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Some properties of the theory of n-categories

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Much interest has recently focused on the problem of comparing different definitions of n-categories. [1] Leinster has made a useful compendium of 10 definitions [8]; May has proposed another definition destined among other things to make comparison easier [9], and he has also led the creation of an umbrella research group including comparison as one of the main research topics; Batanin has started a comparison of his definition with Penon's [2]; and Berger has made a comparison between Batanin's theory and the homotopy theory of spaces, introducing techniques which should allow for other comparisons starting with Batanin's theory [3]. The comparison question was first explicitly mentionned by Grothendieck in [7], in a prescient prediction that many different people would come up with different definitions of n-category; and this theme was again brought up by Baez and Dolan in [1].

The purpose of this short note is to make some observations about properties which one can expect any theory of n-categories to have, and to conjecture that these properties characterize the theory of n-categories. As a small amount of evidence for this conjecture, we show how to go from these properties to the composition law between mapping objects in an n-category.

While we don't give the proofs here, it is not too hard to see that Tamsamani's definition of n-category satisfies the properties listed below. We conjecture that the other definitions satisfy these properties too. This conjecture plus the conjecture of the previous paragraph would give an answer to the comparison question.

Rather than giving all of the references for the various different definitions of n-category, we refer the reader to Leinster's excellent bibliography [8].

The fundamental tool which we use is the Dwyer-Kan localization [5]. This is a generalization of the classical Gabriel-Zisman localization, which keeps higher homotopy data. Dwyer and Kan obtain a mapping space between two objects, where Gabriel and Zisman obtain only the set of homotopy classes of maps i.e. the π_0 of the mapping space. The fundamental observation of Dwyer and Kan is that the mapping spaces (plus their composition and higher homotopy coherence information) are determined by the data of

[1] We always mean "weak n-category".

the original category and the localizing subcategory.

We apply this observation to the problem of comparing definitions of n-categories. In any reasonable theory of n-categories, one starts with a collection of morphisms between objects which are the "strict" morphisms or morphisms strictly respecting the structure; these are of course not expected to be invariant in any way from one theory to the other. They form a usual 1-category $nCat$. Some of these morphisms are "n-equivalences", the n-categorical generalization of equivalences of categories.

Tamsamani in his thesis looked at $Ho - nCat$, the Gabriel-Zisman localization dividing the 1-category $nCat$ out by the n-equivalences. Following the philosophy of Dwyer-Kan, we look instead at $\mathcal{L}_n$ which is the simplicial category obtained by Dwyer-Kan localization dividing the same $nCat$ by the same subcategory of strict n-equivalences. Note that the 1-truncation of this simplicial category (corresponding to replacing the morphism spaces by their π_0) is just the Gabriel-Zisman localization $Ho - nCat$.

Baez and Dolan [1] gave a first precise statement of the comparison problem by saying that in any theory $\mathbf{T}$ of n-categories, one will have an underlying 1-category $nCat_{\mathbf{T}}$ and a notion of (strict) n-equivalence in there. They say that one should ask whether the Gabriel-Zisman localizations for two theories, are the same. Again, following the Dwyer-Kan philosophy, we can reinforce this question as follows: let $\mathcal{L}_{n,\mathbf{T}}$ denote the Dwyer-Kan simplicial localization obtained by dividing $nCat_{\mathbf{T}}$ by the strict n-equivalences. A *comparison* between theories $\mathbf{T}$ and $\mathbf{T}'$ is then an equivalence of simplicial categories [2]

$$\mathcal{L}_{n,\mathbf{T}} \cong \mathcal{L}_{n,\mathbf{T}'}.$$

It would be possible to do everything using the theory of simplicial categories and the notion of morphisms developped by Cordier and Porter [4]. An alternative is to use the theory of Segal categories, which are a type of weak replacement for the notion of simplicial categories. Any Segal category is equivalent to a simplicial category [6], and one assumes that the morphisms as defined in [10] are the same as the morphisms of simplicial categories defined using the coend construction in [4] (I am not sure that anybody has actually checked this yet). There is a notion of adjointness of

[2] To be precise, an "equivalence of simplicial categories" can mean an isomorphism in the Gabriel-Zisman localization obtained by dividing out the category of simplicial categories by their equivalences.

functors between simplicial or Segal categories. See Cordier-Porter [4], and [10].

