GEOMETRICITY OF THE HODGE FILTRATION ON THE ∞-STACK OF PERFECT COMPLEXES OVER X_{DR}

CARLOS SIMPSON

Contents

1. Introduction

We construct a locally geometric ∞-stack $M_{Hod}(X, \mathrm{Perf})$ of perfect complexes on X with λ-connection structure (for a smooth projective variety X). This maps to $\mathcal{A} := \mathbb{A}^1/\mathbb{G}_m$, so it can be considered as a filtration. The stack underlying the filtration, fiber over 1, is $M_{DR}(X, \mathrm{Perf})$ which parametrizes complexes of $\mathcal{D}$-modules which are perfect over $\mathcal{O}_X$. The associated-graded, or fiber over 0, is $M_{Dol}(X, \mathrm{Perf})$ which parametrizes complexes of Higgs sheaves perfect over $\mathcal{O}_X$, whose cohomology is locally free, semistable with vanishing Chern classes. One of the motivations for this question is that if $p : X \to Y$ is a smooth morphism, we can define the higher direct image functor

$$Rp_* : M_{Hod}(X, \mathrm{Perf}) \to M_{Hod}(Y, \mathrm{Perf}),$$

which is a way of saying that the higher direct image functor between de Rham moduli stacks preserves the Hodge filtration.

Key words and phrases. λ-connection, perfect complex, $\mathcal{D}$-module, de Rham cohomology, Higgs bundle, Twistor space, Hochschild complex, Dold-Puppe.

Glueing to $M_{Hod}(\overline{X}, \mathrm{Perf})$ we obtain Hitchin-Deligne's twistor space for perfect complexes. This has prefered sections corresponding to $\mathcal{O}$-perfect mixed Hodge modules over X. Conjecturally, the formal neighborhood of a prefered section should be a nonabelian mixed Hodge structure.

This work is part of a general research project with L. Katzarkov, T. Pantev, B. Toen (and more recently, G. Vezzosi, M. Vaquié) about nonabelian mixed Hodge theory. The main result in the present note is that the moduli stack $M_{Hod}(X, \mathrm{Perf}))$ is locally geometric (Theorem 4.4). Its proof relies heavily on a recent result of Toen and Vaquié that the moduli stack $\mathrm{Perf}(X)$ of perfect complexes on X is locally geometric. We are thus reduced to proving that the morphism

$$M_{Hod}(X, \mathrm{Perf}) \rightarrow \mathrm{Perf}(X \times \mathcal{A}/\mathcal{A}) = \mathrm{Perf}(X) \times \mathcal{A}$$

is geometric. It seems likely that geometricity could be deduced from J. Lurie's representability theorem, and might also be a direct consequence of the formalism of Toen-Vaquié. Nonetheless, it seems interesting to have a reasonably explicit description of the fibers of the map: this means that we fix a perfect complex of $\mathcal{O}$ modules E over X and then describe the possible structures of λ-connection on E. The notion of λ-connection is encoded in a sheaf of rings of differential operators Λ (which is just $\mathcal{D}_X$ when $\lambda = 1$). Our construction works for more general Λ so it should also serve to treat examples such as the case of logarithmic connections.

Our description of the Λ-module structures on E passes through a Kontsevich-style Hochschild weakening of the notion of complex of Λ-modules. In brief, the tensor algebra

$$T\Lambda := \bigoplus \Lambda \otimes_{\mathcal{O}_X} \ldots \otimes_{\mathcal{O}_X} \Lambda$$

has a differential and coproduct, and for $\mathcal{O}$-perfect complexes E and F, this allows us to define the complex

$$Q(E, F) := Hom(T\Lambda \otimes_{\mathcal{O}_X} E, F)$$

with composition. A weak structure is a Maurer-Cartan element $\eta \in Q^1(E, E)$ with $d(\eta) + \eta^2 = 0$. This works on affine open sets, and we need a Čech globalization (again using a Maurer-Cartan equation as done by Toledo-Tong, Hinich, ...) to get to X. The idea that we have to go to weak structures in order to obtain a good computation, was observed by Kontsevich [63] [7] [64], and has now become a classical remark (most recently see [17]). Looking at things in this way was suggested to me by E. Getzler, who was describing his way of looking at some other related questions. The application to weak Λ-module structures is a particularly easy case since everything is almost linear (i.e. there are no higher product structures involved). Our argument is structurally similar to Block-Getzler [14]. An important step in the argument is the calculation of the homotopy fiber product involved in the definition of geometricity, made possible by Bergner's model category structure on the category of simplicial categories [10].

This is a very preliminary version: many proofs are only sketched, and some are left out entirely. At a minimum, at least we have broken up the proof into a collection of more manageable steps which need to be filled in.

2. λ-connections and the Hodge filtration

Suppose X is a smooth projective variety over $\mathbb{C}$. Let $M_{DR}(X, GL(n))$ denote the moduli stack of rank n vector bundles with integrable connection. Let $M_{Higgs}(X, GL(n))$ denote the moduli stack of rank n Higgs bundles, and let $M_{Dol}(X, GL(n))$ denote the open substack of Higgs bundles which are semistable with vanishing Chern classes. Deligne suggested the notion of *λ-connection* as a way of building a bridge between M_{DR} and M_{Dol}. For any parameter $\lambda \in \mathbb{A}^1$ (or any function on the base scheme if we are working in a relative setting), a *λ-connection* ∇ on a vector bundle E is a connection-like operator

$$\nabla : E \to E \otimes_{\mathcal{O}_X} \Omega^1_X$$

which satisfies the Leibniz rule multiplied by λ:

$$\nabla(ae) = \lambda e \otimes da + a\nabla(e).$$

The *curvature* is the tensor ∇^2 and we will mostly (without further specification) consider only flat λ-connections i.e. $\nabla^2 = 0$.

We obtain a moduli stack $M_{Lam}(X, GL(n)) \to \mathbb{A}^1$ whose fiber over λ is the moduli stack of λ-connections which when $\lambda = 0$ are required to be semistable with vanishing Chern classes (more precisely, we make this requirement over any closed point in the base scheme where $\lambda = 0$).

There is a natural action of $\mathbb{G}_m$ on $M_{Lam}(X, GL(n))$ covering its action on $\mathbb{A}^1$. Let $\mathcal{A} := \mathbb{A}^1/\mathbb{G}_m$ denote the quotient stack. Note that $\mathcal{A}$ has just two points, which we denote 0 and 1, corresponding to substacks denoted [0] and [1]. The closed substack [0] is isomorphic to $B\mathbb{G}_m$ whereas the open substack [1] is just $Spec(\mathbb{C})$.

Let $M_{Hod}(X, GL(n))$ denote the quotient stack of M_{Lam} by $\mathbb{G}_m$. We have a morphism

$$M_{Hod}(X, GL(n)) \to \mathcal{A}$$

and the fiber over [1] is $M_{DR}(X, GL(n))$ whereas the fiber over 0 is $M_{Dol}(X, GL(n))$ with its natural action of $\mathbb{G}_m$ (multiplying the Higgs field).

This situation should be thought of as the *Hodge filtration* on M_{DR} with *associated-graded stack* M_{Dol}.

Next we recall Deligne's glueing. Let $\overline{X}$ denote the complex conjugate variety. It is defined by taking the complex conjugates of the coefficients of the equations defining X. Complex conjugation of the coordinates defines a real analytic homoemorphism

$$\gamma : X_B \overset{\cong}{\to} \overline{X}_B$$

where X_B denotes the usual topological space underlying the complex analytic manifold X^{an} (and the same for $\overline{X}$).

Let $M_B(X, GL(n))$ denote the moduli stack of rank n local systems over X. The Riemann-Hilbert correspondence is an analytical isomorphism

$$M_B(X, GL(n))^{\mathrm{an}} \cong M_{DR}(X, GL(n))^{\mathrm{an}}.$$

On the other hand, the homeomorphism given by complex conjugation gives

$$M_B(X, GL(n)) \cong M_B(\overline{X}, GL(n)).$$

Combining these together, we have an analytic isomorphism

$$\gamma_{DR} : M_{DR}(X, GL(n))^{\rm an} \cong M_{DR}(\overline{X}, GL(n))^{\rm an}.$$

Now recall from above that $M_{DR}(X, GL(n))$ is an open substack of $M_{Hod}(X, GL(n))$. Therefore

$$M_{DR}(X, GL(n))^{\rm an} \subset M_{Hod}(X, GL(n))^{\rm an}$$

is an analytic open substack and we can use the isomorphism γ_{DR} to glue together $M_{Hod}(X, GL(n))^{\rm an}$ and $M_{Hod}(\overline{X}, GL(n))^{\rm an}$. This gives the *Deligne glueing* which is the colimit

$$M_{DR}(X, GL(n)) \stackrel{\rightarrow}{\rightarrow} M_{Hod}(X, GL(n))^{\rm an} \sqcup M_{Hod}(\overline{X}, GL(n))^{\rm an} \rightarrow M_{Del}(X, GL(n)).$$

If we let $\mathcal{P}$ denote the stack theoretic quotient of $\mathbb{P}^1$ by the action of $\mathbb{G}_m$ then it can also be expressed as a colimit glueing similar to above:

$$\mathbb{G}_m \stackrel{\rightarrow}{\rightarrow} \mathcal{A} \sqcup \mathcal{A} \rightarrow \mathcal{P}.$$

Glueing the two maps λ from M_{Hod} to $\mathcal{A}$ we obtain a map

$$M_{Del}(X, GL(n)) \rightarrow \mathcal{P}^{\rm an}.$$

Pulling back by the map $\mathbb{P}^1 \rightarrow \mathcal{P}$ we get Deligne's construction of Hitchin's twistor space:

$$M_{Hit}(X, GL(n)) := M_{Del}(X, GL(n)) \times_{\mathcal{P}^{\rm an}} (\mathbb{P}^1)^{\rm an}.$$

It is easier to describe the glueing in this way which doesn't involve the parameter λ, or to put it another way, with our notation that M_{Hod} is the quotient of the stack of λ-connections by the action of $\mathbb{G}_m$, we are implicitly making the identification between λ-connections for $\lambda \neq 0, 1$, with usual connections ($\lambda = 1$). Thus we don't have to go back to this identification at the time we do the glueing.

Recall that a harmonic bundle on X gives what Deligne refers to as a "prefered section" of the twistor space, that is a $(\mathbb{P}^1)^{\rm an} \rightarrow M_{Hit}(X, GL(n))$ such that the composition with the projection back to $(\mathbb{P}^1)^{\rm an}$ is the identity. A complex variation of Hodge structure is a descent of a prefered section to a map $\mathcal{P}^{\rm an} \rightarrow M_{Del}(X, GL(n))$ again projecting as the identity back to $\mathcal{P}^{\rm an}$.

The space $M_{Del}(X, GL(n))$ encapsulates the Hodge filtration and its complex conjugate on the nonabelian cohomology stack (whose easiest Betti version is $M_B(X, GL(n))$). It is natural to ask where the weight filtration fits into this picture. This is a somewhat mysterious question which was attacked in joint work with L. Katzarkov and T. Pantev [59], but for which we don't claim to have a full answer.

The main remark we could make was that the phenomenon of the weight filtration seemed to be concentrated around points of the moduli stack (as opposed to the Hodge filtration which is in a certain sense concentrated at infinity, because the Hodge filtration can be used to compactify M_{DR}). This is seen in the fact that the formal completion of M_{DR} at a point corresponding to a variation of Hodge structure, admits a mixed Hodge structure (closely related to Hain's mixed Hodge structure on the relative Malcev completion [47]). At a general harmonic bundle, the formal completion admits a mixed twistor structure [86].

Our way of thinking of these mixed structures in [59] is inspired by Fulton's construction of "deformation to the normal cone" [31], applied along the prefered section. Consider the scheme T consisting of two crossing copies of $\mathbb{A}^1$, mapping to $\mathbb{A}^1$ by the identity on each factor. Let $\mathcal{T}$ be the quotient by the action of $\mathbb{G}_m$. It consists of two crossing copies of $\mathcal{A}$ and maps to $\mathcal{A}$. Fix one of the copies as the "basepoint" $\mathcal{A} \to \mathcal{T}$. Now if X is any stack and $x \in X$ we can form

$$DN(X;x) := Hom_(\mathcal{T}/\mathcal{A}, X \times \mathcal{A}/\mathcal{A}) \times_{X\times\mathcal{A}} \{x\} \times \mathcal{A}.$$

We have a morphism $DN(X;x) \to \mathcal{A}$ and the fiber over 1 is X whereas the fiber over 0 is the normal cone of X at x, since the fiber of $\mathcal{T} \to \mathcal{A}$ over 0 is $Spec(\mathbb{C}[\varepsilon]/(\varepsilon^2))$. The same construction may be done in the analytic category.

Given a morphism $X \to Y$ and a section $x : Y \to X$ we can form the relative version

$$DN(X/Y;x) \to Y \times \mathcal{A}.$$

If ρ is a preferred section of $M_{Del}(X, GL(n))$ corresponding to a variation of Hodge structure, the mixed Hodge structure along ρ is encoded by

$$DN(M_{Del}(X, GL(n))/\mathcal{P}^{\mathrm{an}}; \rho) \to (\mathcal{P} \times \mathcal{A})^{\mathrm{an}}.$$

The Hodge filtration and its complex conjugate are given by the two copies of $\mathcal{A} \subset \mathcal{P}$ while the weight filtration corresponds to the additional copy of $\mathcal{A}$ introduced by the DN construction. The same construction works along the prefered section of the twistor space corresponding to a harmonic bundle, to give the mixed twistor structure on the formal neighborhood of the prefered section:

$$DN(M_{Hit}(X, GL(n))/(\mathbb{P}^1)^{\mathrm{an}}; \rho) \to (\mathbb{P}^1 \times \mathcal{A})^{\mathrm{an}}.$$

It is obvious that one could say that these constructions vary in a family when we move the prefered section ρ (although it would be necessary to elucidate exactly which good properties this family would have). But other than that, it is not clear what more global version of the weight filtration could be constructed.

