Foundations of Superposition Theory

vol. 1

Superposition Algebra in the Space of Tempered Distributions

and Applications to Economics and Physics

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Contents

I Introductions and preliminaries **11**

0.1 **Motivations** . . . 13
0.2 **Introduction** . . . 15

1 Preliminaries **17**

1.1 **Topological homomorphisms** . . . 18
1.2 **Direct sums** . . . 19
1.3 **Topological supplements** . . . 20
1.4 **Right and Left inverses** . . . 21
1.5 **Homomorphisms among Fréchet spaces** . . . 21
1.6 **Dieudonné-Schwartz theorem** . . . 22
1.7 **Banach-Steinhaus in barreled spaces** . . . 23
1.8 **Tempered distributions** . . . 24
1.8.1 **Test functions** . . . 24
1.8.2 **Tempered distributions** . . . 25
1.9 **Fourier transforms on $\mathcal{S}_n$** . . . 25
1.10 **Fourier transforms on $\mathcal{S}'_n$** . . . 26

II Superpositions **27**

2 Summable families **29**

2.1 **Families of distributions** . . . 29
2.2 **$^{\mathcal{S}}$Families** . . . 30
2.3 **$^{\mathcal{S}}$Family generated by an operator** . . . 31
2.4 **The operator generated by an $^{\mathcal{S}}$family** . . . 32
2.5 **Characterizations of $^{\mathcal{S}}$ families** . . . 33
2.6 **Characterization of transposability** . . . 36
2.7 **Characterizations of $^{\mathcal{D}}$ families (*)** . . . 37
2.8 **^{E}Families and ^{E}summable families** . . . 37

3 Superpositions **41**

3.1 **Introduction** . . . 41
3.1.1 **The wonderful Dirac basis** . . . 41
3.1.2 **A dangerous expression** . . . 42

3.1.3 **Toward a possible solution** . . . 43
3.1.4 **Inadequacy of convolutions** . . . 44
3.1.5 **Conclusions** . . . 45
3.2 **Superpositions of $^{\mathcal{S}}$families in $\mathcal{S}'_n$** . . . 45
3.3 **An alternative definition of superposition** . . . 46
3.4 **Superpositions of an E-family (*)** . . . 48
3.5 **Algebraic properties of superpositions** . . . 49
3.5.1 **Bilinearity of superposition operator** . . . 49
3.5.2 **Selection property of the Dirac distributions** . . . 50
3.5.3 **Linear combination of an $^{\mathcal{S}}$family** . . . 51

4 $^{\mathcal{S}}$Linearity **53**
4.1 **Continuity of superposition operators** . . . 53
4.2 **Superposition operator of a distribution** . . . 54
4.3 **$^{\mathcal{S}}$Linearity of superpositions** . . . 56
4.4 **Generalized distributive laws (*)** . . . 58

5 $^{\mathcal{S}}$Families in $\mathcal{S}_n$ **61**
5.1 **Families in $\mathcal{S}_n$** . . . 61
5.2 **$^{\mathcal{S}}$Families** . . . 63
5.3 **Transpose of an $^{\mathcal{S}}$Family** . . . 64
5.4 **Operator of an $^{\mathcal{S}}$family** . . . 65
5.5 **Continuity of operators of $^{\mathcal{S}}$families** . . . 66

6 Superpositions in $\mathcal{S}_n$ **69**
6.1 **$^{\mathcal{S}}$Linear combinations** . . . 69
6.2 **Superposition operator of $^{\mathcal{S}}$families** . . . 70
6.3 **Summability of $^{\mathcal{S}}$families** . . . 71
6.4 **Transpose family** . . . 72
6.5 **$^{\mathcal{S}}$Linear functional** . . . 72
6.6 **Superpositions of $^{\mathcal{S}}$families in $\mathcal{S}'_n$** . . . 73
6.7 **$^{\mathcal{S}}$Linear superpositions of operators** . . . 75

7 First applications **77**
7.1 **The Fourier expansion theorem** . . . 77
7.2 **Convolution as superpositions** . . . 80
7.3 **Some expressions of Dirac Calculus** . . . 82
7.3.1 **The expansion of a vector in the Dirac basis** . . . 83
7.3.2 **Fourier expansions and the momentum operator** . . . 84
7.4 **Some extensions** . . . 85

III $^{\mathcal{S}}$Linear Algebra and Geometry 89

8 $^{\mathcal{S}}$Linear hulls of $^{\mathcal{S}}$ families **91**
8.1 **$^{\mathcal{S}}$Linear hulls** . . . 91

8.2 **Algebraic properties of $^{\mathcal{S}}$linear hulls** . . . 92
8.3 **Systems of $^{\mathcal{S}}$generators** . . . 94
8.3.1 **Exercises** . . . 94
8.4 **Topological properties of $^{\mathcal{S}}$linear hulls** . . . 96
8.5 **Closedness of $^{\mathcal{S}}$linear hulls** . . . 97
8.5.1 **Examples of systems of $^{\mathcal{S}}$ generators** . . . 98
8.6 **Kernel of an $^{\mathcal{S}}$family** . . . 99
8.6.1 **Application** . . . 100
8.7 **$^{\mathcal{S}}$Linear hull of a subset** . . . 101

9 Bases 103
9.1 **$^{\mathcal{S}}$Linear independence** . . . 103
9.2 **Topology and $^{\mathcal{S}}$linear independence** . . . 104
9.3 **Uniqueness of representation** . . . 106
9.4 **Characterizations of $^{\mathcal{S}}$linear independence** . . . 106
9.5 **$^{\mathcal{S}}$Bases** . . . 107
9.6 **Algebraic characterizations of $^{\mathcal{S}}$ bases** . . . 108
9.6.1 **Example** . . . 108
9.7 **Totality of $^{\mathcal{S}}$bases** . . . 109
9.8 **Topological characterizations of $^{\mathcal{S}}$ bases** . . . 110
9.9 **Equivalent $^{\mathcal{S}}$families** . . . 110

10 $^{\mathcal{S}}$Closedness 113
10.1 **$^{\mathcal{S}}$Closed subsets** . . . 113
10.2 **$^{\mathcal{S}}$Closed hulls** . . . 114
10.3 **Relationships among different hulls** . . . 115
10.3.1 **Relationships among linear hulls** . . . 116
10.3.2 **$^{\mathcal{S}}$Closure of subspaces** . . . 117
10.3.3 **Relationships among linear and $^{\mathcal{S}}$ closed hulls** 117
10.4 **Unions of $^{\mathcal{S}}$closed sets** . . . 119
10.5 **$^{\mathcal{S}}$Closed linear hull of families** . . . 119
10.6 **$^{\mathcal{S}}$Closedness and topology** . . . 120

11 $^{\mathcal{S}}$Connectedness 123
11.1 **$^{\mathcal{S}}$Connected and $^{\mathcal{D}}$ connected sets** . . . 123
11.1.1 **$^{\mathcal{S}}$Families in $\mathcal{S}'_n$ containing a given distribution** . . . 124
11.1.2 **$^{\mathcal{S}}$Families in starshaped subsets of $\mathcal{S}'_n$** . . . 124
11.1.3 **$^{\mathcal{S}}$Connected subsets of $\mathcal{S}'_n$** . . . 126
11.2 **$^{\mathcal{D}_{\mathcal{L}^1}}$Closed sets (*)** . . . 127
11.2.1 **Preliminaries on the space $\mathcal{D}_{\mathcal{L}^1}$** . . . 127
11.3 **Sums of series as superpositions** . . . 129

IV $^{\mathcal{S}}$Linear operators 133

12 $^{\mathcal{S}}$Linear operators 135
12.1 Introduction . . . 135
12.2 $^{\mathcal{S}}$Operators . . . 136
12.3 $^{\mathcal{S}}$Operators defined on $\mathcal{S}'_n$ . . . 137
12.4 Characterization of $^{\mathcal{S}}$operators on $\mathcal{S}'_n$ . . . 138
12.5 $^{\mathcal{S}}$Linear operators on $\mathcal{S}'_n$ . . . 139
12.6 Examples of $^{\mathcal{S}}$linear operators . . . 140
12.6.1 The superposition operator of an $^{\mathcal{S}}$family . . . 140
12.6.2 Transpose operators . . . 141
12.7 Characterization of $^{\mathcal{S}}$linear operators . . . 143

13 Applications of $^{\mathcal{S}}$linear operators 145
13.1 $^{\mathcal{S}}$Bases of subspaces . . . 145
13.2 $^{\mathcal{S}}$Bases of closed subspaces . . . 146
13.3 $^{\mathcal{S}}$Bases of barreled subspaces . . . 148
13.4 Superpositions of $^{\mathcal{S}}$linear operators . . . 148
13.5 $^{\mathcal{S}}$Linear operators and $^{\mathcal{S}}$bases . . . 150
13.6 Invertibility of $^{\mathcal{S}}$linear operators . . . 151
13.7 $^{\mathcal{S}}$Linear operators on subspaces . . . 152
13.8 Compositions of $^{\mathcal{S}}$linear operators . . . 153

14 $^{\mathcal{S}}$Homomorphisms 155
14.1 $^{\mathcal{S}}$Homomorphisms . . . 156
14.2 Injective linear $^{\mathcal{S}}$homomorphisms . . . 157
14.3 Surjective linear $^{\mathcal{S}}$homomorphisms . . . 158
14.4 $^{\mathcal{S}}$Stable families . . . 159
14.5 $^{\mathcal{S}}$Linear operators and $^{\mathcal{S}}$closedness . . . 160
14.6 $^{\mathcal{S}}$Homomorphism and $^{\mathcal{S}}$ closedness . . . 162
14.7 Invertibility of linear $^{\mathcal{S}}$ homomorphism . . . 162
14.8 Left inverse of linear $^{\mathcal{S}}$ homomorphisms . . . 163

15 $^{\mathcal{S}}$Green's families 165
15.1 Introduction . . . 165
15.1.1 Green's functions . . . 165
15.1.2 Motivations for Green's families . . . 166
15.2 Green's families in $\mathcal{S}'_n$ . . . 167
15.3 $^{\mathcal{S}}$Green's families . . . 168
15.4 Application to linear equations . . . 169
15.5 Interpretation of the solution . . . 170
15.6 Characterization of Green's families . . . 172
15.7 Existence of Green's families . . . 173
15.8 Translation invariance and convolutions . . . 174
15.8.1 Example: the Laplacian . . . 175
15.9 $^{\mathcal{S}}$Green's families relative to $^{\mathcal{S}}$bases . . . 176

V Representations **179**

16 $^{\mathcal{S}}$Coordinates **181**
16.1 **Systems of coordinates** 181
16.2 **Coordinate operators** 182
16.3 **Basic properties of coordinate operators** 183
16.3.1 **Linear properties** 183
16.3.2 **$^{\mathcal{S}}$Linear properties** 184
16.3.3 **Topological properties** 186
16.4 **Coordinate operators of $^{\mathcal{S}}$stable families** 187
16.5 **Hulls of $^{\mathcal{S}}$stable families** 188
16.6 **$^{\mathcal{S}}$Linearity of the coordinate operator** 189

17 Applications of $^{\mathcal{S}}$Coordinates **191**
17.1 **Invertibility of $^{\mathcal{S}}$homomorphism** 191
17.2 **Projectors** 192
17.3 **Change of basis** 194
17.4 **Superpositions respect to operators** 195
17.5 **Resolution of the identity** 197

18 $^{\mathcal{S}}$Matrices **199**
18.1 **Introduction** 199
18.2 **Product of families** 200
18.3 **Transpose product of families** 202
18.4 **Invertible families** 203
18.5 **Coordinates and invertible families** 205

19 Representations in QM **211**
19.1 **Representation of $^{\mathcal{S}}$endomorphisms** 211
19.2 **Representation of vector states** 213
19.2.1 **First examples** 213
19.3 **The representation correspondence** 214
19.4 **Representation of $\mathcal{S}$-linear operators** 216
19.5 **Classic examples in QM** 217
19.5.1 **$^{\mathcal{S}}$Matrix representations** 217
19.5.2 **Operator representation** 218

VI $^{\mathcal{S}}$Spectral theory **221**

20 Multiplicative operators in $\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ **223**
20.1 **Introduction** 223
20.2 **$^{\mathcal{O}_M}$Functions** 224
20.2.1 **Topology** 224
20.2.2 **Bounded sets in $\mathcal{O}_M^{(n)}$** 225
20.2.3 **Multiplications by $^{\mathcal{O}_M^{(n)}}$functions** 225

20.2.4 ${}^{\mathcal{S}}$Family of the multiplication operator M_f 226
20.3 Product in $\mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$ by ${}^{\mathcal{O}_M}$functions 226
20.3.1 The ring $\mathcal{O}_M^{(m)}$ 227
20.3.2 The module $\mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$ 228
20.4 Product of ${}^{\mathcal{S}}$families by ${}^{\mathcal{O}_M}$functions 228
20.5 ${}^{\mathcal{O}_M}$Functions and ${}^{\mathcal{S}}$ basis 230
20.6 ${}^{\mathcal{O}_M}$Invertible functions and ${}^{\mathcal{S}}$basis 232

21 Spectral expansions 235
21.1 Spectral ${}^{\mathcal{S}}$expansions 235
21.2 ${}^{\mathcal{S}}$Expansions and ${}^{\mathcal{S}}$ linear equations 237
21.3 Existence of Green families 239
21.4 Superpositions in $\mathcal{O}_M$ 242

22 ${}^{\mathcal{S}}$Diagonalizable operators 245
22.1 ${}^{\mathcal{S}}$Diagonalizable operators $\mathcal{S}'_n$ 245
22.2 ${}^{\mathcal{S}}$Diagonalizable operators on $\mathcal{S}_n$ 247
22.3 Algebra of ${}^{\mathcal{S}}$diagonalizable operators 248
22.4 Building some observables of QM 252
22.4.1 The position operator in one dimension 252
22.4.2 The position operator in three dimensions 252
22.4.3 The momentum operator 253
22.4.4 The kinetic energy in dimension 1 253
22.5 Observables 254
22.5.1 Observables with a singular spectrum 255
22.5.2 The relativistic energy 255

23 Spectrum 257
23.1 Supports and vanishing-laws 257
23.2 Structure of the eigenspectrum 259
23.3 Structure of the spectrum 260

24 Functional Calculus 263
24.1 A vanishing lemma 263
24.2 Transformable ${}^{\mathcal{S}}$diagonalizable operators 265
24.3 Functions of ${}^{\mathcal{S}}$diagonalizable operators 268
24.4 Compatibility with the exponential 270

VII ${}^{\mathcal{S}}$Linear Dynamics 273

25 The Schrödinger equation 275
25.1 Introduction 275
25.2 Differentiable curve 275
25.2.1 Differentiable curves in topological vector spaces . 275
25.2.2 Differentiable curves in $\mathcal{S}'_n$ 277

25.2.3 **Differentiable curves in $\mathcal{O}_M$ (*)** . . . 278
25.2.4 **Pointwise differentiable curve in $\mathcal{O}_M$** . . . 279
25.3 **The Schrödinger's equation in $\mathcal{S}'_n$** . . . 281
25.3.1 **Solutions of the eigen-representation** . . . 281
25.3.2 **The Schrödinger's equation in $\mathcal{S}'_n$** . . . 282
25.3.3 **The evolution group** . . . 285
25.3.4 **The abstract Heat equation on $\mathcal{S}'_n$** . . . 285

26 $^{\mathcal{S}}$Diagonalizable equations **287**
26.1 **Superpositions with respect to curves** . . . 287
26.2 **Solution of the eigenrepresentations** . . . 288
26.3 **The eigenrepresentations** . . . 290
26.4 **Solutions in functional form** . . . 291

27 Feynman propagators **293**
27.1 **Propagators as family-valued functions** . . . 293
27.2 **Evolution operators** . . . 294
27.3 **Operatorial propagators** . . . 297
27.4 **Feynman propagator of a free particle** . . . 298

28 Evolutions in the space $\mathcal{S}'_n$ **301**
28.1 **Introduction** . . . 301
28.2 **Integral of function from $\mathbb{R}^m$ to $\mathcal{S}'_n$** . . . 302
28.3 **Integral of functions of $\mathbb{R}^m$ into $\mathcal{L}(\mathcal{S}'_n)$** . . . 305
28.4 **Curves of $\mathcal{S}$-linear operators** . . . 306
28.5 **The Dyson Formula** . . . 310

Part I

Introductions and preliminaries

0.1 Motivations

Throughout the theoretical development of physics and engineering, it was often necessary to introduce some calculus tools out of the rigorous mathematical framework of the time. One of the uncountable classic examples is the differential calculus introduced by Newton and Leibniz and then formalized by Cauchy. Every time this happened, it encouraged some mathematical research, leading to an expansion of this framework in order to incorporate the new tools. Examples of such tools, in recent times, are the “Heaviside Calculus”, the so called “Dirac Calculus” and Feynman’s path-integrals used in Quantum Mechanics. These tools are formally contradictory and incompatible with modern mathematics. However, the results of this informal methods have proved to be exceptionally adequate in their fields of application. The mathematicians have thus engaged a research for new concepts and structures, inside mathematics, in order to give some rigorous support to the informal calculus used by physicists and engineers. A fundamental step in this direction was made in the ’50 by Laurent Schwartz with his Theory of Distributions. The Theory of Distributions gave a precise mathematical meaning to the “generalized functions”, such as the well known Dirac’s delta function, and so to the whole Heaviside calculus. One of the most interesting points of this theory is that it somewhat justifies the procedures used by the physicists and engineers without changing, in the substance, their mode of operation.

However, in the last years, mathematical research has followed a different path, mainly consisting in the application of methods taken from functional analysis. This approach led to many results of great theoretical value but that are of difficult practical application for physicists and engineers. The reasons of these difficulties are essentially two:

- 1) the results are presented in a formal framework which is substantially different from the one which physicists and engineers are accustomed to;

- 2) they hold only under several technical assumptions, which are often extraneous to the essence of the problems.

But the worst of it's that these results are often less powerful than the non rigorous methods used by physicists and engineers!

As a result the gap between the two languages is getting larger and larger.

The goal of the research introduced in this book is to make a step backward and start again where Schwartz left. Following the spirit of the theory of distributions we'll give a precise mathematical meaning and rigorous support to many calculus methods of Quantum Mechanics, starting from Dirac Calculus, under very few and general conditions. Moreover, not only this approach will give a rigorous justification to the use of this tools substantially "as they are", but, by correct interpretation of these methods in terms of new mathematical entities and concepts, it will help to reach a deeper comprehension of the physical structures studied in Quantum Mechanics. In particular, in this direction we shall give a new definition of the states of a quantum system and of the environment in which they lie. This environment is, often erroneously, identified with a Hilbert space, while, because the physicists need to consider so called "non normalizable" states, it is often obviously something different. Some physicists call it a "physical Hilbert space", without giving a clear mathematical definition. We shall show that the "physical Hilbert space" can smoothly be identified with the space of tempered distributions on a suitable Euclidean space, depending from the nature of the quantum system considered.

0.2 Introduction

It was in 1930 that Paul Dirac published his *Principles of Quantum Mechanics*, in this famous treatise Dirac introduced several "manipulation rules" for vectors and operators in a linear space, which, in their complex, constitute the so called "Dirac Calculus". This Calculus is nothing more than a wide set of formal extensions of the basic properties of the finite-dimensional Linear Algebra to the case of infinite-dimensional vector spaces. The discourse is elegant and surprisingly efficient, but it is far from being a rigorous mathematical argumentation.

The roots of the formal Linear Algebra introduced by Dirac can be found in the symbolic calculus of the engineer Heaviside, and the Linear Algebra of Dirac works good in Quantum Theory as the Analysis of Heaviside works good in the Electromagnetic Theory. It is, then, not amazing that the complete justification of the Dirac's Algebra resides in the *topological-vector structures of the spaces of distributions* as the complete justification of the Heaviside's Analysis lies in the meaningful analysis introduced by Laurent Schwartz in those spaces.

The operation of continuous-superposition is the right tool which allows us to build - in a mathematically rigorous way - the extended Linear Algebra of Dirac in the spaces of distributions, via their natural topological-linear structures. More precisely, the goal we reach is in the following direction: we shall see that the natural algebraic-topological structure of those spaces allows us to define a generalization of the finite linear combinations, when the sets indexing the families of vectors are continuous sets, even in the case in which the systems of coefficients has a continuous-infinity of terms different from zero. Moreover, beside the reconstruction of the Dirac's Calculus, we reread some classic theorems of Functional Analysis in terms of the new extended linear algebra.

Chapter 1

Preliminaries

In this book we shall use some notations. If k is a natural number, $\mathbb{N}_{\leq k}$ is the set of positive integer less than or equal to k. The symbol μ_n shall denote the Lebesgue measure on $\mathbb{R}^n$; $(.) = j_{\mathbb{R}}$ is the canonical immersion of the real line $\mathbb{R}$ into the complex plane $\mathbb{C}$; if X is a non-empty set, $\mathbb{I}_X$, of $(.)_X$, or id_X is the identity map on X.

If X and Y are two topological vector spaces on $\mathbb{K}$ (one of the two fields $\mathbb{R}$ or $\mathbb{C}$), by $\mathrm{Hom}(X,Y)$ we denote the set of all the linear operators from X to Y, $\mathcal{L}(X,Y)$ is the set of all the linear and continuous operators from X into Y, $X^* := \mathrm{Hom}\,(X,\mathbb{K})$ is the algebraic dual of the topological vector space X and $X' = \mathcal{L}\,(X,\mathbb{K})$ is the topological dual of X.

We shall use sometimes the following consequence of the Hahn-Banach theorem.

Proposition. *Let E be a locally convex topological vector space, M a closed subspace of the space E, and let z be a point of the space E which does not belong to the subspace M. Then there exists a continuous linear form f on the space E separating the point z and the subspace M. In other terms there is an element f of the topological dual E' such that $f\,(z) = 1$ and $f\,(x) = 0$ for any point x of the subspace M. Or again, in a more geometric fashion, there is a closed linear hyperplane containing the subspace and not containing the point.*

1.1 Topological homomorphisms

Definition (of topological isomorphism). *A bijective continuous linear map f from a topological vector space E onto a topological vector space F is called a* ***topological isomorphism*** *if the inverse map f^{-1} is continuous (i.e., if f is a homeomorphism).*

In other terms, a mapping among two topological vector spaces is said a topological isomorphism if and only if it is both a linear isomorphism and a homeomorphism.

Two topological vector spaces E and F are said *isomorphic* if there exists a topological isomorphism from E onto F. A topological isomorphism from E onto itself is called a topological *automorphism*.

Definition (of strict injective morphism). *An injective continuous map f from a topological vector space E into a topological vector space F is called a* ***strict injective morphism*** *(or* ***topological monomorphism****) if it is a topological isomorphism from the topological vector space E onto the image $f(E)$ endowed with the topology induced by the topological vector space F.*

Definition (of topological homomorphism). *A continuous linear map f from a topological vector space E into a topological vector space F is said a* ***strict morphism*** *(or* ***topological homomorphism****) iff its associated injection*

$$\overline{f} : E/\ker(f) \to F : x + \ker f \mapsto f(x)$$

is a strict injective morphism.

Theorem. *Let f be a continuous linear map from a topological vector space E into a topological vector space F. Then the following assertions are equivalent*

1) *the linear map f is a strict morphism;*
2) *the linear map f maps every neighborhood of the origin 0_E of E onto a neighborhood of the origin 0_F in the image $f(E)$, with respect to the topology induced on $f(E)$ from that of F;*
3) *the linear map f maps every open set of E onto an open set of the image $f(E)$, open with respect to the topology induced on the image $f(E)$ from the topology of F (the linear map is open from E to $f(E)$).*

Note that, topological homomorphism does not mean that the linear mapping f is open (sending open sets of E onto open sets of F) but that it is open from E into its image $f(E)$.

1.2 Direct sums

Definition (of topological direct sum). *Let E be a topological vector space and $(M_i)_{i=1}^n$ be a finite family of subspaces of E such that the space is algebraic direct sum of the family, i.e. such that*

$$E = \oplus_{i=1}^n M_i.$$

*We say that the space E is **the topological direct sum of the family** M if the algebraic isomorphism*

$$(x_i)_{i=1}^n \mapsto \sum_{i=1}^n x_i$$

is a homeomorphism from the product space

$$\prod_{i=1}^n M_i$$

onto the space E (i.e., it is an isomorphism of the topological vector space structures).

To understand that this definition is meaningful, let us observe that the map

$$x = (x_i)_{i=1}^n \mapsto \sum_{i=1}^n x_i$$

from the product $\prod_{i=1}^n M_i$ onto the space E is always continuous, by the Axiom of continuity of the addition of topological vector spaces, but the inverse map may fail to be continuous.

Proposition. *Let E be a topological vector space and assume that E is the algebraic direct sum of the finite family $M = (M_i)_{i=1}^n$ of subspaces. Then the space E is the topological direct sum of the family M if and only if all the projectors*

$$p_j : E \to M_j : p_j \left(\sum_{i=1}^n x_i \right) = x_j$$

are continuous.

Proof. The inverse of bijection

$$\prod_{i=1}^n M_i \to E : (x_i)_{i=1}^n \mapsto \sum_{i=1}^n x_i$$

is given by the map

$$E \to \prod_{i=1}^n M_i : y \mapsto (p_i (y))_{i=1}^n$$

which is continuous if and only if its component p_i are continuous. ■

1.3 Topological supplements

Definition (of topological supplement). *A linear subspace N of a topological vector space E is said* ***a topological supplement of a linear subspace*** *M iff E is the topological direct sum of M and N. In other terms, if the two subspace are algebraic supplement and the two projectors*

$$p_1 : E \to M : p_1(m+n) = m$$

and

$$p_2 : E \to N : p_1(m+n) = n$$

are continuous.

Theorem. *Let M and N be two algebraic supplements in the topological vector space E, and let $q : E \to N$ be the projector on N corresponding to the decomposition (M, N). Then M and N are topological supplements if and only if the injection*

$$\overline{q} : E/M \to N : x + M \mapsto q(x)$$

associated with q is continuous.

Proof. Indeed, consider the canonical injection $j_N : N \to E$ of N into E and the canonical surjection $\pi : E \to E/M$ of E onto the quotient space E/M, we have

$$q = \overline{q} \circ \pi,$$

in fact

$$x \mapsto^{\pi} \; x + M \mapsto^{\overline{q}} \; q(x),$$

and therefore q is continuous if and only if $\overline{q}$ is. ■

Let us observe that the injection $\overline{q}$ is a bijective linear map (indeed q is surjective) and its inverse $\overline{q}^{-1} = \pi \circ j_N$ is always continuous; hence if $\overline{q}$ is continuous, it is an isomorphism.

It follows that

Theorem. *If M and N are two closed subspace of a Banach space E which are algebraic supplements of each other, then they are also topological supplements.*

Proof. Indeed the bijection $\overline{q}^{-1} : N \to E/M$ is then a continuous bijective linear map from the Banach space N onto the Banach space E/M, and therefore by the Banach Inverse Operator Theorem its inverse $\overline{q}$ is also continuous. ■

The above theorem still holds for a larger class of spaces (e.g., complete metrizable spaces).

1.4 Right and Left inverses

Proposition. *Let E and F be two topological vector spaces and let A be a continuous linear map from E into F. Then, there exists a continuous linear map R from F into E such that $A \circ R$ is the identity map $\mathbb{I}_F$ of F if and only if the linear operator A is a surjective strict morphism and its kernel* $\ker(A)$ *has a topological supplement in E. Namely, if there exists a continuous right inverse R of the surjection A, a topological supplement of the kernel of A is the image of the right inverse R.*

Proof. Indeed, if R is a continuous right inverse of A, the image $R(F)$ is a topological supplement of $\ker(A)$. Conversely, if M is a topological supplement of $\ker A$, the restriction of A to the subspace M is an isomorphism of M onto F whose inverse is a continuous right inverse of A. ∎

Proposition. *Let E and F be two topological vector spaces and A a continuous linear map from E into F. There exists a continuous linear map L from F into E such that $L \circ A$ shall be the identity map $\mathbb{I}_E$ of E if and only if A is a injective strict morphism and its image $A(E)$ has a topological supplement in F.*

1.5 Homomorphisms among Fréchet spaces

Theorem (Banach's homomorphism theorem or the open-map theorem). *Let E and F be two metrizable complete topological vector spaces and f a continuous surjective linear map from E onto F. Then f is a strict surjective morphism.*

Proposition. *Let E and F be two metrizable and complete topological vector spaces and $f : E \to F$ a continuous linear map. Then f is a strict morphism if and only if its image $f(E)$ is closed in F.*

Proposition. *Let E and F be two metrizable and complete topological vector spaces. Then every continuous bijective linear map from E onto F is a topological isomorphism.*

Proposition. *Let E be a metrizable and complete topological vector spaces. If M and N are two closed subspace of E which are algebraic supplements of each other, then they are also topological supplements.*

Proof. The product space $M \times N$ is metrizable and complete, and the map $(x, y) \mapsto x + y$ is continuous, bijective and linear from $M \times N$ onto E, hence an isomorphism by the open mapping theorem. ∎

1.6 Dieudonné-Schwartz theorem

Theorem (Dieudonné-Schwartz). *Let E and F be two Fréchet spaces with topologies $\mathcal{T}_E$ and $\mathcal{T}_F$ respectively, E' and F' their topological duals, and let $u : E \to F$ be a linear and continuous map. Then the following conditions are equivalent:*

1) *the operator u is a topological homomorphism for the topologies $\mathcal{T}_E$ and $\mathcal{T}_F$;*

2) *the operator u is a weak topological homomorphism, that is a topological homomorphism for the weak topologies $\sigma(E, E')$ and $\sigma(F, F')$;*

3) *the image $u(E)$ is closed in F;*

4) *the transpose operator ${}^t u$ is a weakly* strict morphism, that is a topological homomorphism for the weak* topologies $\sigma(F', F)$ and $\sigma(E', E)$;*

5) *the image ${}^t u(F')$ is closed in the dual E' for the weak* topology $\sigma(E', E)$.*

Corollary. *Let E and F be two Fréchet spaces, E' and F' their topological duals, and $u : E \to F$ be a linear continuous map. Then,*

1) *the operator u is an injective strict morphism if and only if its transpose operator is surjective, i.e. if and only if*
$$ {}^t u(F') = E'; $$

2) *the operator u is a surjective strict morphism if and only if the image ${}^t u(F')$ is closed in E' for the weak* topology $\sigma(E', E)$ and the transpose ${}^t u$ is injective;*

3) *the operator u is a topological isomorphism if and only if its transpose operator ${}^t u$ is an isomorphism for the weak* topologies $\sigma(F', F)$ and $\sigma(E', E)$.*

Another useful theorem for us is

Theorem. *Let E and F be two Fréchet spaces, assume E be also a Schwartz space and F be reflexive, let E' and F' be their topological duals, and $u : E \to F$ be a strict morphism. Then, the transpose of u is a strict morphism for the strong* topologies, the β topologies.*

Theorem (Dieudonné-Schwartz). *Let E and F be two Fréchet spaces with topologies $\mathcal{T}_E$ and $\mathcal{T}_F$ respectively, E' and F' their topological duals, and let $u : E \to F$ be a linear and continuous map. Assume that the transpose operator ${}^t u$ is an injective strict morphism for the strong* topologies $\beta(F', F)$ and $\beta(E', E)$. Then the following conditions hold true and are equivalent:*

1) *the operator u is a surjective strict morphism for the topologies $\mathcal{T}_E$ and $\mathcal{T}_F$;*

2) *the operator u is a surjective strict morphism for the weak topologies $\sigma(E, E')$ and $\sigma(F, F')$;*

3) *the transpose operator ${}^t u$ is an injective strict morphism for the strong topologies $\sigma(F', F)$ and $\sigma(E', E)$;*

4) *the image ${}^t u(F')$ is closed in the dual E' for the weak* topology $\sigma(E', E)$.*

5) *the image ${}^t u(F')$ is closed in the dual E' for the weak* topology $\beta(E', E)$.*

1.7 Banach-Steinhaus in barreled spaces

Theorem. *Let E be a barreled space, F a locally convex Hausdorff topological vector space, and $\mathcal{F}$ a filter on $\mathcal{L}(E, F)$ which converges pointwise in E to a linear map u_0 of E into F. Suppose that the filter $\mathcal{F}$ has either one of the following two properties:*

1) *there is a subset H of $\mathcal{L}(E, F)$, belonging to the filter $\mathcal{F}$, which is bounded for the topology of pointwise convergence;*

2) *the filter $\mathcal{F}$ has a countable basis.*

Then the operator u_0 is a continuous linear map of E into F and moreover the filter $\mathcal{F}$ converges to u_0 in the topological vector space $\mathcal{L}_c(E, F)$ (i.e., uniformly on every compact subset of E).

Proof. Suppose that (1) holds. Then the subset H is an equicontinuous set (since a bounded set for the pointwise topology in $\mathcal{L}(E, F)$ is equicontinuous) and the operator u_0 belongs to the closure $\mathrm{cl}_s(H)$ of H in the space $\mathcal{F}_s(E, F)$ of functions from E into F endowed with the pointwise topology (indeed a limit of a filter is also an adherent point of the filter). But the closure $\mathrm{cl}_s(H)$ is an equicontinuous set of linear maps of E into F (since the closure of any equicontinuous set is equicontinuous), hence u_0 is continuous and the filter $\mathcal{F}$ converges to u_0 in the space $\mathcal{L}_s(E, F)$. Now, on an equicontinuous set of linear maps the topology of pointwise convergence coincides with the topology of compact convergence, so the filter $\mathcal{F}$ converges to the operator u_0 in the space $\mathcal{L}_c(E, F)$ (as the set H belongs to the filter $\mathcal{F}$, to say that $\mathcal{F}$ converges to u_0 in the space $\mathcal{L}_c(E, F)$ or that the filter induced by $\mathcal{F}$ on the closure $\mathrm{cl}_s(H)$ converges to u_0 in $\mathrm{cl}_s(H)$ when this set carries the topology of compact convergence, is one and the same thing). Next we suppose that (2) holds. Let $B = (B_k)_{k\in\mathbb{N}}$ be a countable (ordered) basis of the filter $\mathcal{F}$. For each natural k we select an element u_k of B_k.

By hypothesis, for each point x of the space E, the sequence $u(x) = (u_k(x))_{k\in\mathbb{N}}$ converges in the topological vector space F to the point $u_0(x)$. This implies that the set of continuous mappings $\{u_k\}_{k\in\mathbb{N}}$ is bounded in the space $\mathcal{L}_s(E,F)$. Therefore, the filter associated with sequence u has property (1). From the first part of the proof, it follows that u_0 is continuous and that the sequence u converges to the operator u_0 in the space $\mathcal{L}_c(E,F)$. We have now to prove that the filter $\mathcal{F}$ converges to the map u_0 in the space $\mathcal{L}_c(E,F)$, i.e. that any neighborhood of u_0 in $\mathcal{L}_c(E,F)$ should contain a base element. Let, then, U be a neighborhood of the operator u_0 in the space $\mathcal{L}_c(E,F)$: suppose that none of the base sets B_k is contained in the neighborhood U. Then we could find, for each natural k, an element u_k of B_k which is not contained in the neighborhood U. But this would contradict the fact (just proved) that any such sequence u converges to the linear map u_0 in the space $\mathcal{L}_c(E,F)$. Therefore, some set B_k must be contained in the neighborhood U. ■

1.8 Tempered distributions

1.8.1 Test functions

By $\mathcal{S}_n$ we shall denote the space $\mathcal{S}(\mathbb{R}^n, \ \mathbb{K})$, the $(n,\mathbb{K})$-*Schwartz space*, that is to say, the set of all the smooth functions (i.e., of class C^∞) of $\mathbb{R}^n$ into $\mathbb{K}$ *rapidly decreasing at infinity with all their derivatives* (these functions and all their derivatives tend to 0 at $\mp\infty$ faster than the reciprocal of any polynomial).

By $\mathcal{S}_{(n)}$ we shall denote the *standard Schwartz topology* on $\mathcal{S}_n$, and by $(\mathcal{S}_n)$ the topological vector space on the set $\mathcal{S}_n$ with its standard topology. The topology $\mathcal{S}_{(n)}$ is induced by a metric, in fact ($\mathcal{S}_n$ is closed under differentiation and multiplication by polynomials) it is induced by the denumerable family of seminorms $p = (p_k)_{k\in\mathbb{N}_0}$ on $\mathcal{S}_n$ defined by

$$p_k(f) = \sup_{x\in\mathbb{R}^n}\max_{\alpha,\beta\in\mathbb{N}_0^n(\leq k)}\left|x^\beta D^\alpha f(x)\right|,$$

where $\mathbb{N}_0^n(\leq k)$ is the set of all n-dimensional multi-indices with length less or equal to k, for every non-negative integer k. Each seminorm p_k is indeed a norm on the space $\mathcal{S}_n$, and moreover the inequality $p_k(f) \leq p_{k+1}(f)$, for all $f \in \mathcal{S}_n$, holds true. So the pair $(\mathcal{S}_n, p)$ is a countably complete normed space and consequently the topological vector space $(\mathcal{S}_n)$ is a Fréchet space (see also $[Ho]$ and $[Ba]$). The topological vector space $(\mathcal{S}_n)$ is also reflexive, barreled and a Montel space.

1.8.2 Tempered distributions

By $\mathcal{S}'_n$ we shall denote the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n, \mathbb{K})$ from $\mathbb{R}^n$ to $\mathbb{K}$, that is, the topological dual of the topological vector space $(\mathcal{S}_n)$, i.e., $\mathcal{S}'_n = (\mathcal{S}_n)'$. If $x \in \mathbb{R}^n$, δ_x is the *Dirac distribution of* $\mathcal{S}_n$ *centered at the point* x, i.e., the functional

$$\delta_x : \mathcal{S}_n \to \mathbb{K} : \phi \mapsto \phi\,(x)\,.$$

If f belongs to the space $\mathcal{O}_M(\mathbb{R}^n, \mathbb{K})$, where

$$\mathcal{O}_M(\mathbb{R}^n, \mathbb{K}) = \{g \in C^\infty(\mathbb{R}^n, \mathbb{K}) : \forall \phi \in \mathcal{S}_n,\ \phi g \in \mathcal{S}_n\}\,,$$

then the functional

$$[f] = [f]_n : \mathcal{S}_n \to \mathbb{K} : \phi \mapsto \int_{\mathbb{R}^n} f\phi\mu_n$$

is a tempered distribution, called the *regular (tempered) distribution generated by the function* f (see $[Ba]$ page 110). The space $\mathcal{S}'_n$ is reflexive, barreled and a Montel space.

1.9 Fourier transforms on $\mathcal{S}_n$

Let $a, b \in \mathbb{R}_{\neq}$ be two non zero real numbers (by $\mathbb{R}_{\neq}$ we mean the difference $\mathbb{R}\backslash\{0\}$). By $\mathcal{S}_{(a,b)}$ we denote the (a,b)-*Fourier Schwartz transformation*, i.e., the operator $\mathcal{S}_{(a,b)} : \mathcal{S}_n \to \mathcal{S}_n$, defined, for all function $f \in \mathcal{S}_n$ and any point $y \in \mathbb{R}^n$, by

$$\mathcal{S}_{(a,b)}(f)(y) = (1/a)^n \int_{\mathbb{R}^n} f e^{-ib(\cdot|y)}\ \mu_n = \left[(1/a)^n\, e^{-ib(\cdot|y)}\right](f),$$

where $(\cdot|\cdot)$ is the standard scalar product on $\mathbb{R}^n$ and μ_n is the Lebesgue measure on $\mathbb{R}^n$. Moreover, we recall that $\mathcal{S}_{(a,b)}$ is a homeomorphism with respect to the standard topology of $(\mathcal{S}_n)$ and, concerning its inverse, for every $x \in \mathbb{R}^n$ and $g \in \mathcal{S}_n$, we have

$$\begin{aligned} \mathcal{S}^-_{(a,b)}(g)(x) &= \left(\frac{|b|\,a}{2\pi}\right)^n \int_{\mathbb{R}^n} g e^{ib(x|\cdot)}\ \mu_n = \\ &= \mathcal{S}_{(2\pi/(|b|a),-b)}(g)(x). \end{aligned}$$

1.10 Fourier transforms on $\mathcal{S}'_n$

Let $a, b \in \mathbb{R}^{\neq}$, by $\mathcal{F}_{(a,b)}$ we shall denote the (a,b)-*Fourier transformation on the space of tempered distributions*, i.e., the operator $\mathcal{F}_{(a,b)} : \mathcal{S}'_n \to \mathcal{S}'_n$, defined, for all distribution $u \in \mathcal{S}'_n$ and for every test function $\phi \in \mathcal{S}_n$, by

$$\mathcal{F}_{(a,b)}(u)(\phi) = u\left(\mathcal{S}_{(a,b)}(\phi)\right),$$

in other terms it is the transpose of the operator $\mathcal{S}_{(a,b)}$:

$$\mathcal{F}_{(a,b)} = {}^t\left(\mathcal{S}_{(a,b)}\right).$$

Moreover, we recall that $\mathcal{F}_{(a,b)}$ is a homeomorphism in the weak* topology $\sigma(\mathcal{S}'_n)$ (even more it is a topological isomorphism). Moreover, we have

$$\mathcal{F}^{-}_{(a,b)} = \mathcal{F}_{(2\pi/(|b|a),-b)}.$$

Two properties that we shall use are the following ones: for all $\alpha \in \mathbb{N}_0^n$,

$$\mathcal{F}_{(a,b)}(u^{(\alpha)}) = (bi)^{\alpha} \left(\mathbb{I}_{\mathbb{R}^n}\right)^{\alpha} \mathcal{F}_{(a,b)}(u);$$

and

$$\mathcal{F}_{(a,b)}(\left(\mathbb{I}_{\mathbb{R}^n}\right)^{\alpha} u) = \left(\frac{i}{b}\right)^{\alpha} \left(\mathcal{F}_{(a,b)}(u)\right)^{(\alpha)},$$

where, $\mathbb{I}_{\mathbb{R}^n}$ is (as we said) the identity operator on $\mathbb{R}^n$, and where $\left(\mathbb{I}_{\mathbb{R}^n}\right)^{\alpha}$ is the α-th power of the identity in multi-indexed notation.

Moreover, we have

$$\mathcal{F}_{(a,b)}\left(\tau_h u\right) = e^{-ib(h|\cdot)} \mathcal{F}_{(a,b)}(u)$$

under Fourier transforms translations become multiplications by characters, and

$$\mathcal{F}_{(a,b)}\left(e^{ib(h|\cdot)} u\right) = \tau_h\left(\mathcal{F}_{(a,b)}(u)\right)$$

under Fourier transforms multiplications by characters become translations.

For example we have

$$\mathcal{F}_{(1,2\pi)}\left(H_0\right) = \frac{1}{2}(\delta_0 - \frac{i}{\pi}\mathcal{P}(\mathbb{I}^{-1}_{(\mathbb{R},\mathbb{C})}),$$

which implies

$$\begin{aligned} \mathcal{F}_{(1,2\pi)}\left(H_x\right) &= \frac{1}{2}(e^{-i2\pi(x|\cdot)}\delta_0 - \frac{i}{\pi}e^{-i2\pi(x|\cdot)}\mathcal{P}(\mathbb{I}^{-1}_{(\mathbb{R},\mathbb{C})})) = \\ &= \frac{1}{2}(\delta_0 - \frac{i}{\pi}e^{-i2\pi(x|\cdot)}\mathcal{P}(\mathbb{I}^{-1}_{(\mathbb{R},\mathbb{C})})). \end{aligned}$$

Part II

Superpositions

Chapter 2

Summable families

2.1 Families of distributions

Let I be a non-empty set, we shall denote by $(\mathcal{S}'_n)^I$ the space of all the families in the space of tempered distributions $\mathcal{S}'_n$ indexed by the set I, i.e., the set of all the surjective maps from the set I onto a subset of the space $\mathcal{S}'_n$. Moreover, as usual, if v is one of these families, for each index $p \in I$, the distribution $v(p)$ (corresponding to p in the map v) is denoted by v_p, and the family v itself is also denoted by the expressive notation $(v_p)_{p\in I}$.

The set $(\mathcal{S}'_n)^I$ of all families in the space $\mathcal{S}'_n$ indexed by a non-empty set I is a vector space with respect to the following two standard operations of addition

$$+ : (\mathcal{S}'_n)^I \times (\mathcal{S}'_n)^I \to (\mathcal{S}'_n)^I,$$

defined pointwise by

$$v + w := (v_p + w_p)_{p\in I},$$

for any two families v, w, and multiplication by scalars

$$\cdot\ : \mathbb{K} \times (\mathcal{S}'_n)^I \to (\mathcal{S}'_n)^I$$

defined pointwise by

$$av := (av_p)_{p\in I},$$

for any family v and any scalar a. In other words, the family $v + w$ is defined by

$$(v + w)_p = v_p + w_p,$$

for every index p in I, and the family av is defined by

$$(av)_p = av_p,$$

for every p in I.

The basic important consideration for our purposes is the observation that a family of tempered distributions can act on test functions, as specifies the following definition.

Definition (image of a test function by a family of distributions). *Let v be a family in the space $\mathcal{S}'_n$ indexed by a non-empty set I and let $\phi \in \mathcal{S}_n$ be any test function. The mapping*

$$v(\phi) : I \to \mathbb{K}$$

defined by

$$v(\phi)(p) := v_p(\phi),$$

*for each index $p \in I$, is called the **image of the test function ϕ under the family of tempered distributions** v.*

So, to any family v belonging to the space $(\mathcal{S}'_n)^I$ we can associate a mapping from the space of test functions $\mathcal{S}_n$ into the function space $\mathcal{F}(I, \mathbb{K})$. Equivalently, for every test function ϕ, we have a "projection" π_ϕ sending any family of $(\mathcal{S}'_n)^I$ to a scalar family of the product $(\mathbb{K})^I$:

$$\pi_\phi(v) = (v_p(\phi))_{p\in I},$$

for every family v of the space $(\mathcal{S}'_n)^I$.

2.2 $^{\mathcal{S}}$Families

In the Theory of Superpositions on the space of tempered distributions $\mathcal{S}'_n$ the below class of $^{\mathcal{S}}$families plays a basic role.

Definition (family of tempered distributions of class $\mathcal{S}$). *Let v be a family in the space $\mathcal{S}'_n$ indexed by the Euclidean space $\mathbb{R}^m$. The family v is called a **family of class $\mathcal{S}$** or an $^{\mathcal{S}}$**family** if, for each test function $\phi \in \mathcal{S}_n$, the* *image of the test function ϕ by the family v - that is the function $v(\phi) : \mathbb{R}^m \to \mathbb{K}$ defined by*

$$v(\phi)(p) := v_p(\phi),$$

for each index $p \in \mathbb{R}^m$ - belongs to the space of test functions $\mathcal{S}_m$. We shall denote the set of all $^{\mathcal{S}}$families by $\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$.

Example (the Dirac family in $\mathcal{S}'_n$). *The **Dirac family in** $\mathcal{S}'_n$, i.e., the family $\delta := (\delta_x)_{x\in\mathbb{R}^n}$, where δ_x is the Dirac (tempered) distribution centered at the point x of $\mathbb{R}^n$, is a family of class $\mathcal{S}$.*

Proof. Indeed, for each test function $\phi \in \mathcal{S}_n$ and for each index (point) x in $\mathbb{R}^n$, we have

$$\delta(\phi)(x) = \delta_x(\phi) = \phi(x),$$

and hence $\delta(\phi) = \phi$. So the image of the test function ϕ under the family δ is the function ϕ itself, which lies in $\mathcal{S}_n$. ■

It is clear that the space of $^{\mathcal{S}}$families in $\mathcal{S}'_n$, indexed by some Euclidean space I, is a subspace of the vector space $(\mathcal{S}'_n)^I$ of all families in $\mathcal{S}'_n$ indexed by the same index set I.

2.3 $^{\mathcal{S}}$Family generated by an operator

In this section we introduce a wide class of $^{\mathcal{S}}$families. We will see later that this class is indeed the entire class of $^{\mathcal{S}}$ families. We recall that by $\sigma(\mathcal{S}_n)$ we shall denote the weak topology $\sigma(\mathcal{S}_n, \mathcal{S}'_n)$.

Theorem (on the $^{\mathcal{S}}$family generated by a linear and continuous operator). *Let $A : \mathcal{S}_n \to \mathcal{S}_m$ be a linear and continuous operator with respect to the natural topologies of $\mathcal{S}_n$ and $\mathcal{S}_m$ (or equivalently, continuous with respect to the weak topologies $\sigma(\mathcal{S}_n)$ and $\sigma(\mathcal{S}_m)$) and let δ be the Dirac family in $\mathcal{S}'_m$. Then, the family of functionals*

$$A^\vee := (\delta_p \circ A)_{p\in\mathbb{R}^m}$$

is a family of distribution and it is an $^{\mathcal{S}}$family.

Proof. Let $A : \mathcal{S}_n \to \mathcal{S}_m$ be a linear and continuous operator with respect to the natural topologies of $\mathcal{S}_n$ and $\mathcal{S}_m$ (since these topologies are Fréchet-topologies, this is equivalent to assume the operator A be linear and continuous with respect to the weak topologies $\sigma(\mathcal{S}_n)$ and $\sigma(\mathcal{S}_m)$). Let δ be the Dirac family in $\mathcal{S}'_m$ and consider the family

$$A^\vee := (\delta_p \circ A)_{p\in\mathbb{R}^m}.$$

The family $A^\vee$ is a family in $\mathcal{S}'_n$, since each functional $A^\vee_p$ is the composition of two linear and continuous mappings. Moreover, the family $A^\vee$ is of class $\mathcal{S}$, in fact, for every test function ϕ in $\mathcal{S}_n$ and for every index p in $\mathbb{R}^m$, we have

$$\begin{aligned} A^\vee(\phi)(p) &= A^\vee_p(\phi) = \\ &= (\delta_p \circ A)(\phi) = \\ &= \delta_p(A(\phi)) = \\ &= A(\phi)(p), \end{aligned}$$

so that the image of the test function by the family $A^\vee$ is nothing but the image of the test function under the operator A,

$$A^\vee(\phi) = A(\phi),$$

and this image belongs to the space $\mathcal{S}_m$ by the choice of the operator A itself. ■

Definition (of $^{\mathcal{S}}$family generated by a linear and continuous operator). *Let $A : \mathcal{S}_n \to \mathcal{S}_m$ be a linear and continuous operator with respect to the natural topologies of $\mathcal{S}_n$ and $\mathcal{S}_m$ (or equivalently, continuous with respect to the weak topologies $\sigma(\mathcal{S}_n)$ and $\sigma(\mathcal{S}_m)$) and let δ be the Dirac family in $\mathcal{S}'_m$. The family*

$$A^\vee := (\delta_p \circ A)_{p\in\mathbb{R}^m}$$

is called the $\mathcal{S}$-family generated by the operator A.

Remark. We have so constructed the mapping

$$(\cdot)^\vee : \mathcal{L}(\mathcal{S}_n, \mathcal{S}_m) \to \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n) : A \mapsto (\delta_x \circ A)_{x\in\mathbb{R}^m},$$

which we shall call *the canonical representation of the operator space $\mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$ in the family space $\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$*. It is quite simple to prove that this mapping is a linear injection. We shall see, as we already said, that every $^{\mathcal{S}}$family has the form considered above, or in other terms that the above linear mapping is a linear isomorphism.

2.4 The operator generated by an $^{\mathcal{S}}$family

Definition (operator generated by an $^{\mathcal{S}}$family). *Let v be a family of class $\mathcal{S}$ belonging to the space $\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$. We call* ***operator generated by the family*** *v (or* ***associated with the family*** *v) the operator*

$$\widehat{v} : \mathcal{S}_n \to \mathcal{S}_m : \phi \mapsto v(\phi),$$

sending every test function ϕ of $\mathcal{S}_n$ into its image $v(\phi)$ under the family v.

Example (on the Dirac family). *The operator (on $\mathcal{S}_n$) generated by the Dirac family, i.e., by the family $\delta = (\delta_y)_{y\in\mathbb{R}^n}$, is the identity operator on $\mathcal{S}_n$.*

Proof. In fact, for each $y \in \mathbb{R}^n$, we have

$$\begin{aligned} \widehat{\delta}(\phi)(y) &= \delta_y(\phi) = \\ &= \phi(y) = \\ &= \mathbb{I}_{\mathcal{S}_n}(\phi)(y), \end{aligned}$$

for any test function ϕ in $\mathcal{S}_n$. ■

We recall that the set $\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$, of $^{\mathcal{S}}$families indexed by $\mathbb{R}^m$, is a subspace of the vector space $(\mathcal{S}'_n)^{\mathbb{R}^m}$. Moreover, we leave as an exercise to prove that

- *for each family $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$, the operator $\widehat{v}$ associated with the family v is linear and the map*

  $$\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n) \to \mathrm{Hom}\,(\mathcal{S}_n, \mathcal{S}_m) : v \mapsto \widehat{v}$$

  is an injective linear operator.

