On Connes–Kreimer's β-function and RG flow

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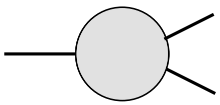
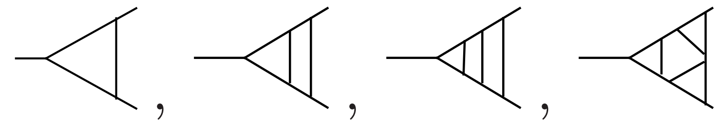
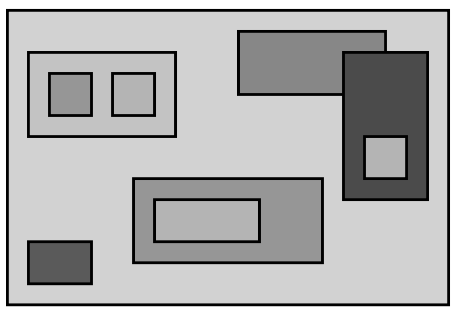
Motivation

Expansion of Green function in 1PI Feynman graphs:



$$= \sum_{\Gamma_{(3\, ext.\, legs)}} \lambda^{|\Gamma|} \frac{1}{sym(\Gamma)}\ \Gamma$$





$$\xrightarrow[rules\ \ \phi]{Feynman} \quad \phi(\Gamma) := \int \cdots \int I\big(\Gamma\big)$$

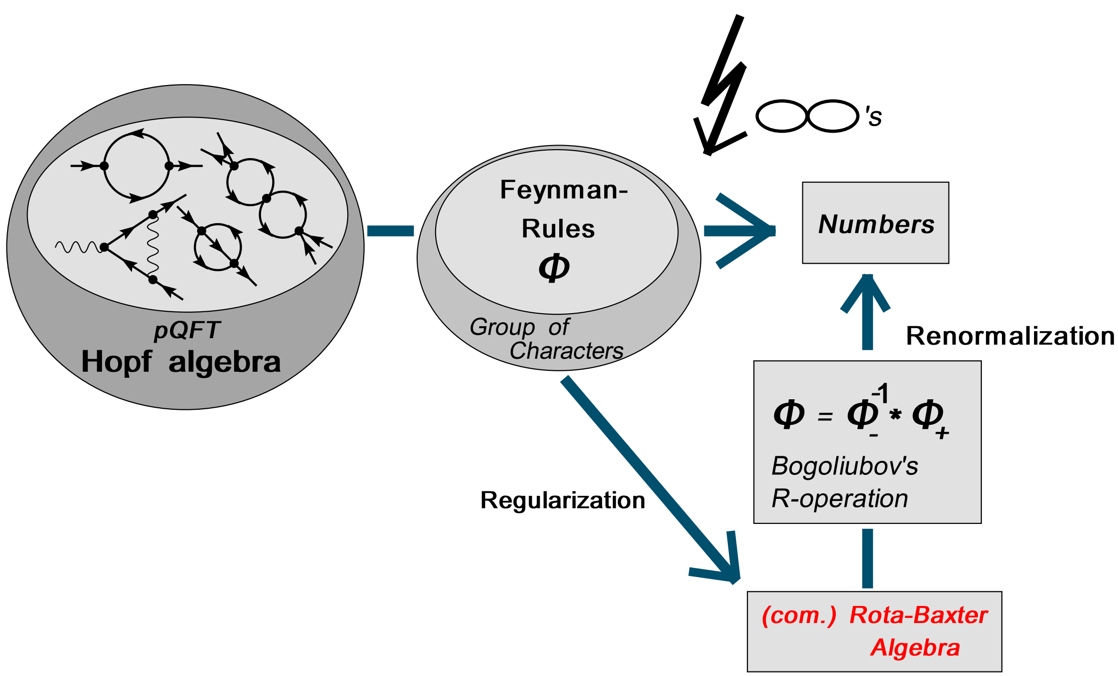
Problem: Ultraviolet divergent amplitudes must be renormalized

"Removing [these] divergencies has been during the decades, the nightmare and the delight of many physicists working in particle physics. [...] This matter of fact even participated to the nobility of the subject..."

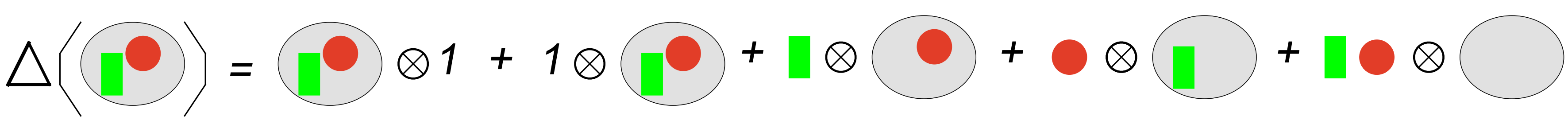
B. Delamotte

Connes-Kreimer's Hopf algebra of renormalization in pQFT

graded connected commutative Hopf algebra of Feynman graphs H



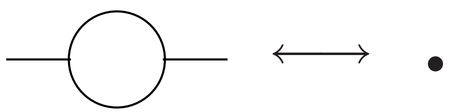
$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma \subset \Gamma} \gamma \otimes \Gamma/\gamma$$

