Towards the boundary of the character variety

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(in progress)

Let X be a compact Riemann surface, $x_0 \in X$. Look at various moduli spaces of local systems on X:

- The character variety $M_B := Hom(\pi_1(X, x_0), SL_r)/SL_r$
- The Hitchin moduli space $M_{Dol} = \{(E, \varphi)\}/\text{S-equiv}$
- The de Rham moduli space $M_{DR} = \{(E, \nabla)\}/\text{S-equiv}$

We have $M_B^{\text{top}} \cong M_{Dol}^{\text{top}} \cong M_{DR}^{\text{top}}$, with furthermore the *Hitchin fibration* $M_{Dol} \to \mathbb{A}^N$.

Example: If $X = \mathbb{P}^1$ with 4 orbifold points, $r = 2$, then
M_B is a cubic surface minus a triangle of lines,
M_{Dol} and M_{DR} are $\mathbb{P}^1 \times \mathbb{P}^1$ blown up 8 times at 4 points on the diagonal, minus some stuff.

The Hitchin fibration is

$$\begin{array}{ccc} J & \to & M_{Dol} \\ & & \downarrow \\ & & \mathbb{A}^1 \end{array}$$

with fiber an elliptic curve J, on which the monodromy acts by -1.

We would like to discuss the neighborhood of the divisor at infinity in a compactification.

M_B has no canonical compactification, indeed the mapping class group couldn't fix any one of them. We choose one and let D_B denote the divisor at infinity.

M_{Dol} and M_{DR} have canonical orbifold compactifications, where the divisor at infinity is

$$D_{DR} = D_{Dol} = M^*_{Dol}/\mathbb{C}^*$$

here M^*_{Dol} is the preimage of $\mathbb{A}^N - \{0\}$ or complement of the nilpotent cone.

In our example, D_B is a triangle formed from three $\mathbb{P}^1$'s, whereas D_{Dol} is $J/\pm 1$ which is $\mathbb{P}^1$ with four orbifold double points.

Comparison: Let $\overline{N}_B$ be a small neighborhood of D_B, let $N_B = \overline{N}_B \cap M_B = \overline{N}_B - D_B$. idem for N_{Dol}, N_{DR}. These have well-defined homotopy types.

Then our homeomorphisms give

$$N_B \sim N_{Dol} \sim N_{DR},$$

and these are well defined up to homotopy.

Write $D_B = \bigcup_i D_i$, and define a simplicial complex with one n-simplex for each connected component of $D_{i_0} \cap \cdots \cap D_{i_n}$. This is called the *incidence complex* and we will denote it by $\mathrm{Step}(M_B)$.
Stepanov, Thuillier: the homotopy type of $|\mathrm{Step}(M_B)|$ is independent of the choice of compactification.
So we can also call it the "Stepanov complex".

We have a map, well-defined up to homotopy, $N_B \to |\mathrm{Step}(M_B)|$.

On the Hitchin side, the Hitchin fibration gives us a map to the sphere at infinity in the Hitchin base

$$N_{Dol} \to S^{2N-1}$$

and by a deformation argument we have the same thing for N_{DR}.

Conjecture: There is a homotopy-commutative diagram

$$\begin{array}{ccc} N_{Dol} & \stackrel{\sim}{\to} & N_B \\ \downarrow & & \downarrow \\ S^{2N-1} & \stackrel{\sim}{\to} & |\mathrm{Step}(M_B)|. \end{array}$$

Motivation: it holds in the example.

It may be viewed as some version of the "$P = W$" conjecture of Hausel *et al*, relating Leray stuff for the Hitchin fibration to weight stuff on the Betti side.

Komyo has shown explicitly $|\mathrm{Step}(M_B)| \cong S^3$ for the case of $\mathbb{P}^1 - 5$ points.

