89] [90] [104] [92]. And [96], [39] discuss a similar question of classification of finite tensor categories (but the word "finite" has a slightly different meaning there).

One might also ask more detailed questions about finite categories which are not invariant under the process of taking subquotients, and we get a situation in many ways analogous to the algebraic-geometric questions discussed above, leading among other things to the question of how to construct finite categories having given properties. We can also think of further questions by analogy with the algebraic-geometric ones. For example, the analogue of the the moduli space M_B could be defined as follows. If Γ is a finite (or even finitely presented) category, define the moduli stack $\mathcal{M}_B(\Gamma)$ as the stack associated to the prestack of functors

$$\mathcal{M}_B(\Gamma)^{\mathrm{pre}}(A) := \underline{\mathrm{Hom}}(\Gamma, \mathbf{Mod}_A^{\mathrm{proj}})$$

where $\mathbf{Mod}_A^{\mathrm{proj}}$ is the category of projective A-modules. This could have variants where we look at all A-modules or even U-coherent sheaves on $Spec(A)$ in the sense of [52]. There would also be n-stack versions where we look at maps into stacks of complexes or other things (and indeed we could fix any ∞-stack $\mathcal{G}$ and look at $\underline{\mathrm{Hom}}(\Gamma, \mathcal{G})$). It isn't our purpose to get into the details of this type of construction here but just to note that these should exist. We can hope in some cases to get geometric stacks—for example the 1-stack $\mathcal{M}_B(\Gamma)$ as defined above is Artin-algebraic (or more precisely its 1-groupoid interior is algebraic). We can also hope that these stacks have natural open substacks with coarse moduli varieties which could be denoted generically by $M_B(\Gamma)$.

Invariants of these moduli varieties (to start with, their dimensions and irreducible components ...) would become invariants of the finite category, and we would like to know something about their distribution, bounds, etc., and also whether we can construct finite categories such that the moduli varieties have given behavior. In the case when Γ is a finitely presented category which is free over a graph, $M_B(\Gamma)$ is the same thing as the moduli space of *quivers*, and in general the moduli space will be a subspace of the space of quivers on the arrows of the category, so there is already a big theory about this (and we can expect semistability for quivers to lead to the open substack required above). It is certainly also related to work by Lusztig, MacPherson and Vilonen and others on combinatorial descriptions of perverse sheaves [72] [73] [44] [116] [24]. Which finite categories arise as specialization categories for stratifications (and particularly naturally arising stratifications)? We can also ask which varieties arise as moduli spaces $M_B(\Gamma)$: this might be relevant as a process for constructing algebraic varieties.

These and any number of similar questions of differing levels of difficulty might provide a good proving ground for tools combining theory and calculation.

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