There are actually two ways of constructing $\mathcal{L}_n$ starting with a theory of n-categories. One way as mentionned above is to have a theory in which one first constructs a 1-category of objects denoted $nCat$, with strict morphisms (in particular the morphisms between objects in $nCat$ are not the real "weak" morphisms, which should form an n-category rather than a set anyway); some of the morphisms in $nCat$ are equivalences of n-categories; and define $\mathcal{L}_n$ to be the Dwyer-Kan localization of $nCat$ dividing by the weak equivalences. The other way is to have a theory in which there is an $n+1$-category $nCAT$ of n-categories, with presumably the correct mapping n-categories between objects. In this case, define $\mathcal{L}_n$ to be the 1-*groupic interior* of $nCAT$, namely the $n+1$-category with the same objects, the same 1-morphisms, but whose i-morphisms for $i \geq 2$ are only the ones which are invertible up to equivalence. Thus if A and B are n-categories,

$$Hom_{\mathcal{L}_n}(A, B) \subset Hom_{nCAT}(A, B)$$

is the *interior* or the sub-n-category with the same objects but with only invertible i-morphisms for $i \geq 1$.

One expects $nCat$ (or some related 1-category) to have a closed model structure. If this is a simplicial closed model structure then $\mathcal{L}_n$ will also be equivalent to the simplicial category of fibrant and cofibrant objects here.

For the theory of Tamsamani n-categories, all of the above descriptions hold and are known to give equivalent Segal categories which we denote by $\mathcal{L}_{n,Ta}$. If $\mathbf{T}$ is another theory of n-categories as in Leinster's list [8] or others [9] etc., then there is always a 1-category $nCat_{\mathbf{T}}$ and a notion of equivalence (an equivalence is a functor which is fully faithful and essentially surjective) and we set $\mathcal{L}_{n,\mathbf{T}}$ equal to the Dwyer-Kan localization of $nCat_{\mathbf{T}}$ divided by the equivalences. One hopes in the case of other theories to have an analogue of the second description as the interior of $nCAT_{\mathbf{T}}$ too.

The rather limited purpose of this note is to consider the categories $\mathcal{L}_n$ obtained as above, as abstract simplicial categories, and to ask what properties they are expected to have. Our comparison conjecture says that our list of properties 1–10 serves to characterize $\mathcal{L}_n$ up to equivalence of simplicial categories. We also make a conjecture about the automorphisms of $\mathcal{L}_n$, which basically says that the equivalence in question will be unique up to the action of $(\mathbf{Z}/2)^n$ obtained using the operations of reversing the directions of the i-arrows for different i.

We leave as an exercise for the reader, to prove properties 1–10 in the case of $\mathcal{L}_{n,Ta}$. For the other theories $\mathbf{T}$ we conjecture that properties 1–10 hold for $\mathcal{L}_{n,\mathbf{T}}$.

So, for the rest of the paper, we will just fix a Segal category $\mathcal{L}_n$ which is supposed to be the Segal category of weak n-categories without reference to which particular theory is concerned, and we will state the properties which it is supposed to have.

Property 1 *The category $\mathcal{L}_n$ is $n+1$-truncated, i.e. the Hom-spaces between any two objects have vanishing homotopy groups in degrees $> n$.*

Property 2 *The category $\mathcal{L}_n$ admits small homotopy limits and colimits.*

More generally it is an ∞-pretopos in the sense of [12] but it isn't clear whether we need to know this.

Notations: Let $*$ denote the final object (which is the empty homotopy limit) and $\emptyset$ the initial object (which is the empty homotopy colimit). Denote by $k*$ the disjoint union of k copies of $*$. If $X \in \mathrm{ob}\mathcal{L}_n$ then put

$$\pi_0(X) := \pi_0 Hom(*, X).$$

A morphism $f : X \to Y$ in $\mathcal{L}_n$ is *essentially surjective* if $\pi_0(f)$ is surjective.