One can furthermore hope to have a more geometrically precise description of the relationship between N_{Dol} and N_B. One should note that it will interchange "small" and "big" subsets. Indeed, in all examples, the neighborhood of a single vertex of D_B in N_B corresponds to a whole chamber in S^{2N-1} and hence in N_{Dol}.

Kontsevich-Soibelman: have a picture where 1-dimensional pieces of D_B correspond to *walls* in S^{2N-1} or equivalently $\mathbb{A}^N$. Their *wallcrossing formulas* express the change of cluster coordinate systems as we go along these one-dimensional pieces.
Kontsevich has a general type of argument saying that in many cases M_B are "cluster varieties", hence log-Calabi-Yau, from which it follows that the incidence complex is a sphere.

Going to the opposite end of the range of dimensions, we therefore expect
divisor components of D_B
$\leftrightarrow$ single directions in the Hitchin base.

Divisor components correspond to *valuations* of the coordinate ring $\mathcal{O}_{M_B}$. However, there are also non-divisorial valuations. We expect more generally that all valuations correspond to directions in the Hitchin base, which in turn correspond to spectral curves $\Sigma \subset T^*X$ (up to scaling).

On the other hand, valuations correspond to harmonic maps to buildings, indeed if K_v is the valued field then the map

$$\pi_1(X, x_0) \to SL_r(\mathcal{O}_{M_B})$$

composes with $\mathcal{O}_{M_B} \subset K_v$ to give

$$\pi_1(X, x_0) \to SL_r(K_v)$$

hence an action of π_1 on the Bruhat-Tits building. One can then take the Gromov-Schoen harmonic map.

We already know the correspondence

harmonic maps to buildings $\leftrightarrow$ spectral curves

indeed, a harmonic map has a differential which is the real part of a multivalued holomorphic form defining a spectral curve.

However, we would like to understand the correspondence with the differential equations picture at the same time.

It turns out that this is closely related to the *spectral networks* which have recently been introduced by **Gaiotto-Moore-Neitzke**, which is the subject of the second half of my talk.

Spectral networks and harmonic maps to buildings

In recent work with L. Katzarkov, A. Noll and P. Pandit in Vienna, we wanted to understand the spectral networks of Gaiotto, Moore and Neitzke from the perspective of euclidean buildings. This should generalize the trees which show up in the SL_2 case. We hope that this can shed some light on the relationship between this picture and moduli spaces of stability conditions as in Kontsevich-Soibelman, Bridgeland-Smith, . . .

We thank many people including M. Kontsevich and F. Haiden for important conversations.

Consider X a Riemann surface, $x_0 \in X$, $E \to X$ a vector bundle of rank r with $\bigwedge^r E \cong \mathcal{O}_X$, and

$$\varphi : E \to E \otimes \Omega^1_X$$

a Higgs field with $\mathrm{Tr}(\varphi) = 0$. Let

$$\Sigma \subset T^*X \xrightarrow{p} X$$

be the spectral curve, which we assume to be reduced.

We have a tautological form

$$\phi \in H^0(\Sigma, p^*\Omega^1_X)$$

which is thought of as a multivalued differential form. Locally we write

$$\phi = (\phi_1, \ldots, \phi_r), \quad \sum \phi_i = 0.$$

The assumption that Σ is reduced amounts to saying that ϕ_i are distinct.

Let $D = p_1 + \ldots + p_m$ be the locus over which Σ is branched, and $X^* := X - D$. The ϕ_i are locally well defined on X^*.

There are 2 kinds of WKB problems associated to this set of data.

(1) The **Riemann-Hilbert** or **complex WKB** problem:

Choose a connection ∇_0 on E and set

$$\nabla_t := \nabla_0 + t\varphi$$

for $t \in \mathbb{R}_{\geq 0}$. Let

$$\rho_t : \pi_1(X, x_0) \to SL_r(\mathbb{C})$$

be the monodromy representation. We also choose a fixed metric h on E.