Example (on the family generated by an operator). The operator associated with the family $A^\vee$ generated by a linear and continuous operator A in $\mathcal{L}\,(\mathcal{S}_n, \mathcal{S}_m)$ is the operator A itself, as can be immediately proved. In other terms we can write $(A^\vee)^\wedge = A$.

2.5 Characterizations of $^{\mathcal{S}}$ families

In the following we shall denote by $\mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$ the set of all the linear and continuous operators among the two topological vector spaces $(\mathcal{S}_n)$ and $(\mathcal{S}_m)$.

Moreover, let consider a linear operator $A : \mathcal{S}_n \to \mathcal{S}_m$, we say that A is *(topologically) transposable* if its algebraic transpose (adjoint) ${}^*A : \mathcal{S}^*_m \to \mathcal{S}^*_n$ (X^* denotes the algebraic dual of a topological vector space X), defined by

$${}^*A(a) = a \circ A,$$

maps the distribution space $\mathcal{S}'_m$ into the distribution space $\mathcal{S}'_n$.

Theorem (basic properties of $^{\mathcal{S}}$families). *Let v be a family of tempered distributions of class $\mathcal{S}$ belonging to the space $\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$. Then, the following assertions hold and are equivalent:*

1) for every tempered distribution $a \in \mathcal{S}'_m$, the composition $u = a \circ \widehat{v}$, i.e., the functional

$$u : \mathcal{S}_n \to \mathbb{K} : \phi \mapsto a(\widehat{v}(\phi)),$$

is a tempered distribution;

2) the operator $\widehat{v}$ is transposable;

3) the operator $\widehat{v}$ is weakly continuous, i.e. continuous from $\mathcal{S}_n$ to $\mathcal{S}_m$ with respect to the pair of weak topologies $(\sigma(\mathcal{S}_n), \sigma(\mathcal{S}_m))$;

4) the operator $\widehat{v}$ is continuous from the space $(\mathcal{S}_n)$ to the space $(\mathcal{S}_m)$.

Proof. We divide the proof in two parts, in the first one we prove the validity of the property (1), in the second we prove that the four properties are equivalent. Note that, after the proof of property (1), if we prove that the other properties are equivalent to (1) then we have proved our theorem. **Proof of property (1).** Let us prove (1). Let $a \in \mathcal{S}'_m$ and let δ be the Dirac family in $\mathcal{S}'_m$. Since the linear hull $\mathrm{span}(\delta)$ of the Dirac family is $\sigma(\mathcal{S}'_m)$-sequentially dense in $\mathcal{S}'_m$ (see $[Bo]$ page 205), there is a sequence of distributions $\alpha = (\alpha_k)_{k\in\mathbb{N}}$ in the linear hull $\mathrm{span}(\delta)$ converging to the distribution a with respect to the weak* topology $\sigma(\mathcal{S}'_m)$, that is such that

$$^{\sigma(\mathcal{S}'_m)}\lim_{k\to+\infty} \alpha_k = a.$$

Now, since for any natural k, the distribution α_k belongs to the linear hull $\mathrm{span}(\delta)$, there exists a finite family $(y_i)_{i=1}^h$ of points in $\mathbb{R}^m$ and there is a finite family of points $(\lambda_i)_{i=1}^h$ in $\mathbb{K}$ such that

$$\alpha_k = \sum_{i=1}^h \lambda_i \delta_{y_i},$$

and consequently, by obvious calculations,

$$\begin{aligned} \alpha_k \circ \widehat{v} &= \sum_{i=1}^h \lambda_i (\delta_{y_i} \circ \widehat{v}) = \\ &= \sum_{i=1}^h \lambda_i v_{y_i}. \end{aligned}$$

Hence, for every index $k \in \mathbb{N}$, the linear functional $\alpha_k \circ \widehat{v}$ belongs to the space $\mathcal{S}'_n$. Let s be the topology of the pointwise convergence in the algebraic dual $(\mathcal{S}_n)^*$, we claim that

$$^{s}\lim_{k\to+\infty} (\alpha_k \circ \widehat{v}) = a \circ \widehat{v}.$$

In fact, for every test function ϕ in $\mathcal{S}_n$, we obtain

$$\begin{aligned}\lim_{k\to+\infty}(\alpha_k\circ\widehat{v})(\phi) &= \lim_{k\to+\infty}\alpha_k(\widehat{v}(\phi)) = \\ &= a(\widehat{v}(\phi)),\end{aligned}$$

so we proved that the sequence of continuous linear functionals $(\alpha_k\circ\widehat{v})_{k\in\mathbb{N}}$ is pointwise converging to the linear functional $a\circ\widehat{v}$; so, by the Banach-Steinhaus theorem (that is applicable since $\mathcal{S}_n$ is barreled), the linear functional $a\circ\widehat{v}$ must be continuous too, i.e. $a\circ\widehat{v}\in\mathcal{S}'_n$. So property (1) holds. **Equivalence of the four properties.** The property (1) is equivalent to property (2) by definition of transposable operator. Property (2) is equivalent to property (3) because the space of linear continuous operators $\mathcal{L}((\mathcal{S}_n)_\sigma,(\mathcal{S}_m)_\sigma)$ is also the space of all the transposable linear operators from the space $(\mathcal{S}_n)$ to $(\mathcal{S}_m)$ (see $[Ho]$, chap. 3, § 12, Proposition 1, page 254). Property (3) is equivalent to property (4). In fact, since the space $(\mathcal{S}_n)$ is an $\mathcal{F}$-space (and then its topology coincides with the Mackey topology $\tau(\mathcal{S}_n,\mathcal{S}'_n)$), the space $\mathcal{L}(\mathcal{S}_n,\mathcal{S}_m)$ contains the above space $\mathcal{L}((\mathcal{S}_n)_\sigma,(\mathcal{S}_m)_\sigma)$, of all weakly linear and continuous operators from $\mathcal{S}_n$ to $\mathcal{S}_m$ (i.e. with respect to the pair of topologies $(\sigma(\mathcal{S}_n),\sigma(\mathcal{S}_m))$, see for this result $[Die, Sch]$ page 91, Corollary or $[Ho]$, page 258, Corollary). Moreover, the space $\mathcal{L}(\mathcal{S}_n,\mathcal{S}_m)$ is contained in the space $\mathcal{L}((\mathcal{S}_n)_\sigma,(\mathcal{S}_m)_\sigma)$, since every continuous linear operator among two Hausdorff locally convex topological vector spaces is weakly continuous (see proposition 3, page 256 of $[Ho]$), so the two spaces must coincide. ■

Corollary (of isomorphism). *The vector spaces $\mathcal{S}(\mathbb{R}^m,\mathcal{S}'_n)$ and $\mathcal{L}(\mathcal{S}_n,\mathcal{S}_m)$ are isomorphic. Namely, the map $(\cdot)^\wedge$ from the space of family $\mathcal{S}(\mathbb{R}^m,\mathcal{S}'_n)$ into the space of operators $\mathcal{L}(\mathcal{S}_n,\mathcal{S}_m)$, associating with each family v its operator $\widehat{v}$, is a vector space isomorphism. Moreover, the inverse of the above isomorphism is the linear mapping*

$$(\cdot)^\vee:\mathcal{L}(\mathcal{S}_n,\mathcal{S}_m)\to\mathcal{S}(\mathbb{R}^m,\mathcal{S}'_n)$$

defined by

$$A\mapsto A^\vee:=(\delta_p\circ A)_{p\in\mathbb{R}^m},$$

i.e. the canonical representation of the operator space $\mathcal{L}(\mathcal{S}_n,\mathcal{S}_m)$ into the family space $\mathcal{S}(\mathbb{R}^m,\mathcal{S}'_n)$, which, as a consequence, is an isomorphism too.

Definition (canonical representation of the space $\mathcal{S}(\mathbb{R}^m,\mathcal{S}'_n)$). *The mapping*

$$(\cdot)^\wedge:\mathcal{S}(\mathbb{R}^m,\mathcal{S}'_n)\to\mathcal{L}(\mathcal{S}_n,\mathcal{S}_m): v\mapsto\widehat{v}$$

is called ***the canonical representation of the family space $\mathcal{S}(\mathbb{R}^m,\mathcal{S}'_n)$ onto the operator space $\mathcal{L}(\mathcal{S}_n,\mathcal{S}_m)$.***

2.6 Characterization of transposability

A way to see that an operator is transposable is given by the following characterization. It is an immediate consequence of the characterization of the $^{\mathcal{S}}$families but we want to prove it independently.

Theorem. *Let $A : \mathcal{S}_n \to \mathcal{S}_m$ be a linear operator and let δ be the Dirac family of the space $\mathcal{S}'_m$. Then, the operator A is (topologically) transposable if and only if, for every point $p \in \mathbb{R}^m$, the composition $\delta_p \circ A$ is a tempered distribution in $\mathcal{S}'_n$.*

Proof. ($\Rightarrow$) The necessity of the condition is obvious. In fact, we have

$$\delta_p \circ A = {}^*A(\delta_p),$$

and so if A is topologically transposable, the composition $\delta_p \circ A$ is continuous.
($\Leftarrow$) Let $a \in \mathcal{S}'_m$ be a tempered distribution; we should prove that the composition $a \circ A$ is continuous. Since the linear hull span(δ) is sequentially dense in the space $\mathcal{S}'_m$ (see $[Bo]$ page 205), there is a sequence of distributions $(\alpha_k)_{k\in\mathbb{N}}$ in the hull span(δ) such that

$$^{\sigma(\mathcal{S}'_m)}\lim_{k\to+\infty} \alpha_k = a.$$

Now, since any distribution α_k lives in the hull span(δ) there exist a finite family $(y_i)_{i=1}^h$ in $\mathbb{R}^m$ and a finite sequence $(\lambda_i)_{i=1}^h$ in $\mathbb{K}$ such that

$$\alpha_k = \sum_{i=1}^h \lambda_i \delta_{y_i},$$

thus we have

$$\begin{aligned} \alpha_k \circ A &= \sum_{i=1}^h (\lambda_i \delta_{y_i}) \circ A = \\ &= \sum_{i=1}^h \lambda_i (\delta_{y_i} \circ A); \end{aligned}$$

hence, for every number $k \in \mathbb{N}$, the composition $\alpha_k \circ A$ belongs to $\mathcal{S}'_n$. Let now s be the topology of pointwise convergence in the algebraic dual $\mathcal{S}^*_n$, we have

$$^{s}\lim_{k\to+\infty} (\alpha_k \circ A) = a \circ A,$$

in fact

$$\begin{aligned} \lim_{k\to+\infty} (\alpha_k \circ A)(\phi) &= \lim_{k\to+\infty} \alpha_k (A(\phi)) = \\ &= a(A(\phi)), \end{aligned}$$

so we have that the sequence (in $\mathcal{S}'_n$) of continuous linear form $(\alpha_k \circ A)_{k\in\mathbb{N}}$ converges pointwise to the linear form $a \circ A$, then, by the Banach-Steinhaus theorem, we conclude that the composition $a \circ A$ is also in the space $\mathcal{S}'_n$. ■

2.7 Characterizations of $^{\mathcal{D}}$ families (*)

As we proved the above theorem, in a perfectly analogous way, it can be proved the following theorem. The $^{\mathcal{D}}$families in $\mathcal{D}'_n$ can be defined analogously to the $^{\mathcal{S}}$families in $\mathcal{S}'_n$ and the Corollary of page 91 of $[Die, Sch]$ holds because $\mathcal{D}'_n$ is an $\mathcal{LF}$-space.

Theorem (basic properties on $^{\mathcal{D}}$families). *Let $v \in \mathcal{D}(\mathbb{R}^m, \mathcal{D}'_n)$ be a family of distributions. Then the following assertions hold and are equivalent:*

1) *for every $a \in \mathcal{D}'_m$ the composition $u = a \circ \widehat{v}$, i.e., the functional*
$$u : \mathcal{D}_n \to \mathbb{K} : \phi \mapsto a\left(\widehat{v}(\phi)\right),$$
is a distribution;

2) *the operator $\widehat{v}$ is transposable;*

3) *the operator $\widehat{v}$ is $(\sigma(\mathcal{D}_n), \sigma(\mathcal{D}_m))$-continuous from $\mathcal{D}_n$ to $\mathcal{D}_m$;*

4) *the operator $\widehat{v}$ is a strongly continuous from $(\mathcal{D}_n)$ to $(\mathcal{D}_m)$.*

2.8 ^{E}Families and ^{E}summable families

Let us begin with a family which is not of class $\mathcal{S}$.

Example (a family that is not of class $\mathcal{S}$). Let u be a distribution in $\mathcal{S}'_n$ and let v be the family in $\mathcal{S}'_n$, indexed by the Euclidean space $\mathbb{R}^m$, defined by $v_y = u$, for each point $y \in \mathbb{R}^m$. Then, if the distribution u is different from zero, v is not of class $\mathcal{S}$. In fact, let $\phi \in \mathcal{S}(\mathbb{R}^n, \mathbb{K})$ be such that $u(\phi) \neq 0$, for every $y \in \mathbb{R}^m$, we have
$$\begin{aligned} v(\phi)(y) &= v_y(\phi) = \\ &= u(\phi)1_{\mathbb{R}^m}(y), \end{aligned}$$
where, $1_{\mathbb{R}^m}$ is the constant $\mathbb{K}$-functional on $\mathbb{R}^m$ of value 1. Thus, the function $v(\phi)$ is a constant $\mathbb{K}$-functional on $\mathbb{R}^m$ different from zero, and so it cannot live in the space $\mathcal{S}_m$.

The preceding example induces us to consider other classes of families in addition to the ${}^{\mathcal{S}}$families, for this reason, we will give the following definitions.

We shall denote by $\mathcal{C}_m$ the space $\mathcal{C}^0(\mathbb{R}^m, \mathbb{K})$ of continuous functions defined on the Euclidean space $\mathbb{R}^m$ and with values in the scalar field $\mathbb{K}$.

Definition (algebraic ^{E}families and ^{E} summuble families). *Let E be a subspace of the function space $\mathcal{F}(\mathbb{R}^m, \mathbb{K})$ (without any topology) containing the space $\mathcal{S}_m$. If v is a family in the distribution space $\mathcal{S}'_n$ indexed by $\mathbb{R}^m$, we say that the family v is an ^{E}**family** if, for every test function ϕ in $\mathcal{S}_n$, the image $v(\phi)$ of the test function by the family v lies in the subspace E. An ^{E}family v is said to be ^{E}**summable** if, for every functional a in the algebraic dual E^*, the functional*

$$\int_{\mathbb{R}^m} av : \phi \mapsto a(v(\phi))$$

is a tempered distribution in $\mathcal{S}'_n$. More generally, if F is a part of E^* we say that the family is ^{F}summable if the above functional is a distribution for every a in F.

With this new definition, the family of the above example is a ${}^{\mathcal{E}_m}$family in $\mathcal{S}'_n$, where by $\mathcal{E}_m$ we (in standard way) denote the space $C^\infty(\mathbb{R}^m, \mathbb{K})$ of smooth function from $\mathbb{R}^m$ into $\mathbb{K}$. Moreover, for every tempered distribution in $\mathcal{E}'_m$, we have

$$\begin{aligned} a(v(\phi)) &= a(u(\phi)1_{\mathbb{R}^m}) = \\ &= u(\phi)a(1_{\mathbb{R}^m}) = \\ &= u(\phi)\int_{\mathbb{R}^m} a, \end{aligned}$$

where we recall that the compact support distributions are integrable and their integral is defined as their value on the constant unit functional, so that

$$\int_{\mathbb{R}^m} av = \left(\int_{\mathbb{R}^m} a\right) u,$$

and the family v is ${}^{\mathcal{E}'_m}$summable.

Remark. In the conditions of the above definition, let w be a Hausdorff locally convex topology on the subspace E.

- If the topological vector space $(\mathcal{S}_m)$ is continuously imbedded in the space E_w, then, the topological dual E'_w is continuously imbedded in the space $\mathcal{S}'_m$. In this case, in Distribution Theory, we say that the dual E'_w is a space of tempered distribution on $\mathbb{R}^m$.

- Moreover, if the topological vector space E_w is continuously imbedded in the space $(\mathcal{C}_m)$, then the dual $\mathcal{C}'_m$ is contained in the dual E'_w. In other terms, every Radon measure with compact support is in the dual E'_w and, in particular, the Dirac family is contained in E'_w; since the Dirac basis in sequentially total in the space of tempered distributions $\mathcal{S}'_m$ and since the space E'_w is continuously imbedded in the space $\mathcal{S}'_m$ itself, the Dirac family shall be sequentially total also in the topological vector space $(E'_w)_\sigma$, that is with respect to the weak* topology $\sigma(E', E)$.

Now we can give two new definitions.

Definition (^{E}families and ^{E}summuble families). *Let E be a subspace of the space $\mathcal{F}(\mathbb{R}^m, \mathbb{K})$ containing the space $\mathcal{S}_m$ and endowed with a locally convex linear topology. If v is a family in the distribution space $\mathcal{S}'_n$ indexed by $\mathbb{R}^m$. We say that the family v is an ^{E}**family** if, for every test function ϕ in $\mathcal{S}_n$, the image $v(\phi)$ of the test function by the family v lies in the subspace E. An ^{E}family is said to be ^{E}**summable** if for every tempered distribution a in the topological dual E' the functional $\phi \mapsto a(v(\phi))$ is a tempered distribution in $\mathcal{S}'_n$.*

Definition (of normal space of test function for $\mathcal{S}'_m$). *We will call a locally convex topological vector space E a **normal space of test functions for the distribution space $\mathcal{S}'_m$** if it verifies the following properties*

- *the space E is an algebraic subspace of the space $\mathcal{C}_m$;*
- *the space E contains the space $\mathcal{S}_m$;*
- *the topological vector space $(\mathcal{S}_m)$ is continuously imbedded and dense in the topological vector space E;*
- *the topological vector space E is continuously imbedded in the space $(\mathcal{C}_m)$.*

In these conditions the dual E' is called a normal space of tempered distributions on $\mathbb{R}^m$.

Theorem (on the ^{E}family generated by a linear and continuous operator). *Let E be a normal space of test function for the space $\mathcal{S}'_m$, let $A : \mathcal{S}_n \to E$ be a linear and continuous operator of the space $(\mathcal{S}_n)$ into the space E and let δ be the Dirac family in $\mathcal{C}'_m$. Then, the family of functionals*

$$A^{\vee} := (\delta_p \circ A)_{p \in \mathbb{R}^m}$$

is a family of distribution in $\mathcal{S}'_n$ and it is an ^{E}family.

We can prove that:

Theorem. *Let E be a normal space of test functions for the distribution space $\mathcal{S}'_m$. Then, every ^{E}family in $\mathcal{S}'_n$ (obviously indexed by the m-dimensional Euclidean space) is ^{E}summable.*

Chapter 3

Superpositions

3.1 Introduction

3.1.1 The wonderful Dirac basis

Often, in Quantum Mechanics treatises, we read

- *it is possible to expand any ket $|f\rangle$ in the position basis, that is, for any ket $|f\rangle$ the following expansion*

$$|f\rangle = \int_{\mathbb{R}} f(x) |x\rangle \, dx,$$

  holds true.

In the above expression reside both the deepest essence and consequence of the Dirac Superposition Principle of Quantum Mechanics:

- the space of states of a quantum system is stable under the continuous (and then discrete) superposition of states;
- any state of a quantum system can be expanded as a continuous superposition of the most elementary states which can be conceived;
- there are systems of vectors capable to generate the entire state space of a quantum system.

3.1.2 A dangerous expression

But, what does the expansion

$$|f\rangle = \int_{\mathbb{R}} f(x)\,|x\rangle\, dx,$$

actually mean from a mathematical point of view?

Quantum physicists justify this claim by recalling the following equality taken from Distribution Theory:

$$f(y) = \int_{\mathbb{R}} f(x)\,\delta(x-y)dx,$$

which is valid for every real y. In the above equality, they consider the Dirac distribution $\delta(x-y)$ as the "base ket" $|x\rangle$, for every real x.

But the above justification is not correct, because in the above equality, x is not an index in the proper sense, in fact:

- the symbol $\delta(x-y)$ is an expression that rigorously represents the Dirac distribution δ_y centered at the point $y \in \mathbb{R}$ (the notation δ_y does not show abuse of notations);
- in the above equality the letter "x" in the expression $\delta(x-y)$ is an abuse of notation, because the distribution δ_y is not defined on the real line $\mathbb{R}$ but on the test function space $\mathcal{D}(\mathbb{R}, \mathbb{C})$.

Consequently, in the equality

$$f(y) = \int_{\mathbb{R}} f(x)\,\delta(x-y)dx,$$

we cannot consider the distribution $\delta(x-y)$ as a vector $|x\rangle$ labeled by x.

More specifically, the vector $|x\rangle = \delta(x-y)$ can be considered but, in this hypotesis, the letter "y" becomes an abuse of notation and the letter x, on the contrary, comes back to be a proper real number.

Concluding, as it is well known:

- when the distribution $\delta(x-y)$ appears in a mathematical relation, we cannot consider both x and y as real numbers simultaneously;
- moreover the equality

$$f(y) = \int_{\mathbb{R}} f(x)\,\delta(x-y)dx,$$

is true only when f is a function. On the contrary this equality does not justifies the expansion

$$\delta(y) = \int_{\mathbb{R}} \delta(x)\, \delta(x-y)dx;$$

further in this latter case the letter y does not represent anything more;

- more generally, if u is a tempered distribution in $\mathcal{S}'_1$, i.e., if u is a tempered distribution on the real line (that is a possible state of quantum system with one degree of freedom), we cannot justify the desired expansion

$$u(y) = \int_{\mathbb{R}} u(x)\, \delta(x-y)dx,$$

by means of the above classic equality of Distribution Theory.

3.1.3 Toward a possible solution

We can try to reconsider the desired expansion in another way, using the whole of the Dirac family $\delta = (\delta_y)_{y\in\mathbb{R}}$, as it is natural in Quantum Mechanics. We want reconstruct a function using the Dirac family, and indeed we have, for every $y \in \mathbb{R}$,

$$f(y) = \int_{\mathbb{R}} f\delta_y,$$

where the function f can live in the space $C^0(\mathbb{R}, \mathbb{C})$, $f\delta_y$ is the product of the function f by the distribution δ_y, δ_y is the Dirac's distribution centered at y and the functional

$$u \mapsto \int_{\mathbb{R}} u$$

is the integral (with respect to the Lebesgue measure) on the space of distributions with compact support $\mathcal{E}'_1 = \mathcal{E}'(\mathbb{R}, \mathbb{C})$, i.e. the functional:

$$\int_{\mathbb{R}} (\cdot)_{\mathcal{E}'_1} : \mathcal{E}'_1 \to \mathbb{C} : u \mapsto u(1_{\mathbb{R}}),$$

(we recall that a distribution with compact support acts on the constant functional

$$1_{\mathbb{R}} : \mathbb{R} \to \mathbb{C} : x \mapsto 1,$$

that is an element of the space $\mathcal{E}_1 = C^\infty(\mathbb{R}, \mathbb{C})$). So we can write

$$f = \int_{\mathbb{R}} f(\delta_y)_{y\in\mathbb{R}} := (\int_{\mathbb{R}} f\delta_y \mu)_{y\in\mathbb{R}},$$

But this is not still exactly what is needed in Quantum Theories: we need an operator

$$\int_{\mathbb{R}} : \mathcal{S}'_1 \times \mathcal{S}^1 \to \mathcal{S}'_1$$

such that

- $\mathcal{S}^1$ is some set of families in the space $\mathcal{S}'_1$ indexed by the real line $\mathbb{R}$;
- $\mathcal{S}^1$ contains exactly the "summable" families;
- the Dirac's family $(\delta_x)_{x\in\mathbb{R}}$ (or $(|x\rangle)_{x\in\mathbb{R}}$) belong to the family space $\mathcal{S}^1$;
- every tempered distribution $u \in \mathcal{S}'_1$ can be expanded as

$$\int_{\mathbb{R}} u\,(|x\rangle)_{x\in\mathbb{R}} = u;$$

Hence we need an operator that to any $\mathcal{S}'$-system of coefficients $a \in \mathcal{S}'_1$ and to any family of distributions $u = (u_k)_{k\in\mathbb{R}}$ in the space $\mathcal{S}^1$ (to be defined) associates a distribution that can be considered the linear superposition of the family u with respect to a in some physical sense.

3.1.4 Inadequacy of convolutions

The mathematical interpretations of the expression

$$u(y) = \int_{\mathbb{R}} u\,(x)\,\delta(x-y)dx,$$

by convolutions is in general inadequate for our purposes. Indeed, the above superposition has the interpretation,

$$u = u * \delta_0,$$

for every distribution $u \in \mathcal{S}'_n$, where δ_0 is the Dirac's distribution centered at 0. This interpretation does not solve the problem of superpositions in the general case, indeed only the element δ_0 of the family $(\delta_y)_{y\in\mathbb{R}}$ appears in the definition of superposition, this depends on the fact that any Dirac distribution δ_y is the translation $\tau_y\,(\delta_0)$ of the Dirac distribution centered at 0. In general, it is impossible to define a linear superposition by convolutions because a family of distributions $v = (v_y)_{y\in\mathbb{R}}$ not necessarily enjoys the property $v_y = \tau_y\,(v_0)$. Concluding, the rigorous version of is not satisfactory for uor general purposes.

3.1.5 Conclusions

Concluding we need an operator $\int_{\mathbb{R}}$, in some way enjoys the properties of the finite combination in a vector space. We recall that, if E is a $\mathbb{K}$-vector space, then it is possible, for each integer $m \in \mathbb{N}$, to define the linear combination operator:

$$\Sigma_m : \mathbb{K}^m \times X^m \to X : (\lambda, v) \mapsto \sum_{i=1}^{m} \lambda_i v_i.$$

We desire an analogous operator in the case in which the families are indexed by $\mathbb{R}$ or $\mathbb{R}^m$ instead of the finite set $\mathbb{N}_{\leq m}$ (that is the set $\{k \in \mathbb{N} : k \leq m\}$). The intention of this chapter is to show that this is possible in the space of tempered distributions and that the operator of superposition enjoys in a quite stitisfacotry manner the principal properties of the finite linear combination in a vector space.

3.2 Superpositions of $^{\mathcal{S}}$families in $\mathcal{S}'_n$

Now we can give a first generalization to the concept of linear combination.

Definition (linear superpositions of an $^{\mathcal{S}}$ family). *Let v be a family of class $\mathcal{S}$ belonging to the space $\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ and let $a \in \mathcal{S}'_m$ be a tempered distribution. The distribution (in $\mathcal{S}'_n$)*

$$a \circ \widehat{v} = {}^t(\widehat{v})(a)$$

is called ***the $^{\mathcal{S}}$linear superposition of the family*** *v* ***with respect to the coefficient distribution (the system of coefficients)*** *a and we denote it by*

$$\int_{\mathbb{R}^m} av.$$

Moreover, if u is a tempered distribution in the space $\mathcal{S}'_n$ and there exists a coefficient distribution $a \in \mathcal{S}'_m$ such that

$$u = \int_{\mathbb{R}^m} av,$$

u is said an ***$^{\mathcal{S}}$linear superposition of the family*** *v.*

As a particular case, we can consider the linear superposition of a family v with respect to the regular distribution generated by the $\mathbb{K}$-constant functional on $\mathbb{R}^m$ of value 1, the distribution $[1_{\mathbb{R}^m}]$, we denote this superposition simply by $\int_{\mathbb{R}^m} v$, and then we have

$$\int_{\mathbb{R}^m} v := \int_{\mathbb{R}^m} [1_{\mathbb{R}^m}]\, v,$$

an we shall call this particular superposition of v simply *the superposition of the family* v.

Example (the Dirac family). Let δ be the Dirac family in $\mathcal{S}'_n$. Then, for each tempered distribution $u \in \mathcal{S}'_n$, we have

$$\begin{aligned} \int_{\mathbb{R}^n} u\delta &= u \circ \widehat{\delta} = \\ &= u \circ \mathbb{I}_{\mathcal{S}_n} = \\ &= u. \end{aligned}$$

Thus any tempered distribution is an $^{\mathcal{S}}$linear superposition of the Dirac family and the coefficient system of this superposition is the distribution u itself: this is a property typycal of the canonical basis of the Euclidean spaces $\mathbb{R}^n$.

3.3 An alternative definition of superposition

An alternative definition of superposition can be obtained defining the superposition of a family of scalars (real or complex numbers) with respect to a distribution system of coefficients.

Definition (superposition of scalar $^{\mathcal{S}}$families). *We say that a family of real or complex numbers $x = (x_i)_{i\in\mathbb{R}^m}$ is a family of class $\mathcal{S}$ if the function $f_x : \mathbb{R}^m \to \mathbb{K}$, defined by $f_x(i) = x_i$, for each i in $\mathbb{R}^m$, is a function of class $\mathcal{S}$. We call f_x the test function associated with the family x. In this conditions, we put*

$$\int_{\mathbb{R}^m} ax := a(f_x),$$

for every tempered distribution $a \in \mathcal{S}'_m$, and we call the number

$$\int_{\mathbb{R}^m} ax$$

superposition of the family x with respect to the distribution coefficient a.

By introducing the canonical bilinear form of the pair $(\mathcal{S}'_n, \mathcal{S}_n)$, the relation between the two kind of superpositions is very natural.

Notation. Let $\langle \cdot, \cdot \rangle$ be the canonical bilinear form on the product $\mathcal{S}'_n \times \mathcal{S}_n$ and let v be an $^{\mathcal{S}}$family of tempered distributions in the space $\mathcal{S}'_n$ indexed by $\mathbb{R}^m$. For every test function $\phi \in \mathcal{S}_n$ by the symbol $\langle v, \phi \rangle$ we denote the family of scalars defined by

$$\langle v, \phi \rangle_i := \langle v_i, \phi \rangle ,$$

for every i in $\mathbb{R}^m$.

Theorem. *Let v be an ${}^{\mathcal{S}}$family of tempered distributions in $\mathcal{S}'_n$ indexed by $\mathbb{R}^m$, let a be a tempered distribution in $\mathcal{S}'_m$ and let $\langle \cdot, \cdot \rangle$ be the canonical bilinear form on $\mathcal{S}'_n \times \mathcal{S}_n$. Then, for every test function $\phi \in \mathcal{S}_n$, we have*

$$\left\langle \int_{\mathbb{R}^m} av, \phi \right\rangle = \int_{\mathbb{R}^m} a \langle v, \phi \rangle .$$

Proof. It's a straightforward computation:

$$\begin{aligned} \left\langle \int_{\mathbb{R}^m} av, \phi \right\rangle &= \left(\int_{\mathbb{R}^m} av \right) (\phi) = \\ &= a(v(\phi)) = \\ &= \int_{\mathbb{R}^m} a \left(v_i \left(\phi \right) \right)_{i \in \mathbb{R}^m} = \\ &= \int_{\mathbb{R}^m} a \langle v, \phi \rangle . \end{aligned}$$

Note, indeed, that the test function associated with the family $\left(v_i \left(\phi \right) \right)_{i \in \mathbb{R}^m}$, i.e. the family $\langle v, \phi \rangle$, is the image $v(\phi)$. ■

We shall see that the preceding result can be restated saying that the canonical bilinear form on the product $\mathcal{S}'_n \times \mathcal{S}_n$ is ${}^{\mathcal{S}}$linear in the first argument.

Definition (the superposition operators). *The bilinear operator*

$$\int_{\mathbb{R}^m} (\cdot, \cdot) : \mathcal{S}'_m \times \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n) \to \mathcal{S}'_n,$$

defined by

$$(a, v) \mapsto \int_{\mathbb{R}^m} av,$$

is called the ***superposition operator in the space $\mathcal{S}'_n$ with coefficient systems in $\mathcal{S}'_m$*** *and the linear operator*

$$\int_{\mathbb{R}^m} (\cdot, v) : \mathcal{S}'_m \to \mathcal{S}'_n,$$

defined by

$$a \mapsto \int_{\mathbb{R}^m} av$$

is called the ***superposition operator associated with (or of) the family*** *v.*

The bilinearity of the superposition operator on $\mathcal{S}'_m \times \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ is immediate, since it is nothing but the operator

$$(a, v) \mapsto {}^t(\widehat{v})(a),$$

and we leave the banal proof as an exercize. We shall see again the bilinearity of the superposition operator in the language of superpositions later.

3.4 Superpositions of an E-family (*)

Definition (superpositions of an E-family). *Let E be a normal space of test functions for $\mathcal{S}'_m$. Let v be an E-family in $\mathcal{S}'_n$ indexed by $\mathbb{R}^m$. We define, for every distribution a in the topological dual of E, the superposition of v with respect to a as the distribution in $\mathcal{S}'_n$ defined by*

$$\left(\int_{\mathbb{R}^m} av \right)(\phi) := a(v(\phi)),$$

for every test function ϕ in $\mathcal{S}_n$.

Remark. Note that, since the Dirac family of $C^{0\prime}(\mathbb{R}^m, \mathbb{K})$ is sequentially dense in the topological dual E' with respect to the weak* topology $\sigma(E', E)$ then, by the Banach-Steinhaus theorem, the superpositions of the above definition belongs indeed to the space $\mathcal{S}'_n$.

Example (of $\mathcal{C}^0$-superposition). Consider a $\mathcal{C}^0$-family v in $\mathcal{S}'_n$ indexed by $\mathbb{R}^m$, and consider a distribution a in $\mathcal{S}'_m$ generated by a Radon measure μ with compact support K. We can consider the superposition

$$\int_{\mathbb{R}^m} av.$$

Since μ is the Radon measure generating the distribution a, then we have

$$\begin{aligned}
\left(\int_{\mathbb{R}^m} av \right)(\phi) &= a(v(\phi)) = \\
&= \mu(v(\phi)) = \\
&= \int_{\mathbb{R}^m} v(\phi)\ \mu = \\
&= \int_K v(\phi)\ \mu,
\end{aligned}$$

for every test function ϕ in $\mathcal{S}_n$.

3.5 Algebraic properties of superpositions

3.5.1 Bilinearity of superposition operator

We already know that the superposition operators are bilinear operators. The present subsection must be considered as nothing but an elementary exercize to acquaint with the language of superpositions.

Proposition (bi-homogeneity). *Let $a \in \mathcal{S}'_m$ be a tempered distribution, $\lambda \in \mathbb{K}$ be a scalar and let v be a family of class $\mathcal{S}$ belonging to the space $\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$. Then,*

$$\begin{aligned} \int_{\mathbb{R}^m} (\lambda a) v &= \lambda \int_{\mathbb{R}^m} av = \\ &= \int_{\mathbb{R}^m} a\,(\lambda v)\,. \end{aligned}$$

Proof. For any $\phi \in \mathcal{S}_n$,

$$\begin{aligned} \left(\int_{\mathbb{R}^m} (\lambda a) v \right)(\phi) &= (\lambda a)\,(\widehat{v}\,(\phi)) = \\ &= \lambda a\,(\widehat{v}\,(\phi)) = \\ &= \lambda \left(\int_{\mathbb{R}^m} av \right)(\phi)\,, \end{aligned}$$

and

$$\begin{aligned} \left(\int_{\mathbb{R}^m} a\,(\lambda v) \right)(\phi) &= a\left(\widehat{\lambda v}\,(\phi) \right) = \\ &= a\,(\lambda \widehat{v}\,(\phi)) = \\ &= \lambda a\,(\widehat{v}\,(\phi)) = \\ &= \lambda \left(\int_{\mathbb{R}^m} av \right)(\phi)\,, \end{aligned}$$

as we desired. ■

Proposition (bi-additivity). *Let $k \in \mathbb{N}$, $(v)_{i=1}^n$ be a finite sequence of families belonging to $\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$, $a = (a_i)_{i=1}^k$ be a finite sequence in the space $\mathcal{S}'_m$ and $b \in \mathcal{S}'_m$ a distribution. Then*

$$\begin{aligned} \int_{\mathbb{R}^m} (\sum_{i=1}^k a_i) v &= \sum_{i=1}^k \int_{\mathbb{R}^m} a_i v, \\ \int_{\mathbb{R}^m} b \sum_{i=1}^k v_i &= \sum_{i=1}^k \int_{\mathbb{R}^m} b v_i. \end{aligned}$$

Proof. It follows immediately by the basic properties of the transpose of a linear operator, but we see it. For any $\phi \in \mathcal{S}_n$, we have

$$\begin{aligned}
\int_{\mathbb{R}^m} \sum_{i=1}^{k} a_i v &= \left(\sum_{i=1}^{k} a_i \right) \circ \widehat{v} = \\
&= \sum_{i=1}^{k} a_i \circ \widehat{v} = \\
&= \sum_{i=1}^{k} \int_{\mathbb{R}^m} a_i v \left(\phi \right).
\end{aligned}$$

For the second relation we have

$$\begin{aligned}
\int_{\mathbb{R}^m} b \sum_{i=1}^{k} v_i &= b \circ \left(\sum_{i=1}^{k} v_i \right)^{\wedge} = \\
&= \sum_{i=1}^{k} b \circ \widehat{v_i} = \\
&= \sum_{i=1}^{k} \int_{\mathbb{R}^m} b v_i,
\end{aligned}$$

as we desired. ■

We shall prove a theorem that gives an infinite-continuous version of the additivity expressed by the above proposition. But we first shall introduce the concept of superposition of an $\mathcal{S}$-family with respect to a family of distributions.

3.5.2 Selection property of the Dirac distributions

The aim of this section is to show that the basic properties of the superpositions extend the basic ones of linear combinations. First we prove a property of the superpositions analogous to the following property:

$$\sum \delta_{(i,\cdot)} v = \sum_{j=1}^{k} \delta_{ij} v_j = v_i,$$

where $\delta : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ is the Kronecker's delta and v is a finite family of vectors.

Theorem (selection property of Dirac distributions). *Let v be a family of class $\mathcal{S}$ belonging to the space $\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$. Then, for each $p \in \mathbb{R}^m$,*

$$\int_{\mathbb{R}^m} \delta_p v = v_p.$$

Proof. For every test function $\phi \in \mathcal{S}_n$ and every index p,

$$\begin{aligned}\delta_p\left(\widehat{v}\left(\phi\right)\right) &= \delta_p\left(v\left(\phi\right)\right) = \\ &= v\left(\phi\right)\left(p\right) = \\ &= v_p\left(\phi\right),\end{aligned}$$

and consequently

$$\begin{aligned}\int_{\mathbb{R}^m} \delta_p v &= \delta_p \circ \widehat{v} = \\ &= v_p.\end{aligned}$$

as we desired. ■

In the sense of the above theorem, the Dirac family is a continuous version of the Kronecker delta.

3.5.3 Linear combination of an $^{\mathcal{S}}$family

The following properties states that every finite linear combination of an $^{\mathcal{S}}$family is an $^{\mathcal{S}}$superposition of the family itself. We will improve considerably (in a certian sense) the following result, proving that every finite linear combination of tempered distributions is in fact a superposition of some $^{\mathcal{S}}$family.

Theorem (about finite linear combinations). *Let v be a family of class $\mathcal{S}$ belonging to the space $\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ and let δ be the Dirac family in $\mathcal{S}'_m$. Then, a tempered distribution u belongs to the linear hull* span(v) *if and only if there exists a distribution Λ in the linear hull* span(δ) *such that*

$$u = \int_{\mathbb{R}^m} \Lambda v.$$

Consequently the superposition operator of the family v transforms the linear hull span(δ) *onto the linear hull* span(v).

Proof. ($\Leftarrow$) If the condition holds true, the tempered distribution u is a finite linear combination of the family v, by the selection property of Dirac distributions and by linearity of the superposition operator. ($\Rightarrow$) Vice versa, let u be a finite linear combination of the family v, then there exist an integer $k \in \mathbb{N}$, a finite sequence $\lambda \in \mathbb{K}^k$ of scalars and a finite family $\alpha \in (\mathbb{R}^m)^k$ of indices of v such that

$$u = \sum_{i=1}^{k} \lambda_i v_{\alpha_i}.$$

Put

$$\Lambda = \sum_{i=1}^{k} \lambda_i \delta_{\alpha_i},$$

then, we have

$$\begin{aligned}
\int_{\mathbb{R}^m} \Lambda v &= \int_{\mathbb{R}^m} \left(\sum_{i=1}^{k} \lambda_i \delta_{\alpha_i} \right) v = \\
&= \sum_{i=1}^{k} \int_{\mathbb{R}^m} \lambda_i \delta_{\alpha_i} v = \\
&= \sum_{i=1}^{k} \lambda_i \int_{\mathbb{R}^m} \delta_{\alpha_i} v = \\
&= \sum_{i=1}^{k} \lambda_i v_{\alpha_i} = \\
&= u,
\end{aligned}$$

as we desired. ■

Chapter 4

$^{\mathcal{S}}$Linearity

4.1 Continuity of superposition operators

Now we pass to properties of continuity.

Theorem. *The bilinear operator of superposition*

$$\int_{\mathbb{R}^m} (\cdot, \cdot) : \mathcal{S}'_m \times \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n) \to \mathcal{S}'_n,$$

with index set $\mathbb{R}^m$, is continuous in the first argument, with respect to the pair of strong topologies $(\beta(\mathcal{S}'_m), \beta(\mathcal{S}'_n))$ and to the pair of weak* topologies $(\sigma(\mathcal{S}'_m), \sigma(\mathcal{S}'_n))$.*

Proof. Note that the operator of superposition with respect to an $^{\mathcal{S}}$family is nothing but the transpose of a linear continuous operator, precisely we have

$$\int_{\mathbb{R}^m} (\cdot, v) = {}^t\widehat{v},$$

and the transpose operator of a linear continuous operator is continuous in the first argument, with respect to the pairs of topologies $(\beta(\mathcal{S}'_m), \beta(\mathcal{S}'_n))$ and $(\sigma(\mathcal{S}'_m), \sigma(\mathcal{S}'_n))$ (see [*Ho*] Corollary, page 256). ■

The continuity of the above operators has many good consequences, for example we can state the following theorem.

Corollary (about denumerable linear combinations). *Let v be a family of class $\mathcal{S}$ belonging to the space $\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$. Let $\sum (a_i)_{i=1}^{\infty}$ be a convergent series of distributions in the space $\mathcal{S}'_m$, with respect to the strong topology $\beta(\mathcal{S}'_m)$. Then, the series of superpositions*

$$\sum \left(\int_{\mathbb{R}^m} a_i v \right)_{i=1}^{\infty}$$

converges in $\mathcal{S}'_n$, with respect to the strong topology $\beta(\mathcal{S}'_n)$, and moreover*

$$^{\beta'_n}\sum_{i=1}^{\infty} \int_{\mathbb{R}^m} a_i v = \int_{\mathbb{R}^m} \left(^{\beta'_m}\sum_{i=1}^{\infty} a_i \right) v.$$

In particular, if δ is the Dirac family of $\mathcal{S}'_m$ and the series

$$\sum (c_i \delta_{p_i})_{i=1}^{\infty}$$

is ${}^{\beta(\mathcal{S}'_m)}$convergent in $\mathcal{S}'_m$, for some selection $p = (p_i)_{i=1}^{\infty}$ in the index set $\mathbb{R}^m$ and some scalar sequence c, then the series

$$\sum (c_i v_{p_i})_{i=1}^{\infty}$$

is ${}^{\beta(\mathcal{S}'_n)}$convergent in the space $\mathcal{S}'_n$ and moreover

$$^{\beta'_n}\sum_{i=1}^{\infty} c_i v_{p_i} = \int_{\mathbb{R}^m} \left(^{\beta'_m}\sum_{i=1}^{\infty} c_i \delta_{p_i} \right) v.$$

4.2 Superposition operator of a distribution

We have already defined the superposition operator of an ${}^{\mathcal{S}}$ family v, as the first section of the superposition operator determined by the family v. Similarly, we can define the superposition operator of a coefficient distribution, as we specify in the following definition.

Definition (superposition operator of a coefficient distribution). *Fix a distribution a in $\mathcal{S}'_m$. We call superposition operator determined by the coefficient distribution a, the following linear operator*

$$\int_{\mathbb{R}^m} (a, .) : \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n) \to \mathcal{S}'_n : v \mapsto \int_{\mathbb{R}^m} av,$$

it is nothing but the second partial section of the superposition operator, with index set $\mathbb{R}^m$, determined by the term a, chosen in the Cartesian first factor of its domain.

Pointwise topology in $\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$. We desire to study the properties of continuity of this operator. To this purpose, we endow the vector space $\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ with the topology of pointwise convergence s and denote the corresponding topological vector space by $\mathcal{S}_s(\mathbb{R}^m, \mathcal{S}'_n)$, namely, this topological vector space is topologically isomorphic to the topological vector space $\mathcal{L}_s(\mathcal{S}_n, \mathcal{S}_m)$ and to the topological vector space $\mathcal{L}_s(\mathcal{S}'_m, \mathcal{S}'_n)$.

Convergence in $\mathcal{S}_s(\mathbb{R}^m, \mathcal{S}'_n)$. To say that a filter $\mathcal{F}$ on the topological vector space $\mathcal{S}_s(\mathbb{R}^m, \mathcal{S}'_n)$ converges to a family v is equivalent to say that, for every test function g in $\mathcal{S}_n$, the filter $\mathcal{F}(g)$ on the space $\mathcal{S}_m$ converges to the test function $v(g)$, with respect to the weak topology $\sigma(\mathcal{S}_m)$. If $(I, \leq)$ is a directed set, a family of ${}^{\mathcal{S}}$families $v = (v(i))_{i\in I}$ converges to the family l in the space $\mathcal{S}_s(\mathbb{R}^m, \mathcal{S}'_n)$, with respect to the filtering order $\leq$, if and only if the equality

$$^{(\leq,\sigma_m)}\lim_{i\in I} v(i)(g) = l(g)$$

holds true, for every test function g in $\mathcal{S}_n$. Fixed a test function g, the last equality means exactly that the scalar equality

$$^{\leq}\lim_{i\in I} a(v(i)(g)) = a(l(g))$$

holds true, for every tempered distribution a in $\mathcal{S}'_m$.

Concerning the continuity of the superposition operator determined by a coefficient distribution we have the following useful theorem.

Theorem. *The superposition operator of a coefficient distribution a in $\mathcal{S}'_n$ is a continuous linear operator from the topological vector space $\mathcal{S}_s(\mathbb{R}^m, \mathcal{S}'_n)$ into the topological vector space $(\mathcal{S}'_n, \sigma(\mathcal{S}'_n))$.*

Proof. We must prove that for every directed set $(I, \leq)$ and for every family of ${}^{\mathcal{S}}$families $v = (v(i))_{i\in I}$ in the space $\mathcal{S}_s(\mathbb{R}^m, \mathcal{S}'_n)$ which converges to some family l in the space $\mathcal{S}_s(\mathbb{R}^m, \mathcal{S}'_n)$, with respect to the filtering order $\leq$, we have

$$^{(\leq,\sigma'_n)}\lim_{i\in I} \int_{\mathbb{R}^m} av(i) = \int_{\mathbb{R}^m} a \ ^{(\leq,s)}\lim_{i\in I} v(i).$$

For every index i of the family v and for every test function g in $\mathcal{S}_n$, we have, by the very definition of superposition, that

$$(\int_{\mathbb{R}^m} av(i))(g) = a(v(i)(g)).$$

Moreover we have

$$\begin{aligned} {}^{\leq}\lim_{i\in I} a(v(i)(g)) &= a(l(g)) = \\ &= (\int_{\mathbb{R}^m} al)(g). \end{aligned}$$

Since the equality

$$^{(\leq,s)}\lim_{i\in I} v(i) = l,$$

means that, for every test function g, the equality

$$^{(\leq,\sigma_m)}\lim_{i\in I} v(i)(g) = l(g),$$

holds too. The preceding equality means that, for every tempered distribution b and for each test function g, the equality

$$^{\leq}\lim_{i\in I} b(v(i)(g)) = b(l(g)),$$

holds, and since this is true for every test function g we deduce exactly

$$^{(\leq,\sigma'_n)}\lim_{i\in I} \int_{\mathbb{R}^m} av(i) = \int_{\mathbb{R}^m} a \ ^{(\leq,s)}\lim_{i\in I} v(i),$$

as we claimed. ■

4.3 $^{\mathcal{S}}$Linearity of superpositions

In this section we generalize the linearity of the operator of superposition, precisely the linearity with respect to the first argument. To this end we have to introduce the concept of superposition of a family with respect to a family, this latter concept will play an importan role in the following development.

Definition (superposition of a family with respect to a family). *Let v be a family in $\mathcal{S}'_m$, indexed by the Euclidean space $\mathbb{R}^k$, and let $w \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be an $\mathcal{S}$family in $\mathcal{S}'_n$. The family in $\mathcal{S}'_n$ defined by*

$$\int_{\mathbb{R}^m} vw := \left(\int_{\mathbb{R}^m} v_p w\right)_{p\in\mathbb{R}^k},$$

*is called **the superposition of** w **with respect to the family** v (note the order in the roles of the two families).*

Theorem ($^{\mathcal{S}}$linearity of the superposition bilinear operator in the first argument). *Let v be a family of class $\mathcal{S}$ belonging to the space $\mathcal{S}(\mathbb{R}^k, \mathcal{S}'_m)$*

and let w be a family of class $\mathcal{S}$ belonging to the space $\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$. Then, the superposition family

$$\int_{\mathbb{R}^m} vw$$

is an $^{\mathcal{S}}$family and its operator is the composition of the operators of the two families, namely

$$\left(\int_{\mathbb{R}^m} vw\right)^{\wedge} = \widehat{v} \circ \widehat{w}.$$

Moreover, the superposition bilinear operator is $^{\mathcal{S}}$linear in the first argument, in the sense that the property

$$\int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^k} av\right) w = \int_{\mathbb{R}^k} a \left(\int_{\mathbb{R}^m} vw\right)$$

holds true, for every coefficient distribution a in $\mathcal{S}'_k$.

Proof. For every test function $\phi \in \mathcal{S}_n$ and for every index q in $\mathbb{R}^k$, we deduce

$$\begin{aligned}
\left(\int_{\mathbb{R}^m} vw\right)(\phi)(q) &= \left(\int_{\mathbb{R}^m} vw\right)_q (\phi) = \\
&= \left(\int_{\mathbb{R}^m} v_q w\right)(\phi) = \\
&= v_q\left(\widehat{w}(\phi)\right) = \\
&= \widehat{v}\left(\widehat{w}(\phi)\right)(q) = \\
&= (\widehat{v} \circ \widehat{w})(\phi)(q);
\end{aligned}$$

so the image of the function ϕ by the superposition family is the function $v(w(\phi))$, which is of class $\mathcal{S}$ since the two families are of class $\mathcal{S}$, and then the superposition $\int_{\mathbb{R}^m} vw$ is an $^{\mathcal{S}}$family; and furthermore

$$\left(\int_{\mathbb{R}^m} vw\right)^{\wedge} = \widehat{v} \circ \widehat{w}.$$

Moreover, for every $\phi \in \mathcal{S}_n$, we have

$$\begin{aligned}
\left(\int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^k} av\right) w\right)(\phi) &= \left(\int_{\mathbb{R}^m} av\right)\left(\widehat{w}\left(\phi\right)\right) = \\
&= a(\widehat{v}\left(\widehat{w}(\phi)\right)) = \\
&= a\left(\left(\widehat{v} \circ \widehat{w}\right)(\phi)\right) = \\
&= a\left(\left(\int_{\mathbb{R}^m} vw\right)^{\wedge}(\phi)\right) = \\
&= \int_{\mathbb{R}^k} a \left(\int_{\mathbb{R}^m} vw\right)(\phi),
\end{aligned}$$

as we desired. ■

4.4 Generalized distributive laws (*)

In this section, we give a generalization of the two distributive laws in the space $\mathcal{S}'_n$.

Let u be in $\mathcal{S}'_n$ and let v be the family in $\mathcal{S}'_n$ defined by $v_y := u$, for every y in $\mathbb{R}^m$. We have already seen that the family v is a smooth family (it sends ${}^{\mathcal{S}}$test functions into smooth test functions, and it is also bounded, in the sense that it sends ${}^{\mathcal{S}}$function to smooth and bounded function). Then, we can consider, for every tempered distribution a in $\mathcal{S}'_m$ with compact support (and more generally, being v smooth and bounded, when a is a summable distribution) the superposition

$$\int_{\mathbb{R}^m} av.$$

We shall generalize firstly the following distributive law

$$\sum_{i=1}^{m}(a_i u) = \left(\sum_{i=1}^{m} a_i\right) u.$$

Theorem (first ${}^{\mathcal{S}}$distributive law). *Let $a \in \mathcal{S}'_m$ be a tempered distribution with compact support (or, more generally, a summable tempered distribution). Let u be a distribution of the space $\mathcal{S}'_n$ and let v be the constant family in $\mathcal{S}'_n$ defined by $v_y := u$, for every y in $\mathbb{R}^m$. Then, we have*

$$\int_{\mathbb{R}^m} av = \left(\int_{\mathbb{R}^m} a\right) u,$$

where $\int_{\mathbb{R}^m} a$ is the Lebesgue measure (integral) of the distribution a (on the whole space $\mathbb{R}^m$).