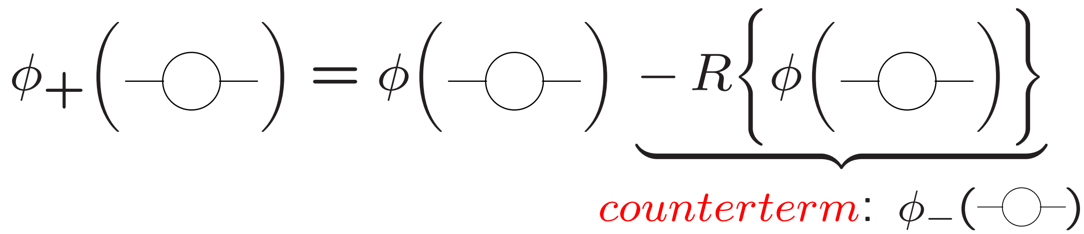


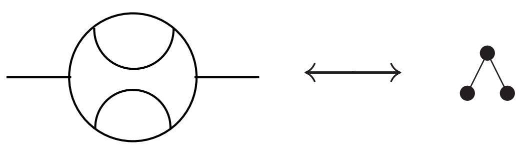
Renormalization: Bogoliubov's R-operation (recursive subtraction)

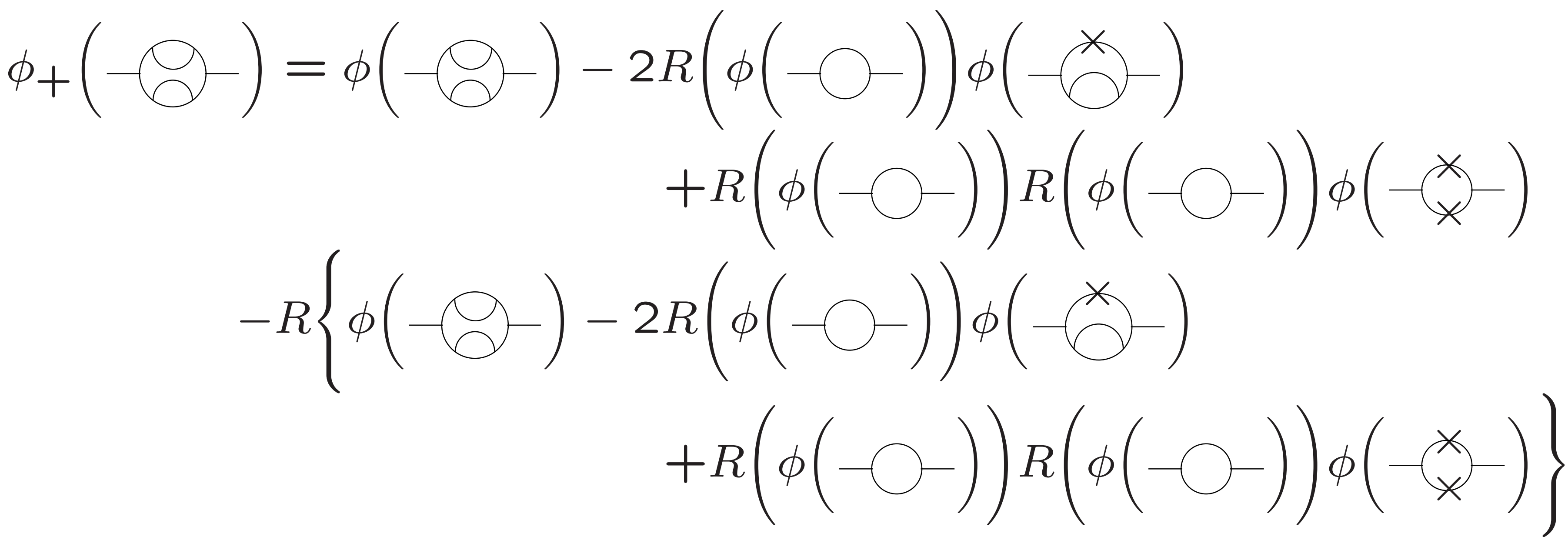
Feynman rules: $\phi : H \to A$

renormalization scheme: $R : A \to A$







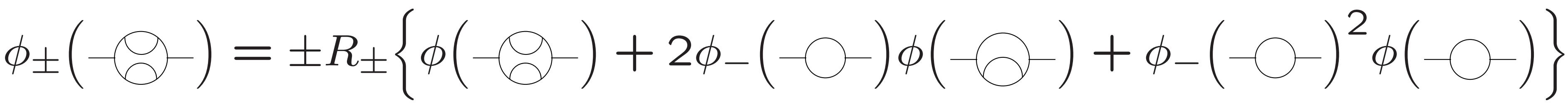


Factorization of Feynman rules

Renormalization process $\longleftrightarrow$ **Factorization problem**

Bogoliubov's preparation map:

$$\Gamma \longrightarrow \bar{\phi}(\Gamma) := \phi(\Gamma) + \sum_{\gamma \subset \Gamma} \phi_-(\gamma)\phi(\Gamma/\gamma) \longrightarrow \begin{cases} \phi_-(\Gamma) := -R\Big(\bar{\phi}(\Gamma)\Big) \\ \phi_+(\Gamma) := (\mathrm{id} - R)\Big(\bar{\phi}(\Gamma)\Big) \end{cases}$$



Counterterm and renormalized character:

$$\phi_\pm(\Gamma_1\Gamma_2) = \phi_\pm(\Gamma_1)\phi_\pm(\Gamma_2)$$

Decomposition: unique for $R_-^2 = R_-$

$$R_-(x)R_-(y) + R_-(xy) = R_-\Big(xR_-(y) + R_-(x)y\Big)$$

$$\boxed{\phi_- \star \phi = \phi_+}$$

Renormalization in a small Nutshell: toy model

Toy-model Dyson–Schwinger equation: expansion of $F(c)$ in powers of the -bare- coupling parameter g.