3. Variation of cohomology—an example

Consider a simple example as in Green and Lazarsfeld [40]. Let A be a three dimensional abelien variety and let $X \subset A$ be a smooth hypersurface. The diamond of Hodge numbers of X looks like

$$\begin{array}{ccc} 3+a & 3 & 1 \\ 3 & 9+b & 3 \\ 1 & 3 & 3+a \end{array}.$$

Look at the moduli space $M_{DR}(X, GL(1))$. This parametrizes line bundles with connection (L, ∇) over X. It is an extension

$$1 \to \mathbb{C}^3 \to M_{DR}(X, GL(1)) \to \widehat{A} \to 1.$$

Let's look at how the de Rham cohomology varies as a function of the point in M_{DR}. If $(L, \nabla) = (\mathcal{O}_X, d)$ then

$$h^i_{DR}(X, (L, \nabla)) = \begin{cases} 1 & i = 0, 4 \\ 6 & i = 1, 3 \\ 15 + 2a + b & i = 2. \end{cases}$$

If we assume that A is simple, then (by the generic vanishing result of Green and Lazarsfeld [40], for example) for any (L,∇) different from $(\mathcal{O}_X,d)$ the de Rham cohomology vanishes except in the middle degree. The Euler characteristic is invariant so in this case

$$h^i_{DR}(X,(L,\nabla)) = \begin{cases} 0 & i=0,1,3,4 \\ 5+2a+b & i=2. \end{cases}$$

The de Rham cohomology bundle in degree 2 defines a vector bundle of rank $5+2a+b$ on

$$M_{DR}(X,GL(1)) - \{0\}.$$

However, in general this bundle will not have an extension across the origin (interestingly enough, it seems that the singularity at the origin depends on the cohomology class $[Y]$).

Clearly what is happening here is that we obtain a natural *perfect complex* $H^{\cdot}(X,-)$ on $M_{DR}(X,GL(1))$, whose restriction to the complement of the origin is a vector bundle of rank $5+2a+b$ placed in degree two, whereas the fiber over the origin is a complex taking up the full interval $[0,4]$.

4. Perfect complexes over X_{DR}

Recall that a *perfect complex* over a scheme Y is a complex of quasicoherent sheaves on Y which is locally quasiisomorphic to a bounded complex of vector bundles. Here, we make the convention that "perfect" always means "of bounded amplitude". The *interval of amplitude* is the smallest interval $[a,b]$ in which the complex of vector bundles can be chosen. It may also be seen as the interval where the cohomology groups of the fibers $E_y := E\otimes_{\mathcal{O}_Y}\mathbf{k}(x)$ are nonzero at closed points $y\in Y$ (on the other hand it is bigger in general than the interval where the cohomology sheaves of E are nonzero).

There are several different options for how to construct the n-stack $\mathrm{Perf}^{[a,b]}$ of perfect complexes with amplitude in $[a,b]$, which was suggested to me by A. Hirschowitz. Here $n=1+b-a$. One possibility is to say that $\mathrm{Perf}^{[a,b]}(Y)$ is the simplicial localization of the subcategory of fibrant and cofibrant objects which correspond to perfect complexes with amplitude in $[a,b]$, in an appropriate closed model category of complexes. In the next section below, we give a different point of view where we apply the Dold-Puppe construction to the differential graded category of perfect complexes, defined for example by Bondal-Kapranov [16]. As a matter of notation, we let Perf denote the union of the $\mathrm{Perf}^{[a,b]}$. Since n depends on a,b, this is an ∞ stack. It may be seen as a Segal 1-stack because the Hom objects are simplicial sets which means ∞-groupoids. Note that even with the notation Perf we are still only considering perfect complexes of bounded amplitude, we are just not specifying which amplitude.

Toen and Vaquié have shown that $\mathrm{Perf}^{[a,b]}$ is a locally geometric n-stack, with geometric charts given by fixing bounds on the dimensions of the cohomology groups [93]. If $\underline{d}(i)$ is a function with $\underline{d}(i)=0$ for i outside the interval $[a,b]$ then let $\mathrm{Perf}(\le\underline{d})$ be the open substack of perfect complexes with $h^i(E_y)\le\underline{d}(i)$. The result of Toen and Vaquié can be stated

Theorem 4.1. *The n-stack* $\mathrm{Perf}(\le\underline{d})$ *is geometric.*

Proof: [93].

In the example of the previous section, the perfect complex of cohomology $H^{\cdot}(X,-)$ may be considered as a map

$$M_{DR}(X, GL(1)) \to \mathrm{Perf}^{[0,4]}.$$

Suppose now that we are in a relative situation, with a smooth projective morphism $p : X \to S$ between smooth projective varieties (ideally one would want to treat open varieties and singular maps too but this is farther off). Given a vector bundle with integrable connection (E, ∇) on X, we will obtain an object $Rp_*(E, \nabla)$ which encodes the family of perfect complexes

$$s \in S \mapsto H^{\cdot}(X_s, (E, \nabla)|_{X_s}).$$

This family has a "Gauss-Manin connection" in a certain sense. It can be seen, for example, as an object in the derived category of $\mathcal{D}$-modules on S, with the higher direct image calculated in terms of $\mathcal{D}$-modules. Another way of looking at it is as a map

$$S_{DR} \to \mathrm{Perf}^{[0,m]}$$

where $m = 2\dim(X/S)$ and S_{DR} is the *de Rham sheaf* $Y \mapsto Hom(Y^{\mathrm{red}}, S)$.

How does the direct image vary as a function of (E, ∇)? The map giving this variance is

$$Rp_* : M_{DR}(X, GL(n)) \to Hom(S_{DR}, \mathrm{Perf}^{[0,m]}).$$

The example in the previous section may be seen as the case where S is a single point.

We would like to think of the higher direct image Rp_* as a map between moduli stacks; this means that we would like to think of $Hom(S_{DR}, \mathrm{Perf}^{[0,m]})$ as being a moduli stack; rename it to

$$M_{DR}(S, \mathrm{Perf}^{[0,m]}) := Hom(S_{DR}, \mathrm{Perf}^{[0,m]}).$$

In order to be precise, the n-stack Hom here is the internal Hom in the world of n-stacks over the etale site of schemes of finite type over $Spec(\mathbb{C})$. In order to give some substance to this notation, our main result (in the de Rham case) is the following theorem.

Theorem 4.2. *If X is a smooth projective variety over $Spec(\mathbb{C})$ then $M_{DR}(X, \mathrm{Perf}^{[0,m]})$ is locally geometric, covered by open geometric substacks $M_{DR}(X, \mathrm{Perf}(\leq \underline{d}))$.*

The higher direct image functor can now be seen as a morphism between locally geometric n-stacks, and indeed it is a good idea to extend its definition to our new moduli stacks so that the map becomes

$$Rp_* : M_{DR}(X, \mathrm{Perf}^{[a,b]}) \to M_{DR}(S, \mathrm{Perf}^{[a,b+2\dim(X/S)]}).$$

It should be stressed that this map is not at all new: it is just the higher derived direct image functor on complexes of $\mathcal{D}$-modules. We are just investigating the properties of some possible choices for the domain and range of the functor.

Points in $M_{DR}(X, \mathrm{Perf}^{[a,b]})$ will be called *perfect complexes over X_{DR}*. In the $\mathcal{D}$-module point of view, these are complexes of $\mathcal{D}$-modules which are perfect as complexes of $\mathcal{O}_X$-modules. The existence of the flat connection guarantees that the cohomology sheaves are actually vector bundles and they inherit flat connections. Thus a perfect complex over X_{DR} is obtained by successive extensions (these can be realized as mapping cones) of vector bundles with integrable connection.

One observation to make here is that for cohomology we are only allowing $\mathcal{D}$-modules which are $\mathcal{O}_X$-coherent. Of course it would be interesting to look at more singular objects too but that goes outside our present scope. On the other hand, *a priori* the $\mathcal{D}$-modules appearing as terms in the complex are only quasicoherent over $\mathcal{O}_X$, indeed the general case is where the terms of the complex are free $\mathcal{D}$-modules. The main idea of our proof will be to use a Hochschild weakening in order to be able to work directly with complexes whose terms are coherent (and locally free) over $\mathcal{O}_X$. This type of reasoning was used by Kontsevich in the calculation of the deformations of various objects by going to their weak versions (e.g. A_∞-categories). Hinich also cites Drinfeld, and historically it also goes back to Stasheff's A_∞-algebras, Toledo and Tong's twisted complexes, the theory of operads and many other things. We had a feeling, in the work of [59], that something of this sort would probably be useful. It was brought more into focus in discussions I recently had with Ezra Getzler upon his visit to Nice. The present argument is only a fairly elementary example of this type of reasoning, applied to perfect complexes which are fairly linear objects (i.e. they don't have any homotopy operations).

Looking at a perfect complex over X_{DR} seems a bit oxymoronic, since the classical purpose of perfect complexes was to speak of resolutions of coherent sheaves which were not vector bundles; whereas here the cohomology sheaves are automatically locally free because of the integrable connection. It was Bertrand Toen who first pointed out to me that perfect complexes over X_{DR} remain interesting objects, essentially because they encode higher cohomological data. He has exploited these objects in his notion of *complex homotopy type* $X \otimes \mathbb{C}$. His basic idea is that tensor product provides the ∞-category of perfect complexes over X_{DR} with a Tannakian structure, and $X \otimes \mathbb{C}$ is the Tannaka dual of this Tannakian ∞-category. As this calls upon notions which are not necessarily well grounded yet, Toen came up with one (or several) more concrete constructions in order to get to $X \otimes \mathbb{C}$ [91] [61]. Nonetheless it is philosophically important to look at the original Tannakian idea. The main problem with the construction is that the category of perfect complexes over X_{DR} (or X_B) is considered as a discretized object. On the Tannaka dual side this gives an algebraic object which is the pro-algebraic completion of the homotopy type of X. This is "too big" in the same sense that the pro-algebraic completion of the fundamental group is too big.

Toen's Tannakian theory could be seen as one motivation for looking at the locally geometric n-stacks $M_{DR}(X, \mathrm{Perf}^{[a,b]})$ we consider here. If one wants to define a tensor product operation it is probably better to go to the locally geometric ∞-stack $M_{DR}(X, \mathrm{Perf})$ of perfect complexes of bounded amplitude over X_{DR} (but where we don't fix the length of the interval). Locally this ∞-stack is an n-stack but n varies depending on the open set. In any case, it is 1-groupic so it can be thought of as a Segal 1-stack. Tensor product provides a monoidal structure on $M_{DR}(X, \mathrm{Perf})$. A basepoint $x \in X$ gives a fiber-functor

$$\omega_{x,DR} : M_{DR}(X, \mathrm{Perf}) \to \mathrm{Perf}.$$

It is unclear what $Aut^{\otimes}(\omega_{x,DR})$ would look like; probably not really anything too good, due to the fact that the basic de Rham spaces of representations $M_{DR}(X, GL(n))$ don't have enough algebraic functions. The corresponding Betti version $Aut^{\otimes}(\omega_{x,B})$ should be better. In particular, starting from the de Rham side it would be better to go to the analytic category before taking the Tannaka dual. These Tannaka duals of continuous objects could be expected to be "discrete" in some sense, or at any rate "more discrete" than the pro-algebraic completion $X \otimes \mathbb{C}$.

Katzarkov, Pantev and Toen provide the schematic homotopy type $X \otimes \mathbb{C}$ with a mixed Hodge structure [60] [61]. This leads to Hodge-theoretic restrictions on the homotopy type of X, and should be related to the nonabelian mixed Hodge structure on the formal completions of $M_{DR}(X, Perf)$ in much the same way as Hain's mixed Hodge structure on the relative Malcev completion is related to the mixed Hodge structure on the formal completion of $M_{DR}(X, GL(n))$.

A natural question is whether the direct image functor Rp_* is compatible with the Hodge filtration. Of course the answer is yes, for purely formal reasons; the only problem is how to state the question. In order to give the statement, we want to introduce the notion of *perfect complex with λ-connection* over X, which is defined to be a map

$$(X_{Hod})_\lambda \rightarrow \mathrm{Perf}^{[a,b]}.$$

In keeping with the fact that the cohomology sheaves of perfect complexes on X_{DR} are vector bundles with integrable connection, we require that over points in the base scheme where $\lambda = 0$, the cohomology sheaves should be locally free, and furthermore semistable Higgs bundles with vanishing Chern classes. Requiring this condition, we obtain a moduli functor which is the relative Hom stack

$$M_{Hod}(X, \mathrm{Perf}^{[a,b]}) := Hom^{\mathrm{lf,se},c_i=0}(X_{Hod}/\mathcal{A}, \mathrm{Perf}^{[a,b]} \times \mathcal{A}/\mathcal{A}) \rightarrow \mathcal{A}.$$

Theorem 4.3. *Suppose $p : X \rightarrow S$ is a smooth morphism between smooth projective varieties. Then the higher direct image functor induces a map of n-stacks (for appropriate n),*

$$Rp_* : M_{Hod}(X, \mathrm{Perf}^{[a,b]}) \rightarrow M_{Hod}(S, \mathrm{Perf}^{[a,b+2\dim(X/S)]}).$$

We don't go into the proof of this here; it is basically just an observation about formal categories as described in [84]. It is the motivation for our main result which is geometricity of the moduli n-stacks entering into the above statement.