Proof. In fact, under the above assumptions, we have

$$\begin{aligned}\left(\int_{\mathbb{R}^m} av\right)(\phi) &= a(v(\phi)) = \\ &= a(u(\phi)1_{\mathbb{R}^m}) = \\ &= a(1_{\mathbb{R}^m})u(\phi) = \\ &= \left(\left(\int_{\mathbb{R}^m} a\right) u\right)(\phi),\end{aligned}$$

for every test function ϕ in $\mathcal{S}_n$, where $1_{\mathbb{R}^m}$ is the constant functional from $\mathbb{R}^m$ to $\mathbb{K}$ of value 1. So we have proved that

$$\int_{\mathbb{R}^m} av = \left(\int_{\mathbb{R}^m} a\right) u,$$

where the integral of a summable distribution is defined, as usual (see [*Sch*]), by the equality

$$\int_{\mathbb{R}^m} a = a(1_{\mathbb{R}^m}),$$

and the proof is completed. ■

Let us see the other distribution law:

$$\sum_{i=1}^{m} av_i = a\sum_{i=1}^{m} v_i.$$

Let k be a real or complex number, with $k_{\mathbb{R}^m}$ we shall denote the constant $\mathbb{K}$-functional of value k on $\mathbb{R}^m$ (in this case, the constant distribution is that of coefficients); consequently, the generated distribution is denoted (in the standard way) by $[k_{\mathbb{R}^m}]$.

Theorem (second $^{\mathcal{S}}$distributive law). *Let v be a smooth and bounded family in $\mathcal{S}_n'$ indexed by $\mathbb{R}^m$ and let k be a scalar. Then, the equality*

$$\int_{\mathbb{R}^m} [k_{\mathbb{R}^m}]\, v = k \int_{\mathbb{R}^m} v$$

holds true (we recall that the superposition of a family is its superposition with respect to the unitary distribution coefficient).

Proof. We have,

$$\begin{aligned} \left(\int_{\mathbb{R}^m} [k_{\mathbb{R}^m}]\, v\right)(\phi) &= [k_{\mathbb{R}^m}]\,(v(\phi)) = \\ &= k\,[1_{\mathbb{R}^m}]\,(v(\phi)) = \\ &= k\left(\int_{\mathbb{R}^m} v\right)(\phi), \end{aligned}$$

for every test function ϕ in $\mathcal{S}_n$, i.e.,

$$\int_{\mathbb{R}^m} [k_{\mathbb{R}^m}]\, v = k \int_{\mathbb{R}^m} v,$$

as we desired. ■

Chapter 5

$^{\mathcal{S}}$Families in $\mathcal{S}_n$

We give in this Chapter (very concisely) the bases of superpositions on test function spaces.

5.1 Families in $\mathcal{S}_n$

A family in $\mathcal{S}_n$, indexed by an Euclidean space, can act on distributions in $\mathcal{S}'_n$ and vice versa. As the following definition is going to specify.

Definition (action of a family of test functions on distributions). *Consider a family g of test functions in the space $\mathcal{S}_n$, indexed by the m-dimensional Euclidean space $\mathbb{R}^m$. For every distribution u in $\mathcal{S}'_n$, we call the scalar family*

$$u(g) := (u(g_p))_{p\in\mathbb{R}^m}$$

***image of the family** g **by the distribution** u (note that the family $u(g)$ is a family in the field $\mathbb{K}$). Moreover, the function $g(u) : \mathbb{R}^m \to \mathbb{K}$, defined by*

$$g(u)(p) = u(g_p),$$

for every point p in $\mathbb{R}^m$, is call the ***image of the distribution*** u ***by the family of test functions*** g.

Remark. The scalar family $u(g)$ is said the family canonically associated with the function $g(u)$; from a purely Set Theory point of view, the family $u(g)$ is nothing but the codomain restriction of the function $g(u)$ to its image $\mathrm{im}(g(u))$, i.e. the surjection canonically associated to the function $g(u)$).

A family g can act also on other objects. Indeed, we have the following definitions:

- we define image of a point x of $\mathbb{R}^n$ by the family g, the function

$$g(x) := g(\delta_x),$$

  that is the scalar function on $\mathbb{R}^m$ defined by

$$g(x)(p) = g_p(x),$$

  for every p in $\mathbb{R}^m$;

- to the family g we can also associate a scalar function of the product $\mathbb{R}^m \times \mathbb{R}^n$, that is the function defined by $(p, x) \mapsto g_p(x)$, for every pair (p, x) in $\mathbb{R}^m \times \mathbb{R}^n$;

- and finally the mapping

$$g_\delta : \mathbb{R}^n \to \mathcal{F}(\mathbb{R}^m, \mathbb{K}) : x \mapsto g(x)$$

  is called the function from $\mathbb{R}^n$ into $\mathcal{F}(\mathbb{R}^m, \mathbb{K})$ canonically associated to the family g;

- if v is a family in $\mathcal{S}'_n$, indexed by $\mathbb{R}^n$, we define image of the family g by the family v, and we shall denote it by $v(g)$, the family $(\langle v_x, g\rangle)_{x\in\mathbb{R}^n}$, in the function space $\mathcal{F}(\mathbb{R}^m, \mathbb{K})$.

Remark. We shall see - after the introduction of the $^{\mathcal{S}}$basis in the spaces of tempered distributions $\mathcal{S}'_n$ - that, fixed an $^{\mathcal{S}}$basis e, indexed by $\mathbb{R}^n$, of the space $\mathcal{S}'_n$, we can associate to every family g the function

$$g_e : \mathbb{R}^n \to \mathcal{F}(\mathbb{R}^m, \mathbb{K}) : x \mapsto g(e_x)$$

on $\mathbb{R}^n$, precisely the function.

5.2 $^{\mathcal{S}}$Families

Also for the superposition in the space $\mathcal{S}_n$ the $^{\mathcal{S}}$families play a central role.

Definition (of $^{\mathcal{S}}$family). *Consider a family g of test functions in the space $\mathcal{S}_n$, indexed by the m-dimensional Euclidean space $\mathbb{R}^m$. We say that the family g is* ***of class*** *$\mathcal{S}$ if, for every distribution u in $\mathcal{S}'_n$ the scalar family image of the family g by the distribution u, that is the family*

$$u(g) := (u(g_p))_{p\in\mathbb{R}^m}$$

($u(g)$ is a family in the field $\mathbb{K}$) is a family of class $\mathcal{S}$. This is equivalent to say that the function $g(u) : \mathbb{R}^m \to \mathbb{K}$, defined by

$$g(u)(p) = u(g_p),$$

for every point p in $\mathbb{R}^m$, is a function of class $\mathcal{S}$.

We can say that

- *a family of test functions (of the space $\mathcal{S}_n$) is an $^{\mathcal{S}}$family if it transforms distributions (of the space $\mathcal{S}'_n$) into test functions (of the space $\mathcal{S}_m$, of the index set of the family).*

Example. Consider the family g in $\mathcal{S}_n$, indexed by the Euclidean space $\mathbb{R}^m$, defined, for every index p in $\mathbb{R}^m$ and for every point x in $\mathbb{R}^n$, by

$$g_p(x) = k(p)h(x),$$

for some choice of a pair (k, h), of test functions, in the product $\mathcal{S}_m \times \mathcal{S}_n$. Note that the family g is associated with the tensor product $k\otimes h$. We will prove that the family g is an $^{\mathcal{S}}$family. For, we have to prove that g transforms distributions into test functions. Let u be a tempered distribution in $\mathcal{S}_n$, we must prove that the function

$$g(u) : p \mapsto u(g_p)$$

lives in $\mathcal{S}_m$. And indeed we have

$$\begin{aligned} g(u)(p) &= u(g_p) = \\ &= u(k(p)h) = \\ &= k(p)u(h), \end{aligned}$$

so that the function $g(u)$ is nothing but the test function k multiplied by the scalar $u(h)$. In particular, for example, consider the family g in $\mathcal{S}_n$, indexed by $\mathbb{R}^m$, defined by

$$g_p(x) = e^{-\|(x,p)\|^2} = e^{-\|p\|^2}e^{-\|x\|^2},$$

for every p in $\mathbb{R}^m$ and x in $\mathbb{R}^n$. We thus have

$$g(u) = u\left(e^{-\|\cdot\|^2_{(n)}}\right) e^{-\|\cdot\|^2_{(m)}},$$

for every u in $\mathcal{S}_n$.

5.3 Transpose of an $^{\mathcal{S}}$Family

Theorem. *Let g be an $^{\mathcal{S}}$family in the space $\mathcal{S}_n$, indexed by the Euclidean space $\mathbb{R}^m$. Then, for every point x in $\mathbb{R}^n$, the function $g(x) : \mathbb{R}^m \to \mathbb{K}$ defined by*

$$g(x)(p) = g_p(x),$$

for every p in $\mathbb{R}^m$, is of class $\mathcal{S}$. Consequently, the family $^t g = (g(x))_{x\in\mathbb{R}^n}$ is a family in $\mathcal{S}_m$.

Proof. If the family g is of class $\mathcal{S}$, then for every tempered distribution u the function $g(u)$ is of class $\mathcal{S}$. So, in particular, the function $g(\delta_x)$ is of class $\mathcal{S}$. The function $g(\delta_x)$ is the function $g(x)$; indeed,

$$\begin{aligned} g(\delta_x)(p) &= \delta_x(g_p) = \\ &= g_p(x) = \\ &= g(x)(p), \end{aligned}$$

so $g(x)$ is of class $\mathcal{S}$. ■

The vice versa is not true in general as the following example shows.

Example. Let g be an $^{\mathcal{S}}$test function on $\mathbb{R}^n$, with non-zero integral, and let $(\tau_p g)_{p\in\mathbb{R}^n}$ be the family of its translations. Then, the value at p of the image of the constant regular distribution $u = [1_{\mathbb{R}^n}]$ is

$$\begin{aligned} g(u)(p) &= u(\tau_p g) = \\ &= \int_{\mathbb{R}^n} \tau_p g = \\ &= \int_{\mathbb{R}^n} g \\ &= u(g), \end{aligned}$$

for every p in $\mathbb{R}^n$, so that the function $g(u)$ is a constant function and cannot be of class $\mathcal{S}$. At this point, it suffices to choose a function g such that the function

$$p \mapsto \tau_p g(x)$$

is of class $\mathcal{S}$, for every x in $\mathbb{R}^n$, and our counterexample is given. For example, the function of $\mathcal{S}_1$, defined by

$$g(x) = e^{-x^2},$$

for every x in $\mathbb{R}^n$. Indeed

$$\tau_p g(x) = e^{-(x-p)^2},$$

and the function

$$p \mapsto e^{-(x-p)^2}$$

is nothing but the translation $\tau_x g$, which belongs to the space $\mathcal{S}_1$, for every x of the real line.

Definition (of transpose). *Let g be an $^{\mathcal{S}}$family in the space $\mathcal{S}_n$, indexed by the Euclidean space $\mathbb{R}^m$ and let, for every point x in $\mathbb{R}^n$, the function $g(x) : \mathbb{R}^m \to \mathbb{K}$ be defined by*

$$g(x)(p) = g_p(x),$$

*for every p in $\mathbb{R}^m$. The family $^t g = (g(x))_{x \in \mathbb{R}^n}$ is called **the transpose family of the family** g.*

5.4 Operator of an $^{\mathcal{S}}$family

Definition (operator canonically associated to an $^{\mathcal{S}}$family). *With the $^{\mathcal{S}}$family g we can associate the operator*

$$\widehat{g} : \mathcal{S}'_n \to \mathcal{S}_m : u \mapsto g(u).$$

*We shall call this operator the **operator canonically associated with the family** g.*

Proposition. *In the condition of the above definition. The operator $\widehat{g}$ is linear.*

Proof. Infact, for every u and v in $\mathcal{S}'_n$, for any two scalars a and b, and for every index p of the family, we have

$$\begin{aligned} g(au+bv)(p) &= \langle g_p, au+bv \rangle = \\ &= a\langle g_p, u\rangle + b\langle g_p, v\rangle = \\ &= ag(u)(p) + bg(v)(p) = \\ &= (ag(u)+bg(v))(p), \end{aligned}$$

for every index p of the family in $\mathbb{R}^m$. ■

5.5 Continuity of operators of ${}^{\mathcal{S}}$families

We have the following important result.

Theorem. *The operator associated canonically with the family g, defined by*

$$\widehat{g} : \mathcal{S}'_n \to \mathcal{S}_m : u \mapsto g(u),$$

is weakly continuous, that is continuous with respect to the pair of topologies (σ'_n, σ_m). Moreover, the topological transpose of the operator is the operator

$${}^t\widehat{g} : \mathcal{S}'_m \to \mathcal{S}''_n : a \mapsto a \circ \widehat{g}.$$

Proof. We shall prove that the linear operator $\widehat{g}$ is topologically transposable, i.e. that its algebraic transpose

$${}^*\widehat{g} : (\mathcal{S}_m)^* \to (\mathcal{S}'_n)^* : a \mapsto a \circ \widehat{g}$$

sends the topological dual $(\mathcal{S}_m)'_\sigma$ into the topological dual $(\mathcal{S}'_n)'_\sigma$. In other terms, we have to prove that, if a is a tempered distribution in $\mathcal{S}'_m$, the linear functional $a\circ\widehat{g}$ is weakly continuous. In fact, the Dirac family of $\mathcal{S}'_m$ is sequentially weakly* dense in $\mathcal{S}'_m$ itself, and the composition $\delta_p \circ \widehat{g}$ is nothing but the linear form $\langle ., g_p\rangle$, which lies in $(\mathcal{S}'_n)'_\sigma$. Indeed,

$$\begin{aligned}
(\delta_p \circ \widehat{g})\,(u) &= \delta_p(\widehat{g}(u)) = \\
&= \delta_p(g(u)) = \\
&= g(u)(p) = \\
&= u(g_p) = \\
&= \langle u, g_p\rangle,
\end{aligned}$$

for every u in $\mathcal{S}'_n$. Consequently, if d is a distribution in the linear hull of the Dirac family, by linearity, the composition $d \circ \widehat{g}$ is a continuous linear form too. Now, in general, let a be a tempered distribution in $\mathcal{S}'_m$. Since the linear hull $\mathrm{span}(\delta)$ of the Dirac family is $\sigma(\mathcal{S}'_m)$-sequentially dense in $\mathcal{S}'_m$ (see [*Bo*] page 205), there is a sequence of distributions $\alpha = (\alpha_k)_{k\in\mathbb{N}}$ in the linear hull $\mathrm{span}(\delta)$ converging to the distribution a with respect to the weak* topology $\sigma(\mathcal{S}'_m)$, that is such that

$${}^{\sigma(\mathcal{S}'_m)}\lim_{k\to+\infty} \alpha_k = a.$$

Now, since for any natural k, the distribution α_k belongs to the linear hull $\mathrm{span}(\delta)$, there exists a finite family $(y_i)_{i=1}^h$ of points in $\mathbb{R}^m$ and there is a finite family of points $(\lambda_i)_{i=1}^h$ in $\mathbb{K}$ such that

$$\alpha_k = \sum_{i=1}^h \lambda_i \delta_{y_i},$$

and consequently, by obvious calculations,

$$\begin{aligned}\alpha_k \circ \widehat{g} &= \sum_{i=1}^{h} \lambda_i (\delta_{y_i} \circ \widehat{g}) = \\ &= \sum_{i=1}^{h} \lambda_i \langle ., g_{y_i} \rangle .\end{aligned}$$

Hence, for every index $k \in \mathbb{N}$, the linear functional $\alpha_k \circ \widehat{g}$ belongs to the space $(\mathcal{S}'_n)'_\sigma$. Let s be the topology of the pointwise convergence in the algebraic dual $(\mathcal{S}'_n)^*$, we claim that

$$^s\lim_{k\to+\infty} (\alpha_k \circ \widehat{g}) = a \circ \widehat{g}.$$

In fact, for every distribution u in $\mathcal{S}'_n$, we obtain

$$\begin{aligned}\lim_{k\to+\infty} (\alpha_k \circ \widehat{g})(u) &= \lim_{k\to+\infty} \alpha_k (\widehat{g}(u)) = \\ &= \left(^s\lim_{k\to+\infty} \alpha_k \right) (\widehat{g}(u)) = \\ &= a(\widehat{g}(u)),\end{aligned}$$

so we proved that the sequence of continuous linear functionals $(\alpha_k \circ \widehat{g})_{k\in\mathbb{N}}$ is pointwise converging to the linear functional $a \circ \widehat{g}$; so, by the Banach-Steinhaus theorem (that is applicable since $\mathcal{S}'_n$ is barreled), the linear functional $a \circ \widehat{g}$ must be continuous too, i.e. $a \circ \widehat{g} \in (\mathcal{S}'_n)'_\sigma$, as we claimed. ■

Chapter 6

Superpositions in $\mathcal{S}_n$

6.1 $^{\mathcal{S}}$Linear combinations

Definition (of superposition). *If g is an $^{\mathcal{S}}$family in the space $\mathcal{S}_n$, we define* ***superposition of the family** g **under a coefficient distribution*** *a, distribution in $\mathcal{S}'_m$, the function, denoted by*

$$\int_{\mathbb{R}^m} ag,$$

from $\mathbb{R}^n$ into the field $\mathbb{K}$ and defined by

$$\left(\int_{\mathbb{R}^m} ag\right)(x) = a(g(\delta_x)),$$

for every point x in $\mathbb{R}^n$, where δ_x is the Dirac distribution of the space $\mathcal{S}'_n$ centered at x.

Example. Consider the family g in $\mathcal{S}_n$, indexed by the Euclidean space $\mathbb{R}^m$, defined, for every index p in $\mathbb{R}^m$ and for every point x in $\mathbb{R}^n$, by

$$g_p(x) = k(p)h(x),$$

for some choice of a pair (k,h), of test functions, in the product $\mathcal{S}_m \times \mathcal{S}_n$. We have proved that the family g is an ${}^{\mathcal{S}}$family. If a is a distribution in $\mathcal{S}'_m$, the value of a at the function $g(u)$ is

$$\begin{aligned} a(g(u)) &= a(u(h)k) = \\ &= u(h)a(k), \end{aligned}$$

so that, in particular we have

$$\begin{aligned} a(g(x)) &= a(g(\delta_x)) = \\ &= h(x)a(k), \end{aligned}$$

so that the function

$$x \mapsto a(g(x))$$

is the function h multiplied by the scalar $a(k)$. In other terms, we have

$$\int_{\mathbb{R}^m} ag = a(k)h,$$

for every a in $\mathcal{S}'_m$.

6.2 Superposition operator of ${}^{\mathcal{S}}$families

Remark. Note that if, for every x in $\mathbb{R}^n$, we denote by $g(x)$ the scalar function on $\mathbb{R}^m$ defined by

$$g(x)(p) = g_p(x),$$

for every p in $\mathbb{R}^m$, the function $g(x)$ is nothing but the ${}^{\mathcal{S}}$function $g(\delta_x)$. In fact, for every x in $\mathbb{R}^n$ and p in $\mathbb{R}^m$, we have

$$\begin{aligned} g(\delta_x)(p) &= \delta_x(g_p) = \\ &= g_p(x) = \\ &= g(x)(p). \end{aligned}$$

So that, we conclude

$$\begin{aligned} \left(\int_{\mathbb{R}^m} ag\right)(x) &= a(g(\delta_x)) = \\ &= a(g(x)), \end{aligned}$$

for every x in $\mathbb{R}^n$. Or, using the transpose of the family g, we have

$$\left(\int_{\mathbb{R}^m} ag\right)(x) = a({}^t g_x),$$

for every x in $\mathbb{R}^n$.

6.3 Summability of $^{\mathcal{S}}$families

We have so constructed the topological traspose of $\widehat{g}$, the operator

$$^{t}\widehat{g} : \mathcal{S}'_m \to \mathcal{S}''_n : a \mapsto a \circ \widehat{g}$$

Well we have the following result.

Theorem. *The topological transpose of $\widehat{g}$ is the superposition operator of g when we identify canonically the bidual $\mathcal{S}''_n$ to the test function space $\mathcal{S}_n$.*

Proof. The weak dual $(\mathcal{S}'_n)'_\sigma$ coincides with the strong dual $\mathcal{S}''_n$, since the space $(\mathcal{S}_n)$ is reflexive. Then, for every linear form L of the dual $(\mathcal{S}'_n)'_\sigma$, there is only one test function $h(L)$ of $\mathcal{S}_n$ such that the functional L is the continuous linear form induced by h on $\mathcal{S}'_n$, i.e. the form

$$L = \langle ., h(L) \rangle.$$

So we can define the operator

$$\mathcal{S}'_m \to \mathcal{S}''_n \to \mathcal{S}_n : a \mapsto a \circ \widehat{g} \mapsto h(a \circ \widehat{g}).$$

So, for every coefficient distribution a in $\mathcal{S}'_m$, the superposition

$$\int_{\mathbb{R}^m} ag$$

is a function belonging to $\mathcal{S}_n$. Thus, the operator associated with the family g is weakly transposable and we have can built the weak transpose

$$^{t}\widehat{g} : \mathcal{S}'_m \to \mathcal{S}_n : a \mapsto \int_{\mathbb{R}^m} ag,$$

which is, as in the case of $^{\mathcal{S}}$families in $\mathcal{S}'_n$, the superposition operator of the family g. ■

Proof. Indeed we have, by definition of weak transpose,

$$\langle {}^{t}\widehat{g}(a), u \rangle_n = \langle a, g(u) \rangle_m,$$

for every u in $\mathcal{S}'_n$ and every a in $\mathcal{S}'_m$, since the two continuous linear functionals

$$a \mapsto \langle {}^{t}\widehat{g}(a), u \rangle_n,$$

and

$$a \mapsto \langle a, g(u) \rangle_m,$$

coincide on the Dirac basis of $\mathcal{S}'_m$, that is total in $\mathcal{S}'_m$. ■

6.4 Transpose family

We then have defined the mapping from $\mathbb{R}^n$ into $\mathcal{S}_m$ defined by $x \mapsto g(x)$ starting from the ${}^{\mathcal{S}}$family $(g_p)_{p\in\mathbb{R}^m}$.

Equivalently, we have defined another ${}^{\mathcal{S}}$family, namely the family $(g(x))_{x\in\mathbb{R}^n}$, that is a family in $\mathcal{S}_m$ indexed by $\mathbb{R}^n$, and which we call the transpose of the family g and denote by ${}^t g$.

Note that the operator canonically associated to the transpose family ${}^t g$ is the operator

$$\widehat{{}^t g} : \mathcal{S}'_m \to \mathcal{S}_n : a \mapsto {}^t g(a).$$

So that

$$\begin{aligned}
\langle \widehat{g}(u)|a\rangle_m &= a(\widehat{g}(u)) = \\
&= a(g(u)) = \\
&= a(u(g)) = \\
&= u(g)(a) = \\
&= u({}^t g(a)) = \\
&= \langle u|{}^t g(a)\rangle_n = \\
&= \langle u|\widehat{{}^t g}(a)\rangle_n,
\end{aligned}$$

and consequently

$${}^w\widehat{g} = \widehat{{}^t g},$$

as we claimed. ■

6.5 ${}^{\mathcal{S}}$Linear functional

Theorem. *Let u be a distribution in $\mathcal{S}'_n$ and g an ${}^{\mathcal{S}}$family of test functions in $\mathcal{S}_n$. Then, we have*

$$u\left(\int_{\mathbb{R}^m} ag\right) = \int_{\mathbb{R}^m} au(g),$$

for everu a in $\mathcal{S}'_m$. note that the last superposition is the superposition of the scalar ${}^{\mathcal{S}}$family $u(g)$ under the system of coefficients a, the scalar family $u(g)$ is a family of class $\mathcal{S}$, by the very definition of test family of class $\mathcal{S}$, since u is a distribution.

Proof. Consider the (strong) transpose

$${}^t\widehat{g} : \mathcal{S}'_m \to \mathcal{S}''_n : a \mapsto a \circ \widehat{g}.$$

The functional ${}^t\widehat{g}(a)$ is defined by

$$ {}^t\widehat{g}(a)(u) = a(g(u)). $$

Since the space $\mathcal{S}_n$ is reflexive there is one and only one function belonging to $\mathcal{S}_n$ representing that functional, the function f such that

$$ u(f) = a(g(u)), $$

for every u, which implies

$$ \begin{aligned} f(x) &= \delta_x(f) = \\ &= a(g(\delta_x)) = \\ &= \left(\int_{\mathbb{R}^m} ag\right)(x), \end{aligned} $$

for every point x of the Euclidean space $\mathbb{R}^n$. So that we can conclude

$$ \begin{aligned} u\left(\int_{\mathbb{R}^m} ag\right) &= u(f) = \\ &= a(g(u)) = \\ &= \int_{\mathbb{R}^m} au(g), \end{aligned} $$

as we claimed. ■

Definition. *A functional $L : \mathcal{S}_n \to \mathbb{K}$ is said an ${}^{\mathcal{S}}$functional if it sends ${}^{\mathcal{S}}$families into ${}^{\mathcal{S}}$families. An ${}^{\mathcal{S}}$functional L is said an ${}^{\mathcal{S}}$linear functional if*

$$ \begin{aligned} L\left(\int_{\mathbb{R}^m} ag\right) &= a(g(L)) = \\ &= \int_{\mathbb{R}^m} aL(g), \end{aligned} $$

for every integer m, for any distribution a in $\mathcal{S}'_m$ and for every ${}^{\mathcal{S}}$family of test functions in $\mathcal{S}_n$.

A distribution u in $\mathcal{S}'_n$ is an ${}^{\mathcal{S}}$functional and it is ${}^{\mathcal{S}}$linear since the above proposition.

6.6 Superpositions of ${}^{\mathcal{S}}$families in $\mathcal{S}'_n$

If $v = (v_q)_{q\in\mathbb{R}^k}$ is a family of families of $\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$, indexed by $\mathbb{R}^k$, we want to superpose v in order to obtain a family of $\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$.

Definition (superposition of a family of families). *Let $v = (v_q)_{q\in\mathbb{R}^k}$ be a family of families of $\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$, indexed by $\mathbb{R}^k$, and let a be a tempered distribution in $\mathcal{S}'_k$. We define* ***superposition of the family** v **by the coefficient distribution*** *a (in a natural way) by*

$$\left(\int_{\mathbb{R}^k} av\right)_p = \int_{\mathbb{R}^k} av(p),$$

for every index p in $\mathbb{R}^m$; where the family

$$v(p) = (v(p)_q)_{q\in\mathbb{R}^k}$$

is (just) defined by

$$v(p)_q = v_q(p),$$

for every q in $\mathbb{R}^k$, and it is assumed of class $\mathcal{S}$, for every index p in $\mathbb{R}^m$.

Concerning its associated operator we have the following result.

Proposition. *We have*

$$\left(\int_{\mathbb{R}^k} av\right)(g) = \int_{\mathbb{R}^k} av(g),$$

where the superposition

$$\int_{\mathbb{R}^k} av$$

of the family of families v with respect to the coefficient distribution a is defined in the previous definition, and the superposition

$$\int_{\mathbb{R}^k} av(g)$$

is the superposition of the family of test functions $v(g) = (v(p)(g))_{p\in\mathbb{R}^m}$ with respect to the same distribution cvoefficient a.

Proof. We have, for every p in $\mathbb{R}^m$,

$$\begin{aligned}
\left(\int_{\mathbb{R}^k} av\right)^{\wedge}(g)(p) &= \left(\int_{\mathbb{R}^k} av\right)_p(g) \\
&= \left(\int_{\mathbb{R}^k} av(p)\right)(g) = \\
&= (a \circ v(p)^{\wedge})(g) = \\
&= a(v(p)(g)) = \\
&= \left(\int_{\mathbb{R}^k} av(g)\right)(p),
\end{aligned}$$

as we claimed. ■

6.7 ${}^{\mathcal{S}}$Linear superpositions of operators

Definition. *Let $A = (A_q)_{q\in\mathbb{R}^k}$ be a family of linear continuous operators from $\mathcal{S}_n$ into $\mathcal{S}_m$. We say that the family A is of class $\mathcal{S}$ if, for every test function g in $\mathcal{S}_n$, the family*

$$A(g) = (A_q(g))_{q\in\mathbb{R}^k}$$

is of class $\mathcal{S}$. In this case we put

$$\left(\int_{\mathbb{R}^k} aA\right)(g) = \int_{\mathbb{R}^k} aA(g),$$

for every test function g.

Theorem. *Let $A = (A_q)_{q\in\mathbb{R}^k}$ be a family of linear continuous operators from $\mathcal{S}_n$ into $\mathcal{S}_m$. Then, the family A is of class $\mathcal{S}$ if and only if the family of corresponding families $A^\vee = (A_q^\vee)_{q\in\mathbb{R}^k}$ is of class $\mathcal{S}$ and in this case we have*

$$\left(\int_{\mathbb{R}^k} aA\right)^\vee = \int_{\mathbb{R}^k} aA^\vee,$$

for every coefficient distribution a.

Proof. The family A is of class $\mathcal{S}$ if and only if the family $A(g)$ is an $\mathcal{S}$-family, for every test function g in $\mathcal{S}_n$; the family $A(g)$ (it lies in the space $\mathcal{S}(\mathbb{R}^k, \mathcal{S}_m)$) is the family of test functions $(A_q(g))_{q\in\mathbb{R}^k}$, the last family is of class $\mathcal{S}$ if and only if, for every p in $\mathbb{R}^m$, the function from $\mathbb{R}^k$ into $\mathbb{K}$ defined by

$$q \mapsto A_q(g)(p)$$

is of class $\mathcal{S}$. Let us pass to the family $A^\vee = (A_q^\vee)_{q\in\mathbb{R}^k}$. It is a family of class $\mathcal{S}$ if and only if the family of distribution

$$A^\vee(p) = (A_q^\vee(p))_{q\in\mathbb{R}^k}$$

is of class $\mathcal{S}$, for every p in $\mathbb{R}^m$, this last family if of class $\mathcal{S}$ if and only if, for every test function g, the function $A^\vee(p)(g)$ is of class $\mathcal{S}$, but this last function is defined exactly by $q \mapsto A_q(g)(p)$, so we conclude the first part. Let us now see that

$$\left(\int_{\mathbb{R}^k} aA\right)^\vee = \int_{\mathbb{R}^k} aA^\vee,$$

for every coefficient distribution a. We have

$$\left(\int_{\mathbb{R}^k} aA\right)^\vee_p (g) \quad = \quad (\delta_p \circ \left(\int_{\mathbb{R}^k} aA\right))(g) =$$

$$
\begin{aligned}
&= (\delta_p \left(\int_{\mathbb{R}^k} aA\right)(g)) = \\
&= \delta_p \left(\int_{\mathbb{R}^k} aA(g)\right) = \\
&= a(A(g)(\delta_p)) = \\
&= a(A(g)(p)) = \\
&= \left(\int_{\mathbb{R}^k} aA^\vee\right)(g)(p),
\end{aligned}
$$

for every coefficient distribution a and every point p, since the operator generate by a family of families v is

$$
\left(\int_{\mathbb{R}^k} av\right)^\wedge (g)(p) = a(v(p)(g)),
$$

and then

$$
\begin{aligned}
\left(\int_{\mathbb{R}^k} aA^\vee\right)(g)(p) &= \left(\int_{\mathbb{R}^k} aA^\vee\right)^\wedge (g)(p) = \\
&= a(A^\vee(p)(g)) = \\
&= a(A^{\vee\wedge}(g)(p)) = \\
&= a(A(g)(p)),
\end{aligned}
$$

as we desired. ■

Chapter 7

First applications

7.1 The Fourier expansion theorem

Definition (the (a, b)-Fourier family). *Let $a, b \in \mathbb{R}_{\neq}$ be two real non-zero numbers. The (a, b)-**Fourier family** in the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n, \mathbb{C})$ is the following family of regular tempered distributions*

$$\left(\left[(1/a)^n e^{-ib(p|\cdot)}\right]\right)_{p\in\mathbb{R}^n}.$$

Remark (the De Broglie family). In the particular case $a = 1$ and $b = -1/\hbar$ (with $\hbar$ the reduced Planck constant) we obtain what we call the De Broglie family, i.e. the family

$$\left(\left[e^{(i/\hbar)(p|\cdot)}\right]\right)_{p\in\mathbb{R}^n}.$$

Proposition (on the operators associated with the Fourier families). *Let $a, b \in \mathbb{R}_{\neq}$ and φ be the (a, b)-Fourier family. Then, the family φ is of class $\mathcal{S}$ and, more precisely, we have*

$$\varphi(\phi) = \mathcal{S}_{(a,b)}(\phi),$$

for each test function $\phi \in \mathcal{S}(\mathbb{R}^n, \mathbb{C})$. Thus the family φ generates the (a, b) Fourier-Schwartz transformation on $\mathcal{S}(\mathbb{R}^n, \mathbb{C})$, i.e., $\widehat{\varphi} = \mathcal{S}_{(a,b)}$.

Proof. For each test function $\phi \in \mathcal{S}(\mathbb{R}^n, \mathbb{C})$ and for each p in $\mathbb{R}^n$, we have

$$\begin{aligned} \varphi(\phi)(p) &= \varphi_p(\phi) = \\ &= \left[(1/a)^n e^{-ib(p|\cdot)}\right](\phi) = \\ &= \int_{\mathbb{R}^n} (1/a)^n e^{-ib(p|\cdot)} \phi \, \mu_n = \\ &= \mathcal{S}_{(a,b)}(\phi)(p), \end{aligned}$$

and thus $\varphi(\phi) = \mathcal{S}_{(a,b)}(\phi)$. Now, since the (a, b) Fourier-Schwartz transform sends $^{\mathcal{S}}$functions into $^{\mathcal{S}}$functions, the function $\varphi(\phi)$ lies in $\mathcal{S}(\mathbb{R}^n, \mathbb{C})$. ■

Example. We have, for any multi-index $\alpha \in \mathbb{N}^n$,

$$\begin{aligned} \mathcal{F}_{(a,b)}(u^{(\alpha)}) &= (bi)^\alpha \left(\mathbb{I}_{\mathbb{R}^n}\right)^\alpha \mathcal{F}_{(a,b)}(u); \\ \mathcal{F}_{(a,b)}(\left(\mathbb{I}_{\mathbb{R}^n}\right)^\alpha u) &= \left(\frac{i}{b}\right)^\alpha \left(\mathcal{F}_{(a,b)}(u)\right)^{(\alpha)}, \end{aligned}$$

where, $\mathbb{I}_{\mathbb{R}^n}$ is the identity operator on $\mathbb{R}^n$, and $\left(\mathbb{I}_{\mathbb{R}^n}\right)^\alpha$ the α-th power of the identity in multi-indexed notation, that is

$$\left(\mathbb{I}_{\mathbb{R}^n}\right)^\alpha = \prod_{j=1}^n \mathrm{pr}_j^{\alpha_j},$$

where pr_j is the canonical projection of the Cartesian power $\mathbb{R}^n$. These two properties can be immediately translated in terms of superpositions. Let φ be the (a, b)-Fourier family. We have, for all $\alpha \in \mathbb{N}_0^n$,

$$\begin{aligned} \int_{\mathbb{R}^n} u^{(\alpha)} \varphi &= (bi)^\alpha \left(\mathbb{I}_{\mathbb{R}^n}\right)^\alpha \int_{\mathbb{R}^n} u\varphi; \\ \int_{\mathbb{R}^n} \left(\mathbb{I}_{\mathbb{R}^n}\right)^\alpha u\varphi &= \left(\frac{i}{b}\right)^\alpha \left(\int_{\mathbb{R}^n} u\varphi\right)^{(\alpha)}. \end{aligned}$$

Example. Moreover, we have

$$\mathcal{F}_{(a,b)}\left(\tau_h(u)\right) = e^{-ib(h|\cdot)} \mathcal{F}_{(a,b)}(u)$$

under Fourier transforms translations become multiplications by characters, and

$$\mathcal{F}_{(a,b)}\left(e^{ib(h|\cdot)} u\right) = \tau_h\left(\mathcal{F}_{(a,b)}(u)\right)$$

under Fourier transforms multiplications by characters become translations. This two properties become

$$\int_{\mathbb{R}^n} \tau_h(u)\varphi = e^{-ib(h|\cdot)} \int_{\mathbb{R}^n} u\varphi$$

and

$$\int_{\mathbb{R}^n} e^{ib(h|\cdot)} u\varphi = \tau_h \left(\int_{\mathbb{R}^n} u\varphi \right).$$

The next theorem affirms that any tempered distribution is an ${}^{\mathcal{S}}$linear combination of the (a, b)-Fourier family, without technical assumptions.

Theorem (Fourier expansion theorem). *Let $u \in \mathcal{S}'(\mathbb{R}^n, \mathbb{C})$ be a tempered distribution and φ be the family of regular tempered distributions*

$$\left(\left[(1/a)^n e^{-ib(p|\cdot)} \right] \right)_{p \in \mathbb{R}^n}.$$

Then, we have

$$u = \int_{\mathbb{R}^n} \mathcal{F}^{-}_{(a,b)}(u)\varphi.$$

In other words, u is the superposition of φ under the system of coefficients $\mathcal{F}^{-}_{(a,b)}(u)$.

Proof. For every test function $\phi \in \mathcal{S}(\mathbb{R}^n,\ \mathbb{C})$, we have

$$\begin{aligned} u(\phi) &= u(\mathcal{S}^{-}_{(a,b)}(\mathcal{S}_{(a,b)}(\phi)) = \\ &= \mathcal{F}^{-}_{(a,b)}(u)(\widehat{\varphi}(\phi)) = \\ &= \left(\int_{\mathbb{R}^n} \mathcal{F}^{-}_{(a,b)}(u)\varphi \right)(\phi), \end{aligned}$$

as we desired. ■

Application. Another consequence of the Fourier ${}^{\mathcal{S}}$expansion theorem is the following pseudo-integral equality:

$$\int_{\mathbb{R}} x e^{-i\xi x} dx = 2\pi i \delta'(\xi).$$

Obviously the above formula must be read in the new sense of superposition, i.e. like this

$$\int_{\mathbb{R}} j_{\mathbb{R}} (e^{-i(\cdot|x)})_{x \in \mathbb{R}} = 2\pi i \delta'_0,$$

where $j_{\mathbb{R}}$ is the immersion of the real line into the complex plane, that is the function defined by $j_{\mathbb{R}}(x) = x$. In fact taking into account that

$$\mathcal{F}_{(a,b)}(j^{\alpha}_{\mathbb{R}} u) = (i/b)^{\alpha} (\mathcal{F}_{(a,b)}(u))^{(\alpha)}$$

and that

$$\mathcal{F}_{(1,1)}(1_{\mathbb{R}}) = 2\pi \delta_0,$$

we conclude

$$\begin{aligned}\int_{\mathbb{R}} [j_{\mathbb{R}}] \, (e^{-i(\cdot|x)})_{x\in\mathbb{R}} &= {}^{t}\mathcal{S}_{(1,1)} \left([j_{\mathbb{R}}]\right) = \\ &= \mathcal{F}_{(1,1)} \left([j_{\mathbb{R}}]\right) = \\ &= \left(\frac{i}{1}\right)^{1} \left(\mathcal{F}_{(1,1)} \, [1_{\mathbb{R}}]\right)' = \\ &= i \, (2\pi\delta_0)' = \\ &= 2\pi i\delta_0',\end{aligned}$$

as we claimed. ■

7.2 Convolution as superpositions

In this section we shall see that the convolution is a particular case of superposition, restoring a common vision on the expression (0). Obviously, not all the superpositions can be viewed as convolutions, only a particular class of them.

Note that, in the language of superpositions, if $a \in \mathcal{S}'_m$ and $b \in \mathcal{S}'_n$ are two tempered distributions, the tensor product $a \otimes b$ is defined, for every function f in the test function space $\mathcal{S}_{m+n}$, by the following numerical superposition

$$(a \otimes b)\,(f) := \int_{\mathbb{R}^m} a \;\; (b(f(p,\cdot)))_{p\in\mathbb{R}^m} \, .$$

Recall now that the convolution $a * b$, with $a \in \mathcal{E}'_n$ and $b \in \mathcal{S}'_n$, is defined, for every ϕ in $\mathcal{D}_n$, by

$$(a * b)\,(\phi) = (a \otimes b)\,(\phi \circ A),$$

where A is the standard addition in $\mathbb{R}^n$, that is the bilinear operator $A : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, defined by $A(x, y) = x + y$, for any two points x, y of $\mathbb{R}^n$.

Theorem (the convolution as superposition). *Let a and b be two tempered distributions, and assume a with compact support. Then, the family of translations $(\tau_p b)_{p\in\mathbb{R}^n}$ is a smooth family, and moreover we have*

$$a * b = \int_{\mathbb{R}^n} a(\tau_p b)_{p\in\mathbb{R}^n},$$

that is the convolution of a and b is the superposition of the (ordered) family of translations of the distribution b with respect to the distribution a.

Proof. Put $v_p = \tau_p(b)$, for every n-tuple p , we see that, for every test function ϕ,

$$\begin{aligned} v(\phi)(p) &= v_p(\phi) = \\ &= \tau_p(b)(\phi) = \\ &= b(\tau_{-p}(\phi)). \end{aligned}$$

Hence, setting $f := \phi \circ A$, i.e.,

$$\begin{aligned} f(p,x) &= \tau_{-p}(\phi)(x) \\ &= \phi(x+p), \end{aligned}$$

for every pair (p,x) in $\mathbb{R}^n \times \mathbb{R}^n$, we read

$$v(\phi)(p) = b(f(p,\cdot)).$$

With standard technics, it can be proved that $v(\phi)$ is a smooth function (in general not of class $\mathcal{S}$), then v is an $\mathcal{E}$-family. Being a a compact support distribution, we can consider the superposition of the family v with respect to the distribution a, obtaining

$$\begin{aligned} \left(\int_{\mathbb{R}^n} av\right)(\phi) &= a(v(\phi)) = \\ &= \int_{\mathbb{R}^n} a \ (b(f(p,\cdot)))_{p\in\mathbb{R}^n} = \\ &= (a \otimes b)(f) = \\ &= (a * b)(\phi), \end{aligned}$$

as we desired. ■

Remark. The family $(\tau_p b)_{p\in\mathbb{R}^n}$ is the image of the distribution b under the one parameter group $\tau = (\tau_p)_{p\in\mathbb{R}^n}$.

Let us see the case in which one distribution is in the convolution operator space $\mathcal{O}'_C(n)$.

Theorem (the convolution as superposition). *Let u and g be two tempered distributions in $\mathcal{S}'_n$, g belonging to the space $\mathcal{O}'_C(n)$. Then, the family of translations $(\tau_p g)_{p\in\mathbb{R}^n}$ is an ${}^{\mathcal{S}}$family, and moreover*

$$u * g = \int_{\mathbb{R}^n} u(\tau_p g)_{p\in\mathbb{R}^n}.$$

Proof. Put $G_p = \tau_p(g)$, for every n-tuple p. For every test function ϕ in $\mathcal{S}_n$, we see that

$$\begin{aligned} G(\phi)(p) &= G_p(\phi) = \\ &= \tau_p(g)(\phi) = \\ &= g(\tau_{-p}(\phi)), \end{aligned}$$

so we can proceed as in the above proof, with u tempered distribution, as soon as we prove that the family $G = (G_p)_{p\in\mathbb{R}^n}$ is an ${}^{\mathcal{S}}$family. To prove that G is an ${}^{\mathcal{S}}$family let us see that its transformation by the Fourier transform $\mathcal{F}_{(a,b)}$ is an ${}^{\mathcal{S}}$family, we can do so because the operator $\mathcal{F}_{(a.b)}$ is a topological automorphism of $\mathcal{S}'_n$. Since the Fourier transform maps the space $\mathcal{O}'_C(n)$ onto the space of regular distributions $[\mathcal{O}_M(n)]$, there exists a slowly increasing function f such that

$$\mathcal{F}_{(a,b)}(g) = [f]\,,$$

so we have

$$\begin{aligned}
\mathcal{F}_{(a,b)}(G_p) &= \mathcal{F}_{(a,b)}(\tau_p g) = \\
&= e^{-ib(p|\cdot)}\mathcal{F}_{(a,b)}(g) = \\
&= e^{-ib(p|\cdot)}\,[f]\,.
\end{aligned}$$

We should notice, now, that the product $e^{-ib(p|\cdot)}\,[f]$ is equal to the product $f\left[e^{-ib(p|\cdot)}\right]$. Note, first of all, that both product has a proper sense, being products of $\mathcal{O}_M(n)$ functions by tempered distributions. For every test function ϕ, we have indeed

$$\begin{aligned}
e^{-ib(p|\cdot)}\,[f]\,(\phi) &= [f]\,(e^{-ib(p|\cdot)}\phi) = \\
&= \int_{\mathbb{R}^n} e^{-ib(p|\cdot)} f\phi\mu_n = \\
&= \left[e^{-ib(p|\cdot)}\right](f\phi) = \\
&= f\left[e^{-ib(p|\cdot)}\right](\phi).
\end{aligned}$$

We have so, for each test function ϕ,

$$\begin{aligned}
\mathcal{F}_{(a,b)}(G_p)(\phi) &= e^{-ib(p|\cdot)}\,[f]\,(\phi) = \\
&= f\left[e^{-ib(p|\cdot)}\right](\phi) = \\
&= \left[e^{-ib(p|\cdot)}\right](f\phi) = \\
&= \mathcal{S}_{(1,b)}(f\phi),
\end{aligned}$$

where $\mathcal{S}_{(1,b)}$ is the $(1,b)$-Fourier-Schwartz transformation on the test function $\mathcal{S}_n$; thus the image of the test function ϕ under the family $\mathcal{F}_{(a,b)}(G_p)$ is a test function in $\mathcal{S}_n$, and consequently the family $\mathcal{F}_{(a,b)}(G_p)$ is an ${}^{\mathcal{S}}$family. ■

7.3 Some expressions of Dirac Calculus

In this section we shall interpret some formulas, used frequently in Dirac Calculus and in Quantum Mechanics, using the new concepts of ${}^{\mathcal{S}}$Linear Algebra introduced up to now.

7.3.1 The expansion of a vector in the Dirac basis

Let u be a tempered distribution on the n-Euclidean space, and let δ be the Dirac family of the space $\mathcal{S}'_n$. The superposition

$$\int_{\mathbb{R}^n} u\delta = u,$$

justifies completely the following formal expression used in Quantum Mechanics (see $[Di]$ page 78)

$$\int_{\mathbb{R}^n} \delta(x-p)\delta(y-x)dx = \delta(y-p).$$

In fact, for $u = \delta_p$, we deduce

$$\int_{\mathbb{R}^n} \delta_p\delta = \delta_p.$$

Critical remark. A correct mathematical interpretation of the left hand side of the formal equality is the convolution of the distribution δ_p with the distribution δ_0, but this interpretation works only because the operation of convolution, which involves only two distributions and not an entire family of distributions, can be viewied as a superposition. According to Dirac the above expression is one of the case belonging to the large class of *continuous expansions of infinite-dimensional vectors in the "basis" δ.*

So the correct interpretation of the above formal equality is the following one:

- *the vector δ_p is the linear superposition of the infinite continuous family of vectors $(\delta_y)_{y\in\,\mathbb{R}^n}$ with respect to the system of coefficients δ_p.*

Hence, for instance, we can ***rigorously*** affirm that:

- *the most general state of a quantum-particle in one dimension (i.e. a complex tempered distribution on $\mathbb{R}$) is a linear superposition of the "eigenstates" of position operator*

  $$Q : \mathcal{S}'(\mathbb{R},\mathbb{C}) \to \mathcal{S}'(\mathbb{R},\mathbb{C}) : u \mapsto j_{\mathbb{R}}u,$$

  where $j_{\mathbb{R}}$ is the immersion of $\mathbb{R}$ into $\mathbb{C}$.

7.3.2 Fourier expansions and the momentum operator

Let us see an expansion which is not a convolution. The Fourier expansion theorem justifies completely another formal expression used in Quantum Mechanics (see [Di] page 38 formula (10)), namely

$$\delta(x-p) = \int_{\mathbb{R}} \frac{1}{2\pi} e^{piy} e^{-iyx} dy.$$

In fact, a classic result on Fourier transform gives

$$\mathcal{F}^{-}_{(a,1)}(\delta_p) = \frac{a}{2\pi}\left[e^{pi(\cdot)}\right],$$

and thus, from the Fourier expansion theorem, setting $a = 1$, we obtain

$$\delta_p = \int_{\mathbb{R}} \frac{1}{2\pi}\left[e^{pi(\cdot)}\right]\left(\left[e^{-i(x|\cdot)}\right]\right)_{x\in\mathbb{R}},$$

for every p in the real line.

So we can read the above expression as follows:

- *the vector δ_p is the linear superposition of the infinite continuous family of vectors $\left(\left[e^{-i(p|\cdot)}\right]\right)_{p\in\mathbb{R}}$ with respect to the system of coefficients $(1/2\pi)\left[e^{pi(\cdot)}\right]$.*

Once more, we can affirm ***rigorously*** that

- *the most general state of a quantum-particle in one dimension (i.e. a complex tempered distribution on $\mathbb{R}$) is a linear superposition of "eigenstates" of the momentum operator*

$$P : \mathcal{S}'(\mathbb{R}, \mathbb{C}) \to \mathcal{S}'(\mathbb{R}, \mathbb{C}) : u \mapsto -i\hbar u'.$$

Another interpretation of the Fourier expansion theorem is the following one:

- *at every time $t \in \mathbb{R}$ a wave $u : \mathbb{R} \to \mathcal{S}'(\mathbb{R}^n, \mathbb{C})$ is an $^{\mathcal{S}}$superposition of the family of the harmonic waves*

$$\left(\left[(1/a)^n e^{-ib(p|\cdot)}\right]\right)_{p\in\mathbb{R}^n}$$

with respect to the system of coefficients $\mathcal{F}^{-}_{(a,b)}(u_t)$.

Note that, for $p = 0$, we have

$$2\pi\delta_0 = \int_{\mathbb{R}} \left(\left[e^{-i(x|\cdot)}\right]\right)_{x\in\mathbb{R}}.$$

So that the Dirac distribution centered at 0 is the superposition of a Fourier family.

7.4 Some extensions

Consider a family v indexed by the m-Euclidean space. Consider a compact set K of the m-space and the sub-family of v indexed by K, namely the family

$$v_{|K} := (v_p)_{p \in K}.$$

We desire to give a meaning to the superpositions of the sub-family $v_{|K}$.

Definition ($^{\mathcal{S}}$linear superpositions of compact sub-families). *Let v be an $^{\mathcal{S}}$family in $\mathcal{S}'_n$ indexed by the Euclidean real m-dimensional space, let K be a compact subset of the Euclidean m-space and let $v_{|K}$ be the restriction of the family v to the compact K, that is the family $(v_p)_{p \in K}$. If a is any tempered distribution in $\mathcal{S}'_m$, with compact support contained in K, we define **superposition of the sub-family $v_{|K}$ with respect to the coefficient distribution** a the superposition*

$$\int_K a v_{|K} := \int_{\mathbb{R}^m} a v.$$

We say also that such superposition is a superposition of the entire family v on the compact K and we write

$$\int_K a v := \int_{\mathbb{R}^m} a v.$$

Moreover, if $\mathcal{S}'_m(K)$ is the set of all tempered distributions in $\mathcal{S}'_m$ with compact support contained in K, we shall define the $^{\mathcal{S}}$linear hull of the sub-family, and we denote it by

$$^{\mathcal{S}}\mathrm{span}(v_{|K}),$$

as the set $\mathcal{S}'_n(K).v$ of all the superpositions

$$\int_K a v,$$

with a in $\mathcal{S}'_n(K)$.

Example. Let p be a point of the m-Euclidean space and $K = \{p\}$. We shall use the fact that every (tempered) distribution with compact support $\{p\}$ is an element of the linear hull of the denumerable family $d = (\delta_p^{(q)})_{q \in \mathbb{N}^m}$. For every $^{\mathcal{S}}$family v, and for every tempered distribution a with compact support K, we have

$$\begin{aligned} \int_K a v &= \int_K a v_{|K} = \\ &= \int_{\mathbb{R}^m} a v = \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{h} c_i \delta_p^{(q_i)} \circ \widehat{v} = \\
&= \sum_{i=1}^{h} (-1)^{|q_i|_1} c_i v_p^{(q_i)},
\end{aligned}
$$

where $q = (q_i)_{i=1}^h$ is some finite family of m-multi-indexes and $c = (c_i)_{i=1}^h$ some finite family of scalars such that . So we have

$$
{}^{\mathcal{S}}\mathrm{span}(v_{|K}) = \mathrm{span}(w),
$$

where w is the family $(v_p^{(q)})_{q\in\mathbb{N}^m}$.

We have already given another natural definition for superpositions on compact sets. We recall it.