$$
\begin{aligned}
F(c) &= 1 + g\int_0^\infty \frac{F(x)\,dx}{x+c} \\
&= 1 + g\int_0^\infty \frac{dx}{x+c} + g^2\int_0^\infty \frac{1}{x+c}\int_0^\infty \frac{dy}{y+x}\,dx + \cdots \\
&= 1 + g\,\phi^{\bullet}(c) + g^2\,\phi^{\substack{\bullet\\ |\\ \bullet}}(c) + \cdots
\end{aligned}
$$

ε - regularization of $F \overset{\varepsilon}{\rightsquigarrow} F_\varepsilon$, $\phi^{(t)}(c) \to \phi^{(t)}(c;\varepsilon)$ and introduce the Z-factor

$$
Z = 1 + \sum_{m>0} g^m \phi_-^{(m)}(\varepsilon)
$$

Let Z enter the g-expansion of F_ε: $F_\varepsilon \to F_{Z,\varepsilon} := Z F_\varepsilon$

$$
\begin{aligned}
F_{Z,\varepsilon}(c) &= Z + g\int_0^\infty \frac{F_{Z,\varepsilon}(x)\,dx}{x+c} \\
&= \Big(1 + \sum_{m>0} g^m \phi_-^{(m)}(\varepsilon)\Big) + \Big(g + \sum_{m>0} g^{m+1}\phi_-^{(m)}(\varepsilon)\Big)\phi^{\bullet}(c;\varepsilon) + \cdots
\end{aligned}
$$

$$
\begin{aligned}
= 1 \;&+\; g\Big(\phi^{\bullet}_{-}(\varepsilon) + \phi^{\bullet}(c;\varepsilon)\Big) \\
&+\; g^2\Big(\phi^{\substack{\bullet\\|\\\bullet}}_{-}(\varepsilon) + \phi^{\bullet}_{-}(\varepsilon)\phi^{\bullet}(c;\varepsilon) + \phi^{\substack{\bullet\\|\\\bullet}}(c;\varepsilon)\Big) \\
&+\; g^3\Big(\phi^{\substack{\bullet\\|\\\bullet\\|\\\bullet}}_{-}(\varepsilon) + \phi^{\substack{\bullet\\|\\\bullet}}_{-}(\varepsilon)\phi^{\bullet}(c;\varepsilon) + \phi^{\bullet}_{-}(\varepsilon)\phi^{\substack{\bullet\\|\\\bullet}}(c;\varepsilon) + \phi^{\substack{\bullet\\|\\\bullet\\|\\\bullet}}(c;\varepsilon)\Big) + \cdots
\end{aligned}
$$

<u>order g</u>:

$$\phi^{\bullet}(c) := \int_0^\infty \frac{dy}{y+c}$$

$\rightsquigarrow$ logarithmic divergent at upper limit.

ε-regularization:

$$\phi^{\bullet}(c;\varepsilon) := \int_0^\infty \frac{\mu^\varepsilon dy}{(y+c)^{1+\varepsilon}} = \frac{1}{\varepsilon} + \log(\mu/c) + O(\varepsilon).$$

Renormalization scheme on (RBA) $\mathcal{A} := \mathbb{C}[\varepsilon^{-1},\varepsilon]]$:

$$\mathsf{R}_{-}\Big(\sum_{k=-N}^{\infty} a_k\varepsilon^k\Big) := \sum_{k=-N}^{-1} a_k\varepsilon^k$$

subtraction: counterterm: $\phi_{-}^{\bullet}(\varepsilon) := -\mathsf{R}_{-}\Big(\phi^{\bullet}(c;\varepsilon)\Big) = \frac{1}{\varepsilon}$

$$
\begin{aligned}
\phi_{+}^{\bullet}(c;\varepsilon) &:= \phi_{-}^{\bullet}(\varepsilon) + \phi^{\bullet}(c;\varepsilon) \\
&= \phi^{\bullet}(c;\varepsilon) - \mathsf{R}_{-}(\phi^{\bullet}(c;\varepsilon)) \\
&= (\mathrm{id} - \mathsf{R}_{-})\phi^{\bullet}(c;\varepsilon) \\
&= -\log(\mu/c)
\end{aligned}
$$

<u>order g^2</u>: nested integrals

$$
\begin{aligned}
\phi^{\substack{\bullet \\ | \\ \bullet}}(c;\varepsilon) &:= \int_0^\infty \frac{\phi^{\bullet}(y;\varepsilon)\mu^\varepsilon dy}{(y+c)^{1+\varepsilon}} = \int_0^\infty \frac{\mu^\varepsilon}{(y+c)^{1+\varepsilon}} \int_0^\infty \frac{\mu^\varepsilon}{(z+y)^{1+\varepsilon}} dz\, dy \\
&= \frac{1}{2\varepsilon^2} + \frac{\log(\mu/c)}{\varepsilon} + \frac{\log^2(\mu/c)}{2!} + O(\varepsilon).
\end{aligned}
$$

(naive) counterterm:

$$
\begin{aligned}
\phi_{-}^{\substack{\bullet \\ | \\ \bullet}}(\varepsilon) &:= -\mathsf{R}_{-}\Big(\phi^{\substack{\bullet \\ | \\ \bullet}}(c;\varepsilon)\Big) \\
&= \frac{-1}{2\varepsilon^2} - \frac{-\log(\mu/c)}{\varepsilon}
\end{aligned}
$$

