The shape of an algebraic variety

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Equations $\Rightarrow$ Topology

X algebraic variety over $\mathbb{C}$

X^{top} associated topological space

$(\mathbb{P}^n)^{\mathsf{top}} = \mathbb{CP}^n$ usual complex projective space in topology

If $X \subset \mathbb{P}^n_{\mathbb{C}}$ given by equations $F_i(Z_0, \ldots, Z_n) = 0$ then $X^{\mathsf{top}} \subset \mathbb{CP}^n$ is the closed subspace, same equations, induced topology

Example: $X \subset \mathbb{P}^2_{\mathbb{C}}$ smooth of degree d
$X^{\mathsf{top}} =$ compact Riemann surface of genus $g = \frac{(d-1)(d-2)}{2}$

History

The study of X^{top} has played an important role in many parts of algebraic geometry:

- Lefschetz, Hodge, Kodaira—
  use real analysis and differential geometry to study X^{top}

- Riemann, Zariski, Artin—
  study of $\pi_1(X^{\text{top}}, x)$, any variety covered by open sets which are $K(\pi, 1)$

- Weil, Serre, Grothendieck, Deligne—
  the etale topology replaces X^{top}
  for algebraic varieties defined over
  finite fields or number fields
  (led to Taylor-Wiles' proof of Fermat)

Basic question

What kinds of topological spaces or homotopy types occur as X^{top}?

Most obstructions we know come from *Hodge theory* (Griffiths, Deligne, . . .)

We know relatively little about the construction of examples

Topological invariants play an important role in classification

Another question: how does the topology of X^{top} relate to the geometry of X?

Hodge theory

Griffiths studies the variation of Hodge structures of the fibers of a family of varieties

Deligne: mixed Hodge structure on $H^i(X^{\text{top}}, \mathbb{Q})$

Rational homotopy theory
(Deligne, Griffiths, Sullivan, Morgan, Hain):

We get a mixed Hodge structure on $\pi_i(X^{\text{top}})$ when X^{top} is simply connected, or for the unipotent completion of $\pi_1(X^{\text{top}})$, later on the relative Malcev completion at a variation of Hodge structures

Johnson-Rees, Gromov use Hodge theory (maybe L^2) to prove that $\pi_1(X^{\text{top}})$ can't have a free product decomposition

Yang-Mills

An important advance was Donaldson's introduction of Yang-Mills equations

Narasimhan-Seshadri generalized to higher-dimensional varieties: classification of unitary representations

Hitchin: inclusion of a *Higgs field* extends Yang-Mills to non-unitary representations

Eells-Sampson, Siu, Carlson-Toledo, Corlette, Donaldson: solutions of harmonic map equations give super-rigidity style restrictions on X^{top}.

Variations of Hodge structures are special types of solutions of Yang-Mills-Higgs/harmonic mapping equations

Rigid representations are variations of Hodge structures—this leads to further restrictions on $\pi_1(X^{\text{top}})$

Moduli of representations

Lubotsky-Magid had introduced the study of the moduli space of representations of π_1

This turned out to be useful in 3-manifold topology (Culler-Shalen), then in number theory (Mazur, Boston, Wiles)

Yang-Mills-Higgs gives a good approach to the study of these moduli spaces for algebraic varieties

Hitchin's moduli space of Higgs bundles has a quaternionic structure

Green-Lazarsfeld, Beauville, Catanese studied the *jump locus*, the subset of local systems L where $\dim H^i(X^{\mathrm{top}}, L) \geq k$: for rank 1 local systems this has the structure of a union of translates of subtori of the moduli space

We would like to unify and extend these points of view

Grothendieck's manuscript "Pursuing Stacks" proposes the notion of *nonabelian cohomology* whose natural coefficients would be "n-stacks"

This fits into a philosophy of "shape theory" (suggestion of Jim Propp)

Shape

Study a space Y by looking at $\underline{Hom}(Y, T)$ for other spaces T

e.g. $H^1(Y, \mathbb{Z}) = \pi_0 \, \underline{Hom}(Y, S^1)$

When $Y = X^{\mathrm{top}}$ try to relate this to algebraic geometry:

- instead of a space, let T be an n-**stack**

- associate a stack to X, for example $X_{DR} := Z \mapsto X(Z^{\mathrm{red}})$

- the n-stack $\underline{Hom}(X_{DR}, T)$ is the *nonabelian de Rham cohomology of* X with coefficients in T

  (cocycle description for $n = 2$ by Brylinski, Hitchin, Breen-Messing)

Higher categories

There are many definitions of n-category (see Leinster's book: Baez-Dolan, Batanin, Street, Trimble, . . .)