Definition. *Let consider a family v in $\mathcal{S}'_n$ indexed by a compact K of the Euclidean m-space, the action of the family v on a test function g in $\mathcal{S}_n$ is a function from K into the scalar field of $\mathcal{S}'_n$. If v is a family of class $C^0(K)$, we define, for every Radon measure μ on K, the superposition $\mu.v$ by*

$$
\mu.v(g) := \mu(v(g)),
$$

for every test function g in $\mathcal{S}_n$. We denote it by

$$
\int_K \mu v,
$$

note that the Radon measure μ is a functional on the space of continuos function $C^0(K)$ and not on the space $\mathcal{S}_m$.

Let us see the relation between the two definitions.

Theorem. *Let v be an ${}^{\mathcal{S}}$family in $\mathcal{S}'_n$ indexed by the Euclidean m-space, let K be a compact subset of the Euclidean m-space and let $v_{|K}$ be the restriction of the family v to the compact K, that is the sub-family $(v_p)_{p\in K}$. Then, if μ is any Radon measure on the compact K, we have*

$$
\int_K [\mu] v = \int_K \mu v_{|K},
$$

where $[\mu]$ is the tempered distribution defined by

$$
[\mu](g) = \mu(g_{|K}),
$$

for every test function g in the space $\mathcal{S}'_m$.

Proof. In the conditions of the first definition, we have that the image $v(g)$ is a function of class $\mathcal{S}_m$ and then a continuous function. The restriction of v to K is a family indexed by K and clearly

$$v_{|K}(g) = v(g)_{|K},$$

and this restriction is continuous on the compact K; so we can consider the superposition $\mu.v_{|K}$, with μ Radon measure on K, so that

$$\begin{aligned} \mu.v_{|K}(g) &= \mu(v(g)_{|K}) = \\ &= \int_K v(g)\mu = \\ &= (\int_{\mathbb{R}^m} [\mu]v)(g) = \\ &= (\int_K ([\mu]v_{|K})(g), \end{aligned}$$

where (as usual) $[\mu]$ is the tempered distribution generated by the Radon measure μ, defined by

$$\begin{aligned} [\mu](g) &= \int_K g\mu = \\ &= \mu(g_{|K}), \end{aligned}$$

for every test function g in $\mathcal{S}_m$. So that

$$\int_K [\mu]v_{|K} = \int_K \mu v_{|K},$$

as we desired. ■

Let c_K be the characteristic function of the compact K. If a is a regular C^0-coefficient distribution generated by a function f, we have

$$\begin{aligned} (\int_K ([fc_K]v_{|K})(g) &= (\int_{\mathbb{R}^m} [fc_K]v)(g) = \\ &= \int_{\mathbb{R}^m} fc_K v(g)\mu_m = \\ &= \int_K fv(g)\mu_m = \\ &= (\int_K [f]v)(g), \end{aligned}$$

where the superposition

$$\int_K [f]v$$

is the superposition of the C^0-family v (if v is an $^{\mathcal{S}}$family it is in particular a C^0 family), with respect to the coefficient measure $[f]$, on the compact K. We can write

$$\int_K ([fc_K]v_{|K} = \int_K [f]v,$$

since the family v is in $\mathcal{S}'_n$.

Part III

$^{\mathcal{S}}$Linear Algebra and Geometry

Chapter 8

$^{\mathcal{S}}$Linear hulls of $^{\mathcal{S}}$ families

8.1 $^{\mathcal{S}}$Linear hulls

In this section we present the $\mathcal{S}$-linear analogous of the concept of linear hull of a finite ordered system of vectors.

Definition (of $^{\mathcal{S}}$linear hull). *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be an $^{\mathcal{S}}$family of tempered distributions. The* ***$^{\mathcal{S}}$linear hull of the family*** *v is the set of all the $\mathcal{S}$-linear combinations of the family v. We shall denote it by $^{\mathcal{S}}\mathrm{span}\,(v)$, or simply by $\mathcal{S}(v)$, and in symbols we have so*

$$^{\mathcal{S}}\mathrm{span}\,(v) := \,^t\widehat{v}(\mathcal{S}'_m),$$

or more explicitly

$$^{\mathcal{S}}\mathrm{span}\,(v) := \{u \in \mathcal{S}'_n : \exists a \in \mathcal{S}'_m : u = \int_{\mathbb{R}^m} av\}.$$

Example (on the Dirac and Fourier families). Let δ be the Dirac family of $\mathcal{S}'_n$, then we have

$$^{\mathcal{S}}\mathrm{span}\,(\delta) = \mathcal{S}'_n.$$

In fact, for any distribution $u \in \mathcal{S}'_n$, we obtain

$$\begin{aligned} u &= u \circ \mathbb{I}_{\mathcal{S}_n} = \\ &= u \circ \widehat{\delta} = \\ &= \int_{\mathbb{R}^n} u\delta. \end{aligned}$$

Let

$$\varphi = \left(\left[(1/a)^n e^{-ib(p|\cdot)}\right]\right)_{p \in \mathbb{R}^n}$$

be the Fourier family, we have again ${}^{\mathcal{S}}\mathrm{span}\,(\varphi) = \mathcal{S}'_n$, as it follows immediately from the Fourier ${}^{\mathcal{S}}$expansion theorem.

8.2 Algebraic properties of ${}^{\mathcal{S}}$linear hulls

Theorem (on the structure of ${}^{\mathcal{S}}$**span).** *Let* $u \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ *be a family of tempered distributions. Then, the* $\mathcal{S}$ *-linear hull* ${}^{\mathcal{S}}\mathrm{span}\,(u)$ *of the family* u *is a subspace of the space* $\mathcal{S}'_n$*, it contains all the elements of the family* u *and, consequently, it contains the linear hull of* u*, that is we have*

$$\mathrm{span}\,(u) \subseteq {}^{\mathcal{S}}\mathrm{span}\,(u)\,.$$

Proof. The fact that the ${}^{\mathcal{S}}$linear hull of a family is a subspace derives immediately from the circumstance that it is the image of a linear operator, but we shall show the fact also in another more explicit way. Let $\lambda \in \mathbb{K}$ be a scalar and let v, w be two vectors in the $\mathcal{S}$-liner hull ${}^{\mathcal{S}}\mathrm{span}\,(u)$, then, there exist two tempered distributions $a, b \in \mathcal{S}'_m$ such that

$$v = \int_{\mathbb{R}^m} au, \; w = \int_{\mathbb{R}^m} bu.$$

Now, we see

$$\begin{aligned} \lambda v + w &= \lambda \int_{\mathbb{R}^m} au + \int_{\mathbb{R}^m} bu = \\ &= \int_{\mathbb{R}^m} (\lambda a + b)\, u, \end{aligned}$$

and so the linear combination $\lambda v + w$ belongs to the hull ${}^{\mathcal{S}}\mathrm{span}\,(u)$ too. On the contrary, the fact that the ${}^{\mathcal{S}}$linear hull contains any element of u is a fact depending more than the preceding one on the ${}^{\mathcal{S}}$linear properties, namely the selection property of Dirac distributions. So, let δ be the Dirac family of the space of coefficient distributions $\mathcal{S}'_m$, we have

$$\int_{\mathbb{R}^m} \delta_p u = u_p,$$

for any index p of the family u, and then any element u_p of the family u is an element of the $^{\mathcal{S}}$linear hull $^{\mathcal{S}}$span (u) too. ■

So, if the (algebraic) linear hull of an $^{\mathcal{S}}$family v is infinite dimensional then the $^{\mathcal{S}}$linear hull of the family v is infinite dimensional too. But we can give another sufficient condition.

Theorem. *Let v be an $\mathcal{S}$-family having an element whose derivatives span (algebraically) an infinite dimensional subspace. Then the $^{\mathcal{S}}$linear hull of the family v is infinite dimensional.*

Proof. Consider the $^{\mathcal{S}}$family v in the space $\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$. The $^{\mathcal{S}}$linear hull of the family v is the image of the space $\mathcal{S}'_m$ under the superposition operator of the family v. The space $\mathcal{S}'_m$ contains, for every m-index p of v, the space M_p of tempered distributions with compact support contained in the singleton $\{p\}$, that is the (algebraic) linear hull of the family of derivatives of the Dirac distribution centered at p. The space M_p is the linear hull of the denumerable linearly independent family $d = (\delta_p^{(q)})_{q\in\mathbb{N}^m}$. Now, it is clear that the $^{\mathcal{S}}$ linear hull $\mathcal{S}'_m.v$ contains the $^{\mathcal{S}}$ linear sub-hull $M_p.v$, for each m-index p. It is also clear that this last sub-hull is infinite dimensional if and only if the (denumerable) family of derivatives $(v_p^{(q)})_{q\in\mathbb{N}^m}$ generates (algebraically) an infinite dimensional space. Indeed, more specifically, we have

$$\begin{aligned} M_p.v &= \operatorname{span}(\delta_p^{(q)} \circ \widehat{v})_{q\in\mathbb{N}^m} = \\ &= \operatorname{span}((-1)^{|q|_1} v_p^{(q)})_{q\in\mathbb{N}^m} = \\ &= \operatorname{span}(v_p^{(q)})_{q\in\mathbb{N}^m}. \end{aligned}$$

So, if the family v has an element v_p whose derivatives span an infinite dimensional subspace, then the $^{\mathcal{S}}$linear hull of the family v is infinite dimensional. ■

The following example shows that there are families with finite dimensional linear span and infinite dimensional $^{\mathcal{S}}$linear span.

Example. Consider a distribution u whose derivatives span an infinite dimensional subspace (for instance the Dirac distribution centered at the origin) and let g be a nonzero test function in $\mathcal{S}_m$. Consider the family $v = (g(p)u)_{p\in\mathbb{R}^m}$. It is clear that the linear hull of the family v is the mono-dimensional subspace generated by the distribution u. On the contrary, if p_0 is a point such that the scalar $g(p_0)$ is different from 0, we have that the subspace generated by the derivatives of the distribution $g(p_0)u$ is infinite dimensional (it coincides exactly with the subspace generated by the derivatives of u), so that the ${}^{\mathcal{S}}$linear hull of the family v is infinite dimensional.

8.3 Systems of ${}^{\mathcal{S}}$generators

Definition (system of ${}^{\mathcal{S}}$generators). *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be an ${}^{\mathcal{S}}$family of tempered distributions. The family v is called* ***a system of ${}^{\mathcal{S}}$generators*** *for a subspace V of the space $\mathcal{S}'_n$ if and only if its ${}^{\mathcal{S}}$linear hull coincides with the subspace V, in symbols if*

$$^{\mathcal{S}}\mathrm{span}(v) = V.$$

Example. The Dirac family and the Fourier families (in the complex case) are systems of ${}^{\mathcal{S}}$generators for the entire space $\mathcal{S}'_n$.

8.3.1 Exercises

Exercise. Let V be the subspace of $\mathcal{S}'_1$ formed by the distributions with compact support contained in some non-degenerate compact interval K (of the real line). Let g be a smooth functional defined on the real line with compact support the interval K and everywhere different from 0 on of the interval K. Consider the family $v = (g(x)\delta_x)_{x\in\mathbb{R}}$ in $\mathcal{S}'_1$. The family v is of class $\mathcal{S}$ (even more, it is of class $\mathcal{D}$), indeed if h is a test function in $\mathcal{S}_1$ we have

$$\begin{aligned} v(h)(x) &= v_x(h) = \\ &= g(x)\delta_x(h) = \\ &= (gh)(x), \end{aligned}$$

for every real number x, so that the function $v(h)$ is of class $\mathcal{S}$ (indeed it is of class $\mathcal{D}$) and the operator associated with the family v is the multiplication operator by the function g. We know (in Functional Analysis) that the transpose of the *multiplication operator by a function* on a the test function space $\mathcal{S}_1$ is the multiplication operator by the same function on the distribution space $\mathcal{S}'_1$. But we desire to see this fact directly. Let us consider now a generic superposition of the family v, if a is a coefficient distribution for the family v, we have

$$\begin{aligned} a.v(h) &= a(v(h)) = \\ &= a(gh) = \\ &= ga(h), \end{aligned}$$

so that $a.v = ga$, as we already knew. The product of a function g of class $\mathcal{D}$ by a distribution is a distribution with compact support contained in the support of the function g, and consequently the $^{\mathcal{S}}$linear hull of the family v is included into the space of distribution with compact support contained in the interval K, the space $\mathcal{S}_1'(K)$, i.e. we have

$$^{\mathcal{S}}\mathrm{span}(v) \subseteq V.$$

Open questions. Is the above inclusion strict, or the $^{\mathcal{S}}$linear hull of the family v coincides with V? In a less general fashion, is the Dirac delta centered at an end point p of the interval K in the $^{\mathcal{S}}$linear hull of the family v or not? Take into account that, if a is such that $a.v = \delta_p$, we have $ga = \delta_p$, so that the distribution a must vanish on the co-level 0 of the function g, that is the distribution a must vanish on the interior of the compact K (indeed from $ga = \delta_p$ follows

$$g^2 a = g(p)\delta_p = 0,$$

and the level 0 of g coincides with the level 0 of g^2).

Example. Let V be the subspace of $\mathcal{S}_1'$ formed by the distributions with compact support contained in some non-degenerate bounded open interval I (of the real line), let us denote by $\mathcal{S}_1'(I)$ this subspace. Let g be a smooth functional defined on the real line with compact support the closure of I and everywhere different from 0 on of the interval I. Consider the above family $v = (g(x)\delta_x)_{x\in\mathbb{R}}$ in $\mathcal{S}_1'$. The product of the function g by the distribution generated by the constant unitary function $1_{\mathbb{R}}$ is the distribution generated by g which has compact support not contained (is the support of the function g) in the open interval I and consequently the $^{\mathcal{S}}$linear hull of the family v is not included into the space of distribution with compact support contained in the open interval I, the space $\mathcal{S}_1'(I)$. Vice versa, if u is a distribution in $\mathcal{S}_1'(I)$, we have $u = a.v$, where a is the distribution in $\mathcal{S}_1'$, such that $ga = u$, defined as follows: let K be the convex envelope of the support of u, this convex envelope is a compact interval contained in I, let f be a smooth function on the real line equal to g on K and different from zero everywhere, we put $a = (1/f)u$ have

$$\begin{aligned} a.v &= ga = \\ &= g(1/f)u = \\ &= u, \end{aligned}$$

since $g(1/h) = 1$ on the support of u (the distribution u is tempered and then of finite order), and then

$$V \subseteq {}^{\mathcal{S}}\mathrm{span}(v).$$

8.4 Topological properties of ${}^{\mathcal{S}}$linear hulls

Notation. In what follows we shall use the notation $\beta(\mathcal{S}'_n)$ for the strong topology $\beta(\mathcal{S}'_n, \mathcal{S}_n)$ and $\sigma(\mathcal{S}'_n)$ for the weak* topology $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$ on the space of tempered distribution $\mathcal{S}'_n$; analogously, we shall use the notation $\sigma(\mathcal{S}_n)$ for the weak topology $\sigma(\mathcal{S}_n, \mathcal{S}'_n)$ on the space of test functions $\mathcal{S}_n$.

Let us see the relation among the $\mathcal{S}$-linear hull of an $\mathcal{S}$-family and the closed linear hull of the same family with respect to the strong topology $\beta(\mathcal{S}'_n)$, or equivalently the weak* one $\sigma(\mathcal{S}'_n)$, since the strong closed subspaces are the weak* ones. Note indeed that, since the topological vector space $\mathcal{S}_n$ is reflexive, it is in particular semi-reflexive and then the linear subspaces of $\mathcal{S}'_n$ are closed in the strong topology $\beta(\mathcal{S}'_n)$ if and only if they are closed in the weak* topology $\sigma(\mathcal{S}'_n)$, so the closed linear hull with respect to the strong topology $\beta(\mathcal{S}'_n)$ of a subset coincides with the $\sigma(\mathcal{S}'_n)$-closed hull of the same set.

Theorem. *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be a system of $\mathcal{S}$-generators for the space $\mathcal{S}'_n$. Then, the family v is a system of topological generators for the space $\mathcal{S}'_n$ with respect to the strong topology $\beta(\mathcal{S}'_n)$, that is we have*

$$\overline{\mathrm{span}}_{\beta(\mathcal{S}'_n)}(v) = \mathcal{S}'_n.$$

Proof. To prove that the linear hull of the family v is dense in the space $\mathcal{S}'_n$, with respect to the strong topology $\beta(\mathcal{S}'_n)$, we shall prove that every linear $\beta(\mathcal{S}'_n)$-continuous form on $\mathcal{S}'_n$ which is zero on the family v, is zero on the whole of the space $\mathcal{S}'_n$. In fact, let L be such a form, since $\mathcal{S}'_n$ is reflexive, there is a test function l in $\mathcal{S}_n$ such that $L(u) = u(l)$, for every distribution u in $\mathcal{S}'_n$. Since the functional L is zero on each member of the family v, for every index i of the family v, we have

$$\begin{aligned} 0 &= L(v_i) = \\ &= v_i(l) = \\ &= v(l)(i) = \\ &= \widehat{v}(l)(i), \end{aligned}$$

and hence the test function $\widehat{v}(l)$ is the origin of the space $\mathcal{S}_m$. Now, the family v is a system of ${}^{\mathcal{S}}$ generators for the entire space $\mathcal{S}'_n$ if and only if the transpose operator ${}^t\widehat{v}$ is surjective (indeed, this transpose operator is nothing but the superposition operator $\int_{\mathbb{R}^m}(., v)$ associated with the family v) and thus, applying the Schwartz-Dieudonné theorem on Fréchet spaces, the operator $\widehat{v}$ is injective, so that the test function l is the origin of the space $\mathcal{S}_n$, and then the linear form L is the origin of the bidual $\mathcal{S}''_n$. ■

This result can be usefully generalized.

Theorem. *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be an $^{\mathcal{S}}$family in $\mathcal{S}'_n$. Then, the following inclusion holds true*

$$^{\mathcal{S}}\mathrm{span}\,(v) \subseteq \overline{\mathrm{span}}_{\beta(\mathcal{S}'_n)}\,(v) = \overline{\mathrm{span}}_{\sigma(\mathcal{S}'_n)}\,(v)\,.$$

Proof. We shall prove that every superposition of the family v is the $\beta(\mathcal{S}'_n)$-limit of a sequence of finite linear combinations of the family v. Let a be in the coefficient space $\mathcal{S}'_m$, then the coefficient distribution a is the $\beta(\mathcal{S}'_m)$-limit of a sequence d of finite combinations of the Dirac family of $\mathcal{S}'_m$, since the Dirac family is $\beta(\mathcal{S}'_m)$-total in $\mathcal{S}'_m$. We have

$$\begin{aligned} \int_{\mathbb{R}^m} av &= \int_{\mathbb{R}^m} \left({}^{\beta(\mathcal{S}'_m)} \lim_{k\to\infty} d_k \right) v = \\ &= {}^{\beta(\mathcal{S}'_n)} \lim_{k\to\infty} \int_{\mathbb{R}^m} d_k v, \end{aligned}$$

by the $(\beta(\mathcal{S}'_m), \beta(\mathcal{S}'_n))$-continuity of the superposition operator ${}^t\widehat{v}$. Moreover, by the selection property of Dirac distributions, the superposition $\int_{\mathbb{R}^m} d_n v$ is a finite linear combination of the family v, and this concludes the proof. ■

8.5 Closedness of $^{\mathcal{S}}$linear hulls

The following theorem shows when the $^{\mathcal{S}}$linear hull $^{\mathcal{S}}\mathrm{span}\,(v)$ of an $^{\mathcal{S}}$family v is closed with respect to the weak* topology $\sigma(\mathcal{S}'_n)$. This result is one of the main justification of the use of $^{\mathcal{S}}$linear hulls.

Theorem. *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be an $^{\mathcal{S}}$family in $\mathcal{S}'_n$. Then, the following conditions are equivalent:*

- *1) the hull $^{\mathcal{S}}\mathrm{span}\,(v)$ is $\sigma(\mathcal{S}'_n)$-closed in $\mathcal{S}'_n$, i.e. it is $\beta(\mathcal{S}'_n)$ -closed;*
- *2) the $^{\mathcal{S}}$linear hull $^{\mathcal{S}}\mathrm{span}\,(v)$ coincides with the $\sigma(\mathcal{S}'_n)$-closed linear hull $\overline{\mathrm{span}}_{\sigma(\mathcal{S}'_n)}\,(v)$;*
- *3) the superposition operator $\int_{\mathbb{R}^m}(\cdot, v)$ is a topological homomorphism for the weak* topologies $\sigma(\mathcal{S}'_m)$ and $\sigma(\mathcal{S}'_n)$;*
- *4) the image $\widehat{v}\,(\mathcal{S}_n)$ is closed in the topological vector space $(\mathcal{S}_m)$;*
- *5) the operator $\widehat{v}$ is a topological homomorphism with respect to the pair of weak topologies $(\sigma(\mathcal{S}_n), \sigma(\mathcal{S}_m))$;*
- *6) the operator $\widehat{v}$ is a topological homomorphism from the topological vector space $(\mathcal{S}_n)$ into the space $(\mathcal{S}_m)$.*

Proof. It is the Dieudonné-Schwartz theorem (see $[Di; Sch]$) reread in our context (note that the two spaces $(\mathcal{S}_n)$ and $(\mathcal{S}_m)$ are two Fréchet spaces), taking into account the preceding theorem. ■

Theorem. *Let the* $^{\mathcal{S}}$*linear hull* $^{\mathcal{S}}\mathrm{span}\,(v)$ *be* $^{\sigma(\mathcal{S}'_n)}$*closed. Then, the superposition operator* $\int_{\mathbb{R}^m}(\cdot, v)$ *is a topological homomorphism for the pair of strong topologies* $\beta(\mathcal{S}'_m)$ *and* $\beta(\mathcal{S}'_n)$*.*

Proof. It follows immediately by proposition 18, page 309 of $[Ho]$. ■

Theorem. *Let* $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ *be an* $^{\mathcal{S}}$*family. Then the following assertions are equivalent*

- *1) the family* v *is a system of* $^{\mathcal{S}}$ *generators for the entire* $\mathcal{S}'_n$*;*
- *2) the superposition operator* $\int_{\mathbb{R}^m}(\cdot, v)$ *is a surjective topological homomorphism for the weak* topologies* $\sigma(\mathcal{S}'_m)$ *and* $\sigma(\mathcal{S}'_n)$*;*
- *3) the superposition operator* $\int_{\mathbb{R}^m}(\cdot, v)$ *is a surjective topological homomorphism for the strong topologies* $\beta(\mathcal{S}'_m)$ *and* $\beta(\mathcal{S}'_n)$*;*
- *4) the operator* $\widehat{v}$ *is an injective topological homomorphism for the weak topologies* $\sigma(\mathcal{S}_n)$ *and* $\sigma(\mathcal{S}_m)$*;*
- *5) the operator* $\widehat{v}$ *is an injective topological homomorphism from the space* $(\mathcal{S}_n)$ *into the space* $(\mathcal{S}_m)$*.*

8.5.1 Examples of systems of $^{\mathcal{S}}$ generators

Example (the family associated with the i**-th component of the position operator).** The mapping $P_i : \mathcal{S}'_n \to \mathcal{S}'_n$ defined by $P_i(u) = (.)_i u$ (where $(.)_i$ is the i-th canonical projection of the Euclidean space $\mathbb{R}^n$) is called the **i-th component of the position operator on the space $\mathcal{S}'_n$**. The operator P_i is a surjective strict morphism (see for example $[Ho]$ page 352), so the associated family

$$v := P_i(\delta) = ((.)_i \delta_x)_{x \in \mathbb{R}^n},$$

is a system of $\mathcal{S}$-generators for the entire space $\mathcal{S}'_n$. Note that if $n = 1$ and

$$u = \int_{\mathbb{R}} av = \int_{\mathbb{R}} bv,$$

then the difference $a - b$ belongs to the subspace $\ker(P_1)$ that is the subspace generated by the Dirac delta δ_0, so that

$$\int_{\mathbb{R}} av = \int_{\mathbb{R}} bv$$

if and only if $b = a + z\delta_0$, for some z in the field $\mathbb{K}$. So, if we consider a hyperplane H of $\mathcal{S}'_1$ supplementary to the line $\operatorname{span}(\delta_0)$, we have that

$$H.v = \mathcal{S}'_1,$$

and that the domain restriction of the operator P_1 to the hyperplane H is a bijection and more precisely a topological isomorphism of the hyperplane H onto the space $\mathcal{S}'_1$.

8.6 Kernel of an $^{\mathcal{S}}$family

Now we see an infinite-dimensional version of a basic theorem of linear algebra, more precisely the following classic result:

Theorem. *Let $v = (v_i)_{i=1}^n$ be a family of linear forms on a vector space X and let w be a linear form vanishing on the kernel of every form v_i of the family. Then, the form w is a linear combination of the family v.*

Note, first of all, that the theorem can be restated as follows.

Terminology and notation. We say kernel of a family $v = (v_i)_{i\in I}$ of linear forms on a vector space X the intersection of all the kernels of the forms of the family v, in symbols

$$\ker v := \bigcap_{i\in I} \ker v_i.$$

Moreover, if Y is a subspace of a vector space X, by $Y^{\perp}$ we denote the orthogonal of Y, i.e., the set of all the linear forms on the space X which vanish on every vector of the subspace Y.

With these notations we can restate the preceding theorem.

Theorem. *Let $v = (v_i)_{i=1}^n$ be a finite family of linear forms on a vector space X and let w be another linear form on the space. Then, the form w vanishes on the kernel of the family v if and only if w is a linear combination of the family v. In other words, the linear hull of the family v coincides with the orthogonal of its kernel:*

$$(\ker v)^{\perp} = \operatorname{span}(v).$$

Finally, we state and prove the $^{\mathcal{S}}$linear version of the above result.

Theorem. *Let $v = (v_p)_{p\in\mathbb{R}^m}$ be an ${}^{\mathcal{S}}$family in the space $\mathcal{S}'_n$. Then the orthogonal of the kernel of the family coincides with the closed linear hull of the family with respect to the weak* topology*

$$(\ker v)^{\perp} = \overline{\operatorname{span}}_{\sigma(\mathcal{S}'_n)}\,(v)\,.$$

In particular, if v is topologically exhaustive - i.e., if the ${}^{\mathcal{S}}$linear hull ${}^{\mathcal{S}}\operatorname{span}(v)$ is $\sigma(\mathcal{S}'_n)$-closed - we have

$$(\ker v)^{\perp} = {}^{\mathcal{S}}\operatorname{span}(v)\,.$$

Proof. A classic theorem on weak duality (see for instance [Die]) affirms that

$$(\ker A)^{\perp} = \overline{(\operatorname{im}({}^tA))}_{\sigma(E',E)},$$

for every weakly continuous operator $A : E \to F$. Now applying this theorem to the operator $\widehat{v}$ generated by the family v, we have

$$\begin{aligned}(\ker \widehat{v})^{\perp} &= \overline{(\operatorname{im}({}^t\widehat{v}))}_{\sigma(\mathcal{S}'_n)} = \\ &= \overline{({}^{\mathcal{S}}\operatorname{span}(v))}_{\sigma(\mathcal{S}'_n)} = \\ &= \overline{\operatorname{span}}_{\sigma(\mathcal{S}'_n)}\,(v)\,.\end{aligned}$$

On the other hand, ϕ belongs to $\ker \widehat{v}$ if and only if $v(\phi)(p) = 0$, for every m-tuple p, and this means that ϕ belongs to the kernel of each tempered distribution v_p, concluding $\ker \widehat{v} = \ker v$. ■

8.6.1 Application

Example. Let V be the subspace of $\mathcal{S}'_1$ formed by the distributions with compact support contained in some non-degenerate compact interval $K = [c, d]$ (of the real line). Let g be a smooth functional defined on the real line with compact support the interval K and everywhere different from 0 on of the interval K. Consider the family $v = (g(x)\delta_x)_{x\in\mathbb{R}}$ in $\mathcal{S}'_1$. As we already know the family v is of class $\mathcal{S}$ (even more, it is of class $\mathcal{D}$). Let us study its kernel, the kernel of v is the set of any test function k such that $v_x(k)$ is zero, for every real x, but this means $g(x)k(x) = 0$ for x in the interior of K, and since g is nowhere 0 on K, the kernel of the family is the set of test functions that are zero on K (since a continuous function that is zero on an open interval is zero on its closure). The orthogonal of the kernel of the family v is so the set of all distributions u which vanish on the complement of K. Recalling the definition of support of a distribution (the complement of the greatest open set on which the distribution vanishes) we deduce that the support of an element u of the orthogonal of the kernel is contained in K, so that by the preceding theorem we conclude

$$\overline{\operatorname{span}}_{\sigma(\mathcal{S}'_n)}\,(v) = (\ker v)^{\perp} = V.$$

8.7 ${}^{\mathcal{S}}$Linear hull of a subset

We propose the following first generalization of the concept of ${}^{\mathcal{S}}$linear hull of a family.

Definition (of ${}^{\mathcal{S}}$linear hull of a subset). *Let X be a subset of the space $\mathcal{S}'_n$. The ${}^{\mathcal{S}}$**linear hull of the subset** X is the intersection of all the ${}^{\mathcal{S}}$linear hulls of ${}^{\mathcal{S}}$families which contain the subset X.*

Note that this linear hull is nonempty since the entire space $\mathcal{S}'_n$ is the ${}^{\mathcal{S}}$linear hull of an ${}^{\mathcal{S}}$ family which contains X. Moreover, the ${}^{\mathcal{S}}$linear hull of the subset X contains X itself.

More precisely, since the intersection of subspaces is a subspace, the ${}^{\mathcal{S}}$linear hull of a subset is a subspace. Since every linear hull (of family) containing a subset X is a subspace (and then containing the subset X it must contain the linear hull of X), the ${}^{\mathcal{S}}$ linear hull of X contains the linear hull of X.

With this definition, the linear hull of an ${}^{\mathcal{S}}$family contains the ${}^{\mathcal{S}}$linear hull of its trace.

Open problem. Find an ${}^{\mathcal{S}}$family whose ${}^{\mathcal{S}}$ hull does not coincide with the ${}^{\mathcal{S}}$hull of its trace.

Open problem. If an ${}^{\mathcal{S}}$family v is a bijection of its index set onto its trace, is the ${}^{\mathcal{S}}$hull of the family equal to the ${}^{\mathcal{S}}$hull of its trace?

A less interesting generalization is the following.

Definition (of interior ${}^{\mathcal{S}}$linear hull of a subset). *Let X be a subset of the space $\mathcal{S}'_n$. The **interior** ${}^{\mathcal{S}}$ **linear hull of the subset** X is the sum of all the ${}^{\mathcal{S}}$linear hulls (of ${}^{\mathcal{S}}$families) which are contained in the subset X.*

The interior ${}^{\mathcal{S}}$linear hull of a subset can be empty, for example the interior ${}^{\mathcal{S}}$linear hull of a nonzero singleton.

Chapter 9

Bases

9.1 $^{\mathcal{S}}$Linear independence

Definition (of $^{\mathcal{S}}$linear independence). *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be an $^{\mathcal{S}}$family of tempered distributions. The family v is said $^{\mathcal{S}}$**linearly independent**, if the relations $a \in \mathcal{S}'_m$ and $\int_{\mathbb{R}^m} av = 0_{\mathcal{S}'_n}$ imply that $a = 0_{\mathcal{S}'_m}$. In other terms the family v is $^{\mathcal{S}}$linearly independent if and only if any zero $^{\mathcal{S}}$linear combination of the family v has necessarily a zero coefficient system.*

Example. *The Dirac family in $\mathcal{S}'_n$ is $^{\mathcal{S}}$linearly independent.* In fact, we have

$$\int_{\mathbb{R}^n} u\delta = u,$$

for all $u \in \mathcal{S}'_n$, and then the relation $\int_{\mathbb{R}^n} u\delta = 0_{\mathcal{S}'_n}$ implies $u = 0_{\mathcal{S}'_n}$.

Example (the Fourier families). *The Fourier families are $^{\mathcal{S}}$linearly independent.* In fact, let φ be the (a,b)-Fourier family, and let $\int_{\mathbb{R}^n} u\varphi = 0_{\mathcal{S}'_n(\mathbb{C})}$. For every $\phi \in \mathcal{S}_n(\mathbb{C})$, we have

$$\begin{aligned} 0 &= \left(\int_{\mathbb{R}^n} u\varphi\right)(\phi) \\ &= u\left(\widehat{\varphi}(\phi)\right) = \\ &= u\left(\mathcal{S}_{(a,b)}(\phi)\right) = \\ &= \mathcal{F}_{(a,b)}(u)(\phi), \end{aligned}$$

i.e.,

$$\mathcal{F}_{(a,b)}(u) = 0_{\mathcal{S}'_n(\mathbb{C})},$$

and thus $u = 0_{\mathcal{S}'_n(\mathbb{C})}$, being $\mathcal{F}_{(a,b)}$ injective.

Theorem. *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be an ${}^{\mathcal{S}}$linearly independent family. Then, v is linearly independent. Consequently, the hull ${}^{\mathcal{S}}\mathrm{span}\,(v)$ is an infinite dimensional subspace of $\mathcal{S}'_n$.*

Proof. Let $k \in \mathbb{N}$ be a positive integer, $\alpha \in (\mathbb{R}^m)^k$ be any k-sequence of points in the m -dimensional Euclidean space, and let $v_\alpha = (v_{\alpha_i})_{i=1}^k$ be the k-sequence of distributions extracted by the family v by means of the index selection α. By contradiction, let assume v_α be a linearly dependent system of $\mathcal{S}'_n$, then there exists a non-zero k-tuple $\lambda \in \mathbb{K}^k$ such that

$$\sum_{i=1}^k \lambda_i v_{\alpha_i} = 0_{\mathcal{S}'_n}.$$

Put $\Lambda = \sum_{i=1}^k \lambda_i \delta_{\alpha_i}$, we have

$$\begin{aligned}
\int_{\mathbb{R}^m} \Lambda v &= \int_{\mathbb{R}^m} \sum_{i=1}^k \lambda_i \delta_{\alpha_i} v = \\
&= \sum_{i=1}^k \lambda_i \int_{\mathbb{R}^m} \delta_{\alpha_i} v = \\
&= \sum_{i=1}^k \lambda_i v_{\alpha_i} = \\
&= 0_{\mathcal{S}'_n}.
\end{aligned}$$

Now, since the distribution Λ is different from the zero distribution $0_{\mathcal{S}'_m}$, the preceding equality contradicts the ${}^{\mathcal{S}}$linear independence of v, against the assumptions. ■

9.2 Topology and ${}^{\mathcal{S}}$linear independence

The last theorem of the above section shows that, for the ${}^{\mathcal{S}}$ families, the ${}^{\mathcal{S}}$linear independence implies the usual linear independence. Actually, the ${}^{\mathcal{S}}$linear independence is more restrictive than the linear independence, as we shall see later by a simple example. On the contrary it is less restrictive than the $\beta(\mathcal{S}'_n)$-topological independence, as it is shown below.

Topological independence. We recall that a system of vectors $v = (v_i)_{i \in I}$ in the space $\mathcal{S}'_n$ is said $\beta(\mathcal{S}'_n)$-topologically free (respectively, $\sigma(\mathcal{S}'_n)$-topologically free) if and only if there exists a family $L = (L_i)_{i \in I}$ of $\beta(\mathcal{S}'_n)$ -continuous (respectively, $\sigma(\mathcal{S}'_n)$-continuous) linear forms on $\mathcal{S}'_n$ such that $L_i(v_k) = \delta_{ik}$, for any pair $(i,k) \in I^2$, where the family $\delta = (\delta_{ik})_{(i,k) \in I^2}$ is the Kronecker family on the square I^2. If the family v is not topologically free it is said topologically bound. If the family v is topologically free, any family L satisfying the above relations is said a dual family of the family v. Note that the above relation can be written as $L \otimes v = \delta$, where δ is the Kronecker family. So, to say that a family v is topologically free is equivalent to say that v has a dual family of linear continuous forms. Recalling that any continuous linear functional on the space $\mathcal{S}'_n$ is canonically and univocally representable by a test function, to say that the family v is topologically free is equivalent to say that the bi-orthonormality condition

$$\langle g_i, v_k \rangle = \delta_{ik},$$

is true, for any pair $(i,k) \in I^2$, for some family g of test functions.

Theorem. *Every ${}^{\mathcal{S}}$family in the space $\mathcal{S}'_n$ is $\beta(\mathcal{S}'_n)$-topologically bound and, thus, $\sigma(\mathcal{S}'_n)$-topologically bound. Consequently, no ${}^{\mathcal{S}}$family has a dual family.*

Proof. Let v be an ${}^{\mathcal{S}}$family in the space $\mathcal{S}'_n$ indexed by $\mathbb{R}^m$. And let L be an arbitrary family in the dual $\mathcal{S}''_n$ indexed by the same index set. Being the Schwartz space ($\mathcal{S}_n$) reflexive, for every i, there is a test function g_i in $\mathcal{S}_n$ such that

$$L_i = \langle ., g_i \rangle \,,$$

that is such that $L_i(u) = u(g)$, for every tempered distribution u in $\mathcal{S}'_n$. Assume the existence of an index i such that $L_i(v_i) = 1$, then we deduce

$$1 = L_i(v_i) = v_i(g_i) = v(g_i)\,(i)\,,$$

being v an ${}^{\mathcal{S}}$family, the function $v(g_i)$ is continuous, then there is a neighborhood U of the point i in which the function $v(g_i)$ is strictly positive. Then, for every point k in the neighborhood U, we have

$$L_i(v_k) = v_k(g_i) = v(g_i)\,(k) > 0,$$

and then L cannot verify the condition of topological independence for v . ■

Note. By the same proof, it is possible to prove that *every C^0-family is strongly topologically bound.* Consequently every smooth family is also strongly topologically bound.

9.3 Uniqueness of representation

It's simple to prove the following property.

Property. *An $^{\mathcal{S}}$family v in $\mathcal{S}'_n$ indexed by $\mathbb{R}^m$ is $^{\mathcal{S}}$linearly dependent if and only if there a point index p in $\mathbb{R}^m$ and a tempered distribution a different from the Dirac delta δ_p such that*

$$v_p = \int_{\mathbb{R}^m} av.$$

Proof. **Necessity.** Indeed, if v_p fulfills that property we have that the $^{\mathcal{S}}$linear combination $(a - \delta_p).v$ is zero with a non-zero coefficient distribution so that the family v is $^{\mathcal{S}}$linearly dependent. **Sufficiency.** Vice versa, let, for every point p, the term v_p of the family be representable in a unique way as the superposition $v_p = \delta_p.v$. Assume v $^{\mathcal{S}}$ linearly dependent, then there is a different from zero such that $a.v = 0$, hence

$$\begin{aligned} v_p &= \delta_p.v - 0 = \\ &= \delta_p.v - a.v = \\ &= (\delta_p - a).v, \end{aligned}$$

since a is a non zero distribution, the sum $\delta_p - a$ is different form δ_p, and so v_p is representable in another different way, against the assumption. ■

9.4 Characterizations of $^{\mathcal{S}}$linear independence

By the Dieudonné-Schwartz theorem we immediately deduce two characterizations.

Theorem. *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be a topologically exhaustive family, that is an $^{\mathcal{S}}$family whose $^{\mathcal{S}}$linear hull $^{\mathcal{S}}\mathrm{span}\,(v)$ is $\sigma(\mathcal{S}'_n)$-closed. Then the following assertions are equivalent*

- *1) the family v is $^{\mathcal{S}}$linearly independent;*
- *2) the superposition operator $\int_{\mathbb{R}^m}(\cdot, v)$ is an injective topological homomorphism for the weak* topologies $\sigma(\mathcal{S}'_m)$ and $\sigma(\mathcal{S}'_n)$;*
- *3) the superposition operator $\int_{\mathbb{R}^m}(\cdot, v)$ is an injective topological homomorphism for the strong topologies $\beta(\mathcal{S}'_m)$ and $\beta(\mathcal{S}'_n)$;*

- *4) the operator* $\widehat{v}$ *is a surjective topological homomorphism for the weak topologies* $\sigma(\mathcal{S}_n)$ *and* $\sigma(\mathcal{S}_m)$*;*

- *5) the operator* $\widehat{v}$ *is an surjective topological homomorphism of the topological vector space* $(\mathcal{S}_n)$ *onto the space* $(\mathcal{S}_m)$*.*

Theorem. *Let* $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ *be an* $^{\mathcal{S}}$*family. Then the following assertions are equivalent*

- *1) the family* v *is* $^{\mathcal{S}}$*linearly independent and the hull* $^{\mathcal{S}}\mathrm{span}\,(v)$ *is* $\sigma(\mathcal{S}'_n)$*-closed;*

- *2) the superposition operator* $\int_{\mathbb{R}^m}(\cdot, v)$ *is an injective topological homomorphism for the weak* topologies* $\sigma(\mathcal{S}'_m)$ *and* $\sigma(\mathcal{S}'_n)$*;*

- *3) the operator* $\widehat{v}$ *is a surjective topological homomorphism for the weak topologies* $\sigma(\mathcal{S}_n)$ *and* $\sigma(\mathcal{S}_m)$*;*

- *4) the operator* $\widehat{v}$ *is a surjective topological homomorphism from* $(\mathcal{S}_n)$ *onto* $(\mathcal{S}_m)$*.*

Remark (on the coordinate operator). In the conditions of the above theorem, if the family v is $^{\mathcal{S}}$linearly independent, we can consider the algebraic isomorphism from the space $\mathcal{S}'_m$ onto the $^{\mathcal{S}}$linear hull $^{\mathcal{S}}\mathrm{span}\,(v)$ that sends every tempered distribution $a \in \mathcal{S}'_m$ to the superposition $\int_{\mathbb{R}^m} av$, that is the restriction of the injection $\int_{\mathbb{R}^m}(\cdot, v)$ to the pair of sets $(\mathcal{S}'_m, {}^{\mathcal{S}}\mathrm{span}\,(v))$. We shall denote the inverse of this isomorphism by the symbol $[\cdot|v]$. It is a consequence of the preceding theorem that

- *the operator* $[\cdot|v] : {}^{\mathcal{S}}\mathrm{span}\,(v) \to \mathcal{S}'_m$ *is a topological isomorphism, with respect to the topology induced by the weak* topology* $\sigma(\mathcal{S}'_n)$ *on the* $^{\mathcal{S}}$ *linear hull* $^{\mathcal{S}}\mathrm{span}\,(v)$ *and to the weak* topology* $\sigma(\mathcal{S}'_m)$*, if and only if the* $^{\mathcal{S}}$*linear hull* $^{\mathcal{S}}\mathrm{span}\,(v)$ *is* $\sigma(\mathcal{S}'_n)$*-closed, that is if the family* v *is topologically exhaustive.*

9.5 $^{\mathcal{S}}$Bases

Definition (of $^{\mathcal{S}}$**basis).** *Let* $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ *be an* $^{\mathcal{S}}$*family in* $\mathcal{S}'_n$ *and let* V *be a subspace of the space* $\mathcal{S}'_n$*. The family* v *is said an* $^{\mathcal{S}}$ ***basis*** *of the subspace* V *if it is* $^{\mathcal{S}}$*linearly independent and it* $^{\mathcal{S}}$*generates* V*, that is if the superposition operator of the family* v *is injective and* $^{\mathcal{S}}\mathrm{span}(v) = V$*.*

The Dirac family δ in $\mathcal{S}'_n$ is an ${}^{\mathcal{S}}$basis of $\mathcal{S}'_n$. We call δ *the canonical $\mathcal{S}$-basis of $\mathcal{S}'_n$* or *the Dirac basis of $\mathcal{S}'_n$*.

Moreover, the following complete version of the Fourier expansion-theorem, allow us to call the Fourier families of $\mathcal{S}'(\mathbb{R}^n, \mathbb{C})$ by the name of *Fourier bases of $\mathcal{S}'(\mathbb{R}^n, \mathbb{C})$*.

Theorem (geometric form of the Fourier expansion theorem). *In the space of complex tempered distributions $\mathcal{S}'_n(\mathbb{C})$ the Fourier families are $\mathcal{S}$-bases (of the entire space $\mathcal{S}'_n(\mathbb{C})$).*

9.6 Algebraic characterizations of ${}^{\mathcal{S}}$ bases

The following is an elementary but meaningful generalization of the Fourier expansion theorem.

Theorem (characterization of an ${}^{\mathcal{S}}$basis). *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be an ${}^{\mathcal{S}}$family. Then,*

- *1) the family v ${}^{\mathcal{S}}$generates the space $\mathcal{S}'_n$ if and only if the superposition operator ${}^t(\widehat{v})$ is surjective;*
- *2) the family v is ${}^{\mathcal{S}}$linearly independent if and only if the superposition operator ${}^t(\widehat{v})$ is injective;*
- *3) the family v is an ${}^{\mathcal{S}}$basis of the space $\mathcal{S}'_n$ if and only if the superposition operator ${}^t(\widehat{v})$ is bijective.*

Proof. First of all the operator ${}^t(\widehat{v})$ is well defined because v is an ${}^{\mathcal{S}}$family. Moreover, it is obvious, by the very definitions, that the family v ${}^{\mathcal{S}}$ generates the space $\mathcal{S}'_n$ if and only if the superposition operator ${}^t(\widehat{v})$ is surjective, and that v is ${}^{\mathcal{S}}$linearly independent if and only if the superposition operator ${}^t(\widehat{v})$ is injective. ■

9.6.1 Example

Example (a system of linearly independent ${}^{\mathcal{S}}$ generators that is not an ${}^{\mathcal{S}}$basis). Let $v = (\delta'_x)_{x\in\mathbb{R}}$ be the family in $\mathcal{S}'_1$ of the first derivatives of the Dirac distributions. The family v is of class $\mathcal{S}$, in fact

$$v(\phi)(x) = v_x(\phi) = \delta'_x(\phi) = -\phi'(x),$$

and $-\phi'$ is an $\mathcal{S}$-function. Consequently, the operator associated with v is the derivation in $\mathcal{S}_n$ up to the sign and, then, ${}^t(\widehat{v})$ is the derivation in $\mathcal{S}'_n$. This last operator is a surjective operator (every tempered distribution has a primitive) but it is not injective (every tempered distribution has many primitives), then v is a system of $\mathcal{S}$-generators for $\mathcal{S}'_1$, but it is not $\mathcal{S}$-linearly independent. Moreover, note that v is linearly independent. In fact, let P be a finite subset of the real line $\mathbb{R}$, and let, for every point p_0 in P, f_{p_0} be a function in $\mathcal{S}_1$ such that $f'_{p_0}(p) = \delta_{p_0 p}$, for every index p in P. If $a = (a_p)_{p\in P}$ is a finite family of scalars such that

$$\sum_{p\in P} a_p v_p = 0_{\mathcal{S}'_1},$$

then

$$0 = \left(\sum_{p\in P} a_p v_p\right)(f_{p_0}) = \sum_{p\in P} a_p \delta_{p_0 p} = a_{p_0},$$

for every index p_0 in P.

9.7 Totality of $^{\mathcal{S}}$bases

Note that another way to express the preceding characterization.

Theorem (characterization of an $^{\mathcal{S}}$basis). *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be an $^{\mathcal{S}}$family. Then,*

- *A family v is a system of $^{\mathcal{S}}$ generators of the entire space if and only if the family v is total in the space $\mathcal{S}_n$, in the sense that if $v_p(g) = 0$ for every p implies $g = 0$.*
- *A family v is $^{\mathcal{S}}$linearly independent if and only it is total in the space $\mathcal{S}'_m$, in the sense that if $a.v = 0$ then $a = 0$.*
- *A family v is an $^{\mathcal{S}}$basis of the space if and only if the family v is total both in the space $\mathcal{S}_n$ and $\mathcal{S}'_m$.*

Proof. Indeed, this means that the operator v is injective, and so $v(g) = 0$ implies $a = 0$. ■

9.8 Topological characterizations of $^{\mathcal{S}}$ bases

By the Dieudonné-Schwartz theorem we immediately take a characterization.

Theorem. *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be a family of tempered distributions. Then the following assertions are equivalent*

- *1) the family v is an $^{\mathcal{S}}$basis of the space $\mathcal{S}'_n$;*
- *2) the superposition operator $\int_{\mathbb{R}^m}(\cdot, v)$ is a topological isomorphism for the weak* topologies $\sigma(\mathcal{S}'_m)$ and $\sigma(\mathcal{S}'_n)$;*
- *3) the superposition operator $\int_{\mathbb{R}^m}(\cdot, v)$ is a topological isomorphism for the strong topologies $\beta(\mathcal{S}'_m)$ and $\beta(\mathcal{S}'_n)$;*
- *4) the operator $\widehat{v}$ is a topological isomorphism for the weak topologies $\sigma(\mathcal{S}_n)$ and $\sigma(\mathcal{S}_m)$;*
- *5) the operator $\widehat{v}$ is a topological isomorphism of the topological vector space $(\mathcal{S}_n)$ onto $(\mathcal{S}_m)$.*

9.9 Equivalent $^{\mathcal{S}}$families

We say that to $^{\mathcal{S}}$families v and w with the same index set are equivalent if there is a diffeomorphism h of the first index set into the second index set such that $v_h = w$, where v_h is the family $v_h(p) = v_{h(p)}$.

First, we have to consider a generalization of composition of a measure with a function. Recall that

Definition (of composition in $\mathcal{S}'_m$ with diffeomorphisms). *Let h be a smooth diffeomorphism for the pair $(\mathbb{R}^m, \mathbb{R}^m)$, that is h is a function from $\mathbb{R}^m$ into $\mathbb{R}^m$ which is bijective and smooth with its inverse. Then, for every test function $g \in \mathcal{S}_m$, the function*

$$g_h = \left(g \circ h^-\right) |\det J_{h^-}|$$

belongs to the space $\mathcal{S}_m$, moreover, for every distribution $a \in \mathcal{S}'_m$, the functional

$$a \circ h : \mathcal{S}_m \to \mathbb{K} : g \mapsto a\left(\left(g \circ h^-\right) |\det J_{h^-}|\right)$$

is a tempered distribution called the composition of u with the diffeomorphism h.

In the conditions of the preceding definition we have

$$\begin{aligned}(a\circ h)(g) &= a(g_h) = \\ &= |\det J_{h^-}|\, a(g\circ h^-).\end{aligned}$$

Let v be an $^{\mathcal{S}}$family and let h be a diffeomorphism, we have

$$\begin{aligned}v_h(g)(p) &= v_{h(p)}(g) = \\ &= v(g)(h(p)) = \\ &= (v(g)\circ h)(p),\end{aligned}$$

so that $v_h(g)$ is of class $\mathcal{S}$ and

$$v_h(g) = v(g)\circ h.$$

Let a be a coefficient distribution in $\mathcal{S}'_m$. We have, for h with unitary determinant,

$$\begin{aligned}(\int_{\mathbb{R}^m} av_h)(g) &= a.v_h(g) \\ &= a(v(g)\circ h) = \\ &= (a\circ h^-)(v(g)) = \\ &= (\int_{\mathbb{R}^m}(a\circ h^-)v)(g),\end{aligned}$$

so that

$$\int_{\mathbb{R}^m} av_h = \int_{\mathbb{R}^m}(a\circ h^-)v.$$

Note that the mapping $a\mapsto a\circ h^-$ is a bijection of $\mathcal{S}'_m$ onto $\mathcal{S}'_m$. So that

$$\mathcal{S}'_m.v = \mathcal{S}'_m.v_h.$$

Moreover, let v $^{\mathcal{S}}$linearly independent. If $a.v_h = 0$ then

$$(a\circ h^-).v = 0$$

and hence $a\circ h^-$ is zero but this implies that $a = 0$.

Recalling the preceding we have that an $^{\mathcal{S}}$family is an $^{\mathcal{S}}$basis if and only if it is equivalent to an $^{\mathcal{S}}$basis.

Chapter 10

${}^{\mathcal{S}}$Closedness

10.1 ${}^{\mathcal{S}}$Closed subsets

A natural kind of stability for subsets of the space $\mathcal{S}'_n$ arises with the definition of ${}^{\mathcal{S}}$linear combinations. As usual we say that a family is contained in a certain subset if its trace is contained in that set.

Definition (of ${}^{\mathcal{S}}$closedness in $\mathcal{S}'_n$). *Let X be a subset of $\mathcal{S}'_n$. The part X is said ${}^{\mathcal{S}}$**closed** or ${}^{\mathcal{S}}$**stable in the space** $\mathcal{S}'_n$ if it contains all the superpositions of the ${}^{\mathcal{S}}$families contained in X . In other words, X is said ${}^{\mathcal{S}}$ closed if, for* *each positive integer $m \in \mathbb{N}$, for each family $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ contained in the part X and for each tempered distribution $a \in \mathcal{S}'_m$, the superposition $\int_{\mathbb{R}^m} av$ lies in the set X.*

Example (trivial ${}^{\mathcal{S}}$closed subsets). The empty set and the entire space $\mathcal{S}'_n$ are ${}^{\mathcal{S}}$closed. The first one does not contain ${}^{\mathcal{S}}$families the second contains all the possible ${}^{\mathcal{S}}$linear combinations.