"Preparation": Bogoliubov's recursive R-map

$$
\begin{aligned}
b[\phi^{\substack{\bullet\\|\\{\color{cyan}\bullet}}}](c;\varepsilon) &:= \phi^{\substack{\bullet\\|\\{\color{cyan}\bullet}}}(c;\varepsilon) + \phi_-^{\bullet}(\varepsilon)\phi^{\bullet}(c;\varepsilon) \\
&= \phi^{\substack{\bullet\\|\\{\color{cyan}\bullet}}}(c;\varepsilon) - \mathsf{R}_-\Big(\phi^{\color{cyan}\bullet}(c;\varepsilon)\Big)\phi^{\bullet}(c;\varepsilon) \\
&= \left(\frac{1}{2\varepsilon^2} + \frac{\log(\mu/c)}{\varepsilon} + \frac{\log^2(\mu/c)}{2!} + O(\varepsilon)\right) \\
&\quad -\left(\frac{1}{\varepsilon^2} + \frac{\log(\mu/c)}{\varepsilon} + O(\varepsilon)\right) \\
&= -\frac{1}{2\varepsilon^2} + \frac{\log^2(\mu/c)}{2} + O(\varepsilon)
\end{aligned}
$$

good counterterm: $\phi_-^{\substack{\bullet\\|\\\bullet}}(\varepsilon) = -\mathsf{R}_-\Big(\phi^{\substack{\bullet\\|\\{\color{cyan}\bullet}}}(c;\varepsilon) + \phi_-^{\bullet}(\varepsilon)\phi^{\bullet}(c;\varepsilon)\Big)$

renormalized expression:

$$
\begin{aligned}
\phi_+^{\substack{\bullet\\|\\{\color{cyan}\bullet}}}(c;\varepsilon) &= \phi_-^{\substack{\bullet\\|\\\bullet}}(\varepsilon) + b[\phi^{\substack{\bullet\\|\\{\color{cyan}\bullet}}}](c;\varepsilon) \\
&= (\mathrm{id} - \mathsf{R}_-)(\mathrm{b}[\phi^{\substack{\bullet\\|\\{\color{cyan}\bullet}}}](\mathrm{c};\varepsilon)) \\
&= \frac{\log(\mu/c)^2}{2!} + O(\varepsilon)
\end{aligned}
$$

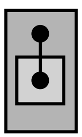
Bogoliubov's recursion map

★ iterated Riemann integrals $\longleftrightarrow$ rooted ladder trees with n vertices: t_n

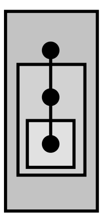
$$\int_c^\infty K[x_1] \int_{x_1}^\infty K[x_2] \cdots \int_{x_{n-1}}^\infty K[x_n] \;\longleftrightarrow\; t_n$$

Bogoliubov's recursion formula for the counterterm of order n: $\phi_-^{(t_n)}$

$$\boxed{\phi_-^{(t_n)} := -\mathrm{R}_-\Big(\, \phi^{(t_n)} + \textstyle\sum_{k=1}^{n-1} \phi_-^{(t_k)}\ \ \phi^{(t_{n-k})}\,\Big)}$$

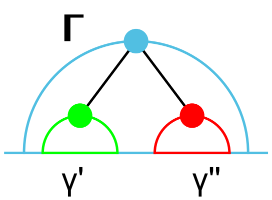


$$\phi_-^{(t_2)} = -\mathrm{R}_-\big(\phi^{(t_2)} + \phi_-^{\bullet}\ \phi^{\bullet}\big)$$



$$\phi_-^{(t_3)} = -\mathrm{R}_-\big(\phi^{(t_3)} + \phi_-^{\bullet}\ \phi^{(t_2)} + \phi_-^{(t_2)}\ \phi^{\bullet}\big)$$

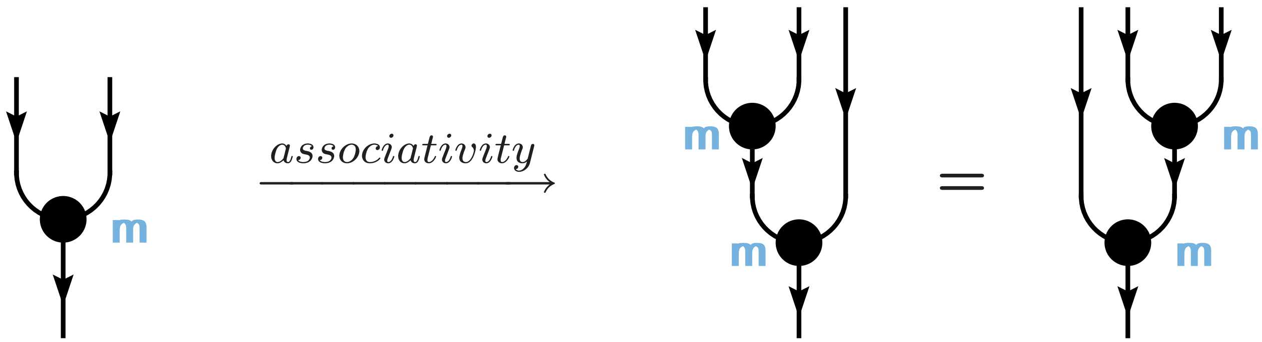
⇝ Generalization to arbitrary Feynman graphs (or rooted trees) Γ:



$$\phi_-(\Gamma) = -\mathrm{R}_-\Big(\phi(\Gamma) + \textstyle\sum_{\gamma \subset \Gamma} \phi_-(\gamma)\phi(\Gamma/\gamma)\Big)$$