Theorem 4.4. *If X is a smooth projective variety over $Spec(\mathbb{C})$ then $M_{Hod}(X, \mathrm{Perf}^{[0,m]})$ is locally geometric, covered by open geometric substacks $M_{Hod}(X, \mathrm{Perf}(\leq \underline{d}))$.*

With this geometricity statement it seems reasonable to make the Deligne glueing in exactly the same way as before, to get an analytic moduli stack

$$M_{Del}(X, \mathrm{Perf}) \rightarrow \mathcal{P}^{\mathrm{an}},$$

whose pullback to $(\mathbb{P}^1)^{\mathrm{an}}$ is the *twistor space of perfect complexes*

$$M_{Hit}(X, \mathrm{Perf}) := M_{Del}(X, \mathrm{Perf}) \times_{\mathcal{P}^{\mathrm{an}}} (\mathbb{P}^1)^{\mathrm{an}}.$$

These have prefered sections which are mixed Hodge modules [76] or mixed twistor modules [75] over X (whose cohomology objects are required to be $\mathcal{O}_X$-coherent). We conjecture that the relative deformation to the normal cone along such a prefered section, should be a variation of nonabelian mixed Hodge structure in the sense of [59]. Saito (in the Hodge case) and Sabbah (in the twistor case) show that the higher direct image functor sends prefered sections to prefered sections.

5. Dold-Puppe of differential graded categories

In order to get the best possible control over the Segal stacks we are working with, the point of view of differential graded categories is optimal. The relationship between complexes and simplicial sets is given by the Dold-Puppe construction. In order to apply this to the enrichment of a category, giving a construction we shall denote by $\widetilde{DP}$ going from differential graded categories to simplicial categories, we have to verify that Dold-Puppe is compatible with the multiplicative structure (used for composition of morphisms in a d.g.c.). Of course this is classical, but I don't know of a good reference beyond [55] which is very succinct, so we do it explicitly here.

By a *chain complex* A, we mean a collection of A_n with differential $d : A_n \to A_{n-1}$. Denote by $\mathbf{s}K$ the *simple* construction of a chain complex out of a simplicial abelian group.

Define the following chain complex which we denote $D(n)$. Let $D(n)_k$ denote the free abelian group generated by the inclusions $\varphi : [k] \hookrightarrow [n]$, for $0 \leq k \leq n$, with $D(n)_k = 0$ otherwise. Denote the generator corresponding to φ by just φ.

Define the differential $d : D(n)_k \to D(n)_{k-1}$ by

$$d\varphi := \sum_{i=0}^{k} (-1)^i (\varphi \circ \partial_i)$$

where $\partial_i : [k-1] \to [k]$ is the face map skipping the ith object. The differential going from $D(n)_0$ to $D(n)_{-1}$ is declared to be zero.

We define a coproduct $\kappa : D(n) \to D(n) \otimes D(n)$ as follows. Let $lt_i : [i] \to [k]$ denote the inclusion of the first $i+1$ objects, and $gt_i : [k-i] \to [k]$ the inclusion of the last $k+1-i$ objects. Note that these overlap, both including object number i.

For $\varphi ; [k] \hookrightarrow [n]$, put

$$\kappa(\varphi) := \sum_{i=0}^{k} (\varphi \circ lt_i) \otimes (\varphi \circ gt_i).$$

This is co-associative: we have $(\kappa \otimes 1) \circ \kappa = (1 \otimes \kappa) \circ \kappa$.

Also κ is compatible with the differentials on $D(n)$ and $D(n) \otimes D(n)$. We have

$$(d \otimes 1 + \sigma \otimes d)(\kappa\varphi) =$$

$$\sum_{i=0}^{k} \sum_{j=0}^{i} (-1)^j (\varphi \circ lt_i \circ \partial_j) \otimes (\varphi \circ gt_i)$$

$$+ \sum_{i=0}^{k} \sum_{j=i}^{k} (-1)^j (\varphi \circ lt_i) \otimes (\varphi \circ gt_i \circ \partial_{j-i}).$$

In this expression the terms $i = 0, j = 0$ in the first line and $i = k, j = k$ in the second line which don't make sense, are defined as zero. This sum contains all of the terms of $\kappa(d\varphi)$ plus some additional terms. The additional terms include ones where various segments are removed; but each removed segment is counted twice, one for each endpoint, with opposite signs, so these terms all cancel. To do this formally, note that

$$lt_i \circ \partial_j = \partial_j \circ lt_{i-1}, \quad j \leq i$$

$$gt_i \circ \partial_{j-i} = \partial_j \circ gt_i, \quad j > i,$$

whereas

$$\partial_j \circ gt_{i-1} = gt_i, \quad j \leq i-1$$

and

$$\partial_j \circ lt_i = lt_i, \quad j > i,$$

finally, when they make sense we have

$$lt_i \circ \partial_i = lt_{i-1}, \quad gt_i \circ \partial_0 = gt_{i+1}.$$

Thus our sum above becomes

$$\sum_{i=1}^{k}\sum_{j=0}^{i-1}(-1)^j(\varphi \circ \partial_j \circ lt_{i-1}) \otimes (\varphi \circ \partial_j \circ gt_{i-1})$$

$$+\sum_{i=0}^{k}\sum_{j=i+1}^{k}(-1)^j(\varphi \circ \partial_j \circ lt_i) \otimes (\varphi \circ \partial_j \circ gt_i)$$

$$+\sum_{i=1}^{k}(-1)^i(\varphi \circ lt_{i-1}) \otimes (\varphi \circ gt_i)$$

$$+\sum_{i=0}^{k-1}(-1)^i(\varphi \circ lt_i) \otimes (\varphi \circ gt_{i+1}).$$

Here the first and second lines combine to give $\kappa(d\varphi)$ while the third and fourth lines cancel.

Our collection of complexes $D(n)$ forms a cosimplicial object in the category of complexes. Namely, for any map $\psi : [n] \to [m]$ we obtain a map $D(n) \to D(m)$. This sends φ to $\psi \circ \varphi$ if the latter is injective, and to zero otherwise. This operation is functorial: suppose we have maps $\psi : [n] \to [m]$ and $\tau : [m] \to [p]$. Then the combined operation sends φ to $\psi \circ \varphi$ or zero; subsequently to $\tau \circ \psi \circ \varphi$ if the latter is injective or zero otherwise. At the intermediate step, if $\tau \circ \psi \circ \varphi$ was injective then $\psi \circ \varphi$ had to be injective too so this works correctly.

On a theoretical level, let N denote the normalized complex functor from simplicial abelian groups to chain complexes of abelian groups. If we let D denote the degenerate subcomplex then $N(A) = \mathbf{s}(A)/D(A)$. The Dold-Kan theorem and Dold-Puppe construction say that N gives an equivalence of categories between simplicial abelian groups and chain complexes. Let $\mathbb{Z}\Delta(n)$ denote the simplicial abelian group generated by the standard n-simplex. Then for any simplicial abelian group A we have that A_n is the set of maps from $\mathbb{Z}\Delta(n)$ to A. By the equivalence of categories this is the same as the set of maps from $N\mathbb{Z}\Delta(n)$ to $N(A)$. Therefore, if we want to invert the functor N we should use the chain complexes $D(n) := N\mathbb{Z}\Delta(n)$ which are exactly those we have described explicitly above. In particular, one should think of $D(n)_k$ as being generated by all $\varphi : [k] \to [n]$, modulo the degenerate ones, which explains the formula for functoriality in the preceding paragraph.

The advantage of the explicit formulation is that we could write down explicitly the coproduct. One should note that the description of the Dold-Puppe operator given in [66] for example relies on formally adjoining degeneracies, which in turn is based on the description of N as a subcomplex of normalized chains rather than the more canonical description as a quotient by $D(A)$. In terms of this description it is hard to write down a nice coproduct.

We can now define the *Dold-Puppe functor* as the functor which to a chain complex A associates the simplicial abelian group which in degree n has

$$DP(A)_n := Hom(D(n), A)$$

(the group of morphisms of complexes from $D(n)$ to A). This is a simplicial object by functoriality of $D(n)$ in n. Elements of $DP(A)_n$ may be considered as functions $a(\varphi) \in A_k$ for each $\varphi : [k] \hookrightarrow [n]$, satisfying $d(a(\varphi)) = \sum (-1)^i a(\varphi \circ \partial_i)$.

We will check that $N(DP(A)) \cong A$ so that DP is a one-sided inverse of N, but as we know from the literature that N has an inverse it follows that our functor DP is the inverse. To verify this, go back to the description of N as the normalized subcomplex. Thus $N(DP(A))_n$ is the subgroup of $Hom(D(n), A)$ consisting of maps a such that $a \circ \partial_i = 0$ for $i = 1, \ldots, n$. Explicitly this means that $a(\varphi \circ \partial_i) = 0$ for $i = 1, \ldots, n$. It follows that the only nonzero values of a are $a(1_{[n]}) \in A_n$ and $a(gt_1) \in A_{n-1}$. On the other hand, the formula saying that a is a map of complexes from $D(n)$ to A says that $a(gt_1) = d(a(1_{[n]}))$, so an element of $N(DP(A))_n$ corresponds to a choice of arbitrary $a(1_{[n]}) \in A_n$. This isomorphism respects the differential, to give the claim and hence the fact that our functor as defined above is inverse to N.

Now the coproduct gives the composed map

$$Hom(D(n), A) \times Hom(D(n), B) \rightarrow Hom(D(n) \otimes D(n), A \otimes B) \rightarrow Hom(D(n), A \otimes B).$$

Therefore we obtain maps of simplicial sets (which however are not linear)

$$DP(A) \times DP(B) \rightarrow DP(A \otimes B).$$

This product is associative, because of co-associativity of κ. Note also that if $\mathbb{Z}[0]$ denotes the complex with $\mathbb{Z}$ in degree 0 then $DP(\mathbb{Z}[0])$ is the constant simplicial abelian group $\mathbb{Z}$. With respect to this isomorphism, the product above is unital too.

W have defined $DP(A)$ for chain complexes, which are the same as cochain complexes concentrated in negative degrees. If A is an unbounded cochain complex then let $\tau_{\leq 0}$ be the intelligent truncation which replaces A^0 by $ker(d : A^0 \rightarrow A^1)$, keeps the A^i for $i < 0$ and replaces A^j by 0 for $j > 0$. Then denote again by $DP(A)$ the result of DP applied to the truncation $\tau_{\leq 0}A$. Note that truncation is also compatible with tensor products: indeed we have

$$\tau_{\leq 0}A \rightarrow A,$$

hence

$$(\tau_{\leq 0}A) \otimes (\tau_{\leq 0}B) \rightarrow A \otimes B$$

and this factors through a map

$$(\tau_{\leq 0}A) \otimes (\tau_{\leq 0}B) \rightarrow \tau_{\leq 0}(A \otimes B).$$

This is associative too. Thus we obtain

$$(DP\tau_{\leq 0}A) \times (DP\tau_{\leq 0}B) \rightarrow (DP(\tau_{\leq 0}A \otimes \tau_{\leq 0}B)) \rightarrow DP(\tau_{\leq 0}(A \otimes B))$$

which is an associative product. This justifies replacing $DP\tau_{\leq 0}A$ by the simpler notation $DP(A)$: we retain our associative product for arbitrary complexes

$$DP(A) \times DP(B) \to DP(A \otimes B).$$

If R is a commutative base ring and A and B are complexes of R-modules then we have a map $A \otimes B \to A \otimes_R B$. This gives

$$DP(A) \times DP(B) \to DP(A \otimes_R B),$$

which remains associative and unital.

These facts allow us to use DP to create a simplicial category out of a differential graded category. A differential graded category over a ring R is a category C enriched in complexes of R-modules. Applying the functor DP to the enrichment, with the product maps constructed above, yields a category which we denote by $\widetilde{DP}(C)$. The objects of this category are the same as the objects of C, whereas if x, y are objects then by definition

$$Hom_{\widetilde{DP}(C)}(x, y) := DP(Hom_C(x, y)).$$

The composition of morphisms is given by the above product maps:

$$DP(Hom_C(x, y)) \times DP(Hom_C(y, z)) \to DP(Hom_C(x, y) \otimes_R Hom_C(y, z))$$

$$\to DP(Hom_C(x, z)).$$

Recall that an *equivalence* in a d.g.c. is a morphism u (that is, an element of $Hom^0(x, y)$ with $d(u) = 0$) which admits a quasi-inverse, that is a morphism v in the other direction such that and homotopies r, s of degree -1 such that $d(r) = uv - 1$ and $d(s) = vu - 1$. Note that the morphisms of $\widetilde{DP}(C)$, i.e. the points in the degree 0 part of the simplicial mapping sets, are the same as the morphisms of C. A morphism of C is an equivalence if and only if its image is an equivalence in $\widetilde{DP}(C)$.

The construction $\widetilde{DP}$ corresponds to forgetting some of the structure of C. For example, the positive-degree parts of the complexes $Hom_C(x, y)$ are truncated off, and also the fact that they are complexes is lost when we forget that $DP(Hom_C(x, y))$ are simplicial abelian groups (or actually simplicial R-modules in this case) and consider these just as simplicial complexes. The product maps for defining the compositions don't respect the abelian group structure of $DP(Hom_C(x, y))$ in any way which is easy to codify (actually they are quadratic maps in a certain sense but we forget that). So, all of this structure on C is lost when we go to the simplicial category $\widetilde{DP}(C)$. Under some circumstances there might be ways to get it back, for example if C has enough shift operators *à la* [16], but we don't consider that here.