Example (${}^{\mathcal{S}}$closed subsets which are not subspaces). Note that a subset X formed by a unique nonzero distribution is not a subspace, but it is ${}^{\mathcal{S}}$stable. Indeed, there are no ${}^{\mathcal{S}}$families in X, since the unique family in X is a constant one and the constant families are not of class $\mathcal{S}$. Consequently, the part X is ${}^{\mathcal{S}}$stable by definition. A subset formed by only two different nonzero points is not a subspace but it is ${}^{\mathcal{S}}$stable. Indeed, there are no ${}^{\mathcal{S}}$families contained in X; since a family v in X must have at most two distinct points, so the images $v(g)$, of a test function g belonging to the co-level 0 of at least one of the distributions of the family, is a nonzero function having at most two values, and hence it cannot be an ${}^{\mathcal{S}}$function (a non zero ${}^{\mathcal{S}}$function must have infinite values).

So an ${}^{\mathcal{S}}$closed set may be not a subspace. We will see later that every subset both ${}^{\mathcal{S}}$closed and star-shaped at the origin is a subspace.

10.2 ${}^{\mathcal{S}}$Closed hulls

The starting point is the following result.

Theorem. *Let F be a family of ${}^{\mathcal{S}}$closed subsets of the space $\mathcal{S}'_n$. Then, the intersection $\cap F$ of the family F is ${}^{\mathcal{S}}$closed.*

Proof. Let $F = (F_i)_{i\in I}$ be our family of ${}^{\mathcal{S}}$closed subsets and let v be an ${}^{\mathcal{S}}$family contained in the intersection $\cap F$ of the family. Then, v is an ${}^{\mathcal{S}}$family in the subset F_i, for every index $i \in I$ of the family (indeed, the family v is a family in the intersection $\cap F$ if and only if $v_p \in F_i$, for every index $p \in \mathbb{R}^m$ and for every $i \in I$). Since the subset F_i is ${}^{\mathcal{S}}$closed, the superposition $\int_{\mathbb{R}^m} av$ must belong to F_i, for every tempered distribution $a \in \mathcal{S}'_m$ and every index $i \in I$, therefore the superposition $\int_{\mathbb{R}^m} av$ must belong to the intersection $\cap F$ of the family F. ■

The above stability property allows us to define the concept of ${}^{\mathcal{S}}$closed hull and ${}^{\mathcal{S}}$closed linear hull of a subset of the space.

Definition (of ${}^{\mathcal{S}}$closed hull). *Let X be a subset of $\mathcal{S}'_n$. The ${}^{\mathcal{S}}$**closed hull of** X is the intersection of all the ${}^{\mathcal{S}}$closed subsets containing X. It is denoted by ${}^{\mathcal{S}}\mathrm{cl}(X)$ or simply by ${}^{\mathcal{S}}\overline{X}$.*

It is clear that the ${}^{\mathcal{S}}$closed hull of a subset X must contain X itself and that ${}^{\mathcal{S}}\mathrm{cl}(X)$ is the smallest (with respect to the inclusion) ${}^{\mathcal{S}}$closed subset containing X.

It is equally clear that the ${}^{\mathcal{S}}$closed hull of an ${}^{\mathcal{S}}$closed subset is the ${}^{\mathcal{S}}$closed subset itself; in particular the ${}^{\mathcal{S}}$closed hull of the ${}^{\mathcal{S}}$closed hull of X is the ${}^{\mathcal{S}}$closed hull of X.

Definition (of ${}^{\mathcal{S}}$closed linear hull). *Let X be a subset of $\mathcal{S}'_n$. The ${}^{\mathcal{S}}$**closed linear hull of** X is the intersection of all the ${}^{\mathcal{S}}$closed subspaces of the space containing X. It is denoted by the symbol ${}^{\mathcal{S}}\overline{\mathrm{span}}(X)$.*

It is clear that the ${}^{\mathcal{S}}$closed linear hull of a subset X must contain X and that it is the smallest (with respect to the inclusion) ${}^{\mathcal{S}}$closed linear subset containing X. Indeed, the intersection of a family of ${}^{\mathcal{S}}$closed subspaces is an ${}^{\mathcal{S}}$closed subspace.

It is equally clear that the ${}^{\mathcal{S}}$closed linear hull of an ${}^{\mathcal{S}}$closed subspace is the ${}^{\mathcal{S}}$closed subspace itself; in particular the ${}^{\mathcal{S}}$closed linear hull of the ${}^{\mathcal{S}}$ closed linear hull of X is the ${}^{\mathcal{S}}$closed linear hull of X.

Concerning the relationship among the two hulls we have the following obvious result.

Proposition. *Let X be a subset of the space $\mathcal{S}'_n$. Then the inclusion*

$$ {}^{\mathcal{S}}\mathrm{cl}(X) \subseteq {}^{\mathcal{S}}\overline{\mathrm{span}}(X) $$

holds true.

Proof. The collection of all ${}^{\mathcal{S}}$closed subsets containing X contains the collection of all ${}^{\mathcal{S}}$closed subspace containing X. ■

10.3 Relationships among different hulls

Given a subset X of the space $\mathcal{S}'_n$, we have some different hulls of the set:

1. *the* ***linear hull of*** X*, denoted by* $\operatorname{span}(X)$*, set of all finite linear combinations of elements of* X*, and also intersections of all subspaces containing* X*;*

2. *the* ${}^{\mathcal{S}}$***linear hull of*** X*, denoted by* ${}^{\mathcal{S}}\operatorname{span}(X)$*, intersection of all the* ${}^{\mathcal{S}}$*linear hulls of* ${}^{\mathcal{S}}$*family which contain the subset* X*;*

3. *the* ${}^{\mathcal{S}}$***closed hull of*** X*, denoted by* ${}^{\mathcal{S}}\operatorname{cl}(X)$*, intersection of all* ${}^{\mathcal{S}}$*closed subsets containing* X*;*

4. *the* ${}^{\mathcal{S}}$***closed linear hull of*** X*, denoted by* ${}^{\mathcal{S}}\overline{\operatorname{span}}(X)$*, intersection of all the* ${}^{\mathcal{S}}$*closed subspaces containing* X*.*

Our aim in this section is to understand the relationships among these hulls.

10.3.1 Relationships among linear hulls

Proposition. *Let* X *be a subset of* $\mathcal{S}'_n$*. Then, the following inclusions*

$$\operatorname{span}(X) \subseteq {}^{\mathcal{S}}\operatorname{span}(X) \quad \textit{and} \quad \operatorname{span}(X) \subseteq {}^{\mathcal{S}}\overline{\operatorname{span}}(X)$$

hold true.

Proof. For the first inclusion, note that the collection of all subspaces containing X contains the collection of all subspaces of the form ${}^{\mathcal{S}}\operatorname{span}(v)$, with v ${}^{\mathcal{S}}$family, containing X. The second inclusion holds for analogous reasons. ■

Open problems. We meet two problems:

- *does the inclusion*
$$ {}^{\mathcal{S}}\operatorname{span}(X) \subseteq {}^{\mathcal{S}}\overline{\operatorname{span}}(X)$$
hold?

- *Is it true that every* ${}^{\mathcal{S}}$*closed subspace of the space* $\mathcal{S}'_n$ *is a subspace of the form* ${}^{\mathcal{S}}\operatorname{span}(v)$*, with* v ${}^{\mathcal{S}}$ *family in* $\mathcal{S}'_n$*?*

10.3.2 $^{\mathcal{S}}$Closure of subspaces

We have the following open problem:

Open problem. *Is the $^{\mathcal{S}}$closed hull of a subspace E still a subspace?*

Note. Let u and v be any two points of the $^{\mathcal{S}}$closed hull of E. If we find an $^{\mathcal{S}}$family w contained in $^{\mathcal{S}}\mathrm{cl}(E)$ and passing through u and v then any linear combination $au + bv$ must stay in the $^{\mathcal{S}}$closed hull of the subspace E, since any linear combination of two elements of an $^{\mathcal{S}}$family is a superposition of the $^{\mathcal{S}}$family. But when such a family does exist?

Question. *Is the linear hull of an $^{\mathcal{S}}$ closed subset still an $^{\mathcal{S}}$closed subspace?*

Answer. The answer is in general negative. Indeed, fixed a point x of the Euclidean n-space, let X be the singleton of the space formed by the Dirac distribution δ_x. As we already have seen X is an $^{\mathcal{S}}$closed subset; on the contrary its linear hull does not. In fact the linear hull of X is the straight line l generated by that Dirac distribution. Let g be a test function on the real line (in $\mathcal{S}_1$) such that $g(0) = 1$. The family $v = (g(y)\delta_x)_{y\in\mathbb{R}}$ is a family in the linear hull l and it is of class $\mathcal{S}$ (the image of a test function h in $\mathcal{S}_n$ by the family v is the test function $h(x)g$ in $\mathcal{S}_1$). The superposition $\delta'_0.v$ (δ'_0 is the derivative of the Dirac distribution on the real line centered at 0) is the distribution δ'_x (the derivative of the Dirac distribution on the Euclidean n-space centered at the point x) which does not belong to the straight line l. ■

10.3.3 Relationships among linear and $^{\mathcal{S}}$ closed hulls

We have the following questions.

Question. *Is the $^{\mathcal{S}}$linear hull of a subset X contained in the $^{\mathcal{S}}$closed hull of that subset? That is, the following inclusion*

$$^{\mathcal{S}}\mathrm{span}(X) \subseteq {}^{\mathcal{S}}\mathrm{cl}(X),$$

does hold?

Answer. The answer, in general, is negative. Indeed, let X be an $^{\mathcal{S}}$closed subset of $\mathcal{S}'_n$ which is not a subspace. Then the $^{\mathcal{S}}$closure of X is X itself and this cannot contain neither its linear hull. ■

Question. *Is the ${}^{\mathcal{S}}$closed hull of a subset X contained in the ${}^{\mathcal{S}}$linear hull of the subset? That is, does the following inclusion*

$$^{\mathcal{S}}\mathrm{cl}(X) \subseteq {}^{\mathcal{S}}\mathrm{span}(X),$$

hold?

Answer. The answer is negative. Indeed, consider a subset X such that its ${}^{\mathcal{S}}$linear hull S is not ${}^{\mathcal{S}}$closed. First of all it is clear that the ${}^{\mathcal{S}}$linear hull of S is S itself (see later). Then consider the ${}^{\mathcal{S}}$closure C of S: it is clear that C contains properly S, indeed the part C is the ${}^{\mathcal{S}}$closure of S and S cannot coincide with C since S is not ${}^{\mathcal{S}}$closed. ■

Concerning the last problem we have the following result.

Proposition. *Let X be a subset of the space $\mathcal{S}'_n$ which is the trace of an ${}^{\mathcal{S}}$family. Then the ${}^{\mathcal{S}}$linear hull of the subset X is contained in the ${}^{\mathcal{S}}$closed hull of the subset X itself. That is, the following inclusion*

$$^{\mathcal{S}}\mathrm{span}(X) \subseteq {}^{\mathcal{S}}\mathrm{cl}(X),$$

holds. As a consequence, in this special case, the linear hull of X is contained in the ${}^{\mathcal{S}}$closed hull of the part X.

Proof. When a subset X of the space $\mathcal{S}'_n$ is the trace of an ${}^{\mathcal{S}}$family v, any finite family of vectors in X is a subfamily of some ${}^{\mathcal{S}}$family in X (just the family v), and then we have

$$\mathrm{span}(X) \subseteq {}^{\mathcal{S}}\mathrm{cl}(X),$$

but we shall obtain this result as a consequence of what follows. We know that the ${}^{\mathcal{S}}$linear hull of the trace of a family v is contained in the ${}^{\mathcal{S}}$linear hull of the family (in fact the family v is a family whose ${}^{\mathcal{S}}$linear hull contains the trace). Let v a family having X as its trace, we shall prove that the ${}^{\mathcal{S}}$closed hull of X contains the ${}^{\mathcal{S}}$linear span of v. Let now u be an element of the ${}^{\mathcal{S}}$linear hull of v, we must prove that u belongs to every ${}^{\mathcal{S}}$closed subset containing X. The family v is a family in X and then in every ${}^{\mathcal{S}}$closed subset C containing X, consequently (by the very definition of ${}^{\mathcal{S}}$closed subset) every superposition $a.v$ of v must stay in every ${}^{\mathcal{S}}$closed subset containing X and then also in the smallest one, as we claimed. ■

We will find again this result in a slightly different form.

Remark. Let us go back again to the above problem: if the inclusion

$$^{\mathcal{S}}\mathrm{cl}(X) \subseteq {}^{\mathcal{S}}\mathrm{span}(X),$$

was true, then, by the preceding result, if v is an $^{\mathcal{S}}$ family, we should have

$$^{\mathcal{S}}\mathrm{cl}(v) = {}^{\mathcal{S}}\mathrm{span}(v),$$

and this implies that the $^{\mathcal{S}}$linear hull of an $^{\mathcal{S}}$ family is always $^{\mathcal{S}}$closed, and this seems to be not true: but we need a counter-example.

10.4 Unions of $^{\mathcal{S}}$closed sets

In this section we study the union of two $^{\mathcal{S}}$closed sets.

Remark (on the union of two $^{\mathcal{S}}$closed sets). The union of two $^{\mathcal{S}}$closed sets need not necessarily to be $^{\mathcal{S}}$closed, neither in the case in which the two $^{\mathcal{S}}$ closed subsets are subspaces. In fact, consider in the space $\mathcal{S}_2'$ the two $^{\mathcal{S}}$linear hulls

$$F_1 = {}^{\mathcal{S}}\mathrm{span}(\delta_{(.,0)}) \ \textit{and} \ F_2 = {}^{\mathcal{S}}\mathrm{span}(\delta_{(0,.)}),$$

and their union $F = F_1 \cup F_2$. The subset F is obviously not a subspace, but it is star-shaped at the origin; in fact, if u is in the union F, then u lies in F_1 or in F_2, and then the segment joining u and the origin is contained in F. Now, a star-shaped $^{\mathcal{S}}$closed set is necessarily a subspace (see later), then F cannot be $^{\mathcal{S}}$closed.

Open problems. Some problems arise:

- If two $^{\mathcal{S}}$closed sets are disjoints, then is their union $^{\mathcal{S}}$closed?
- If v is an $^{\mathcal{S}}$family in the union of two disjoint $^{\mathcal{S}}$closed sets, must v be contained in one and only one of the two component sets?

The last question arises naturally, since the image of the family v is a path-connected subset of $\mathcal{S}_n'$ in the topology weak* topology $\sigma(\mathcal{S}_n', \mathcal{S}_n)$.

10.5 $^{\mathcal{S}}$Closed linear hull of families

Obviously, the $^{\mathcal{S}}$closed linear hull of a family contains the $^{\mathcal{S}}$linear hull of that family, but we can say more.

Theorem. *Let v be an $^{\mathcal{S}}$family in the space $\mathcal{S}_n'$ and let $\mathcal{F}_v$ be the collection of all the $^{\mathcal{S}}$closed subsets of $\mathcal{S}_n'$ containing the family v. Then the $^{\mathcal{S}}$linear hull*

${}^{\mathcal{S}}\mathrm{span}(v)$ *is contained in the intersection* $\cap\mathcal{F}_v$. *In other terms, the* ${}^{\mathcal{S}}$*linear hull of a family is contained in the* ${}^{\mathcal{S}}$*closed hull of the family, that is*

$$ {}^{\mathcal{S}}\mathrm{span}(v) \subseteq {}^{\mathcal{S}}\mathrm{cl}(v). $$

Consequently, the ${}^{\mathcal{S}}$*linear hull* ${}^{\mathcal{S}}\mathrm{span}(v)$ *is* ${}^{\mathcal{S}}$*closed if and only if it coincides with the intersection of the collection* $\mathcal{F}_v$, *that is if and only if*

$$ {}^{\mathcal{S}}\mathrm{span}(v) = \cap\mathcal{F}_v, $$

that is if and only if

$$ {}^{\mathcal{S}}\mathrm{span}(v) = {}^{\mathcal{S}}\mathrm{cl}(v). $$

Proof. Let us index the collection $\mathcal{F}_v$ by a set I and denote this family by the symbol F. Since any member F_i of the family F is ${}^{\mathcal{S}}$closed and contains the family v, we have that the hull ${}^{\mathcal{S}}\mathrm{span}(v)$ must be contained in F_i, for every index $i \in I$, and consequently the hull ${}^{\mathcal{S}}\mathrm{span}(v)$ must be contained in the intersection $\cap F$. If, moreover the hull ${}^{\mathcal{S}}\mathrm{span}(v)$ is ${}^{\mathcal{S}}$closed, since it contains the family v, we conclude that ${}^{\mathcal{S}}\mathrm{span}(v)$ is one of the members of the family F, and hence that the intersection $\cap F$ must be in the hull ${}^{\mathcal{S}}\mathrm{span}(v)$. ■

Remark. Note that, in the conditions of the preceding theorem, in general it is not true that the intersection $\cap F$ is a subspace, however, if ${}^{\mathcal{S}}\mathrm{span}(v)$ is ${}^{\mathcal{S}}$closed then $\cap F$ is necessarily a subspace, since every ${}^{\mathcal{S}}$linear hull is a subspace.

10.6 ${}^{\mathcal{S}}$Closedness and topology

In this section we study the relation between ${}^{\mathcal{S}}$closedness and the closedness with respect to the strong and weak* topologies on the space of tempered distributions.

Theorem. *Let* F *be a* $\beta(\mathcal{S}'_n)$*-closed subspace of* $\mathcal{S}'_n$. *Then* F *is* ${}^{\mathcal{S}}$*closed.*

Proof. Let δ be the Dirac family of the space $\mathcal{S}'_m$ and let v be an ${}^{\mathcal{S}}$family in F, then, for every index p of the family, the superposition $\int_{\mathbb{R}^m} \delta_p v$ belongs to F, being equal to v_p. Now, let $a \in \mathcal{S}'_m$, since the space $(\mathcal{S}_m)$ is reflexive, we know that the closed linear hull of the Dirac family with respect to the strong topology $\beta(\mathcal{S}'_m)$ is the entire space $\mathcal{S}'_m$, i.e.

$$ \overline{\mathrm{span}}_{\beta(\mathcal{S}'_m)}(\delta) = \mathcal{S}'_m; $$

therefore there exists a sequence $\Delta = (\Delta_k)_{k\in\mathbb{N}}$ in the linear hull $\mathrm{span}(\delta)$ converging to the distribution a in the strong topology $\beta(\mathcal{S}'_m)$. We have, by the selection property of the Dirac family, that any superposition $\int_{\mathbb{R}^m} \Delta_k v$ must belong to the linear hull span (v), which is set-included in F since F is a subspace of $\mathcal{S}'_n$. Moreover, being the superposition operator $\int_{\mathbb{R}^m} (\cdot, v)$ continuous with respect to the strong topologies $\beta(\mathcal{S}'_m)$ and $\beta(\mathcal{S}'_n)$, it follows that, for every a in the space $\mathcal{S}'_m$, we have

$$\begin{aligned}
\int_{\mathbb{R}^m} av &= \int_{\mathbb{R}^m} \left({}^{\beta(\mathcal{S}'_m)} \lim_{k \to \infty} \Delta_k \right) v = \\
&= {}^{\beta(\mathcal{S}'_n)} \lim_{k \to \infty} \int_{\mathbb{R}^m} \Delta_k v;
\end{aligned}$$

and the last limit belongs to the subspace F for $\beta(\mathcal{S}'_n)$-closedness. And consequently, F is ${}^{\mathcal{S}}$ closed. ■

Remark. Since the space $(\mathcal{S}_n)$ is semireflexive, a part F of the dual $\mathcal{S}'_n$ is $\sigma(\mathcal{S}'_n)$-closed if and only if it is $\beta(\mathcal{S}'_n)$-closed.

Corollary. *Let X be a part of the space $\mathcal{S}'_n$. Then, we have*

$$^{\mathcal{S}}\overline{\mathrm{span}}\,(X) \subseteq \overline{\mathrm{span}}_{\beta(\mathcal{S}'_n)}\,(X)\,.$$

Chapter 11

${}^{\mathcal{S}}$Connectedness

11.1 ${}^{\mathcal{S}}$Connected and ${}^{\mathcal{D}}$ connected sets

It is interesting to note that every tempered distribution u in $\mathcal{S}'_n$ belongs to many ${}^{\mathcal{S}}$families and to many ${}^{\mathcal{D}}$families.

11.1.1 ${}^{\mathcal{S}}$Families in $\mathcal{S}'_n$ containing a given distribution

Theorem. *For every u in $\mathcal{S}'_n$ and for every not-identically zero test function f belonging to $\mathcal{S}_m$ (respectively, to $\mathcal{D}_m$), there is an ${}^{\mathcal{S}}$family (respectively, a ${}^{\mathcal{D}}$family) v in $\mathcal{S}'_n$ containing u and such that the associated operator $\widehat{v}$ is proportional to the operator $\langle u, \cdot\rangle f$, where $\langle\cdot,\cdot\rangle$ is the canonical bilinear form on $\mathcal{S}'_n \times \mathcal{S}_n$.*

Proof. Let f be an ${}^{\mathcal{S}}$function (respectively ${}^{\mathcal{D}}$function) in $\mathcal{S}_m$ (respectively $\mathcal{D}_m$) not identically 0, and let p_0 be an m-vector such that $f(p_0) \neq 0$. The family defined by

$$v_p = \frac{f(p)}{f(p_0)}u,$$

for every m-vector p, is an ${}^{\mathcal{S}}$family (${}^{\mathcal{D}}$ family) containing u. Indeed, we have $v_{p_0} = u$, and moreover, for every m-index p and any test function ϕ, we obtain

$$\begin{aligned}
v(\phi)(p) &= v_p(\phi) = \\
&= \left(\frac{f(p)}{f(p_0)}u\right)(\phi) = \\
&= \frac{f(p)}{f(p_0)}u(\phi) \\
&= \left(\frac{u(\phi)}{f(p_0)}f\right)(p),
\end{aligned}$$

hence we deduce

$$v(\phi) = (u(\phi)/f(p_0))\, f,$$

and so the function $v(\phi)$ is an ${}^{\mathcal{S}}$function (${}^{\mathcal{D}}$function) in $\mathcal{S}_m$ (respectively $\mathcal{D}_m$). ■

11.1.2 ${}^{\mathcal{S}}$Families in starshaped subsets of $\mathcal{S}'_n$

By the preceding result we immediately deduce a natural sufficient condition in order that a set contains at least one ${}^{\mathcal{D}}$family, or ${}^{\mathcal{S}}$family, through every its point.

Recall that a subset S of a vector space V is said to be *star-shaped at the origin* if it contains, for every s in S, the closed segment joining s with the origin of V. On the other hand, if S contains, for every s, the segment joining s with the origin but not the origin, S is said to be a *blunt star-shaped set at the origin.*

Theorem. *Let S be a (blunt) star-shaped set at the origin of the space $\mathcal{S}'_n$. Then, for every u in S, there is a $^{\mathcal{D}}$family ($^{\mathcal{S}}$family) contained in S and passing through u.*

Proof. It is sufficient to choose a smooth function f defined on $\mathbb{R}^m$, with compact support, real, non-negative, with values lower or equal than 1, and such that $f(0_m) = 1$. Then, for every u in S, the family of distributions indexed by $\mathbb{R}^m$, defined, for every m -index p, by $v_p = f(p)u$, is a $^{\mathcal{D}}$family contained in S (v describes the segment joining u with the origin of $\mathcal{S}'_n$) and containing u. In the blunt case it is necessary to consider a function f of class $\mathcal{S}$, real, non-negative, with values lower or equal than 1, everywhere different from 0 and such that $f(0_m) = 1$. ■

We can see more, as the following result will show.

Theorem. *Every finite linear combinations of distributions can be always viewed as superposition of an $^{\mathcal{S}}$family.*

Proof. Let u_0 and u_1 two tempered distributions in $\mathcal{S}'_n$ and f_0, f_1 two $^{\mathcal{S}}$ functions in $\mathcal{S}_1$, such that $f_i(j) = \delta_{ij}$, for every choice of the indexes i, j in $\{0, 1\}$. Define a family v in $\mathcal{S}'_n$ as follows

$$v_p = f_0(p)u_0 + f_1(p)u_1,$$

for every real number p. We have

$$v_0 = f_0(0)u_0 + f_1(0)u_1 = u_0,$$

and, analogously, $v_1 = u_1$, hence v contains both vectors u_0 and u_1. Moreover, for every index p and any test function ϕ, we obtain

$$\begin{aligned} v(\phi)(p) &= v_p(\phi) = \\ &= (f_0(p)u_0 + f_1(p)u_1)(\phi) = \\ &= f_0(p)u_0(\phi) + f_1(p)u_1(\phi) = \\ &= (u_0(\phi)f_0 + u_1(\phi)f_1)(p), \end{aligned}$$

so the function $v(\phi)$ is a linear combination of f_0 and f_1, and so the function $v(\phi)$ is in $\mathcal{S}_1$. In general, every finite sequence $u = (u_i)_{i=1}^k$ of tempered distributions is a subfamily of many $^{\mathcal{S}}$families in $\mathcal{S}'_n$. It is enough to consider a system $f = (f_i)_{i=1}^k$ of functions in $\mathcal{S}_1$ such that $f_i(j) = \delta_{ij}$ for every choice of the two indexes i, j in $\{1, ..., k\}$, and define a family v in $\mathcal{S}'_n$ as follows

$$v_p = \sum_{i=1}^{k} f_i(p)u_i,$$

for every real number p. In this case, for any test function ϕ, we obtain

$$v(\phi) = \sum_{i=1}^{k} u_i(\phi) f_i,$$

which is a linear combination of the family f. Note that the image of the operator associated with the family v is a subspace of the linear hull of the family f. ■

11.1.3 ${}^{\mathcal{S}}$Connected subsets of $\mathcal{S}'_n$

We can go beyond. But first we give the following definition.

Definition (of ${}^{\mathcal{S}}$connected pair in a subset of $\mathcal{S}'_n$ and ${}^{\mathcal{S}}$ connected subset of $\mathcal{S}'_n$). *Let X be a subset of $\mathcal{S}'_n$ and let $x, y \in X$. The pair (x, y) is said to be an ${}^{\mathcal{S}}$**connected** (${}^{\mathcal{D}}$**connected**) **pair of the subset** X if and only if there is an ${}^{\mathcal{S}}$ family (${}^{\mathcal{D}}$family) v indexed by $\mathbb{R}^m$, for some integer m, containing x and y and contained in X. The part X is said ${}^{\mathcal{S}}$**connected** (${}^{\mathcal{D}}$**connected**) if, for every $x, y \in X$, the pair (x, y) is an ${}^{\mathcal{S}}$connected (${}^{\mathcal{D}}$connected) pair in X.*

Theorem. *Let S be a star-shaped set at the origin of the space $\mathcal{S}'_n$. Then, S is ${}^{\mathcal{D}}$connected and, consequently, ${}^{\mathcal{S}}$ connected.*

Proof. Every finite sequence $(u_i)_{i=1}^k$ of tempered distributions in $\mathcal{S}'_n$ is a subfamily of a particular kind of ${}^{\mathcal{D}}$family. Consider a finite sequence $(f_i)_{i=1}^k$ of functions in the space $\mathcal{D}_1$ such that any function f_i of the sequence is the i-translation of a certain function f_0 $f_i = \tau_i(f_0)$, for every index i, with f_0 a smooth function fulfilling the following properties:

- $f_0(0) = 1$;
- $f_0(x) \in [0, 1]$, for every real x;
- $\mathrm{supp} f_0 = \overline{B}(0, 1/2)$.

Consequently, the functions f_i fullfil the following:

- $f_i(i) = 1$, for every i;
- $f_i(x) \in [0, 1]$, for every real x;
- $\mathrm{supp} f_i = \overline{B}(i, 1/2)$.

Define the family v in $\mathcal{S}'_n$ as

$$v_p = \sum_{i=1}^{k} f_i(p) u_i,$$

for every real number p. It is simple to see that v is a ${}^{\mathcal{D}}$ family contained in S and passing through every u_i. ■

As a consequence, if we say ${}^{\mathcal{S}}$closed a part F of $\mathcal{S}'_n$ such that every superposition of each ${}^{\mathcal{S}}$family in F lies in F, we conclude the following corollary.

Corollary. *Every ${}^{\mathcal{S}}$closed star-shaped subset of $\mathcal{S}'_n$ is a subspace of $\mathcal{S}'_n$.*

11.2 ${}^{\mathcal{D}_{\mathcal{L}^1}}$Closed sets (*)

Following Schwartz, if $p \in [1, +\infty]$, we shall denote by $\mathcal{D}_{\mathcal{L}^p}$ the vector space of the smooth complex functions defined on $\mathbb{R}^n$ whose derivatives belong to $\mathcal{L}^p(\mathbb{R}^n, \mathbb{C})$. The natural topology on this space is, by definition, the topology generated by the family of seminorms $(q_k)_{k \in \mathbb{N}_0^n}$, where, for multi-index k, q_k is defined on $\mathcal{D}_{\mathcal{L}^p}$ by

$$q_k(f) = \left\| f^{(k)} \right\|_{\mathcal{L}^p}.$$

When $\mathcal{D}_{\mathcal{L}^p}$ is endowed with its natural topology, the associated topological vector space is denoted simply by $(\mathcal{D}_{\mathcal{L}^p})$. It is a complete locally convex topological vector space with a denumerable fundamental system of neighborhood of the origin, it is then metrizable and so a Fréchet space. A sequence $f = (f_i)_{i \in \mathbb{N}}$ converges to the zero-function in $(\mathcal{D}_{\mathcal{L}^p})$ if and only if it converges to 0 in the topological vector space $(\mathcal{L}^p)$ with all its derivatives.

11.2.1 Preliminaries on the space $\mathcal{D}_{\mathcal{L}^1}$

Lemma. *Let f be a real C^1-function defined on the non-negative real line $\mathbb{R}_{\geq}$ and of class $\mathcal{L}^1$ with its derivative. Then, the series $\sum (f(k))_{k=1}^{\infty}$ is absolutely convergent, and moreover we have the following inequality for its sum*

$$\sum_{k=1}^{\infty} |f(k)| \leq \|f\|_{\mathcal{L}^1} + \|f'\|_{\mathcal{L}^1}.$$

Proof. Let k be a positive integer, and let m_k be the minimum point of $|f|$ on the interval $[k-1,k]$. For every k, denoted by l the Lebesgue measure on $\mathbb{R}$, we have

$$|f(m_k)|\, l([k-1,k]) \leq \int_{k-1}^{k} |f|\, dl.$$

Hence, for every $n \geq 1$,

$$\sum_{k=1}^{n} |f(m_k)| \leq \sum_{k=1}^{n} \int_{k-1}^{k} |f|\, dl = \int_{0}^{n} |f|\, dl.$$

This implies that the series

$$\sum (|f(m_k)|)_{k=1}^{\infty}$$

is convergent, and that

$$\begin{aligned} \sum_{k=1}^{\infty} |f(m_k)| &\leq \lim_{n\to\infty} \int_0^n |f|\, dl = \\ &= \int_0^{+\infty} |f|\, dl = \\ &= \|f\|_{\mathcal{L}^1}\,. \end{aligned}$$

On the other hand, by the Torricelli-Barrow theorem, for every k,

$$f(k) - f(m_k) = \int_{m_k}^{k} f' dl,$$

and so

$$\begin{aligned} |f(k)| &= \left| f(m_k) + \int_{m_k}^{k} f' dl \right| \leq \\ &\leq |f(m_k)| + \left| \int_{m_k}^{k} f' dl \right| \leq \\ &\leq |f(m_k)| + \int_{k-1}^{k} |f'|\, dl, \end{aligned}$$

by this inequality, the series $\sum(|f(k)|)_{k=1}^{\infty}$ converges, and moreover

$$\begin{aligned} \sum_{k=1}^{\infty} |f(k)| &\leq \sum_{k=1}^{\infty} |f(m_k)| + \sum_{k=1}^{\infty} \int_{k-1}^{k} |f'|\, dl \\ &\leq \|f\|_{\mathcal{L}^1} + \|f'\|_{\mathcal{L}^1}\,, \end{aligned}$$

that is the conclusion. ■

Lemma. *The distribution* $\sum_{i=1}^{\infty} \delta_i$ *belongs to the space* $(\mathcal{D}_{\mathcal{L}^1})'$.

Proof. We have to prove that the distribution $\sum_{i=1}^{\infty} \delta_i$ it is a continuous form on the space $(\mathcal{D}_{\mathcal{L}^1})$. Let $f = (f_j)_{j \in J}$ be a sequence convergent to the zero-function in $(\mathcal{D}_{\mathcal{L}^1})$, then f converges to the zero-function in the topological vector space $(\mathcal{L}^1)$ with all its derivatives. We have to prove that the sequence

$$\left(\left(\sum_{i=1}^{\infty} \delta_i \right) (f_j) \right)_{j \in J}$$

is convergent to 0. By the above lemma we have

$$\sum_{k=1}^{\infty} |f_j(k)| \leq \|f_j\|_{\mathcal{L}^1} + \left\|f_j'\right\|_{\mathcal{L}^1},$$

and, since f converges to the zero-function in $(\mathcal{D}_{\mathcal{L}^1})$, the right hand converges to 0, implying the claim. ■

Remark. Let us see an alternative proof of the second lemma. It is simple to see that every delta-distribution belongs to $(\mathcal{D}_{\mathcal{L}^1})$, and thus every finite linear combination of delta-distributions. So the distribution $\sum_{i=1}^{\infty} \delta_i$ is the punctual limit of a sequence of continuous linear forms on the space $(\mathcal{D}_{\mathcal{L}^1})$. Since $(\mathcal{D}_{\mathcal{L}^1})$ is barreled (it is a Fréchet space) the Banach-Steinhaus theorem holds true, and we conclude, once more, that $\sum_{i=1}^{\infty} \delta_i$ is a continuous linear form on $(\mathcal{D}_{\mathcal{L}^1})$.

When $p = +\infty$ the space $\mathcal{D}_{\mathcal{L}^p}$ is denoted also by $\mathcal{B}_n$. Since a continuous function belonging to $\mathcal{L}^\infty$ is bounded, $\mathcal{B}_n$ is the vector space of the smooth functions that are bounded with all their derivatives. Moreover, $\mathcal{B}_n^o$ denotes the subspace of $\mathcal{B}_n$ containing the function vanishing at infinite with all their derivatives; $(\mathcal{B}_n^o)$ shall be the associated topological vector space endowed with the topology induced by $(\mathcal{B}_n)$. It is clear that $\mathcal{S}_n$ is included in $\mathcal{B}_n^o$, and it is also evident that the topological vector space $(\mathcal{S}_n)$ is continuously imbedded in the space $(\mathcal{B}_n^o)$, consequently $(\mathcal{B}_n^o)' \subset \mathcal{S}_n'$.

11.3 Sums of series as superpositions

Let us see the sum of a convergent series of tempered distribution as a superposition.

Theorem. *Let* $\sum (u_k)_{k=1}^{\infty}$ *be a weakly* convergent series in* $\mathcal{S}_n'$. *Then, there is a* ${}^{\mathcal{B}_n^o}$ *family, more precisely a* ${}^{\mathcal{D}_{\mathcal{L}^1}}$*family, which contains the series as sub-family.*

Proof. Let $\sum (u_k)_{k=1}^{\infty}$ be such series and assume it is weakly* convergent to a tempered distribution u^*. Consider a sequence $f = (f_i)_{i=1}^{\infty}$ of functions in $\mathcal{D}_1$ such that $f_i = \tau_i(f_0)$, for every i, where f_0 a smooth function in $\mathcal{D}_1$ with the following properties:

- $f_0(0) = 1$;
- $f_0(x) \in [0, 1]$, for every real x;
- $\mathrm{supp} f_0 = \overline{B}(0, 1/2)$.

Define the family v in $\mathcal{S}'_n$ as follows

$$v_p := {}^{\sigma(\mathcal{S}'_n)} \sum_{i=1}^{\infty} f_i(p) u_i,$$

for every real number p. Note that the sequence of partial sums of the series $\sum(f_i(p)u_i)_{i=1}^{\infty}$ is definitely constant, and then the series is $\sigma(\mathcal{S}'_n)$-convergent. Moreover, $v_j = u_j$, for every natural j. We have to prove that, if g is a test function of class $\mathcal{S}_n$, then $v(g)$ is of class $\mathcal{B}^o_n$. Let g be a test function in $\mathcal{S}_n$, then for any multi-index p there is an integer j in the closed ball $\overline{B}(p, 1/2)$ such that

$$\begin{aligned} v(g)(p) &= v_p(g) = \\ &= \sum_{i=1}^{\infty} f_i(p) u_i(g) = \\ &= u_j(g) f_j(p). \end{aligned}$$

So the function

$$v(g) = \sum_{i=1}^{\infty} u_i(g) f_i$$

is smooth and vanishing at infinity with all its derivatives. In fact, being the numerical series $\sum(u_i(g))_{i=1}^{\infty}$ convergent, for every test function g, we have

$$\lim_{i\to\infty} |u_i(g)| = 0,$$

hence

$$\begin{aligned} \lim_{p\to\infty} |v(g)(p)| &= \lim_{j\to\infty} |u_j(g) f_j| \leq \\ &\leq \max f_0 \cdot \lim_{i\to\infty} |u_i(g)| = \\ &= 0. \end{aligned}$$

Analogously, for every natural k, we have

$$\begin{aligned} \lim_{p\to\infty} \left| v(g)^{(k)}(p) \right| &= \lim_{j\to\infty} \left| u_j(g) f_j^{(k)} \right| \leq \\ &\leq \max f_0^{(k)} \cdot \lim_{i\to\infty} |u_i(g)| = \\ &= 0. \end{aligned}$$

Hence $v(g)$ belongs to $\mathcal{B}_1^o$. To prove that v is of class $\mathcal{D}_{\mathcal{L}^1}$, note that, for every integer $k \geq 0$,

$$\begin{aligned}
\left| \int_{\mathbb{R}} v(g)^{(k)} dl \right| &= \left| \int_{\mathbb{R}} \sum_{i=1}^{\infty} u_i(g) f_i^{(k)} dl \right| = \\
&= \left| \sum_{i=1}^{\infty} u_i(g) \int_{\mathbb{R}} f_i^{(k)} dl \right| = \\
&= \left| \sum_{i=1}^{\infty} u_i(g) \right| \left| \int_{\mathbb{R}} f_0^{(k)} dl \right|,
\end{aligned}$$

thus $v(g)$ is smooth and of class $\mathcal{L}^1$ with all its derivatives, then, following Schwartz, $v(g)$ belongs to the space $\mathcal{D}_{\mathcal{L}^1}$. ■

Corollary. *Let $\sum (u_k)_{k=1}^{\infty}$ be a weakly* convergent series in $\mathcal{S}'_n$ to a tempered distribution u^*. Then u^* is a superposition of a $\mathcal{D}_{\mathcal{L}^1}$-family. As a consequence, each $\mathcal{D}_{\mathcal{L}^1}$-closed subset of $\mathcal{S}'_n$ is sequentially weakly* closed.*

Proof. Consider the series of distributions $\sum (\delta_i)_{i=1}^{\infty}$ in $(\mathcal{D}_{\mathcal{L}^1})'$; it is convergent in $(\mathcal{D}_{\mathcal{L}^1})'$. In fact, for every s in $\mathcal{D}_{\mathcal{L}^1}$, the series $\sum (s(i))_{i=1}^{n}$ is convergent, and we have

$$\sum_{i=1}^{\infty} \delta_i(s) = \sum_{i=1}^{\infty} s(i).$$

Let v be the family of class $\mathcal{D}_{\mathcal{L}^1}$ built in the proof of the above theorem, we obtain

$$\begin{aligned}
\left(\int_{\mathbb{R}} \sum_{i=1}^{\infty} \delta_i v \right)(g) &= \left(\sum_{i=1}^{\infty} \delta_i \right)(v(g)) = \\
&= \sum_{i=1}^{\infty} v_i(g) = \\
&= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} u_j(g) f_j(i) = \\
&= \sum_{i=1}^{\infty} u_i(g) = \\
&= u^*(g).
\end{aligned}$$

Hence u^* is a $\mathcal{D}_{\mathcal{L}^1}$-superposition of v. Now, a set F is sequentially weakly* closed if and only if contains the sum of every series in F sequentially weakly* convergent, and this concludes the proof. ■

Corollary. *A subset of the space $\mathcal{S}'_n$ is $\mathcal{D}_{\mathcal{L}^1}$-closed if and only if it is sequentially weakly* closed.*

Proof. To prove that every weakly* closed subset F of $\mathcal{S}'_n$ is $\mathcal{D}_{\mathcal{L}^1}$ closed it is sufficient to note that the linear hull of the Dirac family is weakly* dense in $\mathcal{D}'_{\mathcal{L}^1}$, indeed the space $(\mathcal{D}_{\mathcal{L}^1})$ is continuously imbedded in $(\mathcal{S}_n)$ and so the linear hull of the Dirac family is dense in $(\mathcal{D}_{\mathcal{L}^1})'$ that is a subspace of $(\mathcal{S}'_n)$. At this point applying (as usual) the Banach-Steinhaus theorem we conclude that every superposition of a $\mathcal{D}_{\mathcal{L}^1}$-family is in F. ■

- *Let us prove that there are sequences whose associated series are weakly* convergent but which are not rapidly decreasing at infinity.*

Proof. Let u be a tempered distribution and let consider the sequence

$$v = ((1/i^2)u)_{i=1}^{\infty}.$$

The sequence v has a series weakly* convergent but v is not of class $\mathcal{S}$. Indeed, let g be a test function, then the image $v(g)$ is the sequence $((1/i^2)u(g))_{i=1}^{\infty}$ that is not rapidly decreasing at infinity. Now this sequence cannot be a subsequence of an ${}^{\mathcal{S}}$family, since every subsequence of ${}^{\mathcal{S}}$families is a rapidly decreasing sequence. ■

Part IV

$^{\mathcal{S}}$Linear operators

Chapter 12

$^{\mathcal{S}}$Linear operators

12.1 Introduction

Let X and Y be two vector spaces on the field $\mathbb{K}$ (the real field $\mathbb{R}$ or the complex one $\mathbb{C}$). A function f from X into Y is called linear if, for any two points x, y of the space X and for each scalar $\lambda \in \mathbb{K}$, the equality

$$f(\lambda x + y) = \lambda f(x) + f(y),$$

holds true. Equivalently, a mapping f from X into Y is linear if and only if for every integer $k \in \mathbb{N}$, for any k-tuple $x = (x_i)_{i=1}^k$ of points of X and for any k-tuple of scalars $\lambda = (\lambda_i)_{i=1}^k$ of $\mathbb{K}$, setting

$$\sum\nolimits_k \lambda x := \sum_{i=1}^k \lambda_i x_i$$

and $f(x) := (f(x_i))_{i=1}^k$, we have

$$f\left(\sum\nolimits_k \lambda x\right) = \sum\nolimits_k \lambda f(x),$$

i.e., the image of the λ-linear combination of a family x is the λ-linear combination of the image $f(x)$ of the family x under the function f; in indexed notation, we have

$$f\left(\sum_{i=1}^{k}\lambda_i x_i\right)=\sum_{i=1}^{k}\lambda_i f\left(x_i\right).$$

The aim of this chapter is to extend the last definition to the class of ${}^{\mathcal{S}}$families of tempered distributions indexed by the Euclidean space $\mathbb{R}^k$, using, as coefficient systems, locally integrable maps from $\mathbb{R}^k$ to $\mathbb{K}$ and, more generally, Schwartz tempered distributions from $\mathbb{R}^k$ to $\mathbb{K}$ (which, as we already have seen, are so viewed as "non-locally defined" families in $\mathbb{K}$ indexed by $\mathbb{R}^k$). If $v=(v_i)_{i\in\mathbb{R}^k}$ is an ${}^{\mathcal{S}}$family in $\mathcal{S}'_n$, i.e. if for every test function $\phi\in\mathcal{S}_n$, the function

$$v(\phi):\mathbb{R}^k\to\mathbb{K}:i\mapsto v_i(\phi),$$

belongs to $\mathcal{S}_k$, and if $\lambda\in\mathcal{S}'_k$ is a tempered distribution defined on the index set of the family v, we put

$$\int_{\mathbb{R}^k}\lambda v:=\lambda\circ\widehat{v}=\ {}^t\left(\widehat{v}\right)(\lambda),$$

where ${}^t\left(\widehat{v}\right)$ is the transpose of the operator

$$\widehat{v}:\mathcal{S}_n\to\mathcal{S}_k:\phi\mapsto v(\phi).$$

The idea is very natural:

- *an operator $L:\mathcal{S}'_n\to\mathcal{S}'_m$ is said ${}^{\mathcal{S}}$linear if, for every integer $k\in\mathbb{N}$, for every distribution $\lambda\in\ \mathcal{S}'_k$ and for every $v\in\mathcal{S}\left(\mathbb{R}^k,\mathcal{S}'_n\right)$, the image of the ${}^{\mathcal{S}}$family v is an ${}^{\mathcal{S}}$family and the equality*

$$L\left(\int_{\mathbb{R}^k}\lambda v\right)=\int_{\mathbb{R}^k}\lambda L(v),$$

  holds true.

12.2 ${}^{\mathcal{S}}$Operators

First of all we have to act on a family of tempered distribution by means of operators defined on spaces of tempered distribution, the definition is absolutely straightforward.

Definition (image of a family of distributions). *Let W be a subset of the space $\mathcal{S}'_n$, let $A:W\to\mathcal{S}'_m$ be an operator (not necessarily linear) and let $v=(v_p)_{p\in\mathbb{R}^k}$ be a family of tempered distributions in the subset W, i.e. a family* *with trace set* $\{v_p\}_{p\in\mathbb{R}^k}$ *contained in the subset* W. *The* ***image of the family*** v ***under the operator*** A *is by definition the family* $A(v)$ *in* $\mathcal{S}'_m$ *defined by*

$$A(v) = (A(v_p))_{p\in\mathbb{R}^k},$$

i.e., the family $A(v)$ *such that, for all index* $p \in \mathbb{R}^k$, *we have* $A(v)_p = A(v_p)$.

We can read the above definition saying that:

- *the image under an operator of a family of vectors is the family of the images of vectors.*

Definition (operator of class $\mathcal{S}$). *Let* W *be a subset of the space* $\mathcal{S}'_n$ *and let* $L : W \to \mathcal{S}'_m$ *be an operator (not necessarily linear). The operator* L *is said an* $^{\mathcal{S}}$***operator*** *or* ***operator of class*** $\mathcal{S}$ *if, for each natural* k *and for each family* $v \in \mathcal{S}(\mathbb{R}^k, \mathcal{S}'_n)$ *such that the trace set* $\{v_p\}_{p\in\mathbb{R}^k}$ *is contained in* W, *the image family* $L(v)$ *is an* $^{\mathcal{S}}$*family (that is belonging to the space* $\mathcal{S}(\mathbb{R}^k, \mathcal{S}'_m)$*).*

We can read the above definition as follows:

- *An operator* L *is of class* $\mathcal{S}$ *if the image by* L *of an* $^{\mathcal{S}}$*family is an* $^{\mathcal{S}}$*family too.*

12.3 $^{\mathcal{S}}$Operators defined on $\mathcal{S}'_n$

The following property proves that the class of linear $^{\mathcal{S}}$ operators defined on the entire space of tempered distribution contains the class of weakly* continuous linear operators on that space, indeed we shall see that the two class are coincident.

Theorem (the transpose of an operator). *The transpose of a weakly continuous linear operator defined among two spaces of Schwartz test functions is an* $^{\mathcal{S}}$*operator. Consequently, every weakly* continuous linear operator defined among two spaces of tempered distributions is an* $^{\mathcal{S}}$*operator.*

Proof. Let $A : \mathcal{S}_n \to \mathcal{S}_m$ be a continuous linear operator with respect to the pair of weak topologies $(\sigma(\mathcal{S}_n), \sigma(\mathcal{S}_m))$. Then, the operator A is transposable (i.e., for every tempered distribution $a \in \mathcal{S}'_m$, the functional $a \circ A$ lies in the space $\mathcal{S}'_n$) and its transpose is (by definition) the operator

$$^{t}A : \mathcal{S}'_m \to \mathcal{S}'_n : a \mapsto a \circ A.$$

Let $v \in \mathcal{S}(\mathbb{R}^k, \mathcal{S}'_n)$ be an $^{\mathcal{S}}$family, we have, by definition of image of a family,

$$^tA\,(v)_p = {}^tA(v_p),$$

and hence we deduce

$$\begin{aligned} {}^tA\,(v)\,(\phi)\,(p) &= {}^tA\,(v)_p\,(\phi) \\ &= {}^tA(v_p)\,(\phi) \\ &= v_p\,(A\,(\phi)) \\ &= v\,(A\,(\phi))\,(p), \end{aligned}$$

so, taking into account that v is an $^{\mathcal{S}}$family, we deduce that the image

$$^tA\,(v)\,(\phi) = \widehat{v}\,(A\,(\phi))$$

belongs to the space $\mathcal{S}_k$. Concluding, the image family $^tA\,(v)$ is a family of class $\mathcal{S}$ belonging to the space $\mathcal{S}(\mathbb{R}^k, \mathcal{S}'_n)$, and thus the operator ^tA, sending $^{\mathcal{S}}$families into $^{\mathcal{S}}$ families, is an $\mathcal{S}$-operator. ■

Application. Let $L : \mathcal{S}'_n \to \mathcal{S}'_n$ be a differential operator with constant coefficients and let v be an $^{\mathcal{S}}$family in the space $\mathcal{S}'_n$. Then $L(v)$ is an $^{\mathcal{S}}$family, in fact L is the transpose of some differential operator on the space $\mathcal{S}_n$. For instance, the Dirac family $(\delta_x)_{x\in\mathbb{R}^n}$ is obviously an $^{\mathcal{S}}$family, and so the family of i-th derivatives $(\delta_x^{(i)})_{x\in\mathbb{R}^n}$ is an $^{\mathcal{S}}$ family, for every multi-index i.

12.4 Characterization of $^{\mathcal{S}}$operators on $\mathcal{S}'_n$

The following property proves that the class of linear $^{\mathcal{S}}$ operators defined on the entire space of tempered distribution coincides with the class of weakly* continuous linear operators.

Theorem (characterization of $^{\mathcal{S}}$operators). *A linear operator defined among two spaces of tempered distributions is an $^{\mathcal{S}}$operator if and only if it is weakly* continuous.*

Proof. We have already proved that every continuous linear operator defined on a space of tempered distribution is an $^{\mathcal{S}}$operator. Vice versa, if $L : \mathcal{S}'_n \to \mathcal{S}'_m$ is a linear $^{\mathcal{S}}$operator, then the image of the Dirac family $L(\delta)$ is a family of class $\mathcal{S}$. So for every test function h in $\mathcal{S}'_m$, the image of the function h by the family $L(\delta)$ is a family of class $\mathcal{S}$ (belonging to the space $\mathcal{S}'_n$), namely the function

$$L(\delta)(h) : x \mapsto L(\delta_x)(h),$$

so the operator L is weakly topologically transposable, and its transpose is defined by

$$^{t}L : \mathcal{S}_m \to \mathcal{S}_n : h \mapsto L(\delta)(h),$$

or, equivalently, defined by the classic transpose definition

$$\left\langle u, ^{t} L(h) \right\rangle_n = \langle L(u), h \rangle_m ,$$

for every tempered distribution u in the space $\mathcal{S}'_n$ and for every test function h in the space $\mathcal{S}_m$. Since every weak topologically transposable operator is weakly continuous we conclude that every linear ${}^{\mathcal{S}}$operator is weakly continuous. ■

12.5 ${}^{\mathcal{S}}$Linear operators on $\mathcal{S}'_n$

In this section we shall introduce the main concept of the chapter.

Definition (${}^{\mathcal{S}}$linear operators on the entire $\mathcal{S}'_n$). *Let $L : \mathcal{S}'_n \to \mathcal{S}'_m$ be an ${}^{\mathcal{S}}$operator (not necessarily linear). The operator L is called ${}^{\mathcal{S}}$**linear operator** if, for each positive integer k, for each family $v \in \mathcal{S}(\mathbb{R}^k, \mathcal{S}'_n)$ and for every tempered distribution a in $\mathcal{S}'_k$, the equality*

$$L\left(\int_{\mathbb{R}^k} av\right) = \int_{\mathbb{R}^k} aL(v)$$

holds true.

Property (linearity of the ${}^{\mathcal{S}}$linear operators). *An ${}^{\mathcal{S}}$linear operator is linear.*

Proof. Indeed, in the conditions of the above definition, for each couple of scalars b, c and any couple of tempered distributions in the space $\mathcal{S}'_n$, if δ is the Dirac basis of the space $\mathcal{S}'_n$, we have

$$\begin{aligned}
L(au + bw) &= L\left(\int_{\mathbb{R}^k} (au + bw)\delta\right) = \\
&= \int_{\mathbb{R}^k} (au + bw)L(\delta) = \\
&= a\int_{\mathbb{R}^k} uL(\delta) + b\int_{\mathbb{R}^k} wL(\delta) = \\
&= aL\left(\int_{\mathbb{R}^k} u\delta\right) + bL\left(\int_{\mathbb{R}^k} w\delta\right) = \\
&= aL(u) + bL(w),
\end{aligned}$$

as we desired. ■

But we will see more than this preliminary remark about ${}^{\mathcal{S}}$ linear operators.