From the flat structure which depends on t we get a family of maps

$$h_t : \widetilde{X} \to SL_r(\mathbb{C})/SU_r$$

which are ρ_t-equivariant. We would like to understand the asymptotic behavior of ρ_t and h_t as $t \to \infty$.

Definition: For $P, Q \in \widetilde{X}$, let $T_{PQ}(t) : E_P \to E_Q$ be the transport matrix of ρ_t. Define the *WKB exponent*

$$\nu_{PQ} := \limsup_{t\to\infty} \frac{1}{t} \log \|T_{PQ}(t)\|$$

where $\|T_{PQ}(t)\|$ is the operator norm with respect to h_P on E_P and h_Q on E_Q.

Gaiotto-Moore-Neitzke consider a variant on the complex WKB problem, associated with a harmonic bundle $(E, \overline{\partial}, \varphi, \partial, \varphi^\dagger)$ setting

$$d_t := \partial + \overline{\partial} + t\varphi + t^{-1}\varphi^\dagger$$

which corresponds to the holomorphic flat connection $\nabla_t = \partial + t\varphi$ on the holomorphic bundle $(E, \overline{\partial} + t^{-1}\varphi^\dagger)$. We expect this to have the same behavior as the complex WKB problem.

(2) The **Hitchin WKB problem**:

Assume X is compact, or that we have some other control over the behavior at infinity. Suppose (E, φ) is a stable Higgs bundle. Let h_t be the Hitchin Hermitian-Yang-Mills metric on $(E, t\varphi)$ and let ∇_t be the associated flat connection. Let $\rho_t : \pi_1(X, x_0) \to SL_r(\mathbb{C})$ be the monodromy representation.

Our family of metrics gives a family of *harmonic maps*

$$h_t : \widetilde{X} \to SL_r(\mathbb{C})/SU_r$$

which are again ρ_t-equivariant.

We can define $T_{PQ}(t)$ and ν_{PQ} as before, here using $h_{t,P}$ and $h_{t,Q}$ to measure $\|T_{PQ}(t)\|$.

Gaiotto-Moore-Neitzke explain that ν_{PQ} should vary as a function of $P, Q \in X$, in a way dictated by the *spectral networks*. We would like to give a geometric framework.

The basic philosophy is that a WKB problem determines a valuation on $\mathcal{O}_{M_B}$ by looking at the exponential growth rates of functions applied to the points ρ_t. Therefore, π_1 should act on a Bruhat-Tits building and we could try to choose an equivariant harmonic map following Gromov-Schoen.

Recently, **Anne Parreau** has developed a very useful version of this theory, based on work of **Kleiner-Leeb:**

Look at our maps h_t as being maps into a symmetric space with distance rescaled:

$$h_t : \widetilde{X} \to \left(SL_r(\mathbb{C})/SU_r, \frac{1}{t}d \right).$$

Then we can take a "Gromov limit" of the symmetric spaces with their rescaled distances, and it will be a building modelled on the same affine space $\mathbb{A}$ as the SL_r Bruhat-Tits buildings.

The limit construction depends on the choice of *ultrafilter* ω, and the limit is denoted Cone_ω. We get a map

$$h_\omega : \widetilde{X} \to \mathrm{Cone}_\omega,$$

equivariant for the limiting action ρ_ω of π_1 on Cone_ω which was the subject of Parreau's paper.

The main point for us is that we can write

$$d_{\mathsf{Cone}_\omega}(h_\omega(P), h_\omega(Q)) =$$

$$\lim_\omega \frac{1}{t} d_{SL_r\mathbb{C}/SU_r}(h_t(P), h_t(Q)) \, .$$

There are several distances on the building, and these are all related by the above formula to the corresponding distances on $SL_r\mathbb{C}/SU_r$.