5.1. **Homotopy fibers.** In the course of our proof we will need to understand the homotopy fiber of a map between $\widetilde{DP}$ constructions. Suppose $f : A \to B$ is a functor of differential graded categories, and suppose $b \in \mathrm{ob}(B)$. We would like to understand the homotopy fiber of $\widetilde{DP}(f)$ defined as

$$\mathbf{Fib}(\widetilde{DP}(f)/b) := \widetilde{DP}(A) \times^h_{\widetilde{DP}(B)} \{b\}.$$

Recall from [92] that f is *fibrant* if it satisfies the following two conditions:
(1) for each pair of objects x, y of A, the map

$$Hom_A(x,y) \to Hom_B(f(x), f(y))$$

is a surjection of complexes of abelian groups; and
(2) for $x \in \mathrm{ob}(A)$ and $y \in \mathrm{ob}(B)$ and u an equivalence in B between $f(x)$ and y, then u lifts to an equivalence $\tilde{u}$ in A, between x and a lift $\tilde{y}$ of y.

If f is fibrant in the above sense, then $\widetilde{DP}(f)$ is a functor between simplicial categories which is fibrant in Bergner's model structure [10], namely
(1) $\widetilde{DP}(f)$ induces a Kan fibration on mapping complexes; and
(2) equivalences in $\widetilde{DP}(B)$ lift to $\widetilde{DP}(A)$ with one endpoint fixed in the same way as above.

In particular, once we have Bergner's fibrancy condition, we can use $\widetilde{DP}(f)$ directly to form the homotopy fiber product:

$$\mathbf{Fib}(\widetilde{DP}(f)/b) = \widetilde{DP}(A) \times_{\widetilde{DP}(B)} \{b\}.$$

We don't propose an explicit model for the homotopy fiber $\mathbf{Fib}(\widetilde{DP}(f)/b)$ other than in the fibrant case, so it will cause no confusion to say that this formula for the homotopy fiber holds in the case f fibrant. The objects of $\mathbf{Fib}(\widetilde{DP}(f)/b)$ are the objects of A which map to b, and the simplicial mapping sets are the subsets of the mapping sets in $\widetilde{DP}(A)$ of objects mapping to (the degeneracies of) the identity of b.

We can construct the fiber on the level of differential graded categories. However, the fiber is not itself a Dold-Puppe construction but only a closely related affine modification, because of the condition that the morphisms map to the identity of b. In order to define this structure, first say that an *augmented differential graded category* (C, ε) is a dgc C with maps

$$\varepsilon : Hom^0_C(x,y) \to \mathbb{C}$$

for all pairs of objects x, y, such that $\varepsilon(dv) = 0$ when v has degree -1, and such that $\varepsilon(uv) = \varepsilon(u)\varepsilon(v)$ and $\varepsilon(1_x) = 1$. It is the same thing as a functor to the dgc $(*, \mathbf{cc}[\mathbf{0}])$ of one object whose endomorphism algebra is $\mathbb{C}$ in degree 0.

If (A, ε) is an augmented dgc, define the *affine Dold-Puppe* $\widetilde{DP}(A, \varepsilon)$ to be the sub-simplicial category of $\widetilde{DP}(A)$ whose simplices are those which project to degeneracies of the unit $1 \in \mathbb{C}$ in $DP(\mathbb{C}[0])$ which is the constant simplicial group $\mathbb{C}_\Delta$.

If f is fibrant in the above sense, define an augmented dgc $\mathbf{Fib}^{\mathrm{dgc}}(f/b)$ as follows. The objects are the objects of A which map to b. The mapping spaces are the subcomplexes of $Hom_A(x,y)$ consisting of elements which map to 0 in degree $\neq 0$ and which map to a constant multiple of the identity 1_b in degree 0. The augmentation ε is the map to the complex line of multiples of 1_b.

Lemma 5.1. *If $f : A \to B$ is a functor of differential graded categories which is fibrant in the above sense, and if $b \in \mathrm{ob}(B)$, then the homotopy fiber of $\widetilde{DP}(f)$ over b is calculated by the affine Dold-Puppe of the augmented dgc fiber of f,*

$$\mathbf{Fib}(\widetilde{DP}(f)/b) = \widetilde{DP}(\mathbf{Fib}^{\mathrm{dgc}}(f/b), \varepsilon).$$

□

5.2. **Maurer-Cartan stacks.** Suppose Z is an augmented differential graded algebra (i.e. associative but not necessarily unitary) over a commutative ring R, with augmentation $\varepsilon : Z \to \mathbb{C}[0]$ compatible with the product and differential. Then we obtain the differential graded category $\mathbf{MC}(Z, \varepsilon)$ of Maurer-Cartan elements $\eta \in Z^1$ such that $d(\eta) + \eta^2 = 0$. Give this a structure of d.g.c. by defining $Hom_{\mathbf{MC}(Z)}(\eta, \varphi) := Z$ as a graded group; but with differential

$$d_{\eta,\varphi}(a) := d(a) + \varphi a - (-1)^{|a|} a\eta.$$

This has square zero. The composition of morphisms is given by the algebra structure on Z, and it is compatible with the differentials. The augmentation gives an augmentation on the simplicial category. Applying the above construction we get a simplicial category $\widetilde{DP}(\mathbf{MC}(Z), \varepsilon)$. This construction should be basically equivalent to the simplicial Deligne groupoid constructed by Hinich [50].

If B is a commutative R-algebra then we can make the same construction for $Z \otimes_R B$. Let $\mathcal{M}C(Z, \varepsilon)$ denote the Segal stack on the big etale topology over $Spec(R)$, associated to the prestack

$$B \mapsto \widetilde{DP}(\mathbf{MC}(Z \otimes_R B), \varepsilon).$$

We will be interested in this construction under the following hypothesis which we call "quasi-nilpotent accessory cograding". Note first as a matter of notation that we call a *cograding* a decomposition like a grading but with direct product instead of direct sum.

Hypothesis 5.2 (Quasi-nilpotent accessory cograding). *Suppose R is a k-algebra of finite type with k a field of characteristic zero. Suppose that Z is a differential graded algebra over R, with an accessory cograding $Z = \prod_{k \geq 0} Z_k$, such that the differential preserves Z_k and the product structure sends $Z_k \otimes Z_l$ to Z_{k+l}. Assume that ε is an isomorphism from Z_0 to $\mathbb{C}[0]$. The quasi-nilpotence assumption is that there exists k_0 such that Z_k is acyclic for $k > k_0$.*

This hypothesis provides a certain type of nilpotence which replaces the use of Artinian local rings as coefficients in [36] for example. One consequence is homotopy invariance such as in [37].

Lemma 5.3. *Suppose $\psi : Q \to Z$ is a morphism of differential graded R-algebras, such that both Q and Z have quasi-nilpotent accessory cogradings and ψ preserves the accessory cogradings. If ψ is a quasiisomorphism then it induces an equivalence of stacks from $\mathcal{M}C(Q, \varepsilon)$ to $\mathcal{M}C(Z, \varepsilon)$.*

Proof: It suffices to note the same result on the level of differential graded categories $\mathbf{MC}(-)$. The quasi-nilpotence hypothesis allows us to solve the necessary equations automatically at all orders beyond k_0. An example is the proof of quasi-essential surjectivity. Suppose $\eta = (\ldots, \eta_k, \ldots)$ is a Maurer-Cartan element for Z. Construct a corresponding Maurer-Cartan element $\varphi \in Q^1$, along with a morphism called $1 + \alpha \in Z^0$, with φ and α in orders ≥ 1, step by step with respect to the cograding. The equations are

$$d(\varphi) + \varphi^2 = 0,$$

$$d(\alpha) + (\psi\varphi)(1 + \alpha) - (1 + \alpha)\eta = 0.$$

Once we have chosen φ and α up to and including degrees $k-1$, then choose φ_k and α_k so that

$$d(\varphi_k) + (\varphi^2)_k = 0,$$

and

$$d(\alpha_k) + \psi(\varphi_k) - \eta_k + ((\psi\varphi)\alpha - \alpha\eta)_k = 0.$$

This is possible at each step because of the condition that ψ be a quasiisomorphism (it requires some calculation but formally these equations are the same as in the theory of Schlessinger-Stasheff-Deligne-Goldman-Millson [36] [37]). A similar type of argument shows the quasiisomorphisms on the level of mapping complexes. $\square$

Corollary 5.4. *Suppose Z is a differential graded algebra with quasi-nilpotent accessory grading, such that the Z_k are acyclic for $k > k_0$. Let $Z_{\leq k_0}$ be the differential graded algebra obtained by dividing by the ideal of elements whose projections to Z_k are zero for $k \leq k_0$. Then the map $Z \to Z_{\leq k_0}$ induces an equivalence of Maurer-Cartan stacks.*

Proof: Apply the previous lemma; acyclicity exactly means that $Z \to Z_{\leq k_0}$ is a quasiisomorphism. $\square$

Theorem 5.5. *With the accessory cograding hypothesis, suppose that each Z_k is composed of free R-modules of finite rank, concentrated in cohomological degrees $\geq -n$. Then $\mathcal{M}C(Z, \varepsilon)$ is a geometric $n+1$-stack over $Spec(R)$.*

Proof: Using Corollary 5.4 we can assume that $Z_k = 0$ for $k > k_0$. The Hom complexes of the Maurer-Cartan stack are obtained by Dold-Puppe of perfect complexes with amplitude in $[-n, \infty]$. Since DP truncates at 0, the unboundedness in positive degrees is not a problem. By [85] these Hom complexes are geometric. It remains just to be seen how to define the smooth chart. But in this case the equation $d(\eta) + \eta^2 = 0$ defines a closed subvariety V of an affine space of finite dimension over R. The map from

$$V \to \mathcal{M}C(Z, \varepsilon)$$

is obviously surjective. To prove that it is smooth, suppose $Y \to \mathcal{M}C(Z, \varepsilon)$ is another map from an affine scheme Y. It corresponds to a Maurer-Cartan element $\eta \in Z^1 \otimes_R \mathcal{O}_Y(Y)$. We want to show that the map

$$V \times_{\mathcal{M}C(Z,\varepsilon)} Y \to Y$$

is smooth. We already know that this is geometric, so we just need to provide it with a chart which is smooth over Y. Take as chart the set of all $\alpha \in Z^0$ with $\varepsilon(\alpha) = 0$. For any such α, there is a unique solution $\varphi \in V$ with

$$d(\varphi) + \varphi^2 = 0$$

$$d\alpha + \varphi - \eta + (\alpha\varphi - \eta\alpha) = 0.$$

This is by the same argument of induction on the order k that was used in Lemma 5.3 above. One needs to check that the map from this chart to the fiber product is smooth; for this, use the expression of the Hom complexes as perfect complexes over $V \times Y$. The equation for the first term of the truncated perfect complex is exactly the second equation written above. Thus, our chart is actually the same as the standard one for Dold-Puppe of a perfect complex [85]. $\square$

The strategy of the proofs of Theorems 4.2 and 4.4 will consist of letting $Spec(R) \rightarrow Perf(X)$ be a smooth chart, and then expressing the stack $M_{DR} \times_{Perf(X)} Spec(R)$ (or similarly for M_{Hod}) as being Maurer-Cartan stacks of a finite differential graded algebra over R and applying the above theorem.

6. Complexes over sheaves of rings of differential operators

We would like to understand the points of $M_{Hod}(X, \mathrm{Perf})$. A "point" of course means a point with values in a scheme (which we may suppose to be affine) $S = Spec(A)$. Suppose given $\lambda : S \rightarrow \mathbb{A}^1$ (i.e. $\lambda \in A$) and look at S-valued points projecting to λ. By definition, such an S-point is a morphism

$$X_{Hod} \times_{\mathcal{A}} S \rightarrow \mathrm{Perf}.$$

Recall that X_{Hod} is given by a formal groupoid (basically, the "deformation to the normal cone" of the usual formal groupoid corresponding to X_{DR}). The underlying space is $X \times \mathcal{A}$. The fiber product with S over $\mathcal{A}$ is again a formal category, this time with underlying space $X \times S$. It corresponds to a sheaf of rings of differential operators Λ on X. For typical values of λ we have

$$\Lambda = \mathcal{D}_X, \quad \lambda = 1; \quad \Lambda = \mathrm{Sym}^{\cdot}(TX), \quad \lambda = 0.$$

If λ is a general function, we can construct Λ explicitly as follows. It depends on the Rees construction which will also be one of the main techniques of our proof later.

Start with an almost polynomial sheaf of rings of differential operators Λ filtered by Λ_r. The *almost polynomial* condition means that $Gr(\Lambda)$ is isomorphic to a polynomial ring $\mathrm{Sym}^{\cdot}(K)$ for a vector bundle $K = \Lambda_1/\Lambda_0$. We can form the *Rees ring* with a formal variable t:

$$\xi\Lambda := \bigoplus_r t^r \cdot \Lambda_r.$$

This contains the polynomial ring $\mathcal{O}_X[t]$ (corresponding to $\Lambda_0 = \mathcal{O}_X$).

Now, back to the situation of a function $\lambda : S \rightarrow \mathbb{A}^1$, we can construct the sheaf of rings of differential operators corresponding to $X_{Hod} \times_{\mathcal{A}} S$, as

$$\Lambda = (\xi\mathcal{D}_{X\times S/S}) \otimes_{\mathcal{O}_X[t]} \mathcal{O}_X = (\xi\mathcal{D}_{X\times S/S})/(t-\lambda).$$

It is also a split almost polynomial sheaf of rings of differential operators on $X \times S$ (and $\mathcal{O}_S$ is in the center).