Tamsamani's inductive definition:

it's a simplicial $n-1$-category $k \mapsto A_k$
$A_0 = \mathrm{obj}(A)$ is a discrete set
the Segal maps

$$A_k \to A_1 \times_{A_0} \cdots \times_{A_0} A_1$$

are equivalences of $n-1$-categories
("equivalence" is defined inductively)

An n-groupoid is the "same thing" as an n-truncated space $\pi_i(T) = 0$ for $i > n$

We have $n\,\mathrm{CAT}$ with limits, colimits and $\underline{Hom}(A, B)$

Higher stacks

Use the site $\mathsf{Aff}_{\mathbb{C}}$ of affine schemes with the etale topology

An n-prestack is a functor $F : \mathsf{Aff}_{\mathbb{C}} \to n\,\mathsf{CAT}$

It's an n-stack if $F(U) = \lim F(U_i)$ where U_i is in a sieve covering U

$n\,\mathsf{STACK}$ is an $n+1$-stack (with Hirschowitz)

Artin geometric n-stacks T are again defined by induction (Walter):

for $n = 0$ they are algebraic spaces
for any n there should be a smooth surjection $Y \to T$ from a scheme with $Y \times_T Y$ an Artin geometric $n-1$-stack

Nonabelian H^1

The first case is when $T = BG$ for an algebraic group G

We get a 1-stack $\underline{Hom}(X_{\mathsf{DR}}, BG) = M_{\mathsf{DR}}(X, G)$ the moduli stack for (P, ∇) principal G-bundles P with integrable algebraic connection ∇

Replace ∇ by a λ-connection and let $\lambda \to 0$ this gives a deformation to the space M_{Dol} of *Higgs bundles*

$$\begin{array}{ccccc} M_{\mathsf{Dol}} & \hookrightarrow & M_{\mathsf{Hod}} & \hookleftarrow & M_{\mathsf{DR}} \times \mathbb{G}_m \\ \downarrow & & \downarrow & & \downarrow \\ \{0\} & \hookrightarrow & \mathbb{A}^1 & \hookleftarrow & \mathbb{G}_m = \mathbb{A}^1 - \{0\} \end{array}$$

This diagram with the action of $\mathbb{G}_m$ is the *Hodge filtration* on M_{DR}

It is one chart in Deligne's reinterpretation of Hitchin's twistor space

Twistor space

Hitchin: $M_{\mathsf{DR}} \cong M_{\mathsf{Dol}}$ are two complex structures fitting into a quaternionic hyperkähler structure, with a *twistor space* over $\mathbb{P}^1$

Deligne: the twistor space $Tw(X) \to \mathbb{P}^1$ can be constructed by glueing two copies of the Hodge filtration space $M_{\mathsf{Hod}}(X)$ to $M_{\mathsf{Hod}}(\overline{X})$

The glueing map comes from

$$M_{\mathsf{DR}}(X) \cong M_{\mathsf{B}}(X) \cong M_{\mathsf{B}}(\overline{X}) \cong M_{\mathsf{DR}}(\overline{X})$$

$M_{\mathsf{B}}(X) = \mathsf{Rep}(\pi_1(X), G)$, and $X^{\mathsf{top}} \cong \overline{X}^{\mathsf{top}}$

This highlights the importance of the Riemann-Hilbert correspondence "Betti" $\cong$ "de Rham"

Harmonic bundles (Hermitian-Yang-Mills-Higgs solutions) give sections $\mathbb{P}^1 \to Tw(X)$ preserved by the antipodal involution σ

Weights for H^1

The quaternionic structure is a *weight* 1 *property*. It is equivalent to saying that the normal bundle to a preferred section is of the form $\mathcal{O}_{\mathbb{P}^1}(1)^a$, i.e. semistable of slope 1

This means that, locally at least, the map from preferred sections to any of the fibers is an isomorphism

It explains "de Rham" $\cong$ "Higgs"

$$M_{\mathrm{DR}} \overset{\cong}{\leftarrow} \Gamma(\mathbb{P}^1, Tw)^{\sigma}_{\mathrm{pref}} \overset{\cong}{\rightarrow} M_{\mathrm{Dol}}$$

If X is quasiprojective, then the twistor space includes weight 2 directions and

$$\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))^{\sigma} \cong \mathbb{R}^3$$

these three coordinates are the complex residue plus the parabolic weight around singular divisors

Formal stacks

Stacks related to X, originating in crystalline cohomology, p-adic Hodge theory:

X_{B} = the constant n-stack whose values are $\Pi_n(X^{\mathrm{top}})$

X_{DR} = the stack associated to the formal category $Ob = X$, $Mor = (X \times X)^{\wedge}$
in fact it is a sheaf, also given by the formula above $Y \mapsto X(Y^{\mathrm{red}})$

X_{Dol} = the classifying stack for the completion of the zero-section in the tangent bundle,
it results from deformation to the normal cone applied to $Mor(X_{\mathsf{DR}})$

We have a deformation $X_{\mathsf{Hod}} \to \mathbb{A}^1$ from X_{DR} to X_{Dol}

The Deligne-Hitchin glueing is represented by the diagram

$$X^{\mathsf{a}}_{\mathsf{Hod}} \supset (X_{\mathsf{DR}} \times \mathbb{G}_m)^{\mathsf{a}} \to (X_{\mathsf{B}} \times \mathbb{G}_m)^{\mathsf{a}} \leftarrow (\overline{X}_{\mathsf{DR}} \times \mathbb{G}_m)^{\mathsf{a}} \subset \overline{X}^{\mathsf{a}}_{\mathsf{Hod}}$$