• The Euclidean distance $\leftrightarrow$ Usual distance on $SL_r\mathbb{C}/SU_r$

• Finsler distance $\leftrightarrow$ log of operator norm

• Vector distance $\leftrightarrow$ dilation exponents

In the affine space

$$\mathbb{A} = \{(x_1, \ldots, x_r) \in \mathbb{R}^r, \ \sum x_i = 0\} \cong \mathbb{R}^{r-1}$$

the vector distance is translation invariant, defined by

$$\overrightarrow{d}(0, x) := (x_{i_1}, \ldots, x_{i_r})$$

where we use a Weyl group element to reorder so that $x_{i_1} \geq x_{i_2} \geq \cdots \geq x_{i_r}$.

In Cone_ω, any two points are contained in a common apartment, and use the vector distance defined as above in that apartment.

The "dilation exponents" may be discussed as follows: put

$$\overrightarrow{d}(H, K) := (\lambda_1, \ldots, \lambda_k)$$

where

$$\|e_i\|_K = e^{\lambda_i} \|e_i\|_H$$

with $\{e_i\}$ a simultaneously H and K orthonormal basis.

In our situation,

$$\lambda_1 = \log \|T_{PQ}(t)\|,$$

and one can get

$$\lambda_1 + \ldots + \lambda_k = \log \| \bigwedge^k T_{PQ}(t)\|.$$

Define the *ultrafilter exponent*

$$\nu^\omega_{PQ} := \lim_\omega \frac{1}{t} \log \|T_{PQ}(t)\|.$$

We have

$$\nu^\omega_{PQ} \leq \nu_{PQ}.$$

They are equal in some cases:

(a) for any fixed choice of P, Q, there exists a choice of ultrafilter ω such that $\nu^\omega_{PQ} = \nu_{PQ}$. Indeed, we can subordinate the ultrafilter to the condition of having a sequence calculating the $\limsup$ for that pair P, Q. It isn't *a priori* clear whether we can do this for all pairs P, Q at once, though. In our example, it will follow *a posteriori*!

(b) If $\limsup_t \ldots = \lim_t \ldots$ then it is the same as $\lim_\omega \ldots$. This applies in particular for the local WKB case. It would also apply in the complex WKB case, for generic angles, if we knew that $\mathcal{L}T_{PQ}(\zeta)$ didn't have essential singularities.

Theorem ("Classical WKB"):
Suppose $\xi : [0,1] \to \widetilde{X}^*$ is *noncritical* path i.e. $\xi^*\mathrm{Re}\phi_i$ are distinct for all $t \in [0,1]$. Reordering we may assume

$$\xi^*\mathrm{Re}\phi_1 > \xi^*\mathrm{Re}\phi_2 > \ldots > \xi^*\mathrm{Re}\phi_r.$$

Then, for the complex WKB problem we have

$$\frac{1}{t}\overrightarrow{d}\,(h_t(\xi(0)), h_t(\xi(1)) \sim (\lambda_1, \ldots, \lambda_r)$$

where

$$\lambda_i = \int_0^1 \xi^*\mathrm{Re}\phi_i.$$

Corollary: At the limit, we have

$$\overrightarrow{d}_\omega(h_t(\xi(0)), h_t(\xi(1)) = (\lambda_1, \ldots, \lambda_r).$$

Conjecture: The same should be true for the Hitchin WKB problem, also the GMN problem.

Corollary: If $\xi : [0,1] \to \widetilde{X}^*$ is any noncritical path, then $h_\omega \circ \xi$ maps $[0,1]$ into a single apartment, and the vector distance which determines the location in this apartment is given by the integrals:

$$\overrightarrow{d}_\omega(h_t(\xi(0)), h_t(\xi(1)) = (\lambda_1, \ldots, \lambda_r).$$

Corollary: Our map

$$h_\omega : \widetilde{X} \to \mathrm{Cone}_\omega$$

is a harmonic ϕ-map in the sense of Gromov and Schoen. In other words, any point in the complement of a discrete set of points in $\widetilde{X}$

has a neighborhood which maps into a single apartment, and the map has differential Reϕ (no "folding").