Let $k*/\mathcal{L}_n$ denote the Segal category of diagrams $k* \to X$ in $\mathcal{L}_n$. More precisely, if I denotes the Segal category with two objects 0 and 1 and one arrow from 0 to 1, then $k*/\mathcal{L}_n$ is defined to be the fiber over $k*$ of the map "evaluation at 0",

$$Hom(I, \mathcal{L}_n) \to Hom(\{0\}, \mathcal{L}_n) = \mathcal{L}_n.$$

Let $k*//\mathcal{L}_n$ denote the full sub-Segal category of objects $k* \to X$ which are essentially surjective, i.e. where $k*$ surjects onto $\pi_0(X)$.

Property 3 *The inclusion $k*//\mathcal{L}_n \to k*/\mathcal{L}_n$ admits a right homotopy adjoint*

$$tr_k : k*/\mathcal{L}_n \to k*//\mathcal{L}_n.$$

The case $k = 2$ is the most important: $2 * //\mathcal{L}_n$ is the Segal category of n-categories with two objects; the case where the two objects happen to be equivalent is the case of $2* \to X$ with $\pi_0(X) = *$. The notation tr_k refers to the adjoint map as "trimming off the additional objects".

Property 4 *There are four nonempty saturated* [3] *full sub-Segal categories*

$$\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \subset 2 * //\mathcal{L}_n$$

and the set of objects of $2 * //\mathcal{L}_n$ is a disjoint union of the objects of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$. This decomposition has the following properties:
(dec_1) *the only objects of $\mathcal{A}$ are initial objects of $2 * //\mathcal{L}_n$;*
(dec_2) *there are no morphisms from objects in $\mathcal{C}$ or $\mathcal{D}$ to objects in $\mathcal{B}$;*
(dec_3) *there are no morphisms from objects in $\mathcal{B}$ or $\mathcal{D}$ to objects in $\mathcal{C}$;*
(dec_4) *if f is an object of $\mathcal{B}$ and g is an object of $\mathcal{B}$ or $\mathcal{D}$ then there is a morphism from f to g; and*
(dec_5) *if f is an object of $\mathcal{C}$ and g is an object of $\mathcal{C}$ or $\mathcal{D}$ then there is a morphism from f to g.*

Concretely for any specific theory of $\mathcal{L}_n$ these four subcategories are defined as follows:
$\mathcal{A} \subset 2 * //\mathcal{L}_n$ is the full subcategory consisting only of the initial object;
$\mathcal{B} \subset 2 * //\mathcal{L}_n$ is the full subcategory consisting of maps $\{x, y\} \to X$ where X has at least one morphism going from x to y but no morphisms going from y to x;
$\mathcal{C} \subset 2 * //\mathcal{L}_n$ is the full subcategory consisting of maps $\{x, y\} \to X$ where X has at least one morphism going from y to x but no morphisms going from x to y;
$\mathcal{D} \subset 2 * //\mathcal{L}_n$ is the full subcategory consisting of maps $\{x, y\} \to X$ where X has at least one morphism going from x to y and at least one morphism going from y to x.

Lemma 1 *Given any Segal category $\mathcal{M}$, if there exists a decomposition of $\mathcal{M}$ into four saturated full subcategories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ as above (such that the object set of $\mathcal{M}$ is a disjoint union of the object sets of these categories) and if this decomposition satisfies properties $(dec_1) \ldots (dec_5)$ then it is the unique decomposition satisfying those properties, up to permutation of the factors $\mathcal{B}$ and $\mathcal{C}$.*

[3] Saturated means that if an object is in it then so is every equivalent object.