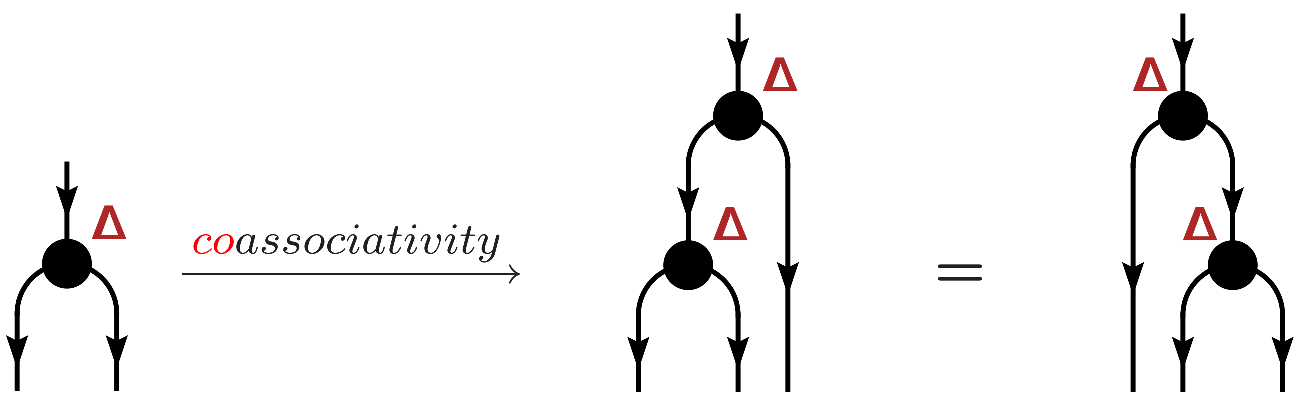
Hopf algebra

associative algebra $\mathcal{A}$: $m : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$



$a \cdot b = c$ $\qquad$ $(a \cdot b) \cdot c \;=\; a \cdot (b \cdot c)$

coassociative algebra $\mathcal{C}$: $\Delta : \mathcal{C} \rightarrow \mathcal{C} \otimes \mathcal{C}$



$\Delta(c)$ $\qquad$ $(\Delta \otimes id)\Delta(c) \;=\; (id \otimes \Delta)\Delta(c)$

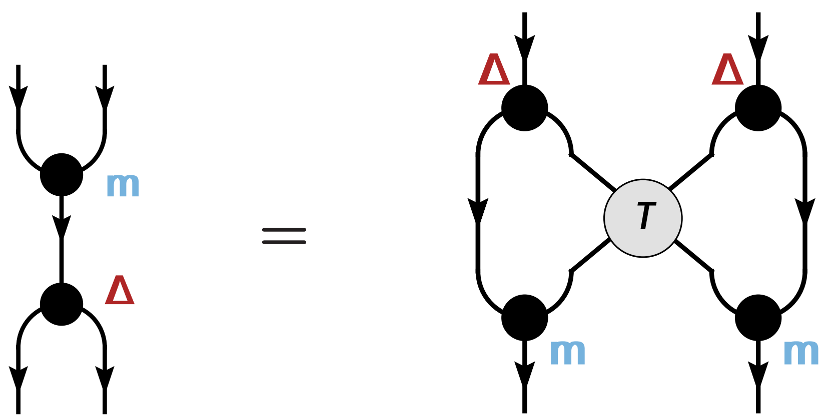
Bialgebra $(\mathsf{H}, \mathsf{m}, \eta, \Delta, \epsilon)$

(H, m, η) algebra: $m : H \otimes H \to H$ and unit $\eta : \mathbb{K} \to H$.

(H, Δ, ϵ) coalgebra: $\Delta : H \to H \otimes H$ and counit $\epsilon : H \to \mathbb{K}$

$$\Delta(a) = \sum_i a'_i \otimes a''_i$$

compatibility: $\Delta(ab) = \Delta(a)\Delta(b)$



$$T(a \otimes b) := b \otimes a$$

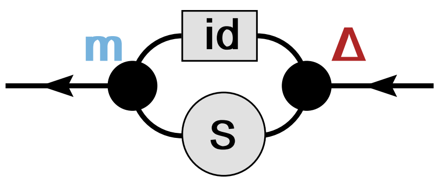
$$\Delta m(a \otimes b) \quad = \quad m \otimes m(id \otimes \tau \otimes id)\Delta \otimes \Delta(a \otimes b)$$

Hopf algebra $\big(\mathsf{H}, \mathsf{m}, \eta, \Delta, \epsilon, \mathsf{S}\big)$

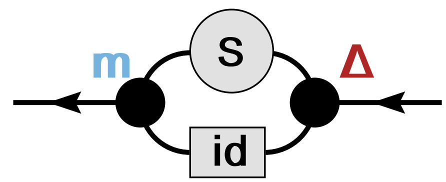
- antipode $S : H \to H$

$$(id \star S)(a) \quad = \quad \eta \circ \epsilon(a) \quad = \quad (S \star id)(a)$$

$$m(id \otimes S)\Delta(a) \quad = \quad \eta \circ \epsilon(a) \quad = \quad m(S \otimes id)\Delta(a)$$



$a = \quad \eta \circ \epsilon(a) \quad =$



a

$$\textstyle\sum_i a_i' S(a_i'') \quad = \quad \eta\, \epsilon(a) \quad = \quad \sum_i S(a_i') a_i''$$