12.6 Examples of $^{\mathcal{S}}$linear operators

In this section we propose two important examples of $^{\mathcal{S}}$linear operators. We note that the first is a particular case of the second one, and indeed we shall see that every $^{\mathcal{S}}$linear operator defined on the entire $\mathcal{S}'_n$ is of the type presented in the second example.

12.6.1 The superposition operator of an $^{\mathcal{S}}$family

Recall that if $v \in s(\mathbb{R}^k, \mathcal{S}'_m)$ is any family of tempered distributions indexed by an Euclidean space $\mathbb{R}^k$ and if $w \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ is any $^{\mathcal{S}}$family of tempered distributions, the family in $\mathcal{S}'_n$ indexed by $\mathbb{R}^k$ and defined by

$$\int_{\mathbb{R}^m} vw := \left(\int_{\mathbb{R}^m} v_p w \right)_{p \in \mathbb{R}^k},$$

is called the superposition of the $^{\mathcal{S}}$family w with respect to the family v.

We have already proved that, if the family v belongs to the space $\mathcal{S}(\mathbb{R}^k, \mathcal{S}'_m)$ then the superposition $\int_{\mathbb{R}^m} vw$ belongs to the space $\mathcal{S}(\mathbb{R}^k, \quad \mathcal{S}'_n)$ and the operator associated with this superposition is the composition of the operators associated with the two families v and w , precisely we have

$$\left(\int_{\mathbb{R}^m} vw \right)^{\wedge} = \widehat{v} \circ \widehat{w}.$$

In this case, sometimes, it is also convenient to denote the superposition

$$\int_{\mathbb{R}^m} vw$$

by the product notation $v.w$ and we call it also the $^{\mathcal{S}}$ product of the family v by the family w.

Proposition. *Let $w \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be an $^{\mathcal{S}}$family of distributions and let $L : \mathcal{S}'_m \to \mathcal{S}'_n$ be the superposition operator of the family w, defined by*

$$L(a) = \int_{\mathbb{R}^m} aw,$$

for all tempered distribution $a \in \mathcal{S}'_m$. Then, the operator L is an $^{\mathcal{S}}$linear operator.

Proof. The operator L is an $^{\mathcal{S}}$operator, indeed we know that L is the transpose of the continuous linear operator associated with v and then it is weakly* continuous. But we want to see this fact directly. If $v \in \mathcal{S}(\mathbb{R}^k, \mathcal{S}'_m)$ is an ${}^{\mathcal{S}}$family then its image by the operator L is $L(v) = v.w$ and the product of two ${}^{\mathcal{S}}$families is an ${}^{\mathcal{S}}$family. Let $a \in \mathcal{S}'_k$ be a tempered distribution and let $v \in \mathcal{S}(\mathbb{R}^k, \mathcal{S}'_m)$ be an ${}^{\mathcal{S}}$family, we have

$$\begin{aligned} L\left(\int_{\mathbb{R}^k} av\right) &= \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^k} av\right) w = \\ &= \int_{\mathbb{R}^k} a \left(\int_{\mathbb{R}^m} vw\right) = \\ &= \int_{\mathbb{R}^k} aL(v). \end{aligned}$$

Note in fact that, for each index $p \in \mathbb{R}^k$, we have

$$\begin{aligned} L(v)_p &= L(v_p) = \\ &= \int_{\mathbb{R}^m} v_p w = \\ &= \left(\int_{\mathbb{R}^m} vw\right)_p, \end{aligned}$$

and the proof is completed. ■

12.6.2 Transpose operators

Lemma (the image under a transpose operator). *Let $B \in \mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$ be a linear continuous operator and let $v \in \mathcal{S}(\mathbb{R}^k, \mathcal{S}'_m)$ be an ${}^{\mathcal{S}}$family. Then, the image of the family v by the transpose operator ${}^tB$ is the product of the family v by the family generated by the operator B, in symbol we have*

$${}^tB(v) = \int_{\mathbb{R}^k} vB^\vee,$$

so in particular, the transpose operator ${}^tB$ is an ${}^{\mathcal{S}}$operator.

Proof. For each index $p \in \mathbb{R}^k$, we have

$$\begin{aligned} \left(\int_{\mathbb{R}^m} vB^\vee\right)_p &= \int_{\mathbb{R}^m} v_p B^\vee = \\ &= v_p \circ (B^\vee)^\wedge = \\ &= v_p \circ B = \\ &= {}^tB(v_p) = \\ &= {}^tB(v)(p), \end{aligned}$$

and hence

$$\int_{\mathbb{R}^m} vB^{\vee} = {}^tB(v),$$

as we desired. ■

Theorem (${}^{\mathcal{S}}$linearity of a transpose operator). *Let $B \in \mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$ be a linear and continuous operator and let $v \in \mathcal{S}(\mathbb{R}^k, \mathcal{S}'_m)$ be an ${}^{\mathcal{S}}$ family. Then, for each tempered coefficient system $a \in \mathcal{S}'_k$, we have*

$${}^tB\left(\int_{\mathbb{R}^k} av\right) = \int_{\mathbb{R}^k} a\, {}^tB(v).$$

Proof. We have

$$\begin{aligned}
{}^tB\left(\int_{\mathbb{R}^k} av\right) &= \left(\int_{\mathbb{R}^k} av\right) \circ B = \\
&= (a \circ \widehat{v}) \circ B = \\
&= a \circ (\widehat{v} \circ B) = \\
&= \int_{\mathbb{R}^k} a(\widehat{v} \circ B)^{\vee} = \\
&= \int_{\mathbb{R}^k} a(\int_{\mathbb{R}^m} vB^{\vee}) = \\
&= \int_{\mathbb{R}^k} a\, {}^tB(v),
\end{aligned}$$

as we desired. ■

Application (derivatives of a distribution). As a simple application, we prove the formula

$$u' = \int_{\mathbb{R}} u\delta',$$

where δ' is the ${}^{\mathcal{S}}$family in $\mathcal{S}'_1$ defined by $\delta' = (\delta'_p)_{p\in\mathbb{R}}$. Let δ be the Dirac family of the space $\mathcal{S}'_1$, then for each tempered distribution $u \in \mathcal{S}'_1$, we have

$$u = \int_{\mathbb{R}} u\delta,$$

and consequently

$$\begin{aligned}
u' &= \partial\left(\int_{\mathbb{R}} u\delta\right) = \\
&= \int_{\mathbb{R}} u\partial(\delta) = \\
&= \int_{\mathbb{R}} u\delta'.
\end{aligned}$$

More generally, in the space $\mathcal{S}'_n$, we have

$$L(u) = \int_{\mathbb{R}} uL(\delta),$$

for every differential operator L, and every tempered distribution u.

12.7 Characterization of $^{\mathcal{S}}$linear operators

Now, we can show the true nature of the $^{\mathcal{S}}$linear operators defined on $\mathcal{S}'_n$.

Theorem (characterization of $^{\mathcal{S}}$linearity). *Let $L : \mathcal{S}'_n \to \mathcal{S}'_m$ be an operator. Then, L is $\mathcal{S}$-linear if and only if there exists a linear and continuous operator $B \in \mathcal{L}(\mathcal{S}_m, \mathcal{S}_n)$ such that $L = {}^t(B)$.*

Proof. Sufficiency. Follows from the above theorem. *Necessity.* Let δ be the Dirac family in the space $\mathcal{S}'_n$, we have

$$\begin{aligned} L(u) &= L\left(\int_{\mathbb{R}^n} u\delta\right) = \\ &= \int_{\mathbb{R}^n} uL(\delta) = \\ &= {}^t\left(L(\delta)^\wedge\right)(u), \end{aligned}$$

so

$$L = {}^t\left(L(\delta)^\wedge\right),$$

as we desire. ■

Before to give the last complete characterization of $^{\mathcal{S}}$linear operators, we recall the following classical definition from Linear Functional Analysis.

Definition (of transposable operator). *A linear operator $L : \mathcal{S}'_n \to \mathcal{S}'_m$ is said to be* ***transposable with respect to the canonical pairings*** $(\mathcal{S}_n, \mathcal{S}'_n)$ ***and*** $(\mathcal{S}_m, \mathcal{S}'_m)$ *if and only if there exists a linear continuous operator $B \in \mathcal{L}(\mathcal{S}_m, \mathcal{S}_n)$ such that $L = {}^t(B)$.*

Recalling that the operator L is weakly continuous if and only if it is strongly continuous if and only if it is transposable, we derive the following definitive characterization.

Theorem (characterization of $^{\mathcal{S}}$linearity). *Let $L : \mathcal{S}'_n \to \mathcal{S}'_m$ be a operator. Then, the following assertions are equivalent*

1) *the operator L is $\mathcal{S}$ -linear;*

2) *there exists an operator* $B \in \mathcal{L}(\mathcal{S}_m, \mathcal{S}_n)$ *such that* $L = {}^t(B)$;

3) *the operator* L *is linear and weakly continuous;*

4) *the operator* L *is linear and strongly continuous;*

5) *the operator* L *is linear and transposable;*

6) *the operator* L *is linear and of class* $\mathcal{S}$.

Chapter 13

Applications of ${}^{\mathcal{S}}$linear operators

13.1 ${}^{\mathcal{S}}$Bases of subspaces

The preceding theorem allow us to state and prove some definitive results about the existence of ${}^{\mathcal{S}}$bases for a subspace of $\mathcal{S}'_n$.

Theorem. *Let V be a subspace of $\mathcal{S}'_n$. Then*

1) the subspace V has a system of ${}^{\mathcal{S}}$generators if and only if there is an ${}^{\mathcal{S}}$linear operator $A : \mathcal{S}'_m \to \mathcal{S}'_n$, for some integer m, such that

$$A(\mathcal{S}'_m) = V;$$

2) the subspace V has an ${}^{\mathcal{S}}$basis if and only if there is an injective ${}^{\mathcal{S}}$ linear operator $A : \mathcal{S}'_m \to \mathcal{S}'_n$, for some integer m, such that

$$A(\mathcal{S}'_m) = V.$$

Proof. 1) It's obvious, because every $^{\mathcal{S}}$family v univocally determines a transposable operator $\widehat{v} : \mathcal{S}_n \to \mathcal{S}_m$ that is univocally determined by the $^{\mathcal{S}}$linear operator $^t\widehat{v} : \mathcal{S}'_m \to \mathcal{S}'_n$, and vice versa. Moreover, for every $^{\mathcal{S}}$family v, we have

$$^{\mathcal{S}}\mathrm{span}\,(v) = {}^t\widehat{v}(\mathcal{S}'_m).$$

2) Remember that

$$^t\widehat{v}(a) = \int_{\mathbb{R}^m} av.$$

Then the conclusion follows immediately from the definition of $^{\mathcal{S}}$linear independence. ■

The preceding result can be reread in the following way.

Theorem. *Let V be a subspace of $\mathcal{S}'_n$. Then*

1) *the subspace V has a system of $^{\mathcal{S}}$generators if and only if there is a continuous linear operator $A : \mathcal{S}_n \to \mathcal{S}_m$, for some integer m, such that*

$$^tA(\mathcal{S}'_m) = V;$$

2) *the subspace V has an $^{\mathcal{S}}$basis if and only if there is a continuous linear operator $A : \mathcal{S}_n \to \mathcal{S}_m$, for some integer m, such that its image* $\mathrm{im}A$ *is dense in $\mathcal{S}_m$ and*

$$^tA(\mathcal{S}'_m) = V.$$

13.2 $^{\mathcal{S}}$Bases of closed subspaces

By the preceding and by the Dieudonné-Schwartz theorem (see later), it follows

Theorem. *Let V be a weakly* closed subspace of $\mathcal{S}'_n$. Then*

1) *the subspace V has a system of $^{\mathcal{S}}$generators if and only if there is a strict morphism $A : \mathcal{S}'_m \to \mathcal{S}'_n$, for some m, such that*

$$A(\mathcal{S}'_m) = V;$$

2) *the subspace V has an $^{\mathcal{S}}$basis if and only if there is an injective strict morphism $A : \mathcal{S}'_m \to \mathcal{S}'_n$, for some m, such that*

$$A(\mathcal{S}'_m) = V.$$

Corollary. *Let V be a weakly* closed subspace of $\mathcal{S}'_n$. Then*

1) *the subspace V has a system of $^{\mathcal{S}}$generators if and only if there is a surjective strict morphism $L : \mathcal{S}'_m \to V$, for some integer m;*

2) *the subspace V has an $^{\mathcal{S}}$basis if and only if it is topologically isomorph with some space $\mathcal{S}'_m$; that is if and only if there a topological isomorphism $L : \mathcal{S}'_m \to V$, for some integer m.*

We recall the classical results we have used.

Theorem (Dieudonné-Schwartz). *Let E and F be two Fréchet spaces with topologies $\mathcal{T}_E$ and $\mathcal{T}_F$ respectively, E' and F' their topological duals, and let $u : E \to F$ be a linear and continuous map. Then the following conditions are equivalent:*

1) *the operator u is a strict morphism for the topologies $\mathcal{T}_E$ and $\mathcal{T}_F$;*

2) *the operator u is a strict morphism for the weak topologies $\sigma(E, E')$ and $\sigma(F, F')$;*

3) *the image $u(E)$ is closed in F;*

4) *the transpose operator $^t u$ is a strict morphism for the weak* topologies $\sigma(F', F)$ and $\sigma(E', E)$;*

5) *the image $^t u(F')$ is closed in the dual E' for the weak* topology $\sigma(E', E)$.*

Corollary. *Let E and F be two Fréchet spaces, E' and F' their topological duals, and $u : E \to F$ be a linear continuous map. Then,*

1) *the operator u is an injective strict morphism if and only if its transpose operator is surjective, i.e. if and only if*

$$^t u(F') = E';$$

2) *the operator u is a surjective strict morphism if and only if the image $^t u(F')$ is closed in E' for the weak* topology $\sigma(E', E)$ and the transpose $^t u$ is injective;*

3) *the operator u is an isomorphism if and only if its transpose operator $^t u$ is an isomorphism for the weak* topologies $\sigma(F', F)$ and $\sigma(E', E)$.*

13.3 $^{\mathcal{S}}$Bases of barreled subspaces

The Pták open mapping theorem allows us to state and prove some definitive results about the existence of $^{\mathcal{S}}$bases for a barreled subspace of $\mathcal{S}'_n$.

Theorem. *Let V be a barreled subspace of $\mathcal{S}'_n$. Then*

1) *the subspace V has a system of $^{\mathcal{S}}$generators if and only if there is a surjective strict morphism $L : \mathcal{S}'_m \to V$, for some integer m;*

2) *the subspace V has an $^{\mathcal{S}}$basis if and only if it is topologically isomorph with some space $\mathcal{S}'_m$; that is if and only if there a topological isomorphism $L : \mathcal{S}'_m \to V$, for some integer m.*

Proof. Indeed, the Pták open mapping theorem affirms that if E is a Pták space and F a barreled space, then every surjective continuous linear operator of E onto F is a topological homomorphism. Now, every space $\mathcal{S}'_m$ is a Pták space, since it is the dual of a Fréchet space and the subspace V is barreled by assumption. Let v be a system of $^{\mathcal{S}}$generators of the subspace V, then the corresponding superposition operator A is a linear continuous operator of $\mathcal{S}'_m$ into $\mathcal{S}'_n$ whose image is V. Let L be the codomain restriction of the operator A to the subspace V, it is a surjective continuous linear operator of the space $\mathcal{S}'_m$ onto V; applying the Pták theorem, we conclude that L is a surjective strict morphism. The second assertion is an obvious consequence of the first one. ■

13.4 Superpositions of $^{\mathcal{S}}$linear operators

In this section we desire to define the superposition of a continuous family of continuous linear operators among two distribution spaces. Let us start immediately with the action of a family of operators on a distribution.

Definition (image of a distribution under a family of operators). *Let A be a family of weakly* continuous linear operators from the space $\mathcal{S}'_n$ into the space $\mathcal{S}'_m$ indexed by the k-dimensional real Euclidean space $\mathbb{R}^k$. For every tempered distribution u in $\mathcal{S}'_n$, we define the action of the family A on the distribution u as the family*

$$A(u) := (A_q(u))_{q\in\mathbb{R}^k}$$

of tempered distributions in $\mathcal{S}'_m$. In other terms, the image of a tempered distribution by a family of operators if the family of the images of the distribution by any single operator of the family.

Definition (family of operators of class $\mathcal{S}$). *We say that the family A of operators is a family of class $\mathcal{S}$ if the family of distribution $A(u)$ is of class $\mathcal{S}$, for any u in $\mathcal{S}'_n$.*

Definition (superpositions of an $^{\mathcal{S}}$family of operators). *If A is a family of operators in the space $\mathcal{L}(\mathcal{S}'_n, \mathcal{S}'_m)$ indexed by $\mathbb{R}^k$ and of class $\mathcal{S}$, we can consider the superposition in $\mathcal{S}'_m$*

$$\int_{\mathbb{R}^k} aA(u),$$

for every distribution u in $\mathcal{S}'_n$ and every distribution a in $\mathcal{S}'_k$, this superposition, we repeat, is a distribution belonging to the space $\mathcal{S}'_m$. So we have constructed an operator, denoted by

$$\int_{\mathbb{R}^k} aA,$$

*from the space $\mathcal{S}'_n$ into $\mathcal{S}'_m$ - that we say **the superposition of the family** A **by the system of coefficients** a - defined by*

$$\left(\int_{\mathbb{R}^k} aA\right)(u) = \int_{\mathbb{R}^k} aA(u),$$

for any distribution u in $\mathcal{S}'_n$.

Is this operator a linear continuous operator? Let us see that it is indeed an $^{\mathcal{S}}$linear operator.

Theorem. *Let A be a family of operators in the space $\mathcal{L}(\mathcal{S}'_n, \mathcal{S}'_m)$ indexed by $\mathbb{R}^k$ and of class $\mathcal{S}$, and let a be a distribution in $\mathcal{S}'_k$. Then, the superposition*

$$\int_{\mathbb{R}^k} aA$$

is a linear continuous operator and (equivalently) an $^{\mathcal{S}}$linear operator.

Proof. Let δ be the Dirac basis of the space $\mathcal{S}'_k$ and let u be a vector of $\mathcal{S}'_n$. For any index q in $\mathbb{R}^k$, we have

$$\begin{aligned}
(\int_{\mathbb{R}^k} \delta_q A)(u) &= \int_{\mathbb{R}^k} \delta_q A(u) = \\
&= A(u)_q = \\
&= A_q(u),
\end{aligned}$$

so the superposition $\int_{\mathbb{R}^k} \delta_q A$ is the operator A_q (as was already clear), which is a continuous operator. Now it is also clear that the superposition $\int_{\mathbb{R}^k} aA$ of the family A with respect to a coefficient distribution in the linear hull $\mathrm{span}(\delta)$

is an operator in the linear hull span(A), hence a linear operator. Since every distribution a in the space $\mathcal{S}'_k$, is the weak* limit (and then a strong limit too) of a sequence d in span(δ), we have that, for any u in $\mathcal{S}'_n$, we have

$$\begin{aligned} {}^{(\beta(\mathcal{S}'_n),\beta(\mathcal{S}'_m))}\lim_{k\to\infty}(\int_{\mathbb{R}^k} d_n A)(u) &= {}^{(\beta(\mathcal{S}'_n),\beta(\mathcal{S}'_m))}\lim_{k\to\infty}\int_{\mathbb{R}^k} d_n A(u) = \\ &= \int_{\mathbb{R}^k} {}^{\beta(\mathcal{S}'_k)}\lim_{k\to\infty} d_n\ A(u) = \\ &= \int_{\mathbb{R}^k} aA(u), \end{aligned}$$

so that the linear operator $\int_{\mathbb{R}^k} aA$ is a pointwise limit (with respect to the pair of strong topologies $(\beta(\mathcal{S}'_n), \beta(\mathcal{S}'_m))$) of a sequence of continuous linear operators, applying the Banach-Steinhaus theorem (we can since it works when the first space of the operators is barreled, taking into account that strong duals of Montel spaces are Montel spaces, that Montel spaces are barreled and that the Schwartz space $\mathcal{S}_n$ is a Montel space) we conclude that this operator must be continuous. ■

13.5 $^{\mathcal{S}}$Linear operators and $^{\mathcal{S}}$bases

Proposition. *Let $A : \mathcal{S}'_n \to \mathcal{S}'_m$ be an $^{\mathcal{S}}$linear operator. Then A sends $^{\mathcal{S}}$linearly dependent families into $^{\mathcal{S}}$linearly dependent families. Consequently, if the image of an $^{\mathcal{S}}$family is $^{\mathcal{S}}$linearly independent the $^{\mathcal{S}}$family must be $^{\mathcal{S}}$linearly independent too.*

Proof. Let v be an $^{\mathcal{S}}$linearly dependent family indexed by $\mathbb{R}^k$, then there exists a nonzero distribution a such that

$$\int_{\mathbb{R}^k} av = 0_{\mathcal{S}'_n},$$

applying the linear operator A we have

$$\begin{aligned} 0_{\mathcal{S}'_m} &= A(0_{\mathcal{S}'_n}) = \\ &= A(\int_{\mathbb{R}^k} av) = \\ &= \int_{\mathbb{R}^k} aA(v), \end{aligned}$$

so the family $A(v)$ is $^{\mathcal{S}}$linearly dependent too. ■

Theorem (basic properties of $^{\mathcal{S}}$linear operators). *We have the following results:*

1) a surjective $^{\mathcal{S}}$*linear operator transforms* $^{\mathcal{S}}$*generating systems of its domain space into* $^{\mathcal{S}}$*generating systems of its codomain space;*

2) an injective $^{\mathcal{S}}$*linear operator sends* $^{\mathcal{S}}$*linearly independent family into* $^{\mathcal{S}}$*linearly independent family;*

3) a bijective $^{\mathcal{S}}$*linear operator transforms* $^{\mathcal{S}}$*bases into* $^{\mathcal{S}}$*bases;*

4) if two $^{\mathcal{S}}$*linear operators defined among the same pair of spaces coincides on an* $^{\mathcal{S}}$ *basis, of their common domain, then they coincide on the entire domain space;*

5) if v *is an* $^{\mathcal{S}}$*basis of a space* $\mathcal{S}'_n$ *and* w *is an* $^{\mathcal{S}}$*family of a space* $\mathcal{S}'_m$ *indexed on the same index-set of the family* v*, then there exists and it is unique an* $^{\mathcal{S}}$*linear operator sending* v *into* w*.*

13.6 Invertibility of $^{\mathcal{S}}$linear operators

Theorem (invertibility of $^{\mathcal{S}}$**linear operators).** *We have the following results:*

1) the inverse of a bijective $^{\mathcal{S}}$ *linear operator is an* $^{\mathcal{S}}$*linear operator too;*

2) an $^{\mathcal{S}}$*linear operator* A *has an* $^{\mathcal{S}}$*linear left inverse if and only if* A *is injective, its image* $A(E)$ *is weakly* closed and* $A(E)$ *has a topological supplement;*

3) an $^{\mathcal{S}}$*linear operator* L *has an* $^{\mathcal{S}}$*linear right inverse if and only if it is surjective and its kernel* $\ker(L)$ *has a topological supplement. Namely, if there exists a continuous right inverse* R *of the surjection* L*, a topological supplement of the kernel of* L *is the image of* R*.*

Proof. (1) Let us prove this one. If $A : \mathcal{S}'_n \to \mathcal{S}'_m$ is such bijective linear continuous operator then its transpose ${}^tA$ is a bijective continuous operator from the Fréchet space $(\mathcal{S}_m)$ onto the Fréchet space $(\mathcal{S}_n)$, by the Banach homomorphism theorem for Fréchet spaces this transpose is a topological isomorphism and then also A will be a topological homomorphism, so the inverse of A is continuous and then an $^{\mathcal{S}}$linear operator. ■

13.7 $^{\mathcal{S}}$Linear operators on subspaces

We recall that an $^{\mathcal{S}}$operator is a not necessarily linear nor continuous operator sending any $^{\mathcal{S}}$family in its domain into another $^{\mathcal{S}}$family (of its image obviously).

Definition (of $^{\mathcal{S}}$linear operator on subspaces). *Let V be an $^{\mathcal{S}}$closed subspace of the space $\mathcal{S}'_n$ and let $L : V \to \mathcal{S}'_m$ be an $^{\mathcal{S}}$operator. We say that the operator L is an $^{\mathcal{S}}$**linear operator** if, for each positive integer k, for any tempered distribution $a \in \mathcal{S}'_k$ and for each $^{\mathcal{S}}$family $v \in \mathcal{S}(\mathbb{R}^k, \mathcal{S}'_n)$ in V, i.e. whose image* im$v$ *is contained in the subspace V, the equality*

$$L\left(\int_{\mathbb{R}^k} av\right) = \int_{\mathbb{R}^k} aL(v),$$

holds.

As in the case of $^{\mathcal{S}}$linear operators defined on the entire space of tempered distribution, the $^{\mathcal{S}}$linear operators defined on subspaces are linear, but the proof is of another nature and requires the $^{\mathcal{S}}$connectedness of the subspaces of the space of tempered distributions.

Theorem (linearity of $^{\mathcal{S}}$linear operators on subspaces). *An $^{\mathcal{S}}$linear operator defined on a subspace V of the space $\mathcal{S}'_n$ is linear, as in the case of $^{\mathcal{S}}$linear operators defined on the entire space $\mathcal{S}'_n$.*

Proof. Indeed, note that a subspace of $\mathcal{S}'_n$ is star-shaped at the origin and so it is $^{\mathcal{S}}$connected. Then, let x, y be any two points of the subspace V and a, b two scalars. Let v be an $^{\mathcal{S}}$family indexed by the real line such that $v_0 = x$ and $v_1 = y$ (it exists by $^{\mathcal{S}}$connectedness), we have

$$\begin{aligned} L(ax+by) &= L\left(a\int_{\mathbb{R}} \delta_0 v + b\int_{\mathbb{R}} \delta_1 v\right) = \\ &= L\left(\int_{\mathbb{R}} (a\delta_0 + b\delta_1)v\right) = \\ &= \int_{\mathbb{R}} (a\delta_0 + b\delta_1)L(v) = \\ &= aL(v_0) + bL(v_1), \end{aligned}$$

so L is linear, as we desired. ■

13.8 Compositions of $^{\mathcal{S}}$linear operators

The composition of two $^{\mathcal{S}}$linear operators (when it exists is $^{\mathcal{S}}$linear).

Theorem. *Consider two $^{\mathcal{S}}$closed subspace V and W of $\mathcal{S}'_n$ and $\mathcal{S}'_m$ respectively and two $^{\mathcal{S}}$linear operators $A : V \to \mathcal{S}'_m$ and $B : W \to \mathcal{S}'_k$ such that $A(V)$ is contained in W. Then the composition BA is an $^{\mathcal{S}}$linear operator.*

Proof. Indeed, if v is an $^{\mathcal{S}}$family in V, since A is an $^{\mathcal{S}}$operator, then the image family $A(v)$ is an $^{\mathcal{S}}$family in W; since B is an $^{\mathcal{S}}$operator the image $B(A(v))$ is an $^{\mathcal{S}}$family and consequently the composition operator BA is an $^{\mathcal{S}}$operator. Now we have

$$\begin{aligned} BA\left(\int_{\mathbb{R}^h} av\right) &= B\left(A\left(\int_{\mathbb{R}^h} av\right)\right) = \\ &= B\left(\int_{\mathbb{R}^h} aA(v)\right) = \\ &= \int_{\mathbb{R}^h} aB(A(v)) = \\ &= \int_{\mathbb{R}^h} aBA(v), \end{aligned}$$

so that the composition BA is also an $^{\mathcal{S}}$linear operator, as we claimed. ■

Chapter 14

$^{\mathcal{S}}$Homomorphisms

In this chapter we study a condition sufficient to guarantee that the $^{\mathcal{S}}$linear hull of an $^{\mathcal{S}}$family is $^{\mathcal{S}}$ closed, and consequently that this $^{\mathcal{S}}$linear hull is the smallest $^{\mathcal{S}}$closed subspace containing the family, i.e., the $^{\mathcal{S}}$closed linear hull of the family. However, the concept of $^{\mathcal{S}}$*homomorphism* is interesting *per se* and it is necessary to study the $^{\mathcal{S}}$linearity of the coordinate operator and for the existence of Green's families of an operator.

14.1 ${}^{\mathcal{S}}$Homomorphisms

We recall that:

- if V and W are two subspaces of the two spaces $\mathcal{S}'_n$ and $\mathcal{S}'_m$, respectively, for every positive integer k and for every family $v = (v_i)_{i\in\mathbb{R}^k}$ in the subspace V, the image of the family v under an operator $A : V \to W$ is the family (in the subspace W) defined by

$$A(v) := (A(v_i))_{i\in\mathbb{R}^k}.$$

- an ${}^{\mathcal{S}}$operator is an operator (not necessarily linear nor continuous), defined among two subspaces of spaces of tempered distributions, which sends ${}^{\mathcal{S}}$families into ${}^{\mathcal{S}}$ families;
- we proved that a linear ${}^{\mathcal{S}}$operator defined on an entire space of tempered distributions is a weakly* continuous operator and (equivalently) and ${}^{\mathcal{S}}$linear operator.

Consider an ${}^{\mathcal{S}}$operator A. If v is an ${}^{\mathcal{S}}$ family in the domain of A, then its image $A(v)$ is an ${}^{\mathcal{S}}$ family in the image of A; but, if we consider an ${}^{\mathcal{S}}$family w in the image of the operator A, the question is:

- *is it possible to find an ${}^{\mathcal{S}}$family v in the domain of A whose image is the family w?*

This problem is a problem of *smooth choice* (in the sense of the *Choice Axiom*); let us explain. If w is a family in the image of A , indexed by some Euclidean space I, then, for every index p in I, there is a maximal (with respect to the inclusion) subset V_p of the domain of A whose image is the singleton $\{w_p\}$, namely the anti-image $A^-(w_p)$ of the element w_p. We so find an ordered family of subsets of the domain of the operator A indexed by the set I, precisely the family $V = (V_p)_{p\in I}$. We know that, by the axiom of choice, there exists a family v indexed by I such that v_p belongs to V_p, for every index p in I - sometimes such families are called choice families of the family V, or simple choice of the family V - the question is:

- *is it possible to find an ${}^{\mathcal{S}}$choice of the family V?*

As we shall see later, the answer in general is negative, therefore we need the following definition.

Definition (of ${}^{\mathcal{S}}$homomorphism). *Let $A : \mathcal{S}'_m \to \mathcal{S}'_n$ be an ${}^{\mathcal{S}}$operator. We say the operator A an ${}^{\mathcal{S}}$**homomorphism** if, for every family u, indexed by some Euclidean space, in the image $A(\mathcal{S}'_m)$ there exists an ${}^{\mathcal{S}}$family a in the space $\mathcal{S}'_m$ such that its image under the operator A is the family u. Analogous* *definition we give for an operator defined among two subspaces of spaces of tempered distributions.*

Motivation. A first motivation for introducing this concept is that, as we shall see, any $^{\mathcal{S}}$homomorphism is $^{\mathcal{S}}$ closed (i.e., the operator A sends $^{\mathcal{S}}$closed subsets into $^{\mathcal{S}}$closed subsets).

Remark. Every $^{\mathcal{S}}$homomorphism is an $^{\mathcal{S}}$ operator and then, if it is linear and defined on the whole of a space of tempered distributions, it is a weakly* continuous operator.

14.2 Injective linear $^{\mathcal{S}}$homomorphisms

Let us see a characterization of injective linear $^{\mathcal{S}}$ homomorphism.

Theorem. *Let $A : \mathcal{S}'_n \to \mathcal{S}'_m$ be a linear operator. Then, the following two conditions are equivalent:*

- *the operator A is an injective linear $^{\mathcal{S}}$homomorphism;*
- *for every family v in the space $\mathcal{S}'_n$, indexed by some Euclidean space, the image of the family v under the operator A is an $^{\mathcal{S}}$family if and only if v is an $^{\mathcal{S}}$family.*

Proof. ($\Rightarrow$) Since the operator A is injective, then for every family w in the image of the operator, indexed by an Euclidean space I, there is only one family v in the domain of the operator such that $A(v) = w$. By definition of $^{\mathcal{S}}$homomorphism, this family v must be an $^{\mathcal{S}}$family, since there is at least one $^{\mathcal{S}}$ choice for the family of anti-image $(A^{-}(w_p))_{p\in I}$. ($\Leftarrow$) Assume that the linear operator A is not injective. It follows that there is a nonzero distribution u belonging to the kernel of A. Let now v the constant family with unique element u, the family v is not an $^{\mathcal{S}}$family but its image is the zero family, which is an $^{\mathcal{S}}$family, and this goes against our assumption that every family having an $^{\mathcal{S}}$image must be an $^{\mathcal{S}}$family. ■

Theorem. *Let $A : \mathcal{S}'_n \to \mathcal{S}'_m$ be an $^{\mathcal{S}}$linear operator. Assume that the operator A has a (surjective) linear continuous left inverse, that is assume that there exists a continuous linear operator $L : \mathcal{S}'_m \to \mathcal{S}'_n$ such that*

$$L \circ A = (\cdot)_{\mathcal{S}'_n}.$$

Then, the operator A is an injective linear $^{\mathcal{S}}$homomorphism.

Proof. Let L be a continuous left inverse of the ${}^{\mathcal{S}}$ linear operator A. The operator A is injective since it has a right inverse. Using the above characterization, we have only to prove that, if v is any family in the domain of A, and if its image $A(v)$ is of class $\mathcal{S}$, then v is of class $\mathcal{S}$ too. Indeed, let v be such a family with image $A(v)$ of class $\mathcal{S}$. Since L is a continuous operator defined on the whole of the space $\mathcal{S}'_m$, it is an ${}^{\mathcal{S}}$operator and hence the image $L(A(v))$ is a family of class $\mathcal{S}$; but the image $L(A(v))$ is exactly the family v and so the condition of ${}^{\mathcal{S}}$homomorphism is verified. ■

14.3 Surjective linear ${}^{\mathcal{S}}$homomorphisms

Let us see a characterization of surjective linear ${}^{\mathcal{S}}$ homomorphism.

Theorem. *Let $A : \mathcal{S}'_n \to \mathcal{S}'_m$ be an ${}^{\mathcal{S}}$linear operator. Then, the following two conditions are equivalent:*

- *the operator A is a surjective linear ${}^{\mathcal{S}}$homomorphism;*
- *the operator A has a (injective) linear continuous right inverse, that is there is a continuous linear operator $R : \mathcal{S}'_m \to \mathcal{S}'_n$ such that*

$$A \circ R = (.)_{\mathcal{S}'_m}.$$

Proof. ($\Rightarrow$) Since the operator A is surjective, the Dirac family μ of $\mathcal{S}'_m$ is in the image of A. Since the operator A is an ${}^{\mathcal{S}}$homomorphism there is an ${}^{\mathcal{S}}$family G in $\mathcal{S}'_n$ (indexed by $\mathbb{R}^m$ since μ is indexed by $\mathbb{R}^m$) such that $A(G) = \mu$. Consider the superposition operator R of the family G, for every tempered distribution a in $\mathcal{S}'_m$, we have

$$\begin{aligned} A \circ \int_{\mathbb{R}^k} (., G)(a) &= A\left(\int_{\mathbb{R}^k} aG\right) = \\ &= \int_{\mathbb{R}^k} aA(G) = \\ &= \int_{\mathbb{R}^k} a\mu = \\ &= a, \end{aligned}$$

and this means exactly that

$$AR = (.)_{\mathcal{S}'_m}.$$

($\Leftarrow$) Vice versa. Let R be a continuous right inverse of the operator A. The operator A is surjective since it has a right inverse. We have only to prove that, if w is any ${}^{\mathcal{S}}$family in the image of A, there exists a reciprocal image of w

which is of class $\mathcal{S}$. Indeed, let w be such a family and let $R(w)$ be its image by the inverse R. Since R is a continuous operator defined on the whole of the space $\mathcal{S}'_m$, it is an ${}^{\mathcal{S}}$operator and hence the image $R(w)$ in a family of class $\mathcal{S}$; moreover, the image $A(R(w))$ is w and so the condition of ${}^{\mathcal{S}}$homomorphism is verified. ■

14.4 ${}^{\mathcal{S}}$Stable families

Definition (of ${}^{\mathcal{S}}$stable family). *We say that an ${}^{\mathcal{S}}$family v, indexed by some $\mathbb{R}^m$ and in the space $\mathcal{S}'_n$, is ${}^{\mathcal{S}}$**stable** if, for every family w, indexed by some Euclidean space $\mathbb{R}^k$ and in the ${}^{\mathcal{S}}$linear hull of v, there is an ${}^{\mathcal{S}}$family $a = (a_i)_{i\in\mathbb{R}^k}$ in $\mathcal{S}'_m$ such that*

$$w = \int_{\mathbb{R}^m} av.$$

In other terms, the family v is said ${}^{\mathcal{S}}$ stable if and only if its superposition operator $\int_{\mathbb{R}^m}(\cdot, v)$ is an ${}^{\mathcal{S}}$homomorphism.

Remark (on the ${}^{\mathcal{S}}$linearly independent ${}^{\mathcal{S}}$stable families). An ${}^{\mathcal{S}}$family v in the space $\mathcal{S}'_n$ and indexed by $\mathbb{R}^m$ is an ${}^{\mathcal{S}}$linearly independent ${}^{\mathcal{S}}$stable family if and only if, for every family $a = (a_i)_{i\in\mathbb{R}^k}$ in the space $\mathcal{S}'_m$, the superposition

$$\int_{\mathbb{R}^m} av$$

(which is necessarily a family indexed by $\mathbb{R}^k$ and in the ${}^{\mathcal{S}}$linear hull of v) is an ${}^{\mathcal{S}}$family if and only if the family a is an ${}^{\mathcal{S}}$family. In other terms, the family v is an ${}^{\mathcal{S}}$linearly independent ${}^{\mathcal{S}}$stable family if and only if the corresponding superposition operator

$$\int_{\mathbb{R}^m} (\cdot, v)$$

is an injective ${}^{\mathcal{S}}$homomorphism.

Theorem. *Let v be an ${}^{\mathcal{S}}$stable family in the space $\mathcal{S}'_n$. Then, the ${}^{\mathcal{S}}$linear hull ${}^{\mathcal{S}}\mathrm{span}(v)$ is ${}^{\mathcal{S}}$closed and hence it coincides with the ${}^{\mathcal{S}}$linear closed hull of the family v and with the ${}^{\mathcal{S}}$closed hull of the family, i.e.,*

$${}^{\mathcal{S}}\mathrm{span}(v) = {}^{\mathcal{S}}\overline{\mathrm{span}}(v) = {}^{\mathcal{S}}\mathrm{cl}(v).$$

Proof. Let the family v be indexed by $\mathbb{R}^m$ and let w be an ${}^{\mathcal{S}}$family in the ${}^{\mathcal{S}}$linear hull of v, indexed by $\mathbb{R}^k$. Then, by definition of ${}^{\mathcal{S}}$stable family, there is an ${}^{\mathcal{S}}$family a such that

$$\int_{\mathbb{R}^m} av = w.$$

Applying the $^{\mathcal{S}}$linearity of the superposition operator of the family v, for every coefficient distribution b in $\mathcal{S}'_k$, we have

$$\begin{aligned} \int_{\mathbb{R}^k} bw &= \int_{\mathbb{R}^k} b\left(\int_{\mathbb{R}^m} av\right) = \\ &= \int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^k} ba\right) v, \end{aligned}$$

and hence any superposition of the family w must belong to the $^{\mathcal{S}}$linear hull of v. ■

Corollary. *Let v be an $^{\mathcal{S}}$ stable family in the space $\mathcal{S}'_n$. Then the $^{\mathcal{S}}$linear hull $^{\mathcal{S}}\mathrm{span}(v)$ is the intersection of all the $^{\mathcal{S}}$closed subset of $\mathcal{S}'_n$ containing the family v, in particular it is $^{\mathcal{S}}$closed.*

Question. *If the $^{\mathcal{S}}$linear hull of an $^{\mathcal{S}}$family is $^{\mathcal{S}}$closed, then is the $^{\mathcal{S}}$family an $^{\mathcal{S}}$linearly independent $^{\mathcal{S}}$stable family?*

Answer. The Answer is in general negative. Indeed, consider a family v such that its superposition operator is surjective but not injective, for instance the derivative of the Dirac family in $\mathcal{S}'_1$. Then the $^{\mathcal{S}}$linear hull of the family v is $^{\mathcal{S}}$ closed (it is the entire space $\mathcal{S}'_1$) but the superposition operator of v is not injective and consequently it cannot be an injective $^{\mathcal{S}}$homomorphism. ■

Question. *If the $^{\mathcal{S}}$linear hull of a family is $^{\mathcal{S}}$closed, then is the family $^{\mathcal{S}}$stable?*

Answer. We will see that if A is a surjective linear $^{\mathcal{S}}$homomorphism, then it is necessarily a surjective topological homomorphism with the kernel admitting a topological supplement. Consider a surjective topological homomorphism A (or equivalently, since its image is topologically closed, a surjective linear continuous operator) which doesn't have a kernel with a topological supplement. Then, the associated family $v = A(\delta)$ is an family whose $^{\mathcal{S}}$linear hull is $^{\mathcal{S}}$closed (the entire space) but that is not $^{\mathcal{S}}$stable (since its superposition operator A is not an $^{\mathcal{S}}$homomorphism). ■

14.5 $^{\mathcal{S}}$Linear operators and $^{\mathcal{S}}$closedness

Proposition. *Let $A : \mathcal{S}'_n \to \mathcal{S}'_m$ be an $^{\mathcal{S}}$linear operator. Then the kernel of the operator A is $^{\mathcal{S}}$closed.*

Proof. Let v be an ${}^{\mathcal{S}}$family in the kernel $\ker A$ of the operator A, indexed by $\mathbb{R}^k$. The image $A(v)$ of the family v is the 0-family in $\mathcal{S}'_m$, so, for any ${}^{\mathcal{S}}$distribution of coefficients a feasible for the family v, we have

$$\begin{aligned} A\left(\int_{\mathbb{R}^k} av\right) &= \int_{\mathbb{R}^k} aA(v) = \\ &= \int_{\mathbb{R}^k} a0 = \\ &= 0_{\mathcal{S}'_m}, \end{aligned}$$

consequently the ${}^{\mathcal{S}}$linear combination

$$\int_{\mathbb{R}^k} av$$

lives in the kernel $\ker A$ of the operator A. ■

More generally, is the preimage of an ${}^{\mathcal{S}}$*closed subspace an* ${}^{\mathcal{S}}$*closed subspace?*

Proposition. *Let $A : \mathcal{S}'_n \to \mathcal{S}'_m$ be an ${}^{\mathcal{S}}$linear operator. Then the preimage of an ${}^{\mathcal{S}}$closed set under the operator A is ${}^{\mathcal{S}}$closed.*

Proof. Let C be an ${}^{\mathcal{S}}$closed subset of the space $\mathcal{S}'_m$. Let v be an ${}^{\mathcal{S}}$family indexed by $\mathbb{R}^k$ in the preimage $A^-(C)$ of the ${}^{\mathcal{S}}$closed C and let a be any coefficient distribution for the family v. We must prove that the superposition

$$\int_{\mathbb{R}^k} av$$

is in the preimage $A^-(C)$. We have that the image $A(v)$ is an ${}^{\mathcal{S}}$family in the subset C, so the superposition

$$\int_{\mathbb{R}^k} aA(v)$$

lives in the subset C too (since C is ${}^{\mathcal{S}}$closed) but the above superposition $a.A(v)$ is equal to the image $A(a.v)$ (by ${}^{\mathcal{S}}$linearity) so that the image $A(a.v)$ belongs to C and this means that the superposition $a.v$ is in the preimage of C by the operator A. ■

Open problem. *Is the image of an ${}^{\mathcal{S}}$ closed subspace, by an ${}^{\mathcal{S}}$linear operator, an ${}^{\mathcal{S}}$closed subspace too?*

Note. Let A be an $^{\mathcal{S}}$linear operator from $\mathcal{S}'_n$ into $\mathcal{S}'_m$. If C is an $^{\mathcal{S}}$closed subspace of the domain of A, consider an $^{\mathcal{S}}$family w in the image $A(C)$. By definition of image of a subset, there exists a family v in C such that $A(v) = w$. But, can we affirm that this family v is an $^{\mathcal{S}}$family? In general an $^{\mathcal{S}}$linear operator transforms $^{\mathcal{S}}$families into $^{\mathcal{S}}$families; but, if the image w of a family v is an $^{\mathcal{S}}$family, we cannot conclude nothing about the family v. For this reason, we shall assume the operator A an $^{\mathcal{S}}$homomorphism, which is such that any $^{\mathcal{S}}$family in its image is the image of at least an $^{\mathcal{S}}$ family. So the answer to above question seems to be negative, *but we need a counterexample.*

14.6 $^{\mathcal{S}}$Homomorphism and $^{\mathcal{S}}$ closedness

We conclude the following result.

Proposition. *Let $A : \mathcal{S}'_n \to \mathcal{S}'_m$ be an $^{\mathcal{S}}$linear operator which is also an $^{\mathcal{S}}$homomorphism. Then the image of an $^{\mathcal{S}}$closed subset of $\mathcal{S}'_n$ is an $^{\mathcal{S}}$closed subset of $\mathcal{S}'_m$. In other terms, we can say that any linear $^{\mathcal{S}}$homomorphism is an $^{\mathcal{S}}$closed operator.*

Proof. If C is an $^{\mathcal{S}}$closed subset of the domain of the operator A, consider an $^{\mathcal{S}}$family w indexed by $\mathbb{R}^k$ in the image $A(C)$. By definition of $^{\mathcal{S}}$homomorphism, there exists an $^{\mathcal{S}}$family v in C such that $A(v) = w$. For any distribution a in $\mathcal{S}'_k$, we have

$$\begin{aligned}\int_{\mathbb{R}^k} aw &= \int_{\mathbb{R}^k} aA(v) = \\ &= A\left(\int_{\mathbb{R}^k} av\right).\end{aligned}$$

The above superposition $a.v$ lies in C (since the subset C is $^{\mathcal{S}}$closed), thus the superposition $a.w$ lies in $A(C)$ and we can conclude that $A(C)$ is $^{\mathcal{S}}$closed. ■

14.7 Invertibility of linear $^{\mathcal{S}}$ homomorphism

We we consider an $^{\mathcal{S}}$homomorphism among $^{\mathcal{S}}$closed subspaces, the linearity of the operator (in general) does not imply the $^{\mathcal{S}}$linearity. We shall call the $^{\mathcal{S}}$linear $^{\mathcal{S}}$homomorphism, simply, $^{\mathcal{S}}$linear homomorphism.

Theorem. *Let V and W be two $^{\mathcal{S}}$closed subspaces of the spaces $\mathcal{S}'_n$ and $\mathcal{S}'_m$ respectively and let $A : V \to W$ be a bijective $^{\mathcal{S}}$linear homomorphism. Then the inverse A^- is a bijective $^{\mathcal{S}}$linear homomorphism.*

Proof. The operator A^- is obviously a linear $^{\mathcal{S}}$ homomorphism. We have only to prove that it is an $^{\mathcal{S}}$linear operator. Let w be an $^{\mathcal{S}}$family in the subspace W, indexed by some k-dimensional Euclidean space. Since A is a bijective $^{\mathcal{S}}$homomorphism, there is an $^{\mathcal{S}}$family v in V such that $A(v) = w$, or equivalently such that $v = A^-(w)$. We have, for every a in $\mathcal{S}'_k$,

$$\begin{aligned} A^-\left(\int_{\mathbb{R}^k} aw\right) &= A^-\left(\int_{\mathbb{R}^k} aA(v)\right) = \\ &= A^-\left(A\left(\int_{\mathbb{R}^k} av\right)\right) = \\ &= \int_{\mathbb{R}^k} av = \\ &= \int_{\mathbb{R}^k} aA^-(w), \end{aligned}$$

as we claimed (note that all the above superpositions live in the domains of the operators by $^{\mathcal{S}}$closedness of the domains themselves). ∎

14.8 Left inverse of linear $^{\mathcal{S}}$ homomorphisms

Let us see a characterization of injective $^{\mathcal{S}}$linear operator with a continuous left inverse. We recall that a projector of a set onto one of its subsets is a surjective and idempotent mapping.

Theorem. *Let $A : \mathcal{S}'_n \to \mathcal{S}'_m$ be an $^{\mathcal{S}}$linear operator. Then, the following two conditions are equivalent:*

1) *the operator A is an injective linear $^{\mathcal{S}}$homomorphism and there is an $^{\mathcal{S}}$ linear projector*
$$p : \mathcal{S}'_m \to A(\mathcal{S}'_n)$$
of the space $\mathcal{S}'_m$ onto the image $A(\mathcal{S}'_m)$;

2) *the operator A has a (surjective) linear continuous left inverse L, that is there is a continuous linear operator $L : \mathcal{S}'_m \to \mathcal{S}'_n$ such that*
$$L \circ A = (.)_{\mathcal{S}'_m}.$$

Proof. Let the assumption *1)* be satisfied. Since A is an $^{\mathcal{S}}$homomorphism, its image W is $^{\mathcal{S}}$closed. Consider the codomain restriction R of the operator A to the image W, i.e. the bijective operator $R : \mathcal{S}'_n \to W$, defined by $R(u) = A(u)$, for every distribution u in $\mathcal{S}'_n$. The inverse operator $R^- : W \to \mathcal{S}'_n$ is defined on an $^{\mathcal{S}}$closed subset of $\mathcal{S}'_m$, is an $^{\mathcal{S}}$operator since A is an $^{\mathcal{S}}$homomorphism and is $^{\mathcal{S}}$linear since it is the inverse of an $^{\mathcal{S}}$linear operator. Consider the composition

$$L = R^- \circ p,$$

this is a mapping of $\mathcal{S}'_m$ into $\mathcal{S}'_n$, composition of two surjective $^{\mathcal{S}}$linear operator, consequently it is a surjective $^{\mathcal{S}}$linear operator from $\mathcal{S}'_m$ onto $\mathcal{S}'_n$ (and hence also continuous). We have only to prove that the composition $L \circ A$ is the identity operator of $\mathcal{S}'_n$. For, let u in $\mathcal{S}'_n$, we have

$$\begin{aligned} L(A(u)) &= R^-(p(A(u))) = \\ &= R^-(A(u)) = \\ &= R^-(R(u)) = \\ &= u, \end{aligned}$$

as we claimed. ■

Corollary (characterization of bijective $^{\mathcal{S}}$ linear homomorphism). *An $^{\mathcal{S}}$linear operator $A : \mathcal{S}'_n \to \mathcal{S}'_m$ is a bijective linear $^{\mathcal{S}}$homomorphism if and only if it is a topological isomorphism.*

Chapter 15

$^{\mathcal{S}}$Green's families

15.1 Introduction

This Chapter is devoted to the concept of Green's family of a linear continuous endomorphism on the space of tempered distributions $\mathcal{S}'_n$. The concept of Green's family is an operative and rigorous version of the formal Green's function used in physics and engineering. The relationship with the modern approach by fundamental solutions is showed, to solve inhomogeneous linear equations.

15.1.1 Green's functions

In this section we give a definition of a Green's function of a linear continuous operator that stays between the classic one (definition which very often in the applications is far from being a rigorous one) and our new definition of $^{\mathcal{S}}$Green's family of a linear endomorphism on the space $\mathcal{S}'_n$.

Definition (semi-classic definition of Green's function). *Let O be an open subset of the Euclidean space $\mathbb{R}^n$ and let $L : \mathcal{D}'(O) \to \mathcal{D}'(O)$ be a* *linear operator, acting on the space of distributions over the open subset O. Any function $G : O^2 \to \mathbb{K}$, such that the section $G(., s)$ is Lebesgue measurable and locally summable, for every point s of the open subset O, is said* ***a Green's function of the linear operator*** *L if it satisfies the following equality*

$$L([G(., s)]) = \delta_s,$$

for any point s of the open subset O, where δ_s is the Dirac delta distribution of $\mathcal{D}'(O)$ centered at the point s and the distribution $[G(., s)]$ is the regular distribution canonically associated with the locally summable function $G(., s)$.

Remark (fundamental solutions at a point). Let O be an open subset of the n-dimensional Euclidean space $\mathbb{R}^n$ and let $L : \mathcal{D}'(O) \to \mathcal{D}'(O)$ be a linear operator. When the equality $L(u) = \delta_s$ is satisfied for some point s of O and some distribution u in $\mathcal{D}'(O)$, we say that the distribution u is a fundamental solution of the operator L at the point s. So, in the conditions of the above definition, we can say that the function G is a Green's function for L if the (distribution associated with the) section $G(., s)$ is a fundamental solution of L at the point s, for every point s of the open subset O.