Proof: Suppose that $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are one such decomposition of $\mathcal{M}$, and suppose that $\mathcal{A}'$, $\mathcal{B}'$, $\mathcal{C}'$ and $\mathcal{D}'$ are another one. Clearly $\mathcal{A} = \mathcal{A}'$ by the saturation condition and (dec_1). Suppose that $x \in \mathcal{B}$. We claim that $x \in \mathcal{B}' \cup \mathcal{C}'$. If not, then $x \in \mathcal{D}'$. In that case, choosing elements $x' \in \mathcal{B}'$ and $y' \in \mathcal{C}'$ respectively, we have maps $x' \to x$ and $y' \to x$ by (dec_4) and (dec_5). But by (dec_2) this implies that $x' \in \mathcal{B}$ and $y' \in \mathcal{B}$. In particular, again by (dec_4) there is a map $x' \to y'$ contradicting (dec_3) for y'. This contradiction to our assumption that $x \in \mathcal{D}'$ proves that $x \in \mathcal{B}' \cup \mathcal{C}'$. Thus $\mathcal{B} \subset \mathcal{B}' \cup \mathcal{C}'$. A similar argument proves the same for $\mathcal{C}$ so we get

$$\mathcal{B} \cup \mathcal{C} \subset \mathcal{B}' \cup \mathcal{C}',$$

and running the argument in the opposite direction gives the opposite inclusion so

$$\mathcal{B} \cup \mathcal{C} = \mathcal{B}' \cup \mathcal{C}'.$$

This implies that $\mathcal{D} = \mathcal{D}'$.

Now fix $x_0 \in \mathcal{B}$. By permuting the factors $\mathcal{B}'$ and $\mathcal{C}'$ if necessary (note that this preserves the hypotheses and conclusion of the theorem) we can assume that $x_0 \in \mathcal{B}'$. Now for any $y \in \mathcal{C}$, we claim that $y \in \mathcal{C}'$; for if not then y would be in $\mathcal{B}'$ so there would be a map $y \to x_0$ by (dec_4), contradicting (dec_2); this proves the claim. Thus $\mathcal{C} \subset \mathcal{C}'$. The same argument in the other direction shows the reverse inclusion so $\mathcal{C} = \mathcal{C}'$ which in turn implies that $\mathcal{B} = \mathcal{B}'$. #

In view of this lemma, the existence of the decomposition is a property, and up to permutation of the factors the full subcategory $\mathcal{B}$ is uniquely determined (to be precise, there are two full subcategories which can play this role).

Property 5 *In the case of the decomposition of* $2 * //\mathcal{L}_n$ *there is an involution of* $\mathcal{L}_n$ *taking a category to its opposite, which permutes the two factors in the above decomposition.*

Thus, without loss of generality we may isolate the factor $\mathcal{B}$ in the decomposition as being well defined.

Let r_0 and r_1 denote the two maps from $2 * /\mathcal{L}_n$ to $*/\mathcal{L}_n$ obtained by retaining only the first or second point. These restrict to maps denoted the same way on $\mathcal{B}$ and we can compose with the "trimming" map tr_1 to obtain

$$tr_1 r_i : \mathcal{B} \to *//\mathcal{L}_n.$$

Let $\mathcal{B}^e$ denote the homotopy fiber of

$$(tr_1 r_0, tr_1 r_1) : \mathcal{B} \rightarrow *//\mathcal{L}_n \times *//\mathcal{L}_n$$

over the object $(*,*)$. Note that this object (plus those equivalent to it) constitutes a full sub-Segal category of $*//\mathcal{L}_n \times *//\mathcal{L}_n$ which is trivial i.e. all mapping spaces are reduced to $*$, so the homotopy fiber $\mathcal{B}^e$ is also a full sub-Segal category of $\mathcal{B}$.

Concretely in the examples, $\mathcal{B}^e$ is the category of n-categories with two objects $0, 1$ and some morphisms between them (going only in one direction, say from 0 to 1), but where the trimmings to either endpoint are trivial. Heuristically, this last condition means that the mapping $n-1$-categories of endomorphisms of either 0 or 1 are trivial, so such an object is determined entirely by the $n-1$-category of morphisms from 0 to 1. The following property makes this precise.