Bialgebra $(H, m, \Delta, \eta, \epsilon)$:

$$\Delta(m(a,b)) = \Delta(a)\Delta(b)$$

Hopf Algebra $(H, m, \Delta, \eta, \epsilon, S)$:

$$m(S \otimes id) \circ \Delta = \eta \circ \epsilon = m(id \otimes S) \circ \Delta$$

graded connected Hopf algebra:

$$H = \bigoplus_{n \geq 0} H^{(n)},$$

$$H^{(0)} = \mathbb{K}, \;\; H^{(n)} H^{(m)} \subseteq H^{(n+m)}$$

$$\epsilon(T) := \begin{cases} 0 & , \; T \in \bigoplus_{n \geq 1} H^{(n)} \\ 1 & , \; else. \end{cases}$$

$$\Delta(H^{(n)}) \subseteq \bigoplus_{k=0}^{n} H^{(n-k)} \otimes H^{(k)}$$

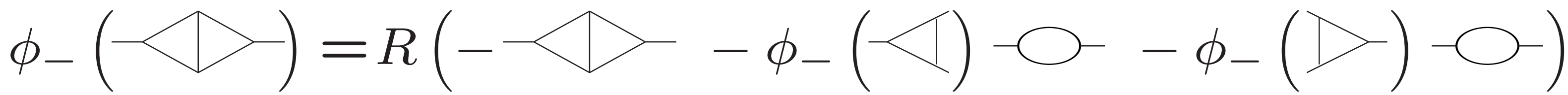
$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum{}^{*} \, \Gamma' \otimes \Gamma''$$

★ Combinatorics of renormalization is captured by a **connected graded commutative Hopf Algebra**:

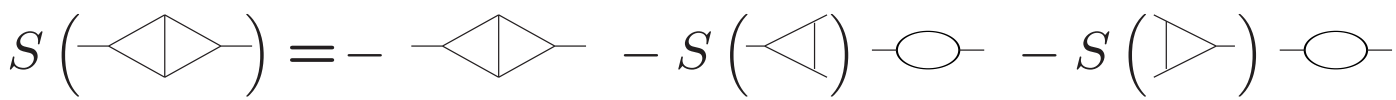
$$\boxed{S(\Gamma) = -\Gamma - \sum{}^{*} \, S(\Gamma') \, \Gamma''}$$

Theorem: Hopf algebra H of Feynman graphs: $H := \big(m, \Delta, \epsilon, \eta, S\big)$
$\rightsquigarrow$ H is a unital, associative, commutative, coassociative, non-cocommutative, connected, graded (e.g. #loops) Hopf algebra, $H = \mathbb{K} \oplus \bigoplus_{n>0} H^{(n)}$.

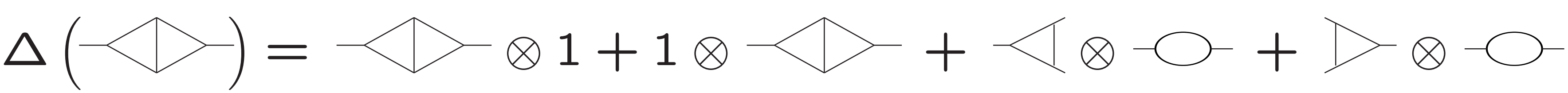
Counterterm: $\phi_- : H \to A$, $R : A \to R$



Antipode: $S : H \to H$



Coproduct: $\Delta : H \to H \otimes H$



Regularization scheme: Rota–Baxter relation

Regularized Feynman Rules: comm., unital Rota–Baxter algebra (A, R)

$$\boxed{\mathsf{R}(x)\mathsf{R}(y) + \theta\mathsf{R}(xy) = \mathsf{R}\Big(\mathsf{R}(x)y + x\mathsf{R}(y)\Big)}$$

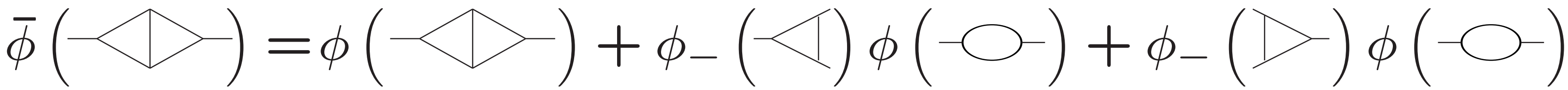
Examples: $\sum_{i=-n}^{\infty} c_i\epsilon^i \in \mathbb{C}[[\epsilon, \epsilon^{-1}]$

$$\mathsf{R}\Big(\sum_{i=-n}^{\infty} c_i\epsilon^i\Big) := \sum_{i=-n}^{-1} c_i\epsilon^i$$

★ A-valued linear functionals: $H \xrightarrow{\mathsf{Hom}(H,A)} (A, \mathsf{R})$

$$f \star g := m_A(f \otimes g)\Delta : H \xrightarrow{\Delta} H \otimes H \xrightarrow{f \otimes g} A \otimes A \xrightarrow{m_A} A$$

$$f \star g(\Gamma) \;=\; f(\Gamma)g(1) + f(1)g(\Gamma) + \sum_{\gamma \subset \Gamma} f(\gamma)g(\Gamma/\gamma).$$



$$\bar{\phi}(\Gamma) = \phi_-(1)\phi(\Gamma) + \sum_{\gamma\subset\Gamma} \phi_-(\gamma)\phi(\Gamma/\gamma)$$
$$= (\phi_- \star \phi)(\Gamma) - \phi_-(\Gamma)\phi(1)$$

For $h \in \bigoplus_{n\geq 0} H_n$: **grading operator**

$$Y : H \to H, \quad Y(h) = \sum_{n\geq 0} n h_n,$$

$$Y(h_1 h_2) = Y(h_1)h_2 + h_1 Y(h_2)$$

$Yf := f \circ Y$: Y extends to a derivation on Hom(H, A), $f, g \in$ Hom(H, A):

$$Y(f \star g) = Yf \star g + f \star Yg$$

Dynkin operator

Group of **regularized characters** $G(A) \ni \ \phi : H \to A$

$$\phi(h_1 h_2) = \phi(h_1)\phi(h_2)$$

Lie algebra of **infinitesimal characters** $g(A) \ni \ \alpha : H \to A$

$$\alpha(h_1 h_2) = \alpha(h_1)e(h_2) + e(h_1)\alpha(h_2)$$

Dynkin operator

$$\boxed{D := S \star Y}$$

Proposition: The Dynkin operator $D = S \star Y$ is an H-valued infinitesimal character of H.