Proposition 6.1. *Suppose S is a base scheme with a function $\lambda : S \rightarrow \mathbb{A}^1$. Let Λ be the sheaf of rings of differential operators corresponding to $X_{Hod} \times_{\mathcal{A}} S$. Then the Segal category of morphisms $X_{Hod} \times_{\mathcal{A}} S \rightarrow \mathrm{Perf}$ is equivalent to the Dold-Puppe of the differential graded category of complexes of Λ-modules on $X \times S$ which are perfect over $\mathcal{O}$,*

$$Hom(X_{Hod} \times_{\mathcal{A}} S, \mathrm{Perf}) \cong \widetilde{DP}(\mathrm{Cpx}_{\mathcal{O}-\mathrm{perf}}(\Lambda)(X \times S)).$$

The Segal category of points of $M_{Hod}(X, \mathrm{Perf})(S)$ lying over λ, is equivalent to the subcategory of complexes whose cohomology sheaves are locally free and semistable with vanishing Chern classes on any fiber X over a point $s \in S$ with $\lambda(s) = 0$.

We don't give the full proof of this here. The basic idea of one possible proof is to note that the cohomology objects of a complex on both sides are the same types of objects, namely U-coherent sheaves on $X \times S$ with λ-connections; and the cohomology groups which govern their extensions are calculated using the same de Rham complex, on both sides (as is well-known in crystalline cohomology, see [12] for example). Another possible proof is to show that points on the left side correspond to weak complexes on the formal category; then have a coherence result showing that these are equivalent to strict complexes, and note that strict complexes on the formal category are the same thing as complexes of Λ-modules.

Another alternative is simply to declare that the statement of Proposition 6.1 is the definition of $M_{Hod}(X, \mathrm{Perf})$.

One important observation is that the condition that the cohomology sheaves be vector bundles, with semistable Higgs structure and vanishing Chern classes over closed points in the base, is an open condition. This is true even in the context of perfect complexes, because when the dimension jumps it means that a quotient of one cohomology bundle is being cancelled by a subobject of the next one; the condition of semistability with vanishing Chern classes implies that both the quotient and subobject must also be semistable with vanishing Chern classes. (One can contrast this to the situation of a mixed Hodge or mixed twistor complex, where we have a perfect complex over $\mathbb{P}^1$ whose components are semistable of different weights—in which case the purity condition is not stable under small deformations.)

In view of the proposition, look at the following somewhat more general situation. It should also serve to cover other cases such as logarithmic connections [71] and Esnault's τ-connections [29] [30] (which are connections over foliations).

Suppose that $X \to S$ is a smooth projective morphism, and Λ is an almost polynomial sheaf of rings of differential operators on X/S (commuting with $\mathcal{O}_S$). Define the presheaf of differential graded categories

$$\mathrm{Cpx}_{\mathcal{O}-\mathrm{perf}}(\Lambda/S)$$

to be the assignment which to an S-scheme Z associates the differential graded category of complexes of $\Lambda|_{X\times_S Z}$-modules on $X \times_S Z$, which are perfect as complexes of $\mathcal{O}$-modules. Let

$$M(\Lambda/S) := \widetilde{DP}(\mathrm{Cpx}_{\mathcal{O}-\mathrm{perf}}(\Lambda/S))$$

be the Segal stack associated to the presheaf of simplicial categories obtained by applying the construction $\widetilde{DP}$ over each object $Z \to S$.

The functor of forgetting the Λ-module structure induces a functor of presheaves of differential graded categories

$$\mathrm{Cpx}_{\mathcal{O}-\mathrm{perf}}(\Lambda/S) \to \mathrm{Perf}_{\mathrm{dgc}}(X/S)$$

to the presheaf which to $Z \to S$ associates the differential graded category of perfect complexes on $X \times_S Z$. Applying $\widetilde{DP}$ we get a morphism of Segal stacks

$$M(\Lambda/S) \to \mathrm{Perf}(X/S).$$

The result of Toen and Vaquié [93] in the relative setting is:

Theorem 6.2. *The Segal stack* $\mathrm{Perf}(X/S)$ *is locally geometric.*

We refer to [93] for the proof. Their proof is based on a Beilinson-style analysis of the situation, expressing the derived category of perfect complexes as equivalent to the derived category of modules over a finite dimensional differential graded algebra on the base S obtained as the endomorphisms of the higher direct image of a generating object.

Our theorem is the following:

Theorem 6.3. *The map*

$$M(\Lambda/S) \to \mathrm{Perf}(X/S)$$

is a geometric morphism of Segal stacks.

The proof will occupy the remainder of the exposition.

Corollary 6.4. *The Segal stack $M(\Lambda/S)$ is locally geometric. The geometric open sets are given as in* [93] *by fixing a bound on the cohomology dimensions.*

Proof: A composition of geometric morphisms is geometric [85]. □

Corollary 6.5. *Fix an interval $[a,b]$ and a function $\underline{r}(i)$ nonzero only for i in $[a,b]$. Set $n = 1+b-a$. Let $M_{Hod}(X, \mathrm{Perf}^{\underline{r}})$ denote the moduli n-stack of perfect complexes on $X_{Hod}/\mathbb{A}^1$, such that in degree i the cohomology object has rank $\leq \underline{r}(i)$ over X, and such that over $\lambda = 0$ the cohomology objects are semistable with vanishing Chern classes. Then $M_{Hod}(X, \mathrm{Perf}^{\underline{r}})$ is a geometric n-stack.*

Proof: By the boundedness results of [83], the cohomology objects in question have bounded dimensions of global cohomology over X; thus $M_{Hod}(X, \mathrm{Perf}^{\underline{r}})$ falls (as an open substack) into one of the geometric open sets of $M(\xi\mathcal{D}_X/\mathbb{A}^1)$ in Corollary 6.4. □

This corollary implies the theorems 4.4 and 4.2 (the latter because M_{DR} is an open substack of M_{Hod}).

We don't directly treat the proof of Theorem 6.3 but first make a reduction via a trick based on the Rees construction. Let

$$\Upsilon := \xi\Lambda = \bigoplus_r \Lambda_r.$$

Recall that this has a subalgebra $\mathcal{O}_X[t] \subset \Upsilon$.

An $\mathcal{O}$-perfect complex of Λ-modules may be considered as an $\mathcal{O}$-perfect complex of Υ-modules such that the element t acts by 1. Thus we can express

$$M(\Lambda/S) = M(\Upsilon/S) \times_{M(\mathcal{O}_X[t]/S)} \mathrm{Perf}(X/S).$$

In particular, if we can prove Theorem 6.3 for Υ and $\mathcal{O}_X[t]$ then we get it for Λ too.

Now Υ has the particularly nice property that it is actually a graded ring. Say that a *graded almost polynomial ring of differential operators* is a sheaf of rings Υ over $\mathcal{O}_X$, graded by pieces $\Upsilon : \bigoplus_k \Upsilon_k$, such that $\Upsilon = \mathcal{O}_X$, such that the pieces are locally free of finite rank for both their left and right module structures, and such that Υ admits an additional filtration F compatible with the grading (i.e. coming from a filtration on each graded piece) such that $Gr^F\Upsilon$ is isomorphic to a polynomial ring over $\mathcal{O}_X$ on $Gr^F\Upsilon_1$.

Lemma 6.6. *If Λ is an almost polynomial ring of differential operators over S in the sense of* [83]*, then its Rees ring $\Upsilon := \xi\Lambda$ is a graded almost polynomial ring of differential operators over X.*

Proof: The grading is given by the Rees construction; the filtration F comes from the original filtration on each graded piece. □

Note of course that $\mathcal{O}_X[t]$ is also a graded almost polynomial ring of differential operators over X. We have reduced to the same statement as 6.3 but for a graded almost polynomial ring.

Theorem 6.7. *Suppose $X \to S$ is a smooth morphism. Suppose Υ is a graded almost polynomial ring of differential operators over X, with $\mathcal{O}_S$ central. Suppose E is a perfect complex of $\mathcal{O}_X$-modules on X. Let $M(\Upsilon/S;E)$ denote the fiber product*

$$M(\Upsilon/S;E) := M(\Upsilon/S) \times_{\mathrm{Perf}(X/S)} \{E\}$$

where $\{E\}$ is the section $S \to \mathrm{Perf}(X/S)$ corresponding to E. Then $M(\Upsilon/S;E)$ is a geometric n-stack (where n is the amplitude of E plus 1).

Lemma 6.8. *Theorem 6.7 implies Theorem 6.3, hence Corollary 6.5 and Theorems 4.4 and 4.2.*

Proof: In the statement of Theorem 6.7 we have just made more explicit the statement of Theorem 6.3 for the case of the ring Υ. As we have seen via the Rees construction above, this special case implies the general case. □

In view of this reduction, we will now concentrate on proving Theorem 6.7. You will see that the grading gives a significant help in many parts of the proof. To understand the places where this helps, it is useful to go back to a basic situation: the moduli stack M_{DR} of vector bundles with connection. This maps to the moduli stack of vector bundles, and the fibers are something like affine spaces. To express them as affine spaces it is useful to introduce the Atiyah bundle construction, which looks like a jet-bundle construction. Another way of putting this is that we can consider a connection as an $\mathcal{O}_X$-linear map

$$(\mathcal{D}_X)_1 \otimes_{\mathcal{O}_X} E \to E,$$

subject to the affine condition that the restriction to $\mathcal{O}_X \subset (\mathcal{D}_X)_1$ be the identity.

Throughout our proof we will be using Maurer-Cartan type equations of the form $d(\eta) + \eta^2 = 0$ (as in the classical equation for a connection of zero curvature). In the Schlessinger-Stasheff-Deligne-Goldman-Millson approach to deformation theory using the Maurer-Cartan equation, some type of nilpotence is necessary in order to insure termination of the calculations involved. In the classical case, this nilpotence is achieved by looking at objects over Artin local rings. The drawback is that we only get information about the formal neighborhood of a point in the moduli space, this way. However, the fact that the fibers of the map $M_{DR} \to Bun$ are affine spaces suggests that we should be able to capture whole fibers in a single coordinate chart. This is indeed the case, and it is basically because of a type of "nilpotence" which says that we only need to consider differential operators of some bounded order (up to order two for the classical case of connections). The same phenomenon will come into play in our proof below. The graded structure of the ring Υ is crucial to controlling and utilising this phenomenon. It eventually will yield a quasi-nilpotent accessory grading on a differential graded algebra (Hypothesis 5.2) which will allow us to apply Theorem 5.5.

7. The Hochschild complex and weak Υ-module structures

We have a fixed perfect complex E and we would like to understand the possible Υ-module structures on E. The obvious definition is that a Υ-module structure on E is a complex of Υ-modules E' plus a quasiisomorphism between E and E' over $\mathcal{O}_X$. This is unwieldy in that it brings in an additional complex E' which will usually be fairly big (like a complex of free Υ-modules). A much better way to attack this question is to try to define a notion of *weak* structure of Υ-modules directly on E itself. For this we follow Kontsevich and use the basic model of Stasheff's definition of A_∞-algebra where the higher homotopies are organized in a natural way. In the present context, calculation of the first few terms quickly leads to a Hochschild-type complex as the natural way to define a weak Υ-module structure. In this section, we investigate this notion over an affine open piece of X, which we call $Spec(R)$.

All the calculations are of course classical.

Suppose R is a commutative $\mathbb{C}$-algebra of finite type and Υ is a graded k-algebra with graded pieces Υ_k such that $\Upsilon_j\Upsilon_k \subset \Upsilon_{j+k}$. Suppose that $\Upsilon_0 = R$.

In particular, Υ (and indeed each Υ_k) has both left and right R-module structures. When we write the tensor product

$$\Upsilon \otimes_R \Upsilon,$$

use the right module structure on the left factor and the left module structure on the right factor; there persist structures on the left (coming from the left structure of the left factor) and on the right (coming from the right structure of the right factor).

We assume throughout that the Υ_k are flat of finite type over R (for both structures). In fact we shall generally assume that there is a further filtration whose associated graded is a polynomial ring over R.

Inductively we obtain the n-fold tensor product

$$T^n\Upsilon := \Upsilon \otimes_R \cdots \otimes_R \Upsilon.$$

Again this has both left and right R-module structures. Let

$$T\Upsilon := \bigoplus_n T^n\Upsilon$$

denote the tensor algebra (in this sense of contracting the interior R-module structures). Again it has left and right R-module structures and we maintain the convention about tensor products with these structures.

This has a Hochschild differential, and a coproduct, which we now describe (these structures are otherwise known as the *bar complex* and the tensor products can be replaced by bars if one wants, although in some notations this induces a shift of indexing from what we are using here). The differential is defined by the formula

$$\delta(u_1 \otimes \cdots \otimes u_n) := \sum_{i=1}^{n-1} (-1)^{i+1} u_1 \otimes \cdots \otimes (u_i u_{i+1}) \otimes \cdots \otimes u_n.$$

Thus,

$$\delta : T^n\Upsilon \to T^{n-1}\Upsilon$$

(so with respect to the grading by powers of T, δ is a homological differential).

A calculation gives $\delta^2 = 0$.

The coproduct

$$\Delta : T\Upsilon \to T\Upsilon \otimes_R T\Upsilon$$

is defined by the formula

$$\Delta(u_1 \otimes \cdots \otimes u_n) := \sum_{i=0}^{n} [u_1 \otimes \cdots \otimes u_i] \otimes [u_{i+1} \otimes \cdots \otimes u_n]$$

where we have put in brackets [] the terms going into the two factors of $T\Upsilon$, and where by convention the empty bracket denotes the object $1 \in T^0\Upsilon$.