Now, we would like to analyse harmonic ϕ-maps in terms of spectral networks.

The main observation is just to note that the reflection hyperplanes in the building, pull back to curves on $\tilde{X}$ which are imaginary foliation curves, including therefore the spectral network curves.

Indeed, the reflection hyperplanes in an apartment have equations $x_{ij} =$ const. where $x_{ij} := x_i - x_j$, and these pull back to curves in $\widetilde{X}$ with equation Re$\phi_{ij} = 0$. This is the equation for the "spectral network curves" of Gaiotto-Moore-Neitzke.

The Berk-Nevins-Roberts (BNR) example

In order to see how the the *collision* spectral network curves play a role in the harmonic map to a building, we decided to look closely at a classical example: it was the original example of Berk-Nevins-Roberts which showed the "collision phenomenon" special to the case of higher-rank WKB problems.

In their case, the spectral curve is given by the equation

$$\Sigma \quad : \quad y^3 - 3y + x = 0$$

where $X = \mathbb{C}$ with variable x, and y is the variable in the cotangent direction.

The differentials ϕ_1, ϕ_2 and ϕ_3 are of the form $y_i dx$ for y_1, y_2, y_3 the three solutions.

Notice that $\Sigma \to X$ has branch points

$$p_1 = 2, \quad p_2 = -2.$$

The imaginary spectral network is as in the accompanying picture.

Here is a summary of what may be seen from the pictures in this example.

• There are two collision points, which in fact lie on the same vertical collision line.

• The spectral network curves divide the plane into 10 regions:
4 regions on the outside to the right of the collision line;
4 regions on the outside to the left of the collision line;
2 regions in the square whose vertices are the singularities and the collisions; the two regions are separated by the interior part of the collision line.

• Arguing with the local WKB approximation, we can conclude that each region is mapped into a single Weyl sector in a single apartment of the building Cone_ω.

• The interior square maps into a single apartment, with a fold line along the "caustic" joining the two singularities. The fact that the whole region goes into one apartment comes from an argument with the axioms of the building. We found the paper of Bennett, Schwer and Struyve about axiom systems for buildings, based on Parreau's paper, to be very useful.

• It turns out, in this case, that the two collision points map to the same point in the building. This may be seen by a contour integral using the fact that the interior region goes into a single apartment.

Therefore, the sectors in question all correspond to sectors in the building with a single vertex. We may therefore argue, in this case, using *spherical buildings* which for SL_3 are just graphs.

Theorem: In the BNR example, there is a *universal building* B^{ϕ} together with a harmonic ϕ-map

$$h^{\phi} :\rightarrow B^{\phi}$$

such that for any other building $\mathcal{C}$ (in particular, $\mathcal{C} = \text{Cone}_{\omega}$) and harmonic ϕ-map $X \to \mathcal{C}$ there is a unique factorization

$$X \to B^{\phi} \xrightarrow{g} \mathcal{C}.$$

Furthermore, on the Finsler secant subset of the image of X, g is an isometry for any of the distances. It depends on the non-folding property of g.

Therefore, we conclude in our example that *distances in $\mathcal{C}$ between points in X are the same as the distances in B^{ϕ}.*

Corollary: In the BNR example, for any pair $P, Q \in X$, the WKB dilation exponent is calculated as the distance in the building B^{ϕ},

$$\overrightarrow{\nu}_{PQ} = \overrightarrow{d}_{B^{\phi}}(h^{\phi}(P), h^{\phi}(Q)).$$

There exist examples (e.g. pullback connections) where we can see that the isometry property cannot be true in general. However, we conjecture that it is true if the spectral curve Σ is smooth and irreducible, and ∇_0 is generic.

We still hope to have a universal ϕ-map to a "building" or building-like object. It should be the higher-rank analogue of the *space of leaves of a foliation* which shows up in the SL_2 case in classical Thurston theory.