Proposition: Right composition with the Dynkin operator D induces a map from $G(A)$ to $g(A)$. Particularly,

$$\gamma \circ D = \gamma \circ (S \star Y) = \gamma^{-1} \star Y\gamma, \quad \gamma \in G(A)$$

Inverse Dynkin operator

Theorem: Right composition with D is a bijective map from $G(A)$ to $g(A)$. The inverse map is given by

$$\Gamma : \alpha \in g(A) \longmapsto \sum_n \sum_{\substack{k_1,\dots,k_l \in \mathbb{N}^* \\ k_1+\cdots+k_l=n}} \frac{\alpha_{k_1} \star \cdots \star \alpha_{k_n}}{k_1(k_1+k_2)\dots(k_1+\cdots+k_n)} \in G(A).$$

The definition of D implies $I \star D = I \star S \star Y = Y$

$$Y_n = nI_n = (I \star D)_n = \sum_{i=1}^{n} I_{n-i} \star D_i.$$

$$I_n \quad = \sum_{\substack{k_1,\dots,k_l \in \mathbb{N}^* \\ k_1+\cdots+k_l=n}} \frac{D_{k_1} \star \cdots \star D_{k_l}}{k_1(k_1+k_2)\dots(k_1+\cdots+k_l)}$$

$$\gamma = e + \gamma \circ \sum_{n>0} I_n = e + \sum_{n \in \mathbb{N}^*} \sum_{\substack{k_1,\dots,k_l \in \mathbb{N}^* \\ k_1+\cdots+k_l=n}} \frac{\gamma \circ D_{k_1} \star \cdots \star \gamma \circ D_{k_l}}{k_1(k_1+k_2)\dots(k_1+\cdots+k_l)}.$$

As D preserves the grading, it follows that Γ is a left inverse to the right composition with D.

We show that Γ is also a right inverse to the composition with D. For any h in the augmentation ideal of H and arbitrary $\alpha \in g(A)$

$$
\begin{aligned}
Y\Gamma(\alpha)(h) &= |h| \sum_{\substack{k_1,\dots,k_l \in \mathbb{N}^* \\ k_1+\dots+k_l=|h|}} \frac{\alpha_{k_1} \star \dots \star \alpha_{k_l}}{k_1(k_1+k_2)\dots(k_1+\dots+k_l)}(h) \\
&= \sum_{\substack{k_1,\dots,k_l \in \mathbb{N}^* \\ k_1+\dots+k_l=|h|}} \frac{\alpha_{k_1} \star \dots \star \alpha_{k_{l-1}}}{k_1(k_1+k_2)\dots(k_1+\dots+k_{l-1})} \star \alpha_{k_l}(h) \\
&= \Gamma(\alpha) \star \alpha\,(h).
\end{aligned}
$$

$$
\Gamma(\alpha) \circ D = \Gamma(\alpha)^{-1} \star Y\Gamma(\alpha) = \alpha \in g(A)
$$

We refrain here from proving that Γ is character-valued, that is, is actually a map from $g(A)$ to $G(A)$.

$$
\Gamma(\alpha)(h_1h_2) = \Gamma(\alpha)(h_1)\Gamma(\alpha)(h_2) \quad \alpha \in g(A).
$$

Decomposition of A-valued characters

(A, R) is a commutative Rota–Baxter algebra with idempotent R

$$A = R(A) \oplus (\mathrm{id} - R)(A).$$

Recall: For any $\phi = \exp(\alpha) \in G(A)$, with $\alpha \in g(A)$, we have unique $\alpha_\pm = g(A_\pm)$, and unique characters $\phi_\pm := \exp(\pm\alpha_\pm) \in G(A_\pm)$ such that:

$$\phi = \phi_-^{-1} \star \phi_+, \qquad \phi_\pm(h) \in A_\pm,\ h \in \bigoplus_{n>0} H_n.$$

It follows that $G(A)$ decomposes

$$G(A) = G_-(A) \star G_+(A),$$

where $G_\pm(A) := G(A_\pm)$.