This is associative in the sense that if we define in the obvious way maps $\Delta \otimes 1$ or $1 \otimes \Delta$ going

$$T\Upsilon \otimes_R T\Upsilon \to T\Upsilon \otimes_R T\Upsilon \otimes_R T\Upsilon,$$

then we have

$$(\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta.$$

This is an easy calculation because all the signs are positive.

The coproduct is also compatible with the differential. More precisely, let σ denote the sign operator which is 1 on $T^{2n}\Upsilon$ and -1 on $T^{2n+1}\Upsilon$. We have a differential $\delta \otimes 1 + \sigma \otimes \delta$ on $T\Upsilon \otimes_R T\Upsilon$, and

$$\Delta \circ \delta = (\delta \otimes 1 + \sigma \otimes \delta) \circ \Delta.$$

To prove this, let's apply it to an element $u_1 \otimes \cdots \otimes u_n$. On the left we get (omitting the tensor product signs in an obvious way)

$$\Delta \sum_{i=1}^{n-1} (-1)^{i+1} u_1 \cdots (u_i u_{i+1}) \cdots u_n$$

$$= \sum_{i=1}^{n-1} \sum_{j=0}^{i-1} (-1)^{i+1} [u_1 \cdots u_j] \otimes [\cdots (u_i u_{i+1}) \cdots u_n]$$

$$+ \sum_{i=1}^{n-1} (-1)^{i+1} [u_1 \cdots (u_i u_{i+1})] \otimes [\cdots u_n]$$

$$+ \sum_{i=1}^{n-1} \sum_{j=i+2}^{n} (-1)^{i+1} [u_1 \cdots (u_i u_{i+1}) \cdots u_j] \otimes [\cdots u_n].$$

The first term on the right is

$$\delta \otimes 1 \sum_{j=0}^{n} [u_1 \cdots u_j] \otimes [u_{j+1} \cdots u_n]$$

$$= \sum_{j=0}^{n} \sum_{i=1}^{j-1} (-1)^{i+1} [u_1 \cdots (u_i u_{i+1}) \cdots u_j] \otimes [\cdots u_n].$$

This accounts for the second and third terms in the left hand side calculated above. The second term on the right is

$$\sigma \otimes \delta \sum_{j=0}^{n} [u_1 \cdots u_j] \otimes [u_{j+1} \cdots u_n]$$

$$= \sum_{j=0}^{n} \sum_{i=j+1}^{n-1} (-1)^{(i-j)+1+j} [u_1 \cdots u_j] \otimes [u_{j+1} \cdots (u_i u_{i+1}) \cdots u_n].$$

This accounts for the first term on the left side as calculated above. Thus we see that the left and right sides are equal.

The grading of Υ induces a second grading of $T\Upsilon$ denoted

$$T\Upsilon = \bigoplus_k (T\Upsilon)_k$$

with

$$(T\Upsilon)_k = \bigoplus_n (T^n\Upsilon)_k$$

for

$$(T^n\Upsilon)_k := \sum_{j_1+\ldots+j_n=k} \Upsilon_{j_1} \otimes_R \cdots \otimes_R \Upsilon_{j_n}.$$

The differential δ preserves this grading:

$$\delta : (T^n\Upsilon)_k \to (T^{n-1}\Upsilon)_k.$$

The coproduct is multiplicative on the grading:

$$\Delta : (T\Upsilon)_k \to (T\Upsilon \otimes_R T\Upsilon)_k$$

where the lower grading of the tensor product is obtained from the lower grading of $T\Upsilon$ as

$$(T\Upsilon \otimes_R T\Upsilon)_k := \bigoplus_{i+j=k} (T\Upsilon)_i \otimes_R (T\Upsilon)_j$$

(in a similar way we obtain the upper grading of the tensor product using the upper grading $T^n\Upsilon$; the differential $\delta \otimes 1 + \sigma \otimes \delta$ is of degree -1 with respect to the upper grading whereas it is of degree 0 with respect to the lower grading).

To simplify notation we will call the degree in the lower grading the *order*, thus an "element of order k" means an element of $(\,)_k$.

We use the above constructions in order to define a differential graded category of *complexes with weak Υ-module structures*. Suppose (E, d) and (F, d) are complexes of R-modules. If necessary in case of confusion we can note the differentials d_E and d_F. Define a complex denoted $(Q'(E, F), d_Q)$ as

$$Q'(E, F) := Hom_R(T\Upsilon \otimes_R E, F)$$

with the differential d_Q combining the differentials on $T\Upsilon$ (defined above) with the differentials d_E and d_F using the usual sign rules. Note that the upper indexing for $T\Upsilon$ with respect to which the differential is homological, is changed to negative indexing with a cohomological differential when we do the construction of Q'. Thus $(Q')^i$, consists of morphisms a such that

$$a(u_1 \otimes \cdots \otimes u_k) : E^j \to F^{i+j-k}.$$

Let $Q \subset Q'$ be the subcomplex of morphisms a such that if $u \in T\Upsilon$ is a tensor of positive degree but order zero (i.e. $u \in (T^i\Upsilon)_0$ for some $i > 0$) then $a(u) = 0$. We use Q rather than Q' in what follows because we would like to maintain some control over the process of "forgetting the weak Υ-module structure". In particular, the complex $Hom_R(E, F)$ appears as a subcomplex (and also a quotient complex) of $(Q(E, F), d_Q)$; it is exactly the subcomplex of elements of order zero.

The coproduct on $T\Upsilon$ gives us an associative product

$$Q(E, F) \times Q(F, G) \to Q(E, G),$$

compatible with the differentials. The product is denoted by simple juxtaposition in what follows. It is also unitary with respect to the identity morphisms of complexes, thought of as elements of order zero in the $Q(E, E)$.

The grading by order is preserved by the differential d_Q, and compatible with product (the order of the product being the sum of the orders). The product is the usual composition on the subcomplexes $Hom_R(E, F)$ of elements of order zero. Denote by $Q(E, F)_{>0}$ the ideal of elements of positive order.

A *Maurer-Cartan element* for a complex of R-modules E is an element

$$\eta \in Q^1(E, E)$$

such that the order-zero term of η vanishes (i.e. $\eta \in Q^1(E, E)_{>0}$), and such that η satisfies the *MC equation*

$$d_Q(\eta) + \eta^2 = 0.$$

If η is a Maurer-Cartan element for E and φ is a Maurer-Cartan element for F, then we define a new differential $d_{\eta,\varphi}$ on $Q(E, F)$ as follows:

$$d_{\eta,\varphi}(a) := d_Q(a) + \varphi a - (-1)^{|a|} a\eta.$$

One can verify using the MC equation that $d^2_{\eta,\varphi} = 0$.

With the new differential, the quotient map

$$(Q(E, F), d_{\eta,\varphi}) \to (Hom_R(E, F), d_{E,F})$$

is still a map of complexes; however $Hom_R(E, F)$ is no longer a subcomplex.

An *MC complex* is a pair (E, η) where E is a complex of R-modules and η is a Maurer-Cartan element. We will usually require that E be bounded (that is, $E^i = 0$ outside of a finite interval) and that E consist of flat R-modules in each degree.

One can verify that if (E, η), (F, φ) and (G, γ) are three MC complexes, the composition map

$$Q(E, F) \times Q(F, G) \to Q(E, G)$$

is still compatible with the new differentials $d_{\eta,\varphi}$, $d_{\varphi,\gamma}$ and $d_{\eta,\gamma}$. In this way, we obtain a differential graded category of MC complexes denoted $\mathrm{Wpx}_{R-\mathrm{perf}}(\Upsilon)$. It has a functor to the differential graded category of perfect complexes of R-modules $\mathrm{Perf}(R)_{\mathrm{dgc}}$ (see [16]) given by the quotient map of morphism complexes considered above.

Almost polynomial hypothesis on Υ: We suppose henceforth that Υ admits a further filtration, compatible with the gradings by degree and order, such that the associated-graded of this filtration is a polynomial ring over R. Thus Υ is a *graded almost-polynomial sheaf of rings of differential operators* in the notation of §6.

Proposition 7.1. *With the graded almost-polynomial hypothesis, the complex $T\Upsilon$ has cohomology which is of finite type over R. In particular, there exists k_0 such that for $k > k_0$, $(T\Upsilon)_k$ is acyclic.*

Proof: This follows from the Hochschild-Kostant-Rosenberg calculation of the Hochschild homology of the polynomial ring [54], using the same spectral sequence argument as in Wodzicki [97]. □

The following theorem should be viewed as a coherence result for weak Υ-module structures. In this sense it is a standard type of thing. Let $\mathrm{Cpx}_{R-\mathrm{perf}}(\Upsilon)$ denote the differential graded category of complexes of Υ-modules which are perfect over R. Use free resolutions to define the differential graded structure.

Theorem 7.2. *The differential graded category of MC-complexes $\mathrm{Wpx}_{R-\mathrm{perf}}(\Upsilon)$ is quasi-equivalent, relative to $\mathrm{Perf}(R)_{\mathrm{dgc}}$, to the differential graded category $\mathrm{Cpx}_{R-\mathrm{perf}}(\Upsilon)$ of complexes of free Υ-modules perfect over R.*

We sketch the proof of the theorem here, but first point out that this should be considered as a "pre-theorem" in the sense of Adams, in that one might have to fiddle with the boundedness conditions on the complexes, or else with the hypotheses on Υ, in order to get all of the details right. In practice for the cases we will need, one can choose bounded resolutions by free Υ-modules so we ignore the question of boundedness of our resolutions in what follows.

The first thing to notice is the invariance of the morphism complexes of our differential graded category, under quasiisomorphisms.

Lemma 7.3. *Suppose $f \in Q^0(E,F)$ such that $d_{\eta,\varphi}(f) = 0$, and such that the underlying morphism of complexes (which is the piece f_0 of order zero) induces a quasiisomorphism from E to F. Suppose as usual that E and F are projective over R. Then for any MC-complex (G,γ), composition with f induces quasiisomorphisms (with respect to the MC-twisted differentials)*

$$Q(F,G) \to Q(E,G)$$

and

$$Q(G,E) \to Q(G,F).$$

Proof: We can filter by a decreasing filtration made out of the grading by order (a level of this filtration will consist of everything with order greater than a certain amount). The fact that the MC elements have order > 0 means that they act trivially on the associated-gradeds for this filtration. Also everything is acyclic in the pieces of high enough order by Proposition 7.1. So, up to passing to associated-graded objects we may assume that the MC elements are zero. In this case the complexes become $Q(E,F) = Hom_R(T\Upsilon \otimes_R E, F)$ and similarly for the other cases. Note that by our hypothesis on Υ, it is a free R-module on either side, so $T\Upsilon$ is also free over R. Thus the hypothesis that E and F are projective gives the result in question. □

Another important type of homotopy invariance (which really explains why we are interested in the notion of weak structure) is the following.

Lemma 7.4. *Suppose (E, η) is an MC complex, and suppose F is a complex of R-modules. Suppose both are projective over R. Suppose $a_0 : E \to F$ is a morphism of complexes of R-modules which is a quasiisomorphism. Then there exists an MC element φ for F, and*

$$a \in Q^0(E, F)$$

such that $d_{\eta,\varphi}(a) = 0$, such that a lifts a_0 in the sense that the piece of a of order 0 is equal to a_0. The same for an equivalence going in the other direction.

Proof: By the condition that the complexes are projective over R, we can choose a homotopy inverse $b_0 : F \to E$ to a_0. In particular there is $K \in End(E)^0$ with $d(K) = b_0a_0 - 1$. Put

$$\varphi := a_0\eta(\sum_{m=0}^{\infty}(K\eta)^m)b_0.$$

We have

$$d(\varphi) = \sum \pm a_0\eta \cdots Kd(\eta)K \ldots \eta b_0 + \sum \pm a_0\eta \cdots \eta d(K)\eta \ldots \eta b_0.$$

Here and in what follows, we leave it to the reader to fill in the signs. Using $d(K) = b_0a_0 - 1$ and $d(\eta) = -\eta^2$, the terms corresponding to b_0a_0 in $d(K)$ give $-\varphi^2$ whereas the terms corresponding to -1 in $d(K)$ and to $d(\eta) = -\eta^2$ cancel. Thus,

$$d(\varphi) = -\varphi^2.$$

Similarly, put

$$a := a_0(\sum_{m=0}^{\infty}(\eta K)^m).$$

The inner terms in $d(a)$ work the same way as before; the term involving $d(K)$ on the right end gives $a\eta$ so we get

$$d(a) + \varphi a - a\eta = 0.$$

This says that a is a morphism from η to φ. □

The next step is to define the functor

$$\mathrm{Cpx}_{R-\mathrm{perf}}(\Upsilon) \to \mathrm{Wpx}_{R-\mathrm{perf}}(\Upsilon).$$

If E is a complex of Υ-modules, define the corresponding MC element by $\eta(u_1) := u_1e$ whereas $\eta(u_1 \otimes \cdots \otimes u_k)e = 0$ for $k \neq 1$. Let 1 denote the unit of $T^0\Upsilon = R$; in a moral sense one should think of $\eta(1)$ as being the differential d_E, however we are keeping the differential distinct in our notation (it came before the notion of MC element), so we require $\eta(1) = 0$ which is the case $k = 0$ in the above condition.