15.1.2 Motivations for Green's families

We are going to introduce the Green's families because the requirement that the "object" G, such that $L(G_s) = \delta_s$, must be a function is too strict in order to give to the concept itself reasonable manageability and effectiveness in the applications. A way to extend the classic definition can be to consider the Green's function G as a distribution on the product $O \times O$, but this choice gives some problem to the definition itself, since we could no longer consider the parameter s as a "true" parameter.

Classically, the definition property of a Green's function can be exploited to solve differential equations of the form

$$L(u) = a,$$

where a is a given regular distribution on the open set O. To solve such equations we need to find a Green's family and in general, an operator L does not need to have a Green's function and if L has a Green's function it can have many Green's functions.

In this chapter:

- we shall define the concepts of ^{S}Green's family and ^{E}Green's family;
- we show how to solve a linear equation (not necessarily a linear differential one) via superpositions, precisely a linear equation $L(.) = a$, when L has an ^{S}Green's family when L is an endomorphism on $\mathcal{S}'_n$;

- we study the existence of $^{\mathcal{S}}$Green's families;
- we shall study the special case of linear operators commuting with the translations.

15.2 Green's families in $\mathcal{S}'_n$

In this section we introduce the main concept of the chapter. Let us start with the classic definition of fundamental solution in our context.

Definition (of fundamental solution for a linear operator). *Let L be a linear endomorphism on the space of tempered distributions $\mathcal{S}'_n$. When the equality $L(u) = \delta_s$ is verified for some point s and some distribution u, we say that the tempered distribution u is a **fundamental solution of the operator** L **at the point** s.*

Now we can define Green's families for linear operators.

Definition (Green's families of tempered distributions). *Let L be a linear endomorphism on the space of tempered distributions $\mathcal{S}'_n$. Any family $G = (G_s)_{s\in\mathbb{R}^n}$ of tempered distributions is said **a Green's family of the linear operator** L if it satisfies the equality*

$$L(G_s) = \delta_s,$$

for any point s of the space $\mathbb{R}^n$, where δ_s is the Dirac distribution centered at the point s. In other terms, a family $G = (G_s)_{s\in\mathbb{R}^n}$ of tempered distributions is said a Green's family of the linear operator L if the distribution G_s is a fundamental solution for the linear operator L at the point s, for every index p of the family.

Note that our definition is not bounded to linear differential operators.

The Green's family of an endomorphism L of $\mathcal{S}'_n$ is the (ordered) family of fundamental solutions, at each point p in $\mathbb{R}^n$, of the operator L.

Theorem (on the existence of Green's families). *Let L be a linear endomorphism on the space of tempered distributions $\mathcal{S}'_n$. We have*

- *if L is a surjective endomorphism on the space $\mathcal{S}'_n$, then L admits a Green's family;*
- *if L is a strict (topological) endomorphism on the space $\mathcal{S}'_n$, then L admits an Green family if and only if L is surjective.*

Proof. The first assertion is evident, since L is surjective, every element of the Dirac basis must have at least an anti-image. For the second one it is enough to prove that the existence of a Green family implies the surjectivity. Indeed, by the Dieudonné-Schwartz theorem, the image of L must be closed and, by definition of Green family, it must contain the Dirac basis; consequently (since the Dirac basis is dense in $\mathcal{S}'_n$) the image must coincide with the entire $\mathcal{S}'_n$. ■

15.3 $^{\mathcal{S}}$Green's families

The $^{\mathcal{S}}$Linear Algebra is necessary for the next step development.

Definition ($^{\mathcal{S}}$Green's families). *Let $L : \mathcal{S}'_n \to \mathcal{S}'_n$ be a linear operator, acting on the space of tempered distributions. An $^{\mathcal{S}}$**Green's family (respectively, a Green's family of class E) of the linear operator** L is a Green's family of the operator L which is of class $\mathcal{S}$ (respectively of class E).*

Sometimes we will call $^{\mathcal{S}}$Green operators the $^{\mathcal{S}}$ linear operators with an $^{\mathcal{S}}$Green family.

Theorem (surjectivity of the $^{\mathcal{S}}$Green operators). *Let L be a continuous endomorphism on the space $\mathcal{S}'_n$ admitting an $^{\mathcal{S}}$Green family, that is an $^{\mathcal{S}}$Green operator. Then*

1. *the operator L is surjective;*
2. *any $^{\mathcal{S}}$Green family of the operator L is $^{\mathcal{S}}$linearly independent.*

Proof. **Surjectivity.** Indeed, if L admits a Green family G then $L(G) = \delta$. Now, for every u in the space $\mathcal{S}'_n$, we have

$$\begin{aligned} L(\int_{\mathbb{R}^n} uG) &= \int_{\mathbb{R}^n} uL(G) = \\ &= \int_{\mathbb{R}^n} u\delta = \\ &= u, \end{aligned}$$

so that any u is the image of some distribution (namely the superposition $u.G$). $^{\mathcal{S}}$**Linear independence.** Let G be any $^{\mathcal{S}}$Green family of the operator L. The image of the family G is an $^{\mathcal{S}}$basis, which is necessarily $^{\mathcal{S}}$linearly independent, consequently the family G is $^{\mathcal{S}}$linearly independent too, because the image of an $^{\mathcal{S}}$linearly dependent family by an $^{\mathcal{S}}$linear operator is $^{\mathcal{S}}$linearly dependent too. ■

15.4 Application to linear equations

In this section we study the linear equation

$$E : L(.) = a$$

associated with a linear continuous endomorphism L, on the topological vector space $(\mathcal{S}'_n)_\sigma$, and with a tempered distribution a belonging to that space. Observe that the set of all solutions of the equation E is simply the reciprocal image

$$L^-(a),$$

but this is far from explain us how to determine this set of solutions or even one particular solution of the equation E. We will show that,

- *if G is an ${}^{\mathcal{S}}$Green's family of the operator L, then we can find explicitly a distribution u that is a solution of the linear equation E, namely, this particular solution will be the superposition*

  $$\int_{\mathbb{R}^n} aG,$$

  that is the superposition of the family G with respect to the coefficient distribution a.

Indeed the above result follows immediately by the theorem on ${}^{\mathcal{S}}$Green operators of the preceding section and by its proof, but we state and prove again in the more explicit way.

Theorem (solution of a linear equation). *Let $L : \mathcal{S}'_n \to \mathcal{S}'_n$ be a continuous endomorphism on the space of tempered distributions $\mathcal{S}'_n$ and let assume there exists an ${}^{\mathcal{S}}$ Green's family $G = (G_s)_{s\in\mathbb{R}^n}$ of the operator L. Then, for every tempered distribution a in $\mathcal{S}'_n$, a (particular) solution u of the equation*

$$E : L(.) = a$$

can be determined by the following superposition

$$u = \int_{\mathbb{R}^n} aG.$$

Moreover, the datum a is the coordinate system of the particular solution u in the ${}^{\mathcal{S}}$linearly independent family G.

Proof. To see that the superposition $\int_{\mathbb{R}^n} aG$ is a solution of the above equation is straightforward. Indeed, applying $^{\mathcal{S}}$linearity of the operator L, we have

$$\begin{aligned} L(\int_{\mathbb{R}^n} aG) &= \int_{\mathbb{R}^n} aL(G) = \\ &= \int_{\mathbb{R}^n} a\delta = \\ &= a, \end{aligned}$$

as we desired. ■

Remark. Note that any solution u of the above equation E verifies the relation

$$L(u) = L(\int_{\mathbb{R}^n} aG).$$

Indeed, since G is an $^{\mathcal{S}}$Green's family for the operator L, then we have

$$L(G) = \delta,$$

from the very definition of Green's family; by superposition, from the preceding equality, we obtain

$$\int_{\mathbb{R}^n} aL(G) = \int_{\mathbb{R}^n} a\delta = a;$$

thus, if u is distribution such that $L(u) = a$, we have

$$L(u) = \int_{\mathbb{R}^n} aL(G);$$

since the operator L is a linear continuous operator, we can use its $^{\mathcal{S}}$linearity, obtaining

$$L(u) = L(\int_{\mathbb{R}^n} aG).$$

In the condition of the above theorem the particular solution $u = a.G$ is said ***the Green solution of the equation E with respect to the Green family G.***

Note that the $^{\mathcal{S}}$linear hull of the family G is exactly the set of all Green solutions obtainable from the family G.

15.5 Interpretation of the solution

Consider the linear continuous equation $L(.) = a$. To find a particular solution of E, we can proceed by steps:

1. we expand the datum a in the Dirac basis δ, obtaining the same equation in a slightly different form
$$L(.) = \int_{\mathbb{R}^n} a\delta;$$
2. then (if possible) we find a solution G_x of the linear equation
$$E_x : L(.) = \delta_x,$$
for every index x in $\mathbb{R}^n$ (this is possible, for instance, when L is surjective);
3. choice the system $G = (G_x)_{x\in\mathbb{R}^n}$ of particular solutions in such a way that it should be an ${}^{\mathcal{S}}$family (this is possible, for instance, when the operator L is surjective and an ${}^{\mathcal{S}}$homomorphism);
4. at last, make the superposition
$$\int_{\mathbb{R}^n} aG$$
of the ${}^{\mathcal{S}}$system G of particular solutions by the weight system a.

Thus, we can obtain a solution u of the equation $L(.) = a$ through knowledge of an ${}^{\mathcal{S}}$Green's family G for the operator L and the source term a. This process results from the ${}^{\mathcal{S}}$linearity of the operator L.

If the operator L is injective the solution is unique.

Let us formalize a first existence result.

Theorem (on the existence of an ${}^{\mathcal{S}}$Green's family). *Let $L : \mathcal{S}'_n \to \mathcal{S}'_n$ be a surjective continuous endomorphism on the space of tempered distributions $\mathcal{S}'_n$ and let assume the operator is an ${}^{\mathcal{S}}$homomorphism. Then, there exists an ${}^{\mathcal{S}}$Green's family G of the operator L.*

Proof. Since L is surjective its image contains the Dirac family. Since L is a weak ${}^{\mathcal{S}}$homomorphism and the Dirac family is an ${}^{\mathcal{S}}$family, there is an ${}^{\mathcal{S}}$family G whose image is the Dirac family, and this implies that G is an ${}^{\mathcal{S}}$Green family of the operator L. ■

15.6 Characterization of Green's families

The problem now lies in finding an ${}^{\mathcal{S}}$Green's family G for an operator L, that is an ${}^{\mathcal{S}}$family G satisfying the equality

$$L(G) = \delta.$$

Not every linear and continuous operator L admits an ${}^{\mathcal{S}}$ Green's family, as we shall see later. More precisely, in this section, we shall obtain a characterization of those ${}^{\mathcal{S}}$linear operators admitting an ${}^{\mathcal{S}}$Green's family.

The key observation to determine the existence of ${}^{\mathcal{S}}$Green's family is the following one:

- *an ${}^{\mathcal{S}}$Green's family determines a continuous right inverse of the operator L and vice versa.*

Theorem. *Let G be an ${}^{\mathcal{S}}$Green's family (necessarily of class $\mathcal{S}$) of a linear continuous operator $L \in \mathcal{L}(\mathcal{S}'_n)$. Then,*

1) *the superposition operator of the family G is a continuous right inverse of L. In other terms, we have*

$$L \circ {}^t\widehat{G} = (.)_{\mathcal{S}'_n},$$

or equivalently

$$L \circ \int_{\mathbb{R}^n} (., G) = (.)_{\mathcal{S}'_n}.$$

2) *Vice versa, if $R : \mathcal{S}'_n \to \mathcal{S}'_n$ is a (not necessarily linear nor continuous) right inverse of a linear operator L then the family $G = R(\delta)$ is a Green's family of the operator L.*

3) *Moreover, let R be an ${}^{\mathcal{S}}$ operator (not necessarily linear nor continuous) which is a right inverse of a linear operator L, then the image family $G = R(\delta)$ is an ${}^{\mathcal{S}}$Green's family of the operator L.*

Proof. 1) For every a in $\mathcal{S}'_n$, we have

$$\begin{aligned}
L \circ \int_{\mathbb{R}^n} (., G)(a) &= L(\int_{\mathbb{R}^n} aG) \\
&= \int_{\mathbb{R}^n} aL(G) = \\
&= \int_{\mathbb{R}^n} a\delta = \\
&= a,
\end{aligned}$$

as we claimed. 2) Vice versa, if R is a right inverse of the operator L we have

$$L(R(\delta_p)) = \delta_p,$$

for every index p of the Dirac family, and this means exactly (by the very definition of Green's family) that the family $R(\delta)$ is a Green's family of the operator L. 3) Moreover, if R is an $^{\mathcal{S}}$ operator, the image $R(\delta)$ of the Dirac $^{\mathcal{S}}$family δ is an $^{\mathcal{S}}$family too. ■

15.7 Existence of Green's families

In this section we solve the problem of existence of $^{\mathcal{S}}$Green's family for a linear and continuous endomorphism on the space $\mathcal{S}'_n$. We start with some elementary considerations.

Proposition. *Let L be a bijective endomorphism of the space $\mathcal{S}'_n$. Then the reciprocal image $L^{-1}(\delta)$ is the unique Green's family of L.*

Theorem. *Let L be a continuous and bijective endomorphism of $\mathcal{S}'_n$. Then the inverse image $L^{-1}(\delta)$ is (not only the unique Green's family of the operator L but also) an $^{\mathcal{S}}$Green's family of the operator L.*

Proof. By the Banach theorem on the inverse of a continuous operator among dual of Fréchet spaces, the inverse operator L^{-1} is linear and continuous too and so the inverse image $L^{-1}(\delta)$ is an $^{\mathcal{S}}$family. ■

From the characterization of the preceding section, using the following theorem, we can deduce the conclusive existence theorem for $^{\mathcal{S}}$ Green's family of an operator.

Theorem. *Let E and F be two topological vector spaces and let L be a continuous linear map from E into F. Then, there exists a continuous linear map R from F into E such that $L \circ R$ is the identity map $\mathbb{I}_F$ from F into itself if and only if the linear operator L is a surjective topological homomorphism and the kernel* $\ker(L)$ *of the operator L has a topological supplement in E. Precisely, if there exists a continuous right inverse R of the surjection L, a topological supplement of the kernel of the operator L is the image of R.*

Theorem (characterization of the existence of $^{\mathcal{S}}$Green's family). *A continuous endomorphism L on the space $\mathcal{S}'_n$ has an $^{\mathcal{S}}$Green family if and only if it is a surjective topological homomorphism and its kernel* $\ker(L)$ *has a topological supplement in $\mathcal{S}'_n$.*

Corollary (characterization of the existence of ${}^{\mathcal{S}}$Green's family). *Let L be a continuous endomorphism L on the space $\mathcal{S}'_n$. The following assertions are equivalent:*

- *the operator L is a linear surjective ${}^{\mathcal{S}}$homomorphism on the space $\mathcal{S}'_n$;*
- *the operator L has an ${}^{\mathcal{S}}$Green family;*
- *the operator L is a surjective topological homomorphism and its kernel* $\ker(L)$ *has a topological supplement in $\mathcal{S}'_n$;*
- *there exists a continuous linear right inverse of L.*

15.8 Translation invariance and convolutions

Theorem. *Let the linear operator $L : \mathcal{S}'_n \to \mathcal{S}'_n$ be translation invariant (in the physical sense), i.e., let the operator L commute with any translations operator τ_p, which it means that*

$$\tau_p L(u) = L(\tau_p u),$$

for any p in $\mathbb{R}^n$. Then,

- *1) if the operator L admits a fundamental solution, a Green's family there exists and can be constructed by that solution (by translations), furthermore this Green family is of class $\mathcal{E}$;*
- *2) if the operator L admits a Green's family, a Green's family can be generated by only one distribution (by translations);*
- *3) if the operator L admits a fundamental solution in the space $\mathcal{O}'_C$, an ${}^{\mathcal{S}}$ Green's family there exists and it can be constructed by that fundamental solution (by translations).*

In any of the above cases the superposition operators of the Green's families are convolution operators.

Proof. 1) Let the operator L commute with translations and assume $g \in \mathcal{S}'_n$ be a fundamental solution of the operator L at 0, that is $L(g) = \delta_0$. We put $G_p = \tau_p g$, for each p, so that

$$\begin{aligned} L(G_p) &= L(\tau_p g) = \\ &= \tau_p L(g) = \\ &= \tau_p \delta_0 = \\ &= \delta_p, \end{aligned}$$

hence G is a Green's family of the operator L. We already have noted that this family is a smooth family, so we can consider its superposition operator

$$\int_{\mathbb{R}^n} (., G) : \mathcal{E}'_n \to \mathcal{S}'_n,$$

we have

$$\begin{aligned} \int_{\mathbb{R}^n} aG &= \int_{\mathbb{R}^n} a(\tau_p g)_{p\in\mathbb{R}^n} = \\ &= a * g, \end{aligned}$$

for every distribution a in $\mathcal{E}'_n$, so that

$$\int_{\mathbb{R}^n} (., G) = (.)_{\mathcal{E}'_n} * g,$$

as we claimed. (2) follows immediately by (1), indeed, if L has a Green family then it has a fundamental solution. (3) the proof is exactly as for (1), noting that, if the fundamental solution g is in the space $\mathcal{O}'_C(n)$, then the family $(\tau_p g)_{p\in\mathbb{R}^n}$ is a family of class $\mathcal{S}$. Moreover in this case the superposition operator of the Green family is defined on the entire space $\mathcal{S}'_n$. ■

15.8.1 Example: the Laplacian

We know that the smooth function

$$-\frac{1}{4\pi \|.\|} : \mathbb{R}^3_{\neq} \to \mathbb{R}$$

defined on the Euclidean 3-dimensional $\mathbb{R}^3$ minus the origin is a locally summable function. Moreover, for any test function ϕ in $\mathcal{S}'_n$, we have

$$\mu(-\frac{\nabla\phi}{4\pi \|.\|}) = \phi(0),$$

where μ is the Lebesgue-Radon measure on $\mathbb{R}^3$. The above equality can be immediately reread as

$$\nabla \left[-\frac{1}{4\pi \|.\|} \right] = \delta_0,$$

so that the regular distribution

$$\left[-\frac{1}{4\pi \|.\|} \right]$$

is a fundamental solution of the Laplacian operator ∇ at 0. The Laplacian ∇ commutes with any translation, so we can write

$$\nabla \left[\frac{-1}{4\pi d(x,.)} \right] = \delta_x,$$

for every point x of the Euclidean space $\mathbb{R}^3$, where (clearly) d denotes the Euclidean distance of the 3-space . Thus we have found a smooth Green's family of the Laplacian ∇ and the distribution defined by the superposition

$$\int_{\mathbb{R}^3} \rho \left(\left[\frac{-1}{4\pi d(x,.)} \right] \right)_{x\in\mathbb{R}^3},$$

is a solution of the Laplace equation

$$\nabla(.) = \rho,$$

for any distribution ρ with compact support.

15.9 ${}^{\mathcal{S}}$Green's families relative to ${}^{\mathcal{S}}$bases

In this section we introduce a generalization of ${}^{\mathcal{S}}$Green's family.

Definition (Green's families with respect to a basis). *Let $L : \mathcal{S}'_n \to \mathcal{S}'_n$ be a linear operator, acting on the space of tempered distributions and let v be an ${}^{\mathcal{S}}$basis of the space $\mathcal{S}'_n$. Any family $G = (G_s)_{s\in\mathbb{R}^n}$ of tempered distributions is said **a Green's family of the linear operator** L **with respect to the** ${}^{\mathcal{S}}$**basis** v if it satisfy the following equality*

$$L(G_s) = v_s,$$

for any point s of the space $\mathbb{R}^n$.

Definition (${}^{\mathcal{S}}$Green's families). *Let $L : \mathcal{S}'_n \to \mathcal{S}'_n$ be a linear operator, acting on the space of tempered distributions. An ${}^{\mathcal{S}}$Green's family of L with respect to a basis is a Green's family for L with respect to that basis of class $\mathcal{S}$.*

Theorem (solution of a linear equation). *Let $L : \mathcal{S}'_n \to \mathcal{S}'_n$ be a linear continuous operator acting on the space of tempered distributions $\mathcal{S}'_n$ and let assume there exists an ${}^{\mathcal{S}}$Green's family $G = (G_s)_{s\in\mathbb{R}^n}$ for the operator L with respect to an ${}^{\mathcal{S}}$basis v. Then, for every tempered distribution a in $\mathcal{S}'_n$, a solution of the equation*

$$E : L(.) = a$$

can be determined by the following superposition

$$u = \int_{\mathbb{R}^n} (a)_v G,$$

where $(a)_v$ is the unique coefficient distribution such that $(a)_v.v = a$.

Proof. To see that the superposition $\int_{\mathbb{R}^n}(a)_v G$ is a solution is straightforward. Indeed, applying ^{S}linearity of the operator L, we have

$$\begin{aligned} L\Big(\int_{\mathbb{R}^n}(a)_v G\Big) &= \int_{\mathbb{R}^n}(a)_v L(G) = \\ &= \int_{\mathbb{R}^n}(a)_v v = \\ &= a, \end{aligned}$$

as we desired. ■

Part V

Representations

Chapter 16

${}^{\mathcal{S}}$Coordinates

16.1 Systems of coordinates

The elementary base remark is the following.

Remark. *If an ${}^{\mathcal{S}}$family v of the space $\mathcal{S}'_n$, indexed by some m -dimensional Euclidean space, is ${}^{\mathcal{S}}$linearly independent and if u is a vector of the ${}^{\mathcal{S}}$linear hull ${}^{\mathcal{S}}\mathrm{span}\,(v)$, then there exists a unique system of coefficients $a \in \mathcal{S}'_m$ such that*

$$u = \int_{\mathbb{R}^m} av.$$

Proof. Indeed the superposition operator of the family v is injective. ■

So, we can give the following definition.

Definition (system of coordinates). *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be an ${}^{\mathcal{S}}$ linearly independent family and let u be a vector of the ${}^{\mathcal{S}}$linear hull ${}^{\mathcal{S}}\mathrm{span}(v)$. The only tempered distribution $a \in \mathcal{S}'_m$ such that*

$$u = \int_{\mathbb{R}^m} av$$

is denoted by $[u|v]$ *or by* $(u)_v$ *and is called* ***the coordinate system of*** u ***in the family*** v.

So, in the conditions of the preceding definition, the coordinate system $[u|v]$ of any distribution u in the $^{\mathcal{S}}$linear hull V of v is the unique distribution in $\mathcal{S}'_m$ such that

$$u = \int_{\mathbb{R}^m} [u|v]\, v.$$

It is also clear that, for any a in the coefficient space $\mathcal{S}'_m$, we have

$$a = [\int_{\mathbb{R}^m} av|v].$$

16.2 Coordinate operators

Definition (of coordinate operator of an $^{\mathcal{S}}$linearly independent family). *Let* $w \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ *be an* $^{\mathcal{S}}$*linearly independent family. The* ***coordinate operator of the family*** w *is the operator*

$$[\cdot \mid w] : \ ^{\mathcal{S}}\mathrm{span}(w) \to \mathcal{S}'_m : u \mapsto [u \mid w]\,,$$

which we will denote also by $(.)_w$.

Example (on the Dirac family and the (a,b)-Fourier family). Let δ be the Dirac family in $\mathcal{S}'_n$. For any tempered distribution $u \in \mathcal{S}'_n$, we have $[u \mid \delta] = u$ (since $u = u.\delta$), and hence the coordinate operator of the Dirac basis is the identity operator on $\mathcal{S}'_n$, i.e. $[\cdot \mid \delta] = (\cdot)_{\mathcal{S}'_n}$. Now, let φ be the (a,b)-Fourier family in the space $\mathcal{S}'_n(\mathbb{C})$. For each $u \in \mathcal{S}'_n$, by the Fourier expansion theorem, we have

$$[u \mid \varphi] = \mathcal{F}^{-1}_{(a,b)}(u),$$

and hence the coordinate operator of the Fourier family is the inverse Fourier transform, i.e.

$$[\cdot \mid \varphi] = \mathcal{F}^{-1}_{(a,b)}.$$

16.3 Basic properties of coordinate operators

16.3.1 Linear properties

Theorem. *Let $w \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be an ${}^{\mathcal{S}}$linearly independent family in the space $\mathcal{S}'_n$. Then, the coordinate operator $[\cdot|w]$ of the family w is a bijective linear operator (an algebraic isomorphism) of the ${}^{\mathcal{S}}$linear hull ${}^{\mathcal{S}}\mathrm{span}(w)$ onto $\mathcal{S}'_m$. Moreover, in this conditions, the reciprocal algebraic isomorphism of the coordinate operator is the codomain restriction of the superposition operator of the family to the ${}^{\mathcal{S}}$linear hull of the family itself.*

Proof. Let $\lambda \in \mathbb{K}$ be a scalar and u, v be two vectors of the ${}^{\mathcal{S}}$linear hull ${}^{\mathcal{S}}\mathrm{span}(w)$, then we have

$$\begin{aligned} u + \lambda v &= \int_{\mathbb{R}^m} [u|w]w + \lambda \int_{\mathbb{R}^m} [v|w]w = \\ &= \int_{\mathbb{R}^m} ([u|w] + \lambda [v|w])w, \end{aligned}$$

and thus, we deduce

$$[u + \lambda v|w] = [u|w] + \lambda [v|w],$$

as we desired. The coordinate operator is surjective because, if a is in $\mathcal{S}'_m$, then the superposition $a.w$ is in the ${}^{\mathcal{S}}$linear hull ${}^{\mathcal{S}}\mathrm{span}(w)$ and moreover

$$(a.w)_w = a.$$

The coordinate operator is injective since from the equality of two systems of coordinates $(u)_w = (v)_w$ we deduce

$$(u)_w.w = (v)_w.w,$$

which is equivalent to $u = v$. The assertion that the codomain restriction

$$M : \mathcal{S}'_m \to V$$

of the superposition operator of the family w, to the ${}^{\mathcal{S}}$ linear hull V of the family itself, is the reciprocal isomorphism of the coordinate operator follows immediately from the two relations:

$$u = \int_{\mathbb{R}^m} [u|w]\, w$$

and,

$$a = [\int_{\mathbb{R}^m} aw|w].$$

for any u in the $^{\mathcal{S}}$linear hull V and for any a in the coefficient space $\mathcal{S}'_m$. ■

Diagrams. Let v be an $^{\mathcal{S}}$linearly independent family indexed by the Euclidean m-space. Denoted by V the $^{\mathcal{S}}$ linear hull of the family v, the relation among the coordinate operator of v and its superposition operator $^t\widehat{v}$ can be expressed by the following commutative diagram

$$
\begin{tikzcd}
\mathcal{S}'_n & V \arrow[l, "j_V"'] \arrow[dl, "(.)_v"] \\
\mathcal{S}'_m \arrow[u, "^t\widehat{v}"] &
\end{tikzcd}
,
$$

where j_V is the immersion of V in $\mathcal{S}'_n$. On the other hand, the invertibility of the coordinate operator $(.)_v$ can be expressed by the following two commutative diagrams

$$
\begin{tikzcd}
V & V \arrow[l, "(.)_V"'] \arrow[dl, "(.)_v"] \\
\mathcal{S}'_m \arrow[u, "M"] &
\end{tikzcd}
,\qquad
\begin{tikzcd}
\mathcal{S}'_m \arrow[r, "M"] \arrow[d, "(.)"] & V \arrow[dl, "(.)_v"] \\
\mathcal{S}'_m &
\end{tikzcd}
,
$$

where M is the codomain restriction of the superposition operator of the family v to the subspace V, $(.)_V$ is the identity mapping of the subspace V and $(.)$ is the identity operator of $\mathcal{S}'_m$.

16.3.2 $^{\mathcal{S}}$Linear properties

Reminder. An $^{\mathcal{S}}$linear operator $A : V \to W$, where V and W are $^{\mathcal{S}}$closed subspaces, is said an $^{\mathcal{S}}$linear isomorphism if and only if it is bijective and its inverse operator is $^{\mathcal{S}}$linear too. We proved that an $^{\mathcal{S}}$linear operator $A : \mathcal{S}'_n \to \mathcal{S}'_m$ is a continuous linear operator (with respect to the pairs of weak* and strong topologies) so that, by the Banach inverse operator theorem for Fréchet spaces, if the operator A is bijective then the operator is a topological isomorphism, and consequently an $^{\mathcal{S}}$linear isomorphism.

Theorem. *Let $w \in \mathcal{S}(\mathbb{R}^k, \mathcal{S}'_n)$ be an $^{\mathcal{S}}$family of tempered distributions and let $A : \mathcal{S}'_n \to \mathcal{S}'_m$ be a bijective $^{\mathcal{S}}$linear operator (and consequently an $^{\mathcal{S}}$linear isomorphism). Then, the following assertions hold true*

1) *the family w is $^{\mathcal{S}}$ linearly independent if and only if the image family $A(w)$ is $^{\mathcal{S}}$linearly independent;*

2) *the $^{\mathcal{S}}$linear hull of the A-image of the family w is the A-image of the $^{\mathcal{S}}$linear hull of the family w, that is*

$$^{\mathcal{S}}\mathrm{span}(A(w)) = A(^{\mathcal{S}}\mathrm{span}(w));$$

3) *if the family w is ${}^{\mathcal{S}}$linearly independent, for each vector u of the image $A({}^{\mathcal{S}}\mathrm{span}(w))$, the system of coordinates of the vector u in the image family $A(w)$ is the coordinate system of the reciprocal image $A^{-1}(u)$ of the vector in the family w:*

$$[u \mid A(w)] = [A^{-1}(u) \mid w].$$

Proof. **Proof of 1).** Let w be ${}^{\mathcal{S}}$linearly independent and let a belong to $\mathcal{S}'_k$ and such that

$$\int_{\mathbb{R}^k} aA(w) = 0_{\mathcal{S}'_m}.$$

Applying A^{-1}, we obtain

$$\begin{aligned} 0_{\mathcal{S}'_n} &= A^{-1}(0_{\mathcal{S}'_m}) = \\ &= A^{-1}(\int_{\mathbb{R}^k} aA(w)) = \\ &= \int_{\mathbb{R}^k} aA^{-1}(A(w)) = \\ &= \int_{\mathbb{R}^k} aw. \end{aligned}$$

Since the family w is ${}^{\mathcal{S}}$linearly independent, we deduce $a = 0_{\mathcal{S}'_k}$, and then the image family $A(w)$ is ${}^{\mathcal{S}}$linearly independent too. **Proof of 2).** Let $u \in A({}^{\mathcal{S}}\mathrm{span}(w))$. Then, there exists an $a \in \mathcal{S}'_k$ such that

$$u = A\left(\int_{\mathbb{R}^k} aw\right).$$

Thus, by ${}^{\mathcal{S}}$linearity, we have

$$u = \int_{\mathbb{R}^k} aA(w),$$

so that the vector u belongs to the ${}^{\mathcal{S}}$linear hull ${}^{\mathcal{S}}\mathrm{span}(Aw)$, and hence we have the first inclusion

$$A({}^{\mathcal{S}}\mathrm{span}(w)) \subseteq {}^{\mathcal{S}}\mathrm{span}(Aw).$$

Vice versa, let u be a point of the ${}^{\mathcal{S}}$linear hull ${}^{\mathcal{S}}\mathrm{span}(Aw)$. Then, there exists an $a \in \mathcal{S}'_k$ such that

$$u = \int_{\mathbb{R}^k} aA(w),$$

which equivalently means

$$u = A\left(\int_{\mathbb{R}^k} aw\right),$$

and hence the vector u belongs to the image $A({}^{\mathcal{S}}\mathrm{span}(w))$, thus we have the second inclusion

$$ {}^{\mathcal{S}}\mathrm{span}(Aw) \subseteq A({}^{\mathcal{S}}\mathrm{span}(w)). $$

Concluding the two ${}^{\mathcal{S}}$linear spans must coincide:

$$ {}^{\mathcal{S}}\mathrm{span}(Aw) = A({}^{\mathcal{S}}\mathrm{span}(\lambda w)). $$

Proof of 3). For every vector u in ${}^{\mathcal{S}}\mathrm{span}(A^{-1}w)$, we have

$$ u = \int_{\mathbb{R}^m} [u \mid A^{-1}w]A^{-1}w, $$

applying the operator A, we obtain

$$ \begin{aligned} A(u) &= \int_{\mathbb{R}^m} [u \mid A^{-1}w]AA^{-1}w = \\ &= \int_{\mathbb{R}^m} [u \mid A^{-1}w]w, \end{aligned} $$

so, the image vector $A(u)$ belongs to ${}^{\mathcal{S}}\mathrm{span}(w)$ and

$$ [A(u) \mid w] = [u \mid A^{-1}w], $$

as we desired. ■

16.3.3 Topological properties

By the Dieudonné-Schwartz theorem we immediately deduce a characterization of those ${}^{\mathcal{S}}$linearly independent families having the ${}^{\mathcal{S}}$linear hull closed with respect to the weak* topology.

Theorem. *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be an ${}^{\mathcal{S}}$linearly independent family of distributions and let V be its ${}^{\mathcal{S}}$ linear hull ${}^{\mathcal{S}}\mathrm{span}\,(v)$. Then the following assertions are equivalent*

1) the ${}^{\mathcal{S}}$linear hull V is closed with respect to the weak topology $\sigma(\mathcal{S}'_n)$;*

2) the superposition operator of the family v is an injective topological homomorphism for the weak topologies $\sigma(\mathcal{S}'_m)$ and $\sigma(\mathcal{S}'_n)$;*

3) the coordinate operator $[\cdot|v]$ of the family v is a topological isomorphism of the ${}^{\mathcal{S}}$linear hull V onto the space $\mathcal{S}'_m$, with respect to the relative topology $\sigma(\mathcal{S}'_n)_{|V}$ and to the weak topology $\sigma(\mathcal{S}'_m)$.*

Proof. The equivalence for 1) and 2) follows immediately by the fact that a continuous linear operator defined among the duals of two Fréchet spaces is a topological homomorphism if and only if its image is weakly* closed. We have, then, to prove only the equivalence between 2) and 3). The superposition operator of v is an injective weakly* topological homomorphism if and only if (by definition of topological homomorphism) the inverse of its restriction to the pair of sets $(\mathcal{S}'_m, V)$, i.e., the coordinate operator $[\cdot|v]$, is a topological isomorphism, with respect to the topology induced by $\sigma(\mathcal{S}'_n)$ on the $^{\mathcal{S}}$linear hull V and to the topology $\sigma(\mathcal{S}'_m)$. So the claimed is proved. ■

16.4 Coordinate operators of $^{\mathcal{S}}$stable families

Theorem. *Let $w \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be an $^{\mathcal{S}}$linearly independent family. Then the following assertions are equivalent:*

- *1) the coordinate operator $[\cdot|w]$ is an $^{\mathcal{S}}$operator;*
- *2) the coordinate operator $[\cdot|w]$ is an $^{\mathcal{S}}$homomorphism;*
- *3) the superposition operator $\int_{\mathbb{R}^m}(\cdot, w)$ is an $^{\mathcal{S}}$homomorphism;*
- *4) the family w is an $^{\mathcal{S}}$stable family.*

Proof. **1) implies 2).** Let the coordinate operator of the family w be an $^{\mathcal{S}}$linear operator. Let v be a family in the $^{\mathcal{S}}$linear hull $^{\mathcal{S}}\mathrm{span}\,(w)$ such that the coordinate family $[v|w]$ is of class $\mathcal{S}$. We have

$$v = \int_{\mathbb{R}^m} [v|w]\, w,$$

and, since the superposition operator $\int_{\mathbb{R}^m}(\cdot, w)$ is an $^{\mathcal{S}}$operator, the family v should be of class $\mathcal{S}$. **2) implies 3).** Let us assume that the coordinate operator of the family w be an $^{\mathcal{S}}$homomorphism. Let a be a family in $\mathcal{S}'_m$ such that the superposition

$$v := \int_{\mathbb{R}^m} aw$$

is a family of class $\mathcal{S}$, we have, by $^{\mathcal{S}}$linear independence of w, that the family a must be the coordinate family $[v|w]$ of v with respect to w. Since the coordinate operator $[\cdot|w]$ is an $^{\mathcal{S}}$operator, the family $[v|w]$ is necessarily of class $\mathcal{S}$. **3) implies 1).** Let the superposition operator of the family w be an $^{\mathcal{S}}$homomorphism.

Let v be an ${}^{\mathcal{S}}$family in the ${}^{\mathcal{S}}$linear hull ${}^{\mathcal{S}}\mathrm{span}\,(w)$, we have to prove that the coordinate family $[v|w]$ is of class $\mathcal{S}$. But, we have again the expansion

$$v = \int_{\mathbb{R}^m} [v|w]\, w,$$

and so, being the superposition operator $\int_{\mathbb{R}^m} (\cdot, w)$ an ${}^{\mathcal{S}}$homomorphism, the coordinate family $[v|w]$ is of class $\mathcal{S}$. **4) is equivalent to 1).** Simply by definition of ${}^{\mathcal{S}}$stable family. ■

16.5 Hulls of ${}^{\mathcal{S}}$stable families

Theorem (${}^{\mathcal{S}}$closedness of ${}^{\mathcal{S}}\mathrm{span}(w)$). *Let $w \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be an ${}^{\mathcal{S}}$linearly independent family such that its coordinate operator $[\cdot|w]$ is an ${}^{\mathcal{S}}$operator. Then, the ${}^{\mathcal{S}}$linear hull ${}^{\mathcal{S}}\mathrm{span}\,(w)$ is ${}^{\mathcal{S}}$closed in $\mathcal{S}'_n$.*

Proof. To prove that the ${}^{\mathcal{S}}$linear hull ${}^{\mathcal{S}}\mathrm{span}(w)$ is ${}^{\mathcal{S}}$closed, let k be a natural number, let $v \in \mathcal{S}(\mathbb{R}^k, \mathcal{S}'_n)$ be a family in the ${}^{\mathcal{S}}$linear hull ${}^{\mathcal{S}}\mathrm{span}(w)$ and let $a \in \mathcal{S}'_k$ be a distribution. Then,

$$v = \int_{\mathbb{R}^m} [v|w]\, w,$$

in fact

$$\begin{aligned} v_p &= \int_{\mathbb{R}^m} [v_p \mid w]\, w = \\ &= \int_{\mathbb{R}^m} [v \mid w]_p\, w = \\ &= \left(\int_{\mathbb{R}^m} [v \mid w]\, w \right)_p , \end{aligned}$$

for any index $p \in \mathbb{R}^k$. Thanks to the ${}^{\mathcal{S}}$linearity of the ${}^{\mathcal{S}}$linear combinations, we have

$$\begin{aligned} \int_{\mathbb{R}^k} av &= \int_{\mathbb{R}^k} a \left(\int_{\mathbb{R}^m} [v \mid w]\, w \right) = \\ &= \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^k} a\, [v \mid w] \right) w, \end{aligned}$$

and thus the superposition $\int_{\mathbb{R}^k} av$ belongs the ${}^{\mathcal{S}}$linear hull ${}^{\mathcal{S}}\mathrm{span}(w)$. ■

Open problem. We do not know if the ${}^{\mathcal{S}}$closedness of the ${}^{\mathcal{S}}$linear hull ${}^{\mathcal{S}}\mathrm{span}\,(w)$ implies that the coordinate operator $[\cdot|w]$ is an ${}^{\mathcal{S}}$operator.

However, we have the following result.

Theorem. *Let $w \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be an $^{\mathcal{S}}$linearly independent family such that its $^{\mathcal{S}}$linear hull $^{\mathcal{S}}\mathrm{span}\,(w)$ is $\sigma(\mathcal{S}'_n)$-closed. Then, the coordinate operator $[\cdot|w]$ is an $^{\mathcal{S}}$operator.*

Proof. Note that, since the $^{\mathcal{S}}$linear hull $^{\mathcal{S}}\mathrm{span}\,(w)$ is $^{\sigma(\mathcal{S}'_n)}$ closed, the operator $\widehat{w}$ is surjective. Consequently, for every $^{\mathcal{S}}$family v in the hull $^{\mathcal{S}}\mathrm{span}\,(w)$, holding the equality

$$\widehat{v}(g) = [v|w]\,(\widehat{w}(g)),$$

we have that the coordinate family $[v|w]$ is an $^{\mathcal{S}}$family, and so the coordinate operator $[\cdot|w]$ is an $^{\mathcal{S}}$operator. ■

Theorem. *Let v be an $^{\mathcal{S}}$ linearly independent family in $\mathcal{S}'_n$, let its coordinate operator $[\cdot|v]$ be an $^{\mathcal{S}}$operator and let F be the collection of all the $^{\mathcal{S}}$closed subsets of $\mathcal{S}'_n$ containing the family v. Then, we have the following equality*

$$^{\mathcal{S}}\mathrm{span}(v) = \cap F,$$

in other terms, the $^{\mathcal{S}}$linear hull of the family v is the smallest $^{\mathcal{S}}$closed subset of the space $\mathcal{S}'_n$ containing the family v itself.

Proof. Let index the collection F by a set I and denote the corresponding family by F itself. Since any member F_i of the family F is $^{\mathcal{S}}$closed and contains v, the $^{\mathcal{S}}$ span(v) is contained in F_i, for every $i \in I$. Consequently the $^{\mathcal{S}}$linear hull $^{\mathcal{S}}\mathrm{span}(v)$ is contained in the intersection $\cap F$. Since the coordinate operator $[\cdot \mid v]$ is an $^{\mathcal{S}}$operator, the $^{\mathcal{S}}$linear hull $^{\mathcal{S}}\mathrm{span}(v)$ is $^{\mathcal{S}}$closed, moreover it contains v, thus $^{\mathcal{S}}\mathrm{span}(v)$ is a member of the family F, and hence we have also the inclusion $\cap F \subseteq {^{\mathcal{S}}\mathrm{span}(v)}$. ■

16.6 $^{\mathcal{S}}$Linearity of the coordinate operator

Theorem ($^{\mathcal{S}}$linearity of the coordinate operator). *Let $w \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be an $^{\mathcal{S}}$linearly independent family such that its coordinate operator $[\cdot|w]$ is an $^{\mathcal{S}}$operator. Then, the coordinate operator $[\cdot|w]$ is also an $^{\mathcal{S}}$linear operator of the $^{\mathcal{S}}$linear hull $^{\mathcal{S}}$ span(w) into the space $\mathcal{S}'_m$.*

Proof. The coordinate operator $[\cdot|w]$ is of class $\mathcal{S}$ for our assumption. For each natural number k, for any distribution $a \in \mathcal{S}'_k$ and for every family $v \in$

$\mathcal{S}(\mathbb{R}^k, \mathcal{S}'_n)$ in the ${}^{\mathcal{S}}$linear hull ${}^{\mathcal{S}}$span(w), we have

$$\begin{aligned} \int_{\mathbb{R}^k} av &= \int_{\mathbb{R}^k} a\left(\int_{\mathbb{R}^m} [v|w]\, w\right) = \\ &= \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^k} a\,[v|w]\right) w, \end{aligned}$$

and thus, by the definition of system of coordinates, we conclude

$$\begin{aligned} [\cdot|w]\left(\int_{\mathbb{R}^k} av\right) &= \left[\int_{\mathbb{R}^k} av|w\right] = \\ &= \int_{\mathbb{R}^k} a\,[v|w]\,, \end{aligned}$$

as we desired. ■

Corollary. *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be an ${}^{\mathcal{S}}$linearly independent family of distributions and let V be its ${}^{\mathcal{S}}$ linear hull ${}^{\mathcal{S}}$*span$\,(v)$*. Then the following assertions are equivalent*

1) the family v is and ${}^{\mathcal{S}}$stable;

2) the superposition operator of the family v is an injective ${}^{\mathcal{S}}$linear homomorphism;

3) the coordinate operator $[\cdot|v]$ of the family v is an ${}^{\mathcal{S}}$linear isomorphism of the ${}^{\mathcal{S}}$linear hull V onto the space $\mathcal{S}'_m$.

Proof. 1) is equivalent to 2). The superposition operator is always ${}^{\mathcal{S}}$linear. A family v is ${}^{\mathcal{S}}$linearly independent if and only if its superposition operator is injective and v is ${}^{\mathcal{S}}$stable if and only iff its superposition operator is an ${}^{\mathcal{S}}$homomorphism. 2) implies 3). We have only to prove that the coordinate operator is ${}^{\mathcal{S}}$linear with its inverse (the coordinate operator is always an algebraic isomorphism). Well, since the superposition operator of v is an ${}^{\mathcal{S}}$homomorphism the coordinate operator is an ${}^{\mathcal{S}}$operator and then ${}^{\mathcal{S}}$ linear by the above theorem. Its inverse is the codomain restriction of the coordinate operator to the ${}^{\mathcal{S}}$linear hull V, and it is so ${}^{\mathcal{S}}$linear. 3) implies 2). We already know that the superposition operator of an ${}^{\mathcal{S}}$linearly independent ${}^{\mathcal{S}}$family is an (injective) ${}^{\mathcal{S}}$homomorphism if and only if its coordinate operator is an ${}^{\mathcal{S}}$operator (and our coordinate operator is ${}^{\mathcal{S}}$linear and then an ${}^{\mathcal{S}}$operator). ■

Chapter 17

Applications of ${}^{\mathcal{S}}$Coordinates

17.1 Invertibility of ${}^{\mathcal{S}}$homomorphism

Theorem. *Let $L : \mathcal{S}'_n \to \mathcal{S}'_m$ be an ${}^{\mathcal{S}}$linear operator such that its image V has an ${}^{\mathcal{S}}$ stable basis v. Then, the operator L is a linear ${}^{\mathcal{S}}$homomorphism if and only if there is an ${}^{\mathcal{S}}$linear right inverse R of the surjection canonically associated to L, that is of the operator $M : \mathcal{S}'_n \to V$ defined, for any tempered distribution in the space $\mathcal{S}'_n$, by $M(u) = L(u)$.*

Proof. ($\Rightarrow$). Assume the basis v be indexed by the k -dimensional Euclidean space $\mathbb{R}^k$. Since the mapping L is an ${}^{\mathcal{S}}$homomorphism there is an ${}^{\mathcal{S}}$family G such that $L(G) = v$. Note that G must be ${}^{\mathcal{S}}$linearly independent since v is ${}^{\mathcal{S}}$linearly independent. Define an operator $R : V \to \mathcal{S}'_n$ by

$$R(a) = \int_{\mathbb{R}^k} (a)_v G,$$

for every a in V, where $(a)_v$ is the only distribution in $\mathcal{S}'_k$ such that

$$\int_{\mathbb{R}^k} (a)_v v = a.$$

We have, for every element a of V,

$$
\begin{aligned}
L(R(a)) &= L\left(\int_{\mathbb{R}^k} (a)_v G\right) = \\
&= \int_{\mathbb{R}^k} (a)_v L(G) = \\
&= \int_{\mathbb{R}^k} (a)_v v = \\
&= a,
\end{aligned}
$$

so R is a right inverse or L. Let as prove that the right inverse R of L is an $^{\mathcal{S}}$linear operator too. Let w be an $^{\mathcal{S}}$ family in the subspace V, since the superposition operator of the basis v is an $^{\mathcal{S}}$homomorphism (the basis v is stable) the family $(w)_v$ is an $^{\mathcal{S}}$family. We have

$$R(w) = (w)_v.G,$$

and the product of two $^{\mathcal{S}}$families (or equivalently the superposition of an family with respect to an family) is an $^{\mathcal{S}}$ family. Moreover, we have

$$R = {}^t\widehat{G} \circ (.)_v,$$

and the composition of two $^{\mathcal{S}}$linear operators is an $^{\mathcal{S}}$linear operator too. ■

Another way to state the above theorem is the following.

Theorem. *Let $L : \mathcal{S}'_n \to \mathcal{S}'_m$ be an $^{\mathcal{S}}$linear operator such that its image V has an $^{\mathcal{S}}$ stable basis v, indexed by some space $\mathbb{R}^k$. Then, the operator L is a linear $^{\mathcal{S}}$homomorphism if and only if there is an $^{\mathcal{S}}$family G in $\mathcal{S}'_n$, indexed by the space $\mathbb{R}^k$, such that the diagram*

$$
\begin{tikzcd}
\mathcal{S}'_n \arrow[r, "M"] & V \arrow[dl, "(.)_v"] \\
\mathcal{S}'_k \arrow[u, "{}^t\widehat{G}"] &
\end{tikzcd}
.
$$

is commutative, where M is the surjection canonically associated to L, that is of the operator $M : \mathcal{S}'_n \to V$ defined, for any tempered distribution in the space $\mathcal{S}'_n$, by $M(u) = L(u)$. In this conditions the composition ${}^t\widehat{G} \circ (.)_v$ is an $^{\mathcal{S}}$linear right inverse of the surjection M.

17.2 Projectors

A linear projector on a subspace V of a vector space E is a linear function $p : E \to V$ such

- for every u in V, we have $p(u) = u$;
- for every u in E, we have $p(p(u)) = u$.

Theorem. *Let V be a subspace of the space $\mathcal{S}'_n$ having an ${}^{\mathcal{S}}$basis e, indexed by the m-dimensional real Euclidean space. Then, the following two assertions are equivalent*

- *the family e is ${}^{\mathcal{S}}$stable and there exists an ${}^{\mathcal{S}}$linear projector $p : \mathcal{S}'_n \to V$;*
- *the subspace V is ${}^{\mathcal{S}}$closed and there exists an ${}^{\mathcal{S}}$linear extension $h : \mathcal{S}'_n \to \mathcal{S}'_m$ of the coordinate operator $[.|e]$ of the ${}^{\mathcal{S}}$basis e.*

Proof. ($\Rightarrow$) Let p be an ${}^{\mathcal{S}}$linear projector of the space $\mathcal{S}'_n$ onto the subspace V. Consider the operator $h : \mathcal{S}'_n \to \mathcal{S}'_m$ defined, for every distribution in $\mathcal{S}'_n$, by

$$h(u) = [p(u)|e].$$

The operator h is the composition of the projector p and of the coordinate operator of e. Since both the preceding operators are ${}^{\mathcal{S}}$linear (the projector by assumption and coordinate operator since the basis e is ${}^{\mathcal{S}}$stable), the operator h is ${}^{\mathcal{S}}$linear. Moreover, it is evident that h is an extension of the coordinate operator e, since, if u is in V then $p(u) = u$. ($\Leftarrow$) Let h be an ${}^{\mathcal{S}}$linear extension of the coordinate operator of the ${}^{\mathcal{S}}$basis e. Since the subspace V is ${}^{\mathcal{S}}$ closed and the coordinate operator of the ${}^{\mathcal{S}}$basis e is the restriction of an ${}^{\mathcal{S}}$linear operator, the coordinate operator itself must be ${}^{\mathcal{S}}$linear; moreover it is in particular ${}^{\mathcal{S}}$linear, and consequently the family e is ${}^{\mathcal{S}}$ stable. Consider the operator $p : \mathcal{S}'_n \to V$ defined, for every tempered distribution u in $\mathcal{S}'_n$, by

$$p(u) = \int_{\mathbb{R}^m} h(u)e.$$

First of all, note that the above distribution $p(u)$ belongs to V, since it is an ${}^{\mathcal{S}}$linear combination of the ${}^{\mathcal{S}}$basis e . The operator p is the composition of the extension h and of the superposition operator of the ${}^{\mathcal{S}}$basis e. Both the preceding operators are ${}^{\mathcal{S}}$linear, so the composition p is ${}^{\mathcal{S}}$linear too. We have to prove that the operator p is a projector. Let u be in the sub space V, since h is an extension of the coordinate operator, we have

$$\begin{aligned} p(u) &= \int_{\mathbb{R}^m} h(u)e = \\ &= \int_{\mathbb{R}^m} [u|e]e = \\ &= u. \end{aligned}$$

Moreover, for every u in $\mathcal{S}'_n$, we have

$$\begin{aligned} p(p(u)) &= p\left(\int_{\mathbb{R}^m} h(p(u))e\right) = \\ &= \int_{\mathbb{R}^m} h(p(u))p(e) = \\ &= \int_{\mathbb{R}^m} h(p(u))e = \\ &= p(u), \end{aligned}$$

and so the operator p is an $^{\mathcal{S}}$linear projector of $\mathcal{S}'_n$ onto V. ■

17.3 Change of basis

Notation (the set of $^{\mathcal{S}}$bases of a subspace). Let X be a subspace of $\mathcal{S}'_n$. In the following we shall use the notation $\mathcal{B}(\mathbb{R}^m, X)$ for the set of all the $^{\mathcal{S}}$bases of the subspace X indexed by $\mathbb{R}^m$, that is for the set

$$\{v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n) : \ \ker({}^t\widehat{v}) = 0 \ \wedge \ {}^t\widehat{v}(\mathcal{S}'_m) = X\}.$$

Definition (the change family). *Let $v \in \mathcal{B}(\mathbb{R}^k, \mathcal{S}'_n)$ and $w \in \mathcal{B}(\mathbb{R}^m, \mathcal{S}'_n)$ be two $^{\mathcal{S}}$bases of the space $\mathcal{S}'_n$. We say* ***change family from*** *v* ***to*** *w the following family*

$$[w|v] := ([w_p|v])_{p\in\mathbb{R}^m}\,.$$

Theorem (change of basis). *Let $v \in \mathcal{B}(\mathbb{R}^k, \mathcal{S}'_n)$ and $w \in \mathcal{B}(\mathbb{R}^m, \mathcal{S}'_n)$ be two $^{\mathcal{S}}$bases of the space $\mathcal{S}'_n$. Then, the change family from w to v is an $^{\mathcal{S}}$basis too, more precisely we have*

$$[v \mid w] \in \mathcal{B}(\mathbb{R}^k, \mathcal{S}'_m).$$

Moreover, for every tempered distribution $u \in \mathcal{S}'_n$, the following equality holds

$$[u \mid w] = \int_{\mathbb{R}^k} [u \mid v]\,[v \mid w]\,.$$

Proof. Since

$$v = \int_{\mathbb{R}^m} [v \mid w]\, w,$$

we have

$$\widehat{v}(\phi) = [v \mid w]\,(\widehat{w}(\phi))\,,$$

for every test function $\phi \in \mathcal{S}'_n$; so, being $\widehat{w}$ surjective (w is an ${}^{\mathcal{S}}$basis and thus it is invertible), $[v \mid w]$ is an ${}^{\mathcal{S}}$family. Moreover, the same equality shows that

$$\widehat{v} \circ (\widehat{w})^{-1} = [v \mid w]^{\wedge},$$

and then $[v \mid w]$ is invertible, that is an ${}^{\mathcal{S}}$ basis. Now, applying the ${}^{\mathcal{S}}$linearity of the ${}^{\mathcal{S}}$ linear combinations, we have

$$\begin{aligned} u &= \int_{\mathbb{R}^k} [u \mid v]\, v = \\ &= \int_{\mathbb{R}^k} [u \mid v] \left(\int_{\mathbb{R}^m} [v \mid w]\, w \right) = \\ &= \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^k} [u \mid v]\, [v \mid w] \right) w, \end{aligned}$$

and thus by definition of system of coordinates in an ${}^{\mathcal{S}}$basis

$$[u \mid w] = \int_{\mathbb{R}^k} [u \mid v]\, [v \mid w],$$

as we desired. ■

17.4 Superpositions respect to operators

Definition (superposition of a family with respect to an operator). *Let $X \subseteq \mathcal{S}'_n$ be a subset of $\mathcal{S}'_n$, $A : X \to \mathcal{S}'_m$ be an operator and $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be a family of distributions. We define* ***superposition of the family*** v ***with respect to the operator*** A, *the operator*

$$\int_{\mathbb{R}^m} Av : X \to \mathcal{S}'_n : u \mapsto \int_{\mathbb{R}^m} A(u)v.$$

Proposition. *Let X be a subspace of $\mathcal{S}'_n$, let $A : X \to \mathcal{S}'_m$ be an operator and let v be an $\mathcal{S}$family in $\mathcal{S}'_n$ indexed by $\mathbb{R}^m$. In these conditions,*

1) *if the operator A is a linear operator, the superposition*

$$\int_{\mathbb{R}^m} Av$$

of the family v with respect to the coefficient operator A is also a linear operator;

2) *if the operator A is a continuous operator with respect to the pair of topologies (τ, α_m), where τ is a topology on X and α_m is the strong (respectively, the weak* topology) on $\mathcal{S}'_m$, then the superposition*

$$\int_{\mathbb{R}^m} Av$$

is continuous with respect to the pair of topologies (τ, α_n), where τ is a topology on X and where α_n is the strong (respectively the weak topology) on $\mathcal{S}'_n$;*

3) *if X is an $^{\mathcal{S}}$closed subspace and the operator A is $^{\mathcal{S}}$linear, then the superposition*

$$\int_{\mathbb{R}^m} Av$$

is also $^{\mathcal{S}}$linear.