For any $\phi = \exp(\alpha)$ the unique characters $\phi_\pm := \exp(\pm\alpha_\pm) \in G_\pm(A)$ in the previous corollary solve the equations:

$$\phi_\pm(h) = \Big(e \pm R_\pm(\phi_- \star (\phi - e))\Big)(h).$$

Dimensional regularization

Feynman amplitude for a graph $F \in H$

$$F \xrightarrow{Feynman\,rules} \phi(F)(p) = \Big[\int \prod_{l=1}^{|F|} d^{\mathcal{D}} k_l \Big] I_F(p,k).$$

In **dimensional regularization** one introduces a complex parameter $\varepsilon \in \mathbb{C}$ by changing the integral measure

$$d^{\mathcal{D}} k \xrightarrow{\text{dim.-reg.}} \mu^{\varepsilon}\, d^{\mathsf{D}} k, \qquad \varepsilon = (\mathcal{D} - \mathsf{D})$$

$$F \longrightarrow \phi(\varepsilon,\mu)(F)(p) = \mu^{|F|\varepsilon} \Big[\int \prod_{l=1}^{|F|} d^{\mathsf{D}} k_l \Big]\, I_F(p,k).$$

We define on the group of A-valued characters $G(A)$ a **one-parameter action** of $\mathbb{C}^* \ni t$:

$$\boxed{\phi^t(\varepsilon,\mu)(h) := t^{\varepsilon|h|}\phi(\varepsilon,\mu)(h)}$$

Physically: replacing the $\mu^{\varepsilon|h|}$ by $(\mu t)^{\varepsilon|h|}$; that is, the mass scale is changed from μ to $t\mu$: $t^{\varepsilon|h|}\phi(\varepsilon,\mu)(h) = \phi(\varepsilon,t\mu)(h)$

$G(A) \ni \phi \to \phi^t$ is still a character, and $(\phi_1 \star \phi_2)^t = \phi_1^t \star \phi_2^t$. For any t and any homogeneous $h \in H$ we have $t^{\varepsilon|h|} \in A_+ := \mathbb{C}[[\varepsilon]]$.

$$\phi \in G_+(A) \mapsto \phi^t \in G_+(A).$$

$$t\frac{\partial}{\partial t}\phi^t = \varepsilon|h|\phi^t(\varepsilon, \mu)(h) = \varepsilon Y \phi^t \quad \text{such that} \quad t\frac{\partial}{\partial t}\Big|_{t=1} \phi^t = \varepsilon Y \phi.$$

We have for the regularized character $\phi^t \in G(A)$

$$\phi^t = (\phi^t)_-^{-1} \star (\phi^t)_+.$$

Theorem: (**locality**) Let ϕ be a dimensionally regularized Feynman rule character. The counterterm character in $\phi^t = (\phi^t)_-^{-1} \star (\phi^t)_+$ satisfies

$$t\,\frac{\partial(\phi^t)_-}{\partial t} = 0$$

Or $(\phi^t)_-$ is equal to ϕ_-, i.e. independent of t. We say the A-valued characters with this property are **local** characters: $\phi \in G^{\text{loc}}(A) \subset G(A)$.

Proposition: $G^{\text{loc}}(A)$ decomposes into the product $G^{\text{loc}}_-(A) \star G_+(A)$.

Theorem: The map $\phi \mapsto \varepsilon(\phi \circ D)$, with D the Dynkin operator, sends $G^{\mathrm{loc}}(A)$ to $g(A_+)$ and $G_-^{\mathrm{loc}}(A)$ to $g(\mathbb{C})$; explicitly, in the second case:

$$G_-^{\mathrm{loc}}(A) \ni \phi \mapsto \varepsilon(\phi \circ D) = Y\mathrm{Res}\phi \in g(\mathbb{C}).$$

Definition: **Beta function** $\beta(\phi) := \varepsilon(\phi \circ D) = Y\mathrm{Res}\phi \in g(\mathbb{C})$.

Now let $\beta \in g(\mathbb{C})$ be a *scalar*-valued infinitesimal character. Notice that β/ε can be regarded as an element of $g(A_-)$.

Theorem: With Γ the inverse of D, we find:

$$\phi_\beta \;:=\; \Gamma(\beta/\varepsilon) \in G_-^{\mathrm{loc}}(A).$$

$$\phi_\beta = \Gamma(\beta/\varepsilon) = \sum_n \left(\sum_{k_1,\ldots,k_n \in (N)^*} \frac{\beta_{k_1} \star \cdots \star \beta_{k_n}}{k_1(k_1+k_2)\ldots(k_1+\cdots+k_n)} \right) \frac{1}{\varepsilon^n}$$

We conclude that for $\phi \in G_-^{\mathrm{loc}}(A)$ one has:

$$\Gamma\left(\frac{Y\mathrm{Res}\phi}{\varepsilon}\right) = \phi$$

Theorem: For the renormalized character $\phi_{\text{ren}}(t) := (\phi^t)_+(\varepsilon = 0)$ it holds

$$t\frac{\partial}{\partial t}\phi_{\text{ren}}(t) = (Y\text{Res}\phi) * \phi_{\text{ren}}(t),$$

the abstract RG equation.

This equation is solved using the beta function $\beta(\phi) := Y\text{Res}\phi \in g(\mathbb{C})$:

$$\phi_{\text{ren}}(t) = \exp(\ln(t)\beta(\phi)) \star \phi_{\text{ren}}(1).$$

$$\begin{aligned}\lim_{\varepsilon\to 0}\phi_- \star (\phi_-^{-1})^t &= \lim_{\varepsilon\to 0}(\phi^t)_+ \star ((\phi_+)^t)^{-1}\\ &= \phi_{\text{ren}}(t) * \phi_{\text{ren}}^{-1}(1)\\ &= \exp(\ln(t)\beta(\phi)).\end{aligned}$$

The scalar-valued characters

$$\Omega_t(\phi) := \exp(\ln(t)\beta(\phi)) \in G(\mathbb{C})$$

obviously form a one-parameter subgroup in $G(A)$: $\Omega_{t_1}(\phi) * \Omega_{t_2}(\phi) = \Omega_{t_1 t_2}(\phi)$, generated by the beta function and controlling the flow of the renormalized Feynman rule character with respect to the mass scale.

THANK YOU!!