If E and F are complexes of Υ-modules then we obtain a map

$$Hom_{\Upsilon}(E, F) \to Q(E, F)$$

sending a morphism a to the same element $a \in Hom_R(T^0\Upsilon \otimes_R E, F)$. One calculates that this respects the differential on the left and the $d_{\eta,\varphi}$ on the right where η and φ are the MC elements corresponding as above to the module structure. This gives the functor from $\mathrm{Cpx}_{R-\mathrm{perf}}(\Upsilon)$ to $\mathrm{Wpx}_{R-\mathrm{perf}}(\Upsilon)$.

The following lemma shows that this functor is a quasi-fully-faithful.

Lemma 7.5. *Suppose E and F are bounded complexes of Υ-modules, such that in each degree E is a free Υ-module. Then the above map*

$$Hom_\Upsilon(E, F) \to (Q(E, F), d_{\eta,\varphi})$$

is a quasiisomorphism.

Proof: It suffices to prove this for E and F being just single modules concentrated in degree 0, with $E \cong \Upsilon$. The map in question is the first map in the complex

$$F \to Hom_R(T^1\Upsilon, F) \to Hom_R(T^2\Upsilon, F) \to \ldots .$$

Note that the tensor powers are increased by one because of tensoring with $E = \Upsilon$. A calculation shows that the differential of this complex (including the first map which is the map in question) has the formula (up to a potential sign error)

$$(da)(u_0 \otimes \cdots \otimes u_k) = u_0 a(u_1 \otimes \cdots \otimes u_k) + \sum_{i=0}^{k-1} (-1)^i a(\cdots \otimes u_i u_{i+1} \otimes \cdots).$$

We claim that this complex is acyclic. To see this, write down the homotopy

$$(Ka)(u_1 \otimes \cdots \otimes u_k) := \sum_{j=0}^{k} (-1)^j (\cdots \otimes u_j \otimes 1 \otimes u_{j+1} \otimes \cdots).$$

Up to a sign error (which I am ignoring because this is a standard type of argument) we have that $Kd + dK$ is the identity. Thus the full complex is acyclic, and the map in question which is the map from F to the rest of the complex, is a quasiisomorphism. □

To complete the proof, we have to argue that the functor is essentially surjective. Thus we want to show that any MC complex is equivalent to a complex of free Υ-modules. For this part, note first that there is a notion of *mapping cone* for $\mathrm{Wpx}_{R-\mathrm{perf}}(\Upsilon)$. If (E, η) and (F, φ) are MC complexes, and if $\beta \in Q^1(E, F)$ with $d_{\eta,\varphi}\beta = 0$ then $E \oplus F$ can be given the MC element whose matrix is triangular with entries η and φ on the diagonal and β in the corner. This construction functions as a mapping cone. If we are given a map $E \to F$ (that is, an element $\alpha \in Q^0(E, F)$ with $d_{\eta,\varphi}\alpha = 0$) then we can form the mapping cone $\mathrm{Cone}(\alpha) = F \oplus E[1]$ with a triangular MC element as described just before. We can think of this as satisfying a universal property, in the sense that for any MC complex G we have

$$Q(G, \mathrm{Cone}(\alpha)) = \mathrm{Cone}(Q(G, \alpha))$$

where $Q(G, \alpha)$ is the map $Q(G, E) \to Q(G, F)$. Similarly

$$Q(\mathrm{Cone}(\alpha), G) = \mathrm{Cone}(Q(\alpha, G)).$$

The cone constructions on the right in these two statements are kernels equal to cokernels in the differential graded category of complexes, and in this sense the cone construction in our

$\mathrm{Wpx}_{R-\mathrm{perf}}(\Upsilon)$ can be viewed as being defined, up to homotopy, by a universal property. We don't go into the theory necessary to make this precise. Kontsevich pointed out in a talk in Luminy that Bondal and Kapranov had shown that triangles were intrisically defined in the differential graded category of complexes; we later noticed that Gabriel and Zisman in their book had made much the same observation although in a somewhat 2-truncated way.

For our present purposes we will just be happy with having the cone construction and its effect on morphism complexes.

Here is how to prove essential surjectivity. It is done by induction on the length of the amplitude interval. If this length is one, then the object is just a Υ-module. Suppose now that we have treated the case of amplitude $n-1$ and E is a MC complex of amplitude n. Using the invariance property 7.4, we can replace E up to quasiisomorphism (which is the same as quasi-equivalence by Lemma 7.3) by a complex which is bounded at the high end of the interval of amplitude. Then choose a surjection from a free Υ-module Υ^I to the last cohomology module (note that the cohomology modules are Υ-modules). We can make this correspond in a trivial way to a map α of MC complexes. The mapping cone $\mathrm{Cone}(\alpha)$ has amplitude $n-1$, so it is equivalent to a complex B of free Υ-modules. We can get back E as the cone on the map from B to Υ^I. By Lemma 7.5, this map is equivalent to an actual map of complexes of Υ-modules, and mapping cones of homotopic maps are equivalent, so E becomes equivalent to the cone on a map of complexes of free Υ-modules. This completes the induction, proving essential surjectivity and hence finishing the proof of Theorem 7.2. □

We finish this section by noting that the functor of dgc's

$$\mathrm{Wpx}_{R-\mathrm{perf}}(\Upsilon) \to \mathrm{Perf}^{\mathrm{dgc}}(R)$$

is fibrant in the sense of §5.1. The first condition is clear by construction, indeed the Hom complexes for $\mathrm{Perf}^{\mathrm{dgc}}$ are split subcomplexes of the Hom complexes of $\mathrm{Wpx}_{R-\mathrm{perf}}(\Upsilon)$. The second condition is exactly Lemma 7.4.

The construction of the Maurer-Cartan differential graded category made in §5.2 is identical to the construction we have made above, except that above we consider all different underlying complexes, and we include the complex of morphisms between underlying complexes. These differences go away when we go to $\mathbf{Fib}^{\mathrm{dgc}}$. However, we should use as differential graded algebra the ideal $Q(E,E)_{>0}$ plus the augmentation $\mathbb{C}\cdot 1$.

Lemma 7.6. *The dgc fiber* $\mathbf{Fib}^{\mathrm{dgc}}$ *of the functor*

$$\mathrm{Wpx}_{R-\mathrm{perf}}(\Upsilon) \to \mathrm{Perf}^{\mathrm{dgc}}(R)$$

over E is equal to the augmented Maurer-Cartan dgc

$$\mathbf{MC}(\mathbb{C}\cdot 1 \oplus Q(E,E)_{>0}, \varepsilon).$$

□

8. Čech globalization

The arguments of the previous section concerned the case of an affine Zariski-open subset of X. We obtained a new expression for the differential graded category of complexes of Υ-modules. In this section we show how to put these together into a global expression over

X; and at the end how to reduce to a finite dimensional differential graded algebra so as to apply Theorem 5.5. These topics are closely related to Hinich's work [49] [51] as well as to the twisted complexes of Toledo and Tong [95].

There is a multiplicative Čech resolution for a sheaf of differential graded algebras.

Suppose X is a topoligical space, and $\mathcal{U}$ is an open covering. Think of $\mathcal{U}$ as being the semicategory of multiple intersections in the covering, where the morphisms are only the nontrivial inclusions (throw out the identity inclusions). Note that there is at most one morphism between elements of $\mathcal{U}$, so we leave the morphisms out of our notation.

Suppose A is a complex of sheaves on X.

Define the *local sections of A over $\mathcal{U}$* to be the following complex of groups denoted $G = G_{\mathcal{U}}A$. Let G^i be the set of functions

$$g(U_0, U_1, \ldots, U_k) \in A^{i-k}(U_0)$$

defined whenever $U_0 \subset U_1 \subset \ldots \subset U_k$ is a strictly increasing sequence of objects of $\mathcal{U}$. Set

$$(dg)(U_0, \ldots, U_k) := d(g(U_0, \ldots, U_k)) + \sum_{j=0}^{k} (-1)^j g(U_0, \ldots, \widehat{U_j}, \ldots, U_k)|_{U_0}.$$

The restriction to U_0 is necessary only for the single term $j = 0$ where the value starts out in $A(U_1)$ and needs to be restricted to U_0. This defines a differential with $d^2 = 0$.

We have a product

$$\mu : G_{\mathcal{U}}(A) \otimes G_{\mathcal{U}}(B) \to G_{\mathcal{U}}(A \otimes B)$$

defined by

$$\mu(f \otimes g)(U_0, \ldots, U_k) := \sum_{j=0}^{k} f(U_0, \ldots, U_j) \otimes g(U_j, \ldots, U_k)|_{U_0}.$$

This is associative, and compatible with the differential (for the same reason as before). In particular, if A is a presheaf of differential graded algebras, then $G_{\mathcal{U}}(A)$ has a natural structure of differential graded algebra.

We can also define a sheaf-theoretic version of this construction, obtained by replacing $A^{i-k}(U_0)$ by the direct image $j_{U_0/X}(A^{i-k}|_{U_0})$ in the above definition. Call this $\mathcal{G}_{\mathcal{U}}(A)$. A section a of A^i gives a section of $\mathcal{G}_{\mathcal{U}}(A)$ obtained by setting $g(U_0) := a$ and $g(U_0, \ldots, U_k) := 0$ for $k > 0$. This is compatible with the differential, so it gives a map of complexes of presheaves

$$A \to \mathcal{G}_{\mathcal{U}}(A).$$

It is easy to verify that this is a quasiisomorphism. Indeed, if X appears as part of the covering $\mathcal{U}$ then $A(X) \to G_{\mathcal{U}}(A)$ is a quasiisomorphism by a classical calculation. Therefore in general the above map of presheaves of complexes induces a quasiisomorphism on any open subset contained in some element of the covering.

Suppose X is a quasi-separated scheme and $\mathcal{U}$ is an affine open covering (in particular all of the open sets in $\mathcal{U}$ which includes the multiple intersections, are affine). Suppose that A is a complex of quasicoherent sheaves. Then the elements of $\mathcal{G}_{\mathcal{U}}(A)$ are direct images of quasicoherent sheaves via affine inclusions, so they are acyclic. In particular,

$$A \to \mathcal{G}_{\mathcal{U}}(A)$$

is an acyclic resolution. Again if A is a sheaf of quasicoherent differential graded algebras then $\mathcal{G}_{\mathcal{U}}(A)$ is a quasicoherent differential graded algebra whose components are acyclic, and the map is a map of sheaves of dga's.

Note that $G_{\mathcal{U}}(A)$ is the complex (or dga) of global sections of $\mathcal{G}_{\mathcal{U}}(A)$.

It is instructive to write down this resolution for the example of an open covering with two elements denoted U, V, for the complex $A := \mathcal{O}$. Denote by UV the intersection. Denote for example by $\mathcal{O}_{UV}$ the direct image from UV to X of the sheaf $\mathcal{O}$. Then our resolution $\mathcal{G}_{\mathcal{U}}(\mathcal{O})$ takes the form

$$\mathcal{O}_U \oplus \mathcal{O}_V \oplus \mathcal{O}_{UV} \to \mathcal{O}_{UV} \oplus \mathcal{O}_{UV}.$$

Thus it is a little bit bigger than the standard Čech resolution, but equivalent (the difference is the acyclic complex $\mathcal{O}_{UV} \to \mathcal{O}_{UV}$).

In a similar way we will define a Čech globalization of a presheaf of differential graded categories. Suppose C is a presheaf of dgc's. Then define a differential graded category $G_{\mathcal{U}}(C)$ as follows. An object can be denoted (E, η) where E is a collection of objects $E(U) \in \mathrm{ob}(C(U))$, and where for any strictly increasing sequence $U_0 \subset \ldots \subset U_k$ in $\mathcal{U}$, we have $\eta(U_0, \ldots, U_k)$ an element of $Hom^{1-k}_{C(U_0)}(E(U_0), E(U_k)|_{U_0})$ (for $k \geq 1$) subject to a Maurer-Cartan equation of the form $d(\eta) + \eta^2 = 0$, where the differential and product are defined much as previously. Define the complex of morphisms

$$Hom_{G_{\mathcal{U}}(C)}((E, \eta), (F, \varphi))$$

to be the complex whose piece of degree i consists of collections of functions $a(U_0, \ldots, U_k) \in Hom^{i-k}(E(U_0), F(U_k)|_{U_0})$, with differential denoted $d_{\eta,\varphi}$ obtained by a formula analogous to the previous ones, and with composition product defined as before also.

Let $G^{\mathrm{eq}}_{\mathcal{U}}(C)$ denote the subcategory of objects where the principal restriction maps are equivalences, i.e. the $\eta(U_0, U_1)$ are equivalences from $E(U_0)$ to $E(U_1)|_{U_0}$ in the dgc $C(U_0)$.

If (C, ε) is a presheaf of augmented dgc's then define the *augmented globalization* $G_{\mathcal{U}}(C, \varepsilon)$ to be the subcategory of objects of $G_{\mathcal{U}}(C)$ such that the transition maps are mapped to 1 by the augmentation; and with morphisms being those which map to a constant in $\mathbb{C}$ by the augmentation. In our applications this condition will automatically put us into $G^{\mathrm{eq}}_{\mathcal{U}}(C)$.

Suppose now that we have a presheaf of augmented differential graded algebras Q. We make the following hypthesis: that Q has an accessory grading preserved by the differential and compatible with the product, for which $Q_0 = \mathbb{C}[0]$ via the augmentation. We can compare the globalization of the Maurer-Cartan dgc with the Maurer-Cartan dgc of the globalization of the dga.