Let $\mathcal{A}+\mathcal{B}^e$ denote the saturated full subcategory of $2*//\mathcal{L}_n$ whose object set is the union of the object sets of $\mathcal{A}$ and $\mathcal{B}^e$ (i.e. we add to $\mathcal{B}^e$ the initial object which corresponds in the examples to cases where the morphism $n-1$-category from 0 to 1 is empty).

Property 6 *The Segal category $\mathcal{A}+\mathcal{B}^e$ is equivalent to $\mathcal{L}_{n-1}$.*

Choose an equivalence between $\mathcal{A}+\mathcal{B}^e$ and $\mathcal{L}_{n-1}$. Denote by

$$\Upsilon : \mathcal{L}_{n-1} \rightarrow 2*//\mathcal{L}_n \rightarrow 2*/\mathcal{L}_n$$

the composed functor. [4]

Conjecture 2 *The Segal groupoid of auto-equivalences $Aut(\mathcal{L}_n)$ is equal to $(\mathbf{Z}/2)^n$ with the first factor being the involution referred to in Property 5 above and with the remaining factors restricting via Υ to the automorphisms of $\mathcal{L}_{n-1}$.*

This is stated as a conjecture because even for $\mathcal{L}_{n,Ta}$ it is a more advanced exercise. Nevertheless, it explains why the problem of choice of an equivalence in Property 6 is not too serious.

[4] In the case $\mathcal{L}_n = \mathcal{L}_{n,Ta}$ it is possible to make the choice of equivalence in Property 6 so that the functor Υ is just the localization of the functor of the same name used for example in [11].

Property 7 *The functor Υ has a right adjoint*

$$H : 2*/\mathcal{L}_n \to \mathcal{L}_{n-1}.$$

This is to be viewed as the functor which to a diagram $\{x,y\} \to X$ associates the $n-1$-category of morphisms from x to y in X. In cases where it makes sense to talk about this $n-1$-category of morphisms, it is a property that H has this interpretation. [5]

Property 8 *The adjunction morphism for $E \in \mathcal{L}_{n-1}$ is an equivalence*

$$E \stackrel{\cong}{\to} H(\Upsilon(E)).$$

Say that a morphism $f : X \to Y$ in $\mathcal{L}_n$ is *fully faithful* if for every $u : \{x,y\} \to X$ the resulting map

$$H(X,u) \to H(Y, f\circ u)$$

is an equivalence in $\mathcal{L}_{n-1}$.

This notion coincides with the previously existing notions of fully faithful map in $\mathcal{L}_n$.

Property 9 *Equivalence in $\mathcal{L}_n$ is characterized using this notion: a morphism f is an equivalence if and only if it is essentially surjective and fully faithful in the above senses.*

In particular, this means that an object X with $\pi_0(X) = \emptyset$ is equivalent to the initial object $\emptyset$. It is interesting to note that the Segal category $\mathcal{M}$ of locally constant n-stacks on the $n+1$-sphere satisfies most of the other properties we write down here, but there are n-stacks classified by nonzero cohomology classes in degree $n+1$ which have no global sections, thus have $\pi_0 = \emptyset$ but are not equivalent to $\emptyset$. Property 9 is an important way of ruling out that type of example.

[5] That is to say, in $\mathcal{L}_{n,Ta}$ with the choice referred to in the previous footnote, this interpretation is true; and for any theory $\mathbf{T}$ in which we have an $n-1$-category of morphisms, it is a conjecture that for an appropriate choice of equivalence in Property 6 above, this interpretation is true.

Given $E_1, \ldots, E_k \in \mathcal{L}_{n-1}$, define the homotopy coproduct

$$\Upsilon^k(E_1, \ldots, E_k) := \Upsilon(E_1) \cup^{\{1\}} \ldots \cup^{\{k-1\}} \Upsilon(E_k)$$

where we have denoted the $k+1$ objects $*$ by $0, \ldots, k$. These are organized as follows: the first and last objects of the term $\Upsilon(E_i)$ are $i-1$ and i. Thus this coproduct represents "composing arrows head to tail". The main case is

$$\Upsilon^2(E, F) = \Upsilon(E) \cup^{\{1\}} \Upsilon(F).$$

The morphism $\Upsilon(E_i) \to *$ induces a morphism of the coproduct

$$\Upsilon^k(E_1, \ldots, E_k) \to \Upsilon^{k-1}(E_1, \ldots, \widehat{E_i}, \ldots, E_k).$$