Coefficients

To complete the nonabelian cohomology or shape-theory picture, we need to specify what kinds of stacks T will be allowed as coefficients

For simplicity assume that $\pi_0(T) = *$

In order to get a good GAGA result comparing de Rham and Betti cohomology, we ask that

- $\pi_1(T, t)$ be an affine algebraic group

- for $i \geq 2$, $\pi_i(T)$ be a vector space with π_1 acting algebraically

Under these hypotheses,
$\underline{Hom}(X_{\mathsf{DR}}, T)$ and $\underline{Hom}(X_{\mathsf{B}}, T)$ are Artin geometric n-stacks, and their associated analytic stacks are isomorphic

Example: Consider a fibration

$$\begin{array}{ccc} K(V,n) & \to & T \\ & & \downarrow \\ & & BG \end{array}$$

where G acts on a representation V; then

$$\underline{Hom}(X_{\mathsf{DR}}, T) = \{(P, \nabla, \alpha)\}$$

where (P, ∇) is a principal G-bundle with connection, and

$$\alpha \in H^n(X_{\mathsf{DR}}, P \times^G V)$$

is a de Rham cohomology class in the associated representation

We recover the "jump loci" in this way

$$\underline{Hom}(X_{\mathsf{DR}}, T) \to M_{\mathsf{DR}}(X, G)$$

The Hodge filtration

$\underline{Hom}^{\mathrm{se},c_i=0}(X_{\mathrm{Dol}}, T)$ is Artin-geometric

This is the fiber over $\lambda = 0$ of the Hodge-filtration deformation

$$\underline{Hom}^{\mathrm{se},c_i=0}(X_{\mathrm{Hod}}/\mathbb{A}^1, T) \to \mathbb{A}^1$$

whose general fiber is $\underline{Hom}(X_{\mathrm{DR}}, T)$

Glueing together with the complex conjugate chart we get a *Higher twistor space*

$$Tw(X, T) \to \mathbb{P}^1$$

which, in the case $T = BG$, gives back Hitchin's twistor space

Furthermore there is an action of $\mathbb{G}_m$ giving the "Hodge structure"

The weight filtration?

We would like to define a notion of weight filtration in this situation, and obtain a notion of mixed Hodge structure on the nonabelian cohomology

In joint work with Katzarkov and Pantev, we gave a conjectural definition

Current work, also with Toen and Vezzosi (. . .) aims to give a better definition using the notion of *derived stack*

(Kontsevich, Kapranov, Ciocan-Fontanine, Hinich, Toen, Vezzosi, Lurie, . . .)

Representability

If X is simply connected, then the shape is representable by universal maps

$$X_{\mathsf{B}} \to T_{\mathsf{B}} \quad X_{\mathsf{DR}} \to T_{\mathsf{DR}}$$

to the *complex Betti or de Rham homotopy type of* X
These are calculated by the dga's of rational homotopy theory

The above considerations provide T_{DR} with a canonical Hodge filtration,
a Gauss-Manin connection for the rational homotopy types of fibers in a family
(Navarro-Aznar),
Griffiths transversality

We would like also to explain the weight filtration (previous slide)
And define a notion of "variation of nonabelian mixed Hodge structure"

Schematic homotopy types

For the non-simply connected case, Toen has shown that the shape is in a certain sense *pro-representable* by the "schematic homotopy type" of X

Katzarkov-Pantev-Toen endow the schematic homotopy type with a mixed Hodge structure, Pridham extends this to the case of singular varieties, gives a $\mathbb{Q}_\ell$ analogue,
Olsson does a p-adic Hodge theory analogue

These imply formality, degeneration of the Curtis spectral sequence

The schematic homotopy type doesn't see all of the shape: it discretizes the mapping stacks $\underline{Hom}(X_{\mathsf{DR}}, T)$, in other words it misses that representations move in families

Geometry

The structures described above give strong restrictions on $\mathbf{Shape}(X^{\mathsf{top}})$ for complex varieties: restrictions on rational homotopy type, on π_1, properties of the jump loci and other loci defined by homotopy structures such as cup products

One of the main problems in the subject is understanding examples of what kinds of X^{top} can occur

It would be good to have the notions of nonabelian cohomology and "shape" participate, by serving to organize the search

We should look for what kinds of answers $\underline{Hom}(X_{\mathsf{DR}}, T)$ can occur, depending on the coefficient stack T

These answers could be used to cut up the classification problem in new ways

An example of classification for nonabelian H^1 (with Corlette): maps $X_{\mathsf{DR}} \to BGL(2)$ factor as

$$X_{\mathsf{DR}} \to Y_{\mathsf{DR}} \to BGL(2)$$

for Y either a curve, or a certain kind of Shimura modular stack

Far future: could it be possible to classify the possible homotopy types of algebraic varieties?