Lemma 8.1. *With the above notations, the globalization $G_{\mathcal{U}}(U \mapsto \mathbf{MC}(Q(U), \varepsilon), \varepsilon)$ is equal to the differential graded category of Maurer-Cartan elements of the globalization $\mathbf{MC}(G_{\mathcal{U}}(Q), \varepsilon)$.*

□

Lemma 8.2. *If C, C' are two presheaves of augmented dgc's and if $C \to C'$ induces a quasiequivalence at least over elements of the covering $\mathcal{U}$, then $G_{\mathcal{U}}(C, \varepsilon) \to G_{\mathcal{U}}(C', \varepsilon)$ is a quasiequivalence.*

Proof: Since the covering is finite, there is a finite amount of data of the form $\eta(U_0, \ldots, U_k)$ to consider. Also we only consider strict inclusions of open sets so the multiplication (even of the degree zero piece) in something like the formula $d(\eta) + \eta^2$, is nilpotent. Using this one can obtain the invariance. $\square$

Remark: Clearly one can define $\mathcal{G}_{\mathcal{U}}(C, \varepsilon)$ when C is defined only for the elements of the covering $\mathcal{U}$, for example C might only be defined for affine Zariski-open sets of a quasi-separated scheme. The invariance result of Lemma 8.2 holds also in this case.

It is tempting to use the Čech globalization as a way of defining the notion of *differential graded stack*. This would be a presheaf of differential graded categories C such that for any open set U and open covering $\mathcal{U}$ of U, the morphism $C(U) \rightarrow \mathcal{G}_{\mathcal{U}}(C)$ is a quasi-equivalence. One would then like a compatibility result with the notion of Segal 1-stack, namely that $\widetilde{DP}$ of a differential graded stack should be a Segal stack. A related point is that $\mathcal{G}_{\mathcal{U}}(C)$ which we have defined explicitly here should be seen as a homotopy limit in the model category of d.g.c.'s (see [92]). A similar remark we should have made earlier is that the Čech globalization of a presheaf of d.g.a.'s should be the homotopy limit in Hinich's model category [49].

We don't get into the details of these compatibility statements here. Instead, we make do with the following much more concrete result (but which basically says the same thing in the case we are interested in). See Toledo-Tong [95].

Lemma 8.3. *Let $\underline{\mathrm{Cpx}}_{\mathcal{O}-\mathrm{perf}}(\Upsilon)$ denote the presheaf of d.g.c.'s on the Zariski topology of X, which associates to $U \subset X$ the d.g.c. of complexes of $\Upsilon|_U$-modules which are $\mathcal{O}$-perfect. Then for any affine open covering $\mathcal{U}$, the map*

$$\underline{\mathrm{Cpx}}_{\mathcal{O}-\mathrm{perf}}(\Upsilon)(U) \rightarrow G^{\mathrm{eq}}_{\mathcal{U}}(\underline{\mathrm{Cpx}}_{\mathcal{O}-\mathrm{perf}}(\Upsilon))$$

is a quasi-equivalence of differential graded categories.

Proof: Given a Čech-twisted complex, one can define in a natural way its complex of sections over an open set. This has a structure of complex of Υ-modules, and the natural map from the original object to the new one is seen to be a quasiisomorphism over any open subset contained in some element of the covering (according to the usual principle that Čech-type resolutions including the full space are acyclic). $\square$

This result fits in with the fact that $\widetilde{DP}(\underline{\mathrm{Cpx}}_{\mathcal{O}-\mathrm{perf}}(\Upsilon))$ is a Segal stack [53].

The Čech globalization commutes with the dgc fiber when we have a collection of fibrant functors of dgc's over the open sets of the covering.

Lemma 8.4. *If $A \rightarrow B$ is a morphism of presheaves of dgc's which is fibrant over each open set of $\mathcal{U}$, then $G_{\mathcal{U}}(A) \rightarrow G_{\mathcal{U}}(B)$ is fibrant. For a global section E of $B(X)$, the fiber and globalization operations commute:*

$$\mathbf{Fib}^{\mathrm{dgc}}(G^{\mathrm{eq}}_{\mathcal{U}}(A) \rightarrow G^{\mathrm{eq}}_{\mathcal{U}}(B); E) = G^{\mathrm{eq}}_{\mathcal{U}}(U \mapsto \mathbf{Fib}^{\mathrm{dgc}}(A(U) \rightarrow B(U); E|_U); \varepsilon).$$

$\square$

8.1. **A finite-dimensional replacement.** The last step of the proof is to reduce from a Čech complex to a complex involving sections with a bounded number of poles along the complementary divisors of the affine open sets. This will give a finite dimensional complex.

Suppose $X \to Spec(R)$ is a smooth projective map. (with R a commutative $\mathbb{C}$-algebra of finite type). Suppose that the open covering is defined by open sets $U_i = X - D_i$ where D_i is the divisor of a section s_i of a fixed very ample line bundle $\mathcal{O}_X(1)$. Thus the multiple intersections $U_I = U_{i_1,\ldots,i_m}$ are the complements of the divisors D_I given by sections s_I of $\mathcal{O}_X(m)$ where $m = |I|$. Let $\mathcal{D}$ denote this collection of divisors.

Suppose that each Q_k is locally free over X.

We will fix a function $\underline{m}(k)$. In the definition of $\mathcal{G}_{\mathcal{U}}(Q)$, replace $(Q_k)_{U_I}$ by

$$Q_k(m(k)D_I) := \mathcal{O}_X(m(k)D_I) \otimes_{\mathcal{O}_X} Q_k.$$

The $\underline{m}(k)$ can be chosen so that the product map is still defined here, because of the condition that our graded ring Υ is almost-polynomial. We get an intermediate Beilinson-style globalization called $\mathcal{G}_{\underline{m}\mathcal{D}}(Q)$ which fits in between Q and the full Čech globalization

$$Q \to \mathcal{G}_{\underline{m}\mathcal{D}}(Q) \to \mathcal{G}_{\mathcal{U}}(Q).$$

These are quasiisomorphisms of complexes of sheaves (this is standard, in view of the fact that we are basically dealing with Čech complexes plus pieces which are homotopic to zero). We can assume that $\underline{m}(k)$ is big enough so that the terms of $\mathcal{G}_{\underline{m}\mathcal{D}}(Q)$ are acyclic (and their global sections are locally free over k). Then the local quasiisomorphism on the right induces a quasiisomorphism on the dga's of global sections

$$G_{\underline{m}\mathcal{D}/X}(Q) \stackrel{\mathrm{qis}}{\to} G_{\mathcal{U}}(Q).$$

Furthermore, we can cut off after k_0. With this cutoff, $G_{\underline{m}\mathcal{D}/X}(Q/A)$ becomes a differential graded R-algebra which is locally free of finite rank over R. Its differential graded category of Maurer-Cartan elements will be the geometric stack we are looking for.

8.2. **The proof of Theorem 6.7.** The problem is to show geometricity of

$$M(\Upsilon/S; E) := M(\Upsilon/S) \times_{\mathrm{Perf}(X/S)} \{E\}.$$

This is the Segal 1-stack which associates to an affine scheme $Y \to S$ the homotopy fiber of

$$\widetilde{DP}\mathrm{Cpx}_{\mathcal{O}-\mathrm{perf}}(\Upsilon|_{X_Y}) \to \mathrm{Perf}(X_Y)$$

over $E_Y := E|_{X_Y}$ where $X_Y := X \times_S Y$. Recall that $\mathrm{Perf}(X_Y) = \widetilde{DP}(\mathrm{Perf}^{\mathrm{dgc}}(X_Y)$.

Fix an affine open covering $\mathcal{U}$ of X. We can be assuming that S is affine, so we get an affine open covering $\mathcal{U}_Y$ of X_Y. Now, by Lemma 8.3,

$$\mathrm{Cpx}_{\mathcal{O}-\mathrm{perf}}(\Upsilon|_{X_Y}) \stackrel{\cong}{\to} G^{\mathrm{eq}}_{\mathcal{U}_Y}(U \mapsto \mathrm{Cpx}_{\mathcal{O}-\mathrm{perf}}(\Upsilon|_U)).$$

We have a similar equivalence on the other side of the arrow (which is exactly [95])

$$\mathrm{Perf}^{\mathrm{dgc}}(X_Y) \stackrel{\cong}{\to} G^{\mathrm{eq}}_{\mathcal{U}_Y}(U \mapsto \mathrm{Perf}^{\mathrm{dgc}}(U)).$$

On the other hand by Lemma 8.2 and Theorem 7.2

$$G^{\mathrm{eq}}_{\mathcal{U}_Y}(U \mapsto \mathrm{Cpx}_{\mathcal{O}-\mathrm{perf}}(\Upsilon|_U)) \cong G^{\mathrm{eq}}_{\mathcal{U}_Y}(U \mapsto \mathrm{Wpx}_{\mathcal{O}-\mathrm{perf}}(\Upsilon|_U)).$$

Thus $M(\Upsilon/S; E)(Y/S)$ is equivalent to the homotopy fiber over E_Y of

$$\widetilde{DP}G^{\mathrm{eq}}_{\mathcal{U}_Y}(U \mapsto \mathrm{Wpx}_{\mathcal{O}-\mathrm{perf}}(\Upsilon|_U)) \to \widetilde{DP}G^{\mathrm{eq}}_{\mathcal{U}_Y}(U \mapsto \mathrm{Perf}^{\mathrm{dgc}}(U)).$$

The underlying functor of dgc's here is fibrant in the sense of §5.1. Therefore, by Lemma 5.1, $M(\Upsilon/S;E)(Y/S)$ is given by the augmented Dold-Puppe of the dgc fiber of

$$G^{\rm eq}_{\mathcal{U}_Y}(U\mapsto \mathrm{Wpx}_{\mathcal{O}-\mathrm{perf}}(\Upsilon|_U))\rightarrow G^{\rm eq}_{\mathcal{U}_Y}(U\mapsto \mathrm{Perf}^{\rm dgc}(U)).$$

This in turn is equivalent, by Lemma 8.4, to the augmented Dold-Puppe of the augmented globalization

$$G_{\mathcal{U}_Y}(U\mapsto \mathbf{Fib}^{\rm dgc}(\mathrm{Wpx}_{\mathcal{O}-\mathrm{perf}}(\Upsilon|_U)\rightarrow \mathrm{Perf}^{\rm dgc}(U)/E_Y),\varepsilon).$$

Recall that when we pass to the augmented globalization the superscript $G^{\rm eq}$ is no longer necessary (it is automatic).

Let $Q(E_Y,E_Y)$ be the sheaf of differential graded algebras defined in §7. It has an ideal $Q(E_Y,E_Y)_{>0}$. Let $Q_Y:=\mathbb{C}\cdot 1\oplus Q(E_Y,E_Y)_{>0}$. This is an augmented sheaf of dga's on X_Y. On each open set U, we have (Lemma 7.6)

$$\mathbf{Fib}^{\rm dgc}(\mathrm{Wpx}_{\mathcal{O}-\mathrm{perf}}(\Upsilon|_U)\rightarrow \mathrm{Perf}^{\rm dgc}(U)/E)=\mathbf{MC}(Q_Y(U),\epsilon).$$

Thus, up to now

$$M(\Upsilon/S;E)(Y/S)\cong \widetilde{DP}(G_{\mathcal{U}_Y}(U\mapsto \mathbf{MC}(Q_Y(U),\varepsilon),\varepsilon),\varepsilon).$$

Of course in this notation each ε is the augmentation of the object in question; they all correspond to each other but are not actually the same.

By Lemma 8.1,

$$M(\Upsilon/S;E)(Y/S)\cong \widetilde{DP}(\mathbf{MC}(G_{\mathcal{U}_Y}(Q_Y,\varepsilon),\varepsilon),\varepsilon)=\widetilde{DP}(\mathbf{MC}(G_{\mathcal{U}_Y}(Q_Y,\varepsilon),\varepsilon),\varepsilon).$$

Finally, by the previous subsection, for an appropriate $\underline{m}$ we have a quasiisomorphism of dga's

$$G_{\mathcal{U}_Y}(Q_Y,\varepsilon)\cong G_{\underline{m}D_Y}(Q_Y,\varepsilon).$$

Thus

$$M(\Upsilon/S;E)(Y/S)\cong \widetilde{DP}(\mathbf{MC}(G_{\underline{m}D_Y}(Q_Y,\varepsilon),\varepsilon),\varepsilon).$$

The sheaf of dga's Q_Y is the pullback to X_Y of $Q:=Q_X$ on X. Similarly, $G_{\underline{m}D_Y}(Q_Y,\varepsilon)$ is the pullback to Y of the dga $G_{\underline{m}D}(Q,\varepsilon)$ over S. Thus

$$\widetilde{DP}(\mathbf{MC}(G_{\underline{m}D_Y}(Q_Y,\varepsilon),\varepsilon),\varepsilon)=\mathcal{M}(G_{\underline{m}D}(Q,\varepsilon))(Y).$$

Now $\mathcal{G}_{\underline{m}D}(Q,\varepsilon)$ satisfies the hypothesis of Theorem 5.5, so we can apply that theorem to conclude that the Segal stack

$$Y\mapsto \mathcal{M}(\mathcal{G}_{\underline{m}D}(Q,\varepsilon))(Y)$$

is geometric. We have shown in the preceding paragraphs that this Segal stack is equivalent to $M(\Upsilon/S;E)$, so this concludes the proof of Theorem 6.7.

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CNRS, Laboratoire J. A. Dieudonné, UMR 6621, Université de Nice-Sophia Antipolis, 06108 Nice, Cedex 2, France
E-mail address: carlos@math.unice.fr
URL: http://math.unice.fr/~carlos/