In particular, composing many of these we get down to morphisms

$$\Upsilon^k(E_1, \ldots, E_k) \to \Upsilon(E_j)$$

for any $1 \leq j \leq k$. [6] Given that $H(\Upsilon(E)) \cong E$ by Property 8, we obtain a morphism

$$H(\Upsilon^k(E_1, \ldots, E_k), \{0, k\}) \to E_1 \times \ldots \times E_k$$

where $\times$ denotes the direct product in $\mathcal{L}_{n-1}$.

Property 10 (MAIN PROPERTY) *The above morphism is an equivalence:*

$$H(\Upsilon^k(E_1, \ldots, E_k), \{0, k\}) \stackrel{\cong}{\to} E_1 \times \ldots \times E_k.$$

This finishes our list of properties.

Conjecture 3 *The above properties 1–10 characterize the Segal category $\mathcal{L}_n$ up to equivalence (the choice of equivalence is governed by Conjecture 2).*

This conjecture would give a way of approaching the comparison problem: to compare two definitions of n-category, if we could establish properties 1–10 for both of them, and if we knew the conjecture, then we would get equivalence of the definitions at least in the sense of equivalence between the Dwyer-Kan localized homotopy categories.

[6] In the case of $\mathcal{L}_{n,Ta}$ with the good choice made in Property 6, the construction Υ^k and the various maps are induced by those considered in [11].

To finish this note, we just show how the above properties lead to the composition map

$$H(X,\{x,y\})\times H(X,\{y,z\})\rightarrow H(X,\{x,z\}).$$

Suppose X is an object of $\mathcal{L}_n$ and suppose $x,y,z:\ast\rightarrow X$ are three maps. Let $(X,\{x,y\})$ denote the object of $2\ast/\mathcal{L}_n$ corresponding to $(x,y):2\ast\rightarrow X$ and so forth. Put

$$E:=H(X,\{x,y\}),\quad F:=H(X,\{y,z\}).$$

By adjunction we obtain maps $\Upsilon(E)\rightarrow X$ and $\Upsilon(F)\rightarrow X$ sending $0,1,2$ to x,y,z respectively (here the two objects of $\Upsilon(F)$ are denoted $1,2$ as per the convention above). These induce a map on the coproduct

$$\Upsilon^2(E,F)=\Upsilon(E)\cup^{\{1\}}\Upsilon(F)\stackrel{\alpha}{\rightarrow}X.$$

On the other hand, Property 10 says that the map

$$H(\Upsilon^2(E,F),\{0,2\})\stackrel{\beta}{\rightarrow}E\times F$$

induced by the maps $\Upsilon^2(E,F)\rightarrow\Upsilon(E)$ and $\Upsilon^2(E,F)\rightarrow\Upsilon(F)$, is an equivalence. The map α induces a map

$$H(\Upsilon^2(E,F),\{0,2\})\rightarrow H(X,\{x,z\}),$$

and composing with the inverse of the equivalence β we obtain a map

$$E\times F\rightarrow H(X,\{x,y\}).$$

Recalling the definitions of E and F this becomes a map

$$\gamma(x,y,z):H(X,\{x,y\})\times H(X,\{y,z\})\rightarrow H(X,\{x,z\}).$$

This map is the "composition" map between the mapping objects. The definition of the mapping object functor H and this composition have been constructed using only the structure of the simplicial category $\mathcal{L}_n$.

We expect that, after solving the homotopy-coherence problems involved, one would obtain more generally a functor from objects of $\mathcal{L}_n$ to simplicial objects in $\mathcal{L}_{n-1}$ having discrete 0-th stage and satisfying the Segal properties; this would allow one to obtain (by induction) a functor

$$\mathcal{L}_n\rightarrow\mathcal{L}_{n,Ta}$$

and in order to prove Conjecture 3 it would suffice to prove that this map is an equivalence.

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