Proof. 1-2) In fact, note that, for every tempered distribution u in the domain of the operator A, we have

$$\begin{aligned} (\int_{\mathbb{R}^m} A\, v)(u) &= \int_{\mathbb{R}^m} A\,(u)\ v = \\ &= {}^t\widehat{v}(A(u)), \end{aligned}$$

so we have that the above superposition is nothing but the composition of the superposition operator of the family v with the operator A, in symbols

$$\int_{\mathbb{R}^m} A\, v = \int_{\mathbb{R}^m} (\cdot, v) \circ A.$$

Since the superposition operator $\int_{\mathbb{R}^m}(\cdot, v)$ is continuous (with respect to the pairs of topologies strong-strong and weak*-weak*) and equivalently $^{\mathcal{S}}$linear, if A is continuous then the superposition

$$\int_{\mathbb{R}^m} Av$$

is also continuous. Let us prove that, if the subspace X is an $^{\mathcal{S}}$closed subspace and A is $^{\mathcal{S}}$linear, then the superposition $A.v$ is also $^{\mathcal{S}}$linear. Indeed

$$\begin{aligned} A.v\left(\int_{\mathbb{R}^k} aw\right) &= \left(\int_{\mathbb{R}^m} Av\right)\left(\int_{\mathbb{R}^k} aw\right) = \\ &= \left(\int_{\mathbb{R}^m} (\cdot, v) \circ A\right)\left(\int_{\mathbb{R}^k} aw\right) = \\ &= \left(\int_{\mathbb{R}^m} (\cdot, v)\right)\left(A\left(\int_{\mathbb{R}^k} aw\right)\right) = \end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^m} A\left(\int_{\mathbb{R}^k} aw\right) v = \\
&= \int_{\mathbb{R}^m} (\int_{\mathbb{R}^k} aA(w))v = \\
&= \int_{\mathbb{R}^k} a \int_{\mathbb{R}^m} A(w)v = \\
&= \int_{\mathbb{R}^k} a(\int_{\mathbb{R}^m} Av)(w) = \\
&= \int_{\mathbb{R}^k} a\ A.v(w),
\end{aligned}
$$

as we desired. ■

17.5 Resolution of the identity

This section is devoted to the resolution of the identity, that is to the expansion of the identity operator of a subspace of the space of tempered distribution $\mathcal{S}'_n$ as a superposition of a $^{\mathcal{S}}$linearly independent family of tempered distributions with respect to its coordinate operator. This "resolution" is widely used in Quantum Mechanics.

Theorem (resolution of the identity). *Let X be a subspace of the space $\mathcal{S}'_n$ and let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be an $^{\mathcal{S}}$linearly independent family in X. Then, the family v is an $^{\mathcal{S}}$basis of X, belonging to $\mathcal{B}(\mathbb{R}^m, X)$, if and only if*

$$j_X = \int_{\mathbb{R}^m} [\cdot \mid v]v,$$

where j_X is the canonical injection of the subspace X into $\mathcal{S}'_n$.

Proof. ($\Rightarrow$) If $v \in \mathcal{B}(\mathbb{R}^m, X)$, then, the subspace X is the $^{\mathcal{S}}$linear hull of the family v and the coordinate operator is thus actually defined on the subspace X. Moreover, for any distribution $u \in X$, we have

$$u = \int_{\mathbb{R}^m} [u \mid v]v,$$

i.e.,

$$j_X(u) = \left(\int_{\mathbb{R}^m} [\cdot \mid v]v\right)(u),$$

and thus

$$j_X = \int_{\mathbb{R}^m} [\cdot \mid v]v.$$

($\Leftarrow$) If the immersion of X in $\mathcal{S}'_n$ is the superposition

$$j_X = \int_{\mathbb{R}^m} [\cdot \mid v]v,$$

then this two operators must have the same domain, so the subspace X must coincide with the $^{\mathcal{S}}$linear hull $^{\mathcal{S}}(v)$. Moreover, for each point $u \in X$, we have

$$\begin{aligned} u &= j_X\,(u) = \\ &= \left(\int_{\mathbb{R}^m} [\cdot \mid v]v\right)(u) = \\ &= \int_{\mathbb{R}^m} [u \mid v]v, \end{aligned}$$

and we conclude. ■

Chapter 18

$\mathcal{S}$ Matrices

This chapter is devoted to the interpretation of $\mathcal{S}$-families as matrices associated to linear and continuous operators. We are dealing with our standard pairings of the type $(\mathcal{S}_n, \mathcal{S}'_n)$, and we shall associate to every linear continuous operator in the space $\mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$ and $\mathcal{L}(\mathcal{S}'_n, \mathcal{S}'_m)$ an $\mathcal{S}$-family in order to represent those monoids (with respect to the composition) into two different monoids having the same set $\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ as underlying set of the structures.

18.1 Introduction

An ordered basis of the vector space $\mathbb{R}^n$ is a n-tuple $e = (e_i)_{i=1}^n$ of linearly independent vectors. With any ordered basis we can associate a square matrix, exactly the following one

$$E = ((e_i)_j)_{i,j=1}^n,$$

where $E_{ij} = (e_i)_j$ is the j-th component of the vector e_i. This matrix is invertible, in fact, its determinant is different from zero because its row vectors $e_1, ..., e_n$ are linearly independent. Conversely, let $A = (A_{ij})_{i,j=1}^n$ be a matrix in $\mathbb{R}^{n,n}$, it is obvious that the n-tuple of the rows of A

$$(R_i)_{i=1}^n$$

is an ordered basis if and only if A is invertible. Thus, there exists a bijective correspondence among the family of the ordered bases of $\mathbb{R}^n$ and the set of the square invertible matrices on $\mathbb{R}^{n,n}$. In other words, let $(\mathbb{R}^n)^n$ be the set of all the n-tuples of vectors of $\mathbb{R}^n$, for each n-tuple $x \in (\mathbb{R}^n)^n$ we can associate the square matrix

$$\psi(x) = ((x_i)_j)_{i,j=1}^n,$$

i.e., the matrix having the vector x_i as i-th row. At this point:

- *there exists an operation . on $(\mathbb{R}^n)^n$ such that the pair $((\mathbb{R}^n)^n, .)$ is a semigroup and moreover it is transformed by the bijection*

  $$\psi : (\mathbb{R}^n)^n \to \mathbb{R}^{n,n}$$

  in the row-column multiplication on $\mathbb{R}^{n,n}$? So, there exists an operation

  $$. : (\mathbb{R}^n)^n \times (\mathbb{R}^n)^n \to (\mathbb{R}^n)^n,$$

  such that for every $x, y \in \mathbb{R}^{n,n}$ we have $\psi(x.y) = \psi(x)\psi(y)$?

The answer is already known, we can put for every index $i \in \mathbb{N}_{\leq n}$,

$$(x.y)_i = \sum_{j=1}^{n} x_{ij} y_j .$$

And it's obvious that the collection of invertible elements with respect to . is transformed by the bijection ψ in the group $GL(\mathbb{R}, n)$ of the invertible matrices on $\mathbb{R}^{n,n}$ (those having determinant different from zero). The aim of this chapter is to study in detail the multiplication in the space of all the $\mathcal{S}$-families (which first of all are ordered sets of distributions, i.e., ordered sets of "non-local defined" systems of elements belonging to the field $\mathbb{K}$. We can evict, that this multiplication is a generalization of the operation . to the infinite dimensional space $\mathcal{S}(\mathbb{R}^n, \mathcal{S}'_n)$. This is very interesting from the theoretical point of view and also in the applications of the Theory of Distributions.

18.2 Product of families

Definition (product of two ${}^{\mathcal{S}}$families). *Let $v \in \mathcal{S}(\mathbb{R}^k, \mathcal{S}'_m)$ and $w \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be two families of distributions.* ***The product of the family** v **by the family** w* *is the family of superpositions (in $\mathcal{S}'_n$) defined by*

$$v.w = (\int_{\mathbb{R}^m} v_p w)_{p \in \mathbb{R}^k} .$$

It is nothing but the superposition $\int_{\mathbb{R}^m} vw$ *of the family* w *with respect to the family* v*. Hence, for each index* $p \in \mathbb{R}^k$*, we have*

$$(v.w)_p = (\int_{\mathbb{R}^m} vw)_p = \int_{\mathbb{R}^m} v_p w.$$

Example. Let v be a family in $\mathcal{S}'_n$ indexed by $\mathbb{R}^m$ and ${}_k\delta$ be the Dirac family in $\mathcal{S}'_k$. Then,

- if δ is the Dirac family in $\mathcal{S}'_m$, we have

$$\delta.v = (\int_{\mathbb{R}^m} \delta_p v)_{p\in\mathbb{R}^m} = v;$$

- if δ is the Dirac family in $\mathcal{S}'_n$, we have

$$v.\delta = (\int_{\mathbb{R}^m} v_p \delta)_{p\in\mathbb{R}^m} = v;$$

- if, in particular, $n = m$ and if δ is the Dirac family in $\mathcal{S}'_n$, we have

$$\delta^2 := \delta.\delta = (\int_{\mathbb{R}^n} \delta_p \delta)_{p\in\mathbb{R}^n} = \delta.$$

We have already proved that the product of two $\mathcal{S}$-families is an $\mathcal{S}$-family and that

$$(v \cdot w)^\wedge = \widehat{v} \circ \widehat{w},$$

Moreover the correspondence

$$\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n) \to \mathcal{L}(\mathcal{S}_n, \mathcal{S}_m) : v \mapsto \widehat{v}$$

is bijective, and so, in the particular case $m = n$, it is a bijective representation of the unitary semigroup $(\mathcal{S}(\mathbb{R}^n, \mathcal{S}'_n), \cdot)$ onto the unitary semigroup $(\mathcal{L}(\mathcal{S}_n), \circ)$.

Definition. *If* A *is an operator belonging to* $\mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$ *the family* $A^\vee$ *is called the* ***canonical*** $\mathcal{S}$***-matrix representation of the operator*** A*.*

The product among families is the natural product when we shall deal with the families associated with linear continuous operators among spaces of test functions.

18.3 Transpose product of families

The transpose product among families will be the natural product when we shall deal with the families associated with a linear continuous operator among spaces of tempered distributions.

Definition (transpose product of two $\mathcal{S}$- families). *Let $v \in \mathcal{S}(\mathbb{R}^k, \mathcal{S}'_m)$ and $w \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be two families of distributions.* ***The transpose product of the family*** *w* ***by the family*** *v is the family (in $\mathcal{S}'_n$) defined by*

$$wv = \left(\int_{\mathbb{R}^m} v_p w \right)_{p \in \mathbb{R}^k}.$$

This new product is nothing but the opposite or transpose of the product of two families, indeed, for each $p \in \mathbb{R}^k$, we have

$$(wv)_p = (v.w)_p = \int_{\mathbb{R}^m} v_p w.$$

The semigroup $(\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n), .)$ has a transpose (or opposite, see Bourbaki, Algebra) semigroup, namely the semigroup $(\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n), \ ^t.)$, where the transpose operation $^t.$, denoted just by a juxtaposition, is defined by

$$vw = w.v,$$

for every pair of families.

Theorem. *Let $v \in \mathcal{S}(\mathbb{R}^k, \mathcal{S}'_m)$ and $w \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be two families of distributions. Then we have*

$$^t(wv)^\wedge = \ ^t(\widehat{w}) \circ \ ^t(\widehat{v}).$$

In particular the correspondence

$$\mathcal{S}(\mathbb{R}^n, \mathcal{S}'_n) \to \mathcal{L}(\mathcal{S}'_n) : v \mapsto \ ^t\widehat{v}$$

is a bijective representation of the semigroup $(\mathcal{S}(\mathbb{R}^n, \mathcal{S}'_n), \ ^t\cdot)$ onto the semigroup $(\mathcal{L}(\mathcal{S}'_n), \circ)$.

Proof. We have

$$\begin{aligned} ^t(wv)^\wedge &= \ ^t(v \cdot w)^\wedge = \\ &= \ ^t(\widehat{v} \circ \widehat{w}) = \\ &= \ ^t(\widehat{w}) \circ \ ^t(\widehat{v}), \end{aligned}$$

and the remaining part is evident. ∎

Definition (the canonical representation of linear continuous operators). *The correspondence* $\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n) \to \mathcal{L}(\mathcal{S}'_m, \mathcal{S}'_n)$ *defined by* $v \mapsto {}^t\widehat{v}$ *is a linear isomorphism and is called* ***the canonical representation of the space*** $\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ *onto the space* $\mathcal{L}(\mathcal{S}'_m, \mathcal{S}'_n)$*, its inverse is called* ***the canonical representation of the space*** $\mathcal{L}(\mathcal{S}'_m, \mathcal{S}'_n)$ *onto the space* $\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$.

18.4 Invertible families

Definition (of invertible $^{\mathcal{S}}$family). *Let* $a \in \mathcal{S}(\mathbb{R}^n, \mathcal{S}'_n)$ *be a square family of tempered distribution. The family* a *is called* ***invertible*** *if it is an invertible element of the semigroup* $(\mathcal{S}(\mathbb{R}^n, \mathcal{S}'_n), .)$, *i.e. if there exists a family* $b \in \mathcal{S}(\mathbb{R}^n, \mathcal{S}'_n)$ *such that* $a \cdot b = b \cdot a = \delta$.

Remark. We have to note that:

- To say that a square family v is invertible with respect to the product of families it's one and the same thing to say that it is invertible with respect to the transpose product of families.
- Moreover, if b is the inverse of a with respect to one of the two product it is the inverse of a with respect to the other one.
- It's easy to prove that, for each invertible family $a \in \mathcal{S}(\mathbb{R}^n, \mathcal{S}'_n)$, there exists only a square family $b \in \mathcal{S}(\mathbb{R}^n, \mathcal{S}'_n)$ such that $a \cdot b = b \cdot a = \delta$, this is a simple fact of semigroups. This family is denoted by a^-. It derives also immediately from the fact that the semigroup of square families is isomorphic to the semigroup of linear continuous endomorphisms on the space $\mathcal{S}'_n$ (or $\mathcal{S}_n$) with respect to the composition.
- Moreover, it can be proved that the operator generated by a is invertible and $(a^-)^\wedge = (\widehat{a})^-$. And this depends again on the above isomorphism. We can prove it directly as an exercise. In fact, for each test function $\phi \in \mathcal{S}_n$, we have

$$\begin{aligned}(a^-)^\wedge \circ \widehat{a} &= \left(a^- \cdot a\right)^\wedge(\phi) \\ &= \delta^\wedge = \\ &= (\cdot)_{\mathcal{S}_n}\end{aligned}$$

  Analogously we have $\widehat{a} \circ (a^-)^\wedge = (\cdot)_{\mathcal{S}_n}$, and hence $\widehat{a}$ is invertible and $(a^-)^\wedge = (\widehat{a})^-$.

Theorem. *Let* $v, w \in \mathcal{S}(\mathbb{R}^n, \mathcal{S}'_n)$ *be two* $\mathcal{S}$*-linearly independent family. Then, their product* $v \cdot w$ *is* $\mathcal{S}$ *-linearly independent. Moreover, if* w *is invertible (i.e. an* $\mathcal{S}$*-basis) we have*

$$[u|w.v] = \int_{\mathbb{R}^n} [u|v]w^- = [u|v].w^-.$$

Proof. Let $a \in \mathcal{S}'_n$ be such that $\int_{\mathbb{R}^n} a(v \cdot w) = 0_{\mathcal{S}'_n}$, we have

$$\begin{aligned} 0_{\mathcal{S}'_n} &= \int_{\mathbb{R}^n} a(v \cdot w) = \\ &= a \circ (v \cdot w)^{\wedge} = \\ &= a \circ (\widehat{v} \circ \widehat{w}) = \\ &= (a \circ \widehat{v}) \circ \widehat{w} = \\ &= \int_{\mathbb{R}^n} (\int_{\mathbb{R}^n} av) w. \end{aligned}$$

Since w is $\mathcal{S}$-linearly independent, we have $\int_{\mathbb{R}^n} av = 0_{\mathcal{S}'_n}$. And because v is $\mathcal{S}$ -linearly independent, we have $a = 0_{\mathcal{S}'_n}$. So $v \cdot w$ is $\mathcal{S}$-linearly independent. If w is invertible then $\widehat{w}$ is invertible and we have

$$\begin{aligned} u &= \int_{\mathbb{R}^n} [u|v]v = \\ &= [u|v] \circ \widehat{v} = \\ &= [u|v] \circ \widehat{w}^{-} \circ \widehat{w} \circ \widehat{v} \\ &= \left([u|v] \circ \widehat{w}^{-}\right) \circ (\widehat{w} \circ \widehat{v}) = \\ &= ([u|v] \circ \widehat{w}^{-}) \circ (w \cdot v)^{\wedge} = \\ &= \int_{\mathbb{R}^n} \left([u|v] \circ \widehat{w}^{-}\right) (w \cdot v), \end{aligned}$$

and hence

$$\begin{aligned} [u|w.v] &= [u|v] \circ \widehat{w}^{-} = \\ &= [u|v] \circ \left(w^{-}\right)^{\wedge} = \\ &= \int_{\mathbb{R}^n} [u|v] w^{-}, \end{aligned}$$

as we desired. ■

Example. Let $w \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be an $\mathcal{S}$-linearly independent family and let $v \in s(\mathbb{R}^k, \mathcal{S}'_n)$ be such that $v_p \in {}^{\mathcal{S}}\mathrm{span}\,(w)$, for each $p \in \mathbb{R}^k$. Then, for each $p \in \mathbb{R}^k$, we have

$$v_p = \int_{\mathbb{R}^m} [v_p \mid w]\, w,$$

i.e.,

$$v = \int_{\mathbb{R}^m} [v|w]w = [v|w].w,$$

where $[v|w]$ is the family in $\mathcal{S}'_m$ defined by

$$[v|w] := ([v_p|w])_{p \in \mathbb{R}^k}.$$

In fact

$$\begin{aligned}(\int_{\mathbb{R}^m}[v|w]w)(p) &= \int_{\mathbb{R}^m}[v|w]_p w \\ &= \int_{\mathbb{R}^m}[v_p|w]w \\ &= v_p.\end{aligned}$$

Remark. Obviously, in the conditions of the above definition, if $v \in \mathcal{S}(\mathbb{R}^k, \mathcal{S}'_m)$, we have

$$\int_{\mathbb{R}^m} vw = v \cdot w,$$

and thus $\int_{\mathbb{R}^m} vw \in \mathcal{S}(\mathbb{R}^k, \mathcal{S}'_n)$.

18.5 Coordinates and invertible families

Definition (of representation of a family in an $\mathcal{S}$-linearly independent family). *Let $w \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be an $\mathcal{S}$-linearly independent family and let v be a family in ${}^{\mathcal{S}}\mathrm{span}(w)$ indexed by the set I. The family in $\mathcal{S}'_m$*

$$[v|w] := ([v_p|w])_{p\in I}$$

*is called the **representation of the family** v **in the family** w.*

If the family w is invertible we have $[v|w] = v.w^-$, indeed $v = [v|w].w$. In general, if w is linearly independent and the coordinate representation is of class $\mathcal{S}$ we have from the same expansion equality the following equality for superposition operators

$${}^t\widehat{v} = {}^t\widehat{w} \circ {}^t[v|w]^\wedge.$$

If L is a left inverse of the superposition operator of the family w we have

$$L \circ {}^t\widehat{v} = {}^t [v|w]^\wedge,$$

and if L is continuous (if we assume the $\mathcal{S}$-hull of w closed in the weak* topology we have both the conditions satisfied) there is a right inverse with respect to the product of families of w, say l, and we have

$$[v|w] = v.l = lv.$$

Example. Let δ be the Dirac basis in $\mathcal{S}'_n$ and v be a family in $\mathcal{S}'_n$, we have $[v \mid \delta] = v$.

Remark. More generally, if v is a family in $\mathcal{S}'_m$ indexed by a set I and $A : \mathcal{S}'_m \to \mathcal{S}'_n$ is an operator, we denote by $A(v)$ the family $(A(v_p))_{p\in I}$. Now, let $w \in \mathcal{S}(\mathbb{R}^h, \mathcal{S}'_m)$ be a family in ${}^{\mathcal{S}}\mathrm{span}(w)$. One has

$$[v|w]_p = [v_p|w] = [.|w](v_p) = [.|w](v)(p),$$

and hence $[v|w] = [.|w](v)$.

Theorem (invertible version of the "change of basis theorem"). *Let $v \in \mathcal{S}(\mathbb{R}^n, \mathcal{S}'_n)$ be an $\mathcal{S}$-basis and let $w \in \mathcal{S}(\mathbb{R}^n, \mathcal{S}'_n)$ be an invertible square family. Then, the family $[v \mid w]$ is an $\mathcal{S}$-family and we have*

$$[\cdot \mid w] = \int_{\mathbb{R}^n} [\cdot \mid v][v \mid w].$$

Proof. First of all we prove that $[v \mid w]$ is of class $\mathcal{S}$. Recalling the definition of superposition of a family with respect to a family, we have

$$v = \int_{\mathbb{R}^n} [v \mid w]w$$

in fact, for each $p \in \mathbb{R}^n$, we have

$$\begin{aligned} v_p &= \int_{\mathbb{R}^n} [v_p \mid w]w = \\ &= \int_{\mathbb{R}^n} [v \mid w]_p w = \\ &= \left(\int_{\mathbb{R}^n} [v \mid w]w\right)(p). \end{aligned}$$

And hence, for all $\phi \in \mathcal{S}_n$, we have

$$\begin{aligned} \widehat{v}(\phi)(p) &= v_p(\phi) = \\ &= \left(\int_{\mathbb{R}^n} [v_p \mid w]w\right)(\phi) = \\ &= [v_p \mid w](\widehat{w}(\phi)), \end{aligned}$$

now, for all $\psi \in \mathcal{S}_n$, because $\widehat{w}$ is surjective, there exists a $\phi \in \mathcal{S}_n$ such that $\psi = \widehat{w}(\phi)$, so

$$\begin{aligned} [v| w](\psi)(p) &= [v_p| w](\psi) = \\ &= [v_p| w](\widehat{w}(\phi)) = \\ &= \widehat{v}(\phi)(p) \end{aligned}$$

i.e. $[v| w](\psi) = \widehat{v}(\phi) \in \mathcal{S}_n$, and concluding $[v| w]$ is a family of class $\mathcal{S}$. Moreover

$$\begin{aligned} \widehat{v}(\phi) &= [v| w](\widehat{w}(\phi)) = \\ &= [v| w]^\wedge(\widehat{w}(\phi)) = \\ &= ([v| w]^\wedge \circ \widehat{w})(\phi) \end{aligned}$$

and so $\widehat{v} = [v|\, w]^{\wedge} \circ \widehat{w}$, applying to both sides $(\widehat{w})^{-}$, we have $[v|\, w]^{\wedge} = \widehat{v} \circ (\widehat{w})^{-}$, and hence $[v|\, w]^{\wedge}$ is a continuous operator, so $[v|\, w]$ is of class $\mathcal{S}$. The "change of basis theorem" completes the proof. ■

Remark (important). Thanks to the Dieudonné-Schwartz theorem, we proved that every invertible family is an $\mathcal{S}$-basis and vice versa.. Then, the preceding result can be stated in a more elegant way, assuming both v and w be $\mathcal{S}$-bases. Nevertheless, in the form in which we gave the statement and proof, we need no resort to the characterization of invertible families.

Corollary *Let* $v \in \mathcal{B}(\mathbb{R}^n, \mathcal{S}'_n)$ *. Then, we have* $[v|\, v^-] = v^2$ *and*

$$[u|\, v^-] = \int_{\mathbb{R}^n} [u|\, v]v^2.$$

Proof. We have

$$[v|v^-] = v.v = v^2;$$

and, from the change of basis theorem, we deduce

$$\begin{aligned} [u|\, v^-] &= \int_{\mathbb{R}^n} [u|v][v|v^-] = \\ &= \int_{\mathbb{R}^n} [u|v]v^2, \end{aligned}$$

as we desired. ■

Theorem. *Let* $v \in \mathcal{B}(\mathbb{R}^n, \mathcal{S}'_n)$ *be an invertible family. Then, the coordinate operator* $[.|v]$ *is invertible and moreover we have*

$$[\cdot \mid v]^- = [\cdot \mid v^-].$$

Proof. Let $u \in \mathcal{S}'_n$, we have

$$\begin{aligned} \big([\cdot \mid v] \circ [\cdot \mid v^-]\big)(u) &= [\cdot \mid v]([\cdot \mid v^-](u)) = \\ &= [\cdot \mid v]([u \mid v^-]) = \\ &= [[u|v^-]|v]. \end{aligned}$$

Now, we have already seen that,

$$[u|w^-] = \int_{\mathbb{R}^n} uw,$$

and thus

$$\begin{aligned} ([\cdot|\, v^-] \circ [\cdot|\, v^-](u)) &= [[u \mid v^-] \mid v] = \\ &= \int_{\mathbb{R}^n} [u \mid v^-]v^- = \\ &= u. \end{aligned}$$

Analogously, we have

$$\begin{aligned}
\left([\cdot|\, v^-] \circ [\cdot|\, v]\right)(u) &= [\cdot|\, v^-]([\cdot|\, v](u)) = \\
&= [\cdot|\, v^-]([u|\, v]) = \\
&= [[u|\, v] \mid v^-] = \\
&= \int_{\mathbb{R}^n} [u|\, v]v = \\
&= u.
\end{aligned}$$

So, the operator $[\cdot \mid v]$ is invertible and $[\cdot \mid v]^- = [\cdot|\, v^-]$. ■

Theorem. *Let $v \in \mathcal{S}(\mathbb{R}^n, \mathcal{S}'_n)$ be an ${}^{\mathcal{S}}$basis such that the family $[\delta|\, v]$ is of class $\mathcal{S}$ and the operator $[\cdot \mid v]$ is invertible. Then, v is invertible, we have $v^- = [\delta|\, v]$. Moreover*

$${}^t(\widehat{v}) = [\cdot \mid v]^-.$$

Proof. First of all we see that v is invertible. For each $p \in \mathbb{R}^n$, we have

$$\delta_p = \int_{\mathbb{R}^n} [\delta \mid v]_p v = ([\delta \mid v] \cdot v)(p),$$

and hence $[\delta \mid v] \cdot v = \delta$. On the other hand, by the "change of basis theorem" we have

$$\begin{aligned}
(v \cdot [\delta|\, v])(p) &= \int_{\mathbb{R}^n} v_p\, [\delta|\, v] \\
&= \int_{\mathbb{R}^n} [v_p|\, \delta]\, [\delta|\, v] \\
&= [v_p|\, v] \\
&= \delta_p
\end{aligned}$$

and thus $v \cdot [\delta \mid v] = \delta$, so v has a left and a right inverse and then $v \in \mathcal{B}(\mathbb{R}^n, \mathcal{S}'_n)$ and moreover

$$v^- = [\delta \mid v].$$

Now, because v is invertible, the operator $[\cdot|\, v]$ is invertible. We shall prove that

$$[\cdot|\, v]^- = {}^t(\widehat{v}).$$

For each $u \in \mathcal{S}'_n$,

$$u = \int_{\mathbb{R}^n} [u \mid v]v,$$

i.e.

$$\begin{aligned}
u &= [u \mid v] \circ \widehat{v} \\
&= {}^t(\widehat{v})([u \mid v]) \\
&= ({}^t(\widehat{v}) \circ [\cdot \mid v])(u),
\end{aligned}$$

i.e., $(\cdot)_{\mathcal{S}'_n} = {}^t(\widehat{v}) \circ [\cdot \mid v]$, and hence ${}^t(\widehat{v}) = [\cdot \mid v]^{-}$. ■

Chapter 19

Representations in QM

In the present section we give a rigorous and greatly enriched version of the representation theory introduced by Dirac in [*Di*] (page 66). A *pure state* σ of a quantum system is a mono-dimensional subspace of the space $\mathcal{S}'_n$, each $\psi \in \sigma$ is a *vector-state representing* σ.

19.1 Representation of $^{\mathcal{S}}$endomorphisms

Let $\psi = (\psi_p)_{p\in\mathbb{R}^n}$ be an $\mathcal{S}$-basis of the space $\mathcal{S}'_n$ and let $A \in \mathcal{L}(\mathcal{S}'_n)$ be an $\mathcal{S}$-linear endomorphism of $\mathcal{S}'_n$. For every index $p \in \mathbb{R}^n$ of the $\mathcal{S}$ -basis, by definition of coordinate operator, we have

$$A(\psi_p) = \int_{\mathbb{R}^n} [A(\psi_p) \mid \psi]\, \psi.$$

Following Dirac, we call the family of coordinate distributions

$$(A)_\psi = ([A(\psi_p) \mid \psi])_{p\in\mathbb{R}^n}\,,$$

the matrix representation of the operator A *in the* $^{\mathcal{S}}$*basis* ψ.

Interpretation. The $\mathcal{S}$-family $(A)_\psi$ must be interpreted as the $\mathcal{S}$-matrix having for columns the systems of coordinates of the images of the elements of the $\mathcal{S}$-basis in the $\mathcal{S}$-basis itself. Of course, we shall have

$$(A)_\psi (u)_\psi = (A(u))_\psi,$$

as soon as we define suitably the representation of a vector in an $\mathcal{S}$-basis and the product of an $\mathcal{S}$-matrix by a vector, and we will see this in early.

Concerning the matrix representation of the product of two operators we have the following result.

Theorem. *The family representing the product of two $\mathcal{S}$-endomorphism on $\mathcal{S}'_n$ is the transpose product of the families representing the two operators.*

We recall that the transpose product is defined as follows.

Let v and w be two families in $\mathcal{S}(\mathbb{R}^n, \mathcal{S}'_n)$. The transpose product of v by w is the family vw (note: there is no dot) defined as the superposition of the family v with respect to the family w, i.e. by

$$vw := w.v = \int_{\mathbb{R}^n} wv,$$

where $w.v$ is the product of the family w by the family v.

Proof. Let A, B be two $\mathcal{S}$-linear operators on $\mathcal{S}'_n$, we have

$$\begin{aligned} A(B(\psi_p)) &= A(\int_{\mathbb{R}^n} [B(\psi_p) \mid \psi]\, \psi) = \\ &= \int_{\mathbb{R}^n} (B)_\psi^{\ p}\, A(\psi) = \\ &= \int_{\mathbb{R}^n} (B)_\psi^{\ p} \int_{\mathbb{R}^n} (A)_\psi\, \psi = \\ &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} (B)_\psi^{\ p}\, (A)_\psi \right) \psi, \end{aligned}$$

so it follows

$$(AB)_\psi = (A)_\psi\, (B)_\psi\,,$$

as we claimed. ■

Interpretation. The family representing an operator $A \in \mathcal{L}(\mathcal{S}'_n)$ takes the place of the matrix representing a linear operator among two finite dimensional vector spaces. For this reason we shall call the $\mathcal{S}$-families also with the name $\mathcal{S}$ -matrices.

19.2 Representation of vector states

For any tempered distribution u in the space $\mathcal{S}'_n$, we have, moreover, that

$$u = \int_{\mathbb{R}^n} [u \mid \psi]\, \psi.$$

We call the non-locally defined family (tempered distribution)

$$(u)_\psi := [u \mid \psi]\,,$$

the representation of the vector u in the basis ψ.

We have

$$\begin{aligned} A(u) &= \int_{\mathbb{R}^n} (u)_\psi\, A(\psi) = \\ &= \int_{\mathbb{R}^n} (u)_\psi \int_{\mathbb{R}^n} (A)_\psi\, \psi = \\ &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} (u)_\psi\, (A)_\psi \right) \psi, \end{aligned}$$

thus we proved that

$$(A(u))_\psi = \int_{\mathbb{R}^n} (u)_\psi\, (A)_\psi\,.$$

19.2.1 First examples

A simple kind of observables often used in Quantum Mechanics is that of the so called "multiplication operators by a scalar", let us see in the following example their definition and their matrix representation.

Example (the multiplication by a scalar). If we regard, following Dirac, the multiplication by a scalar c as the observable (endomorphism) M_c defined by $M_c(u) = cu$, for any distribution u in $\mathcal{S}'_n$, we have

$$\begin{aligned} M_c(u) &= cu = \\ &= c \int_{\mathbb{R}^n} (u)_\psi\, \psi = \\ &= \int_{\mathbb{R}^n} (u)_\psi\, (c\psi) = \\ &= \int_{\mathbb{R}^n} c\,(u)_\psi\, \psi. \end{aligned}$$

Then we have

$$\begin{aligned} M_c(\psi_p) &= \int_{\mathbb{R}^n} c(\psi_p)_\psi \psi = \\ &= \int_{\mathbb{R}^n} (c\delta_p)\psi, \end{aligned}$$

so the $\mathcal{S}$-family representing the observable M_c is the diagonal family $(c\delta_p)_{p\in\mathbb{R}^n}$, i.e., the family $c\delta$.

More generally we can consider the observable "multiplication by a smooth function" more precisely by an $\mathcal{O}_M$ function.

Example (the multiplication by a smooth function). If we regard, following Dirac, the multiplication by a scalar function f as the observable (endomorphism) M_f defined by $M_f(u) = fu$, for any distribution u in $\mathcal{S}'_n$, we have

$$\begin{aligned} M_f(\delta_p) &= f\delta_p = \\ &= \int_{\mathbb{R}^n} (f\delta_p)_\delta \delta = \\ &= \int_{\mathbb{R}^n} (f(p)\delta_p)\delta, \end{aligned}$$

so the $\mathcal{S}$-family representing the observable M_f in the Dirac basis is the diagonal family $(f(p)\delta_p)_{p\in\mathbb{R}^n}$, i.e., the family $f\delta$.

19.3 The representation correspondence

Theorem. *The correspondence that sends every $\mathcal{S}$-endomorphism to its corresponding $\mathcal{S}$-matrix, that is the representation*

$$(\cdot)_\psi : \mathcal{L}(\mathcal{S}'_n) \to \mathcal{S}(\mathbb{R}^n, \mathcal{S}'_n),$$

is bijective.

Proof. In fact, $(A)_\psi = (B)_\psi$ implies, for every u in $\mathcal{S}'_n$,

$$\begin{aligned} (A(u))_\psi &= (A)_\psi (u)_\psi = \\ &= \int_{\mathbb{R}^n} (u)_\psi (A)_\psi = \\ &= \int_{\mathbb{R}^n} (u)_\psi (B)_\psi = \\ &= (B)_\psi (u)_\psi = \\ &= (B(u))_\psi , \end{aligned}$$

and thus $Au = Bu$, for every u in $\mathcal{S}'_n$, i.e. $A = B$, so the representation is injective. ■

Theorem. *The representation* $(\cdot)_\psi$ *is also surjective.*

Proof. In fact, if $v = (v_p)_{p\in\mathbb{R}^n}$ is an $\mathcal{S}$ -family, we put

$$A\,(u) = \int_{\mathbb{R}^n} (u)_\psi \, (\int_{\mathbb{R}^n} v\psi),$$

we have

$$\begin{aligned} A(\psi_p) &= \int_{\mathbb{R}^n} (\psi_p)_\psi \left(\int_{\mathbb{R}^n} v\psi\right) = \\ &= \int_{\mathbb{R}^n} \delta_p \left(\int_{\mathbb{R}^n} v\psi\right) = \\ &= \left(\int_{\mathbb{R}^n} v\psi\right)_p = \\ &= \int_{\mathbb{R}^n} v_p\psi, \end{aligned}$$

and thus $(A)_\psi = v$. ■

Let $(\cdot)^-_\psi$ the inverse of $(\cdot)_\psi$. In the above proof we have deduced that

$$(v)^-_\psi\,(u) = \int_{\mathbb{R}^n} (u)_\psi \left(\int_{\mathbb{R}^n} v\psi\right).$$

If we choose the canonical basis δ, we have (as we already know)

$$\begin{aligned} (v)^-_\delta\,(u) &= \int_{\mathbb{R}^n} (u)_\delta \left(\int_{\mathbb{R}^n} v\delta\right) = \\ &= \int_{\mathbb{R}^n} uv. \end{aligned}$$

If we put (as in the finite-dimensional case)

$$vu := (v)^-_\delta\,(u) = \int_{\mathbb{R}^n} uv,$$

(vu is called the image of the vector u under the matrix v, or the transpose product of v by u) we obtain

$$\begin{aligned} (A\,(u))_\psi &= \int_{\mathbb{R}^n} (u)_\psi\,(A)_\psi = \\ &= (A)_\psi\,(u)_\psi\,, \end{aligned}$$

as we claimed in the first section.

19.4 Representation of $\mathcal{S}$-linear operators

The generalization to the case $A \in \mathcal{L}(\mathcal{S}'_n, \mathcal{S}'_m)$ is, at this point, very natural.

Let ψ an $\mathcal{S}$-basis of $\mathcal{S}'_n$ and φ be an $^{\mathcal{S}}$ basis of $\mathcal{S}'_m$. We define $^{\mathcal{S}}$matrix associated with A in the pair of basis (ψ, φ) the $\mathcal{S}$-family $(A)_{(\psi,\varphi)}$ in $\mathcal{S}(\mathbb{R}^n, \mathcal{S}'_m)$ defined by

$$(A\,(u))_{\varphi} = \int_{\mathbb{R}^n} (u)_{\psi}\,(A)_{(\psi,\varphi)}\,.$$

We have

$$(A)_{(\psi,\varphi)} = (A(\psi))_{\varphi} := ((A(\psi_q))_{\varphi})_{q\in\mathbb{R}^n},$$

indeed

$$\begin{aligned}\int_{\mathbb{R}^n} (\psi_{q'})_{\psi}(A(\psi))_{\varphi} &= \int_{\mathbb{R}^n} \delta_{q'}(A(\psi))_{\varphi} = \\ &= (A(\psi_{q'}))_{\varphi},\end{aligned}$$

and, using $\mathcal{S}$-linearity, if it is true for a basis its is true for every vector.

Or, equivalently using the transpose product, the $\mathcal{S}$-matrix such that

$$(A\,(u))_{\varphi} = (A)_{(\psi,\varphi)}\,(u)_{\psi}\,.$$

Remark. Note that on the contrary the family associated to and endomorphism $A \in \mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$ is the family

$$(A)_{(\psi,\varphi)} = (\varphi_p \circ A \circ \widehat{\psi}^{-})_{p\in\mathbb{R}^m}.$$

And we have

$$(B \circ A)_{(\alpha,\gamma)} = (B)_{(\beta,\gamma)} \cdot (A)_{(\alpha,\beta)}\,,$$

where $\cdot$ is the product of two families (and no the transpose product!).

It's simple to prove that ψ is an $\mathcal{S}$-basis of the entire space if and only if ${}^t\widehat{\psi}$ is bijective. In this case we have

$$u_{\psi} = {}^t(\widehat{\psi})^{-}(u) = {}^t(\widehat{\psi}^{-})(u).$$

And moreover, denoted by ψ^{-} the families associated with the operator $\widehat{\psi}^{-}$, the following decomposition holds

$$(A)_{\psi}^{\;p} = \int_{\mathbb{R}^n} A\psi_p\psi^{-}.$$

This relations will be used in the following classic examples.

19.5 Classic examples in QM

19.5.1 $^{\mathcal{S}}$Matrix representations

Example (the representation of the position operator in the momentum basis). Let

$$X : \mathcal{S}_1' \to \mathcal{S}_1' : u \mapsto (\cdot)u$$

be the position operator and let φ be the $(1, -1/\hbar)$-Fourier family, then we have

$$\begin{aligned}
(X)_\varphi^{\ p} &= \int_{\mathbb{R}} X\varphi_p\varphi^- = \\
&= \int_{\mathbb{R}} \mathbb{I}_{\mathbb{R}}\varphi_p\varphi^- = \\
&= \left(\frac{i}{1/\hbar}\right)^1 \left(\int_{\mathbb{R}} \varphi_p\varphi^-\right)' = \\
&= i\hbar\,(\varphi_p)_\varphi^{\ \prime} = \\
&= i\hbar\delta_p'.
\end{aligned}$$

Example (the representation of the momentum operator in the momentum basis). Let

$$P : \mathcal{S}_1' \to \mathcal{S}_1' : u \mapsto -i\hbar u'$$

be the momentum operator of a particle. We have

$$\begin{aligned}
(P)_\varphi^{\ p} &= \int_{\mathbb{R}} P\varphi_p\varphi^- = \\
&= \int_{\mathbb{R}} p\varphi_p\varphi^- = \\
&= p(\varphi_p)_\varphi = \\
&= p\delta_p.
\end{aligned}$$

and hence $(P)_\varphi = \mathbb{I}_{\mathbb{R}}\delta$.

Example (the representation of the kinetic energy operator in the momentum basis). Let

$$T : \mathcal{S}_1' \to \mathcal{S}_1' : u \mapsto \left(\hbar^2/2m\right) u'' = (1/2m)\, P^2(u)$$

be the kinetic energy operator of a nonrelativistic particle, we have

$$(T)_\varphi^{\ p} = \int_{\mathbb{R}} T\varphi_p\varphi^- =$$

$$\begin{aligned}
&= \int_{\mathbb{R}} \frac{1}{2m} P^2 \varphi_p \varphi^- = \\
&= \frac{1}{2m} \int_{\mathbb{R}} p^2 \varphi_p \varphi^- = \\
&= \frac{1}{2m} p^2 \left(\varphi_p\right)_\varphi = \\
&= \frac{1}{2m} p^2 \delta_p.
\end{aligned}$$

19.5.2 Operator representation

Definition (of representation of an operator in an invertible family). *Let $A : \mathcal{S}'_n \to \mathcal{S}'_n$ be an operator and $v \in \mathcal{B}(\mathbb{R}^n, \mathcal{S}'_n)$. We define* ***(operator) representation of the operator A in the basis v*** *the following operator*

$$[A]_v := [A \mid v] \circ [\cdot \mid v^-].$$

Example (a representation of the momentum operator). Let v be the family in $\mathcal{S}'_1$ defined by

$$v_p = \frac{1}{\sqrt{2\pi\hbar}} \left[e^{(i/\hbar)p(\cdot)}\right],$$

for every $p \in \mathbb{R}$, i.e., the $(\sqrt{2\pi\hbar}, -1/\hbar)$-Fourier family. Let $u \in \mathcal{S}'_1$, from the Fourier expansion theorem we have

$$u = \int_{\mathbb{R}} \mathcal{F}^-_{(a,b)}(u) v,$$

where $a = \sqrt{2\pi\hbar}$ and $b = -1/\hbar$, we have

$$\begin{aligned}
\mathcal{F}^-_{(a,b)} &= \mathcal{F}_{(2\pi/|b|a, -b)} = \\
&= \mathcal{F}_{(2\pi/(|-1/\hbar|\sqrt{2\pi\hbar}), 1/\hbar)} = \\
&= \mathcal{F}_{(\sqrt{2\pi\hbar}, 1/\hbar)} = \\
&= \mathcal{F}_{(a,-b)},
\end{aligned}$$

and hence

$$[u \mid v] = \mathcal{F}_{(\sqrt{2\pi\hbar}, 1/\hbar)}(u).$$

Moreover, let

$$P : \mathcal{S}'_1 \to \mathcal{S}'_1 : u \mapsto -i\hbar u'$$

be the momentum operator, we have

$$\begin{aligned}
[P(u)|\,v] &= [-i\hbar\partial(u)\mid v] = \\
&= \mathcal{F}_{(\sqrt{2\pi\hbar},1/\hbar)}(-i\hbar\partial(u)) = \\
&= -i\hbar\mathcal{F}_{(\sqrt{2\pi\hbar},1/\hbar)}(\partial(u)) = \\
&= -i\hbar(i/\hbar)^1\,(\cdot)^1\,\mathcal{F}_{(a,-b)}(u) = \\
&= (\cdot)\mathcal{F}_{(a,-b)}(u),
\end{aligned}$$

now let

$$X : \mathcal{S}'_1 \to \mathcal{S}'_1 : u \mapsto (\cdot)\,u$$

be the position operator, we have

$$\begin{aligned}
[P]_v &= [P\mid v]\circ[\cdot\mid v^-] = \\
&= (\cdot)\,\mathcal{F}_{(a,-b)}\circ\mathcal{F}_{(a,b)} = \\
&= \left(X\circ\mathcal{F}_{(a,-b)}\right)\circ\mathcal{F}_{(a,b)} = \\
&= X\circ\left(\mathcal{F}_{(a,-b)}\circ\mathcal{F}_{(a,b)}\right) = \\
&= X.
\end{aligned}$$

Example (the representation of the position operator in a Fourier basis). Let

$$X : \mathcal{S}'_1 \to \mathcal{S}'_1 : u \mapsto (\cdot)u$$

be the position operator, we have

$$[X]_v = [X\mid v]\circ[\cdot\mid v^-];$$

now

$$\begin{aligned}
[X(u)|\,v] &= \mathcal{F}_{(\sqrt{2\pi\hbar},1/\hbar)}(X(u)) = \\
&= \mathcal{F}_{(\sqrt{2\pi\hbar},1/\hbar)}((\cdot)\,u) = \\
&= \left(\frac{i}{1/\hbar}\right)^1\left(\mathcal{F}_{(\sqrt{2\pi\hbar},1/\hbar)}(u)\right)' = \\
&= -P\left(\mathcal{F}_{(\sqrt{2\pi\hbar},1/\hbar)}(u)\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
[X]_v &= -\left(P\circ\mathcal{F}_{(a,-b)}\right)\circ\mathcal{F}_{(a,b)} = \\
&= -P\circ\left(\mathcal{F}_{(a,-b)}\circ\mathcal{F}_{(a,b)}\right) = \\
&= -P = \\
&= -(-i\hbar D) = \\
&= i\hbar D.
\end{aligned}$$

Concluding
$$[X]_v = -P = i\hbar D.$$

Example (a representation of the kinetic energy operator). Let
$$T : \mathcal{S}_1' \to \mathcal{S}_1' : u \mapsto -\frac{\hbar^2}{2m} u''$$
be the kinetic energy operator of a nonrelativistic quantum particle. And let v the $(\sqrt{2\pi\hbar}, -1/\hbar)$-Fourier family. One has

$$\begin{aligned}
[T(u)|\, v] &= \mathcal{F}_{(\sqrt{2\pi\hbar},1/\hbar)}(T(u)) = \mathcal{F}_{(\sqrt{2\pi\hbar},1/\hbar)}\left(-\frac{\hbar^2}{2m}u''\right) = \\
&= -\frac{\hbar^2}{2m}(1/\hbar)^2 i^2 (\cdot)^2 \mathcal{F}_{(\sqrt{2\pi\hbar},1/\hbar)}(u) = \\
&= \frac{(\cdot)^2}{2m}\mathcal{F}_{(\sqrt{2\pi\hbar},1/\hbar)}(u) = \\
&= \frac{X^2}{2m} \circ \mathcal{F}_{(\sqrt{2\pi\hbar},1/\hbar)}(u).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
[T|\, v] \circ [\cdot|\, v^-] &= \left(\frac{X^2}{2m} \circ \mathcal{F}_{(\sqrt{2\pi\hbar},1/\hbar)}\right) \circ \mathcal{F}_{(\sqrt{2\pi\hbar},-1/\hbar)} = \\
&= \frac{X^2}{2m} \circ \left(\mathcal{F}_{(\sqrt{2\pi\hbar},1/\hbar)} \circ \mathcal{F}_{(\sqrt{2\pi\hbar},-1/\hbar)}\right) = \\
&= \frac{X^2}{2m} \circ (\cdot)_{\mathcal{S}_1'} = \\
&= \frac{X^2}{2m}.
\end{aligned}$$

Concluding
$$[T]_v = \frac{X^2}{2m}.$$

Part VI

$^{\mathcal{S}}$Spectral theory

Chapter 20

Multiplicative operators in $\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$

20.1 Introduction

In the Spectral Theory of ${}^{\mathcal{S}}$linear operators, the eigenvalues corresponding to the elements of certain ${}^{\mathcal{S}}$families have fundamental importance. If L is an ${}^{\mathcal{S}}$linear operator and v is an ${}^{\mathcal{S}}$family, the family v is defined an eigenfamily of the operator L if there exists a real or complex function l - defined on the set of indices of the family v - such that the relation

$$L(v_p) = l(p)v_p,$$

holds for every index p of the family v. As we already have seen, in the context of ${}^{\mathcal{S}}$linear operators, it is important how the operator L acts on the entire family v. Taking into account the above definition, it is natural to consider the image family $L(v)$ as the product - in pointwise sense - of the family v by the function l, but:

- *is this pointwise product a binary operation in the space of ${}^{\mathcal{S}}$families?*
- *what kind of properties are satisfied by this product?*

In this chapter we define and study the properties of such product.

20.2 $\mathcal{O}_M$ Functions

We recall, for convenience of the reader, some basic notions from theory of distributions.

Definition (of slowly increasing smooth function). *We denote by* $\mathcal{O}_M(\mathbb{R}^n, \mathbb{K})$*, or more simply by* $\mathcal{O}_M^{(n)}$*, the subspace of all smooth functions* f *belonging to the space* $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{K})$ *such that, for every test function* $\phi \in \mathcal{S}_n$ *the product* ϕf *lives in* $\mathcal{S}_n$*. The space* $\mathcal{O}_M(\mathbb{R}^n, \mathbb{K})$ *is said to be the* ***space of smooth functions from*** $\mathbb{R}^n$ ***into the field*** $\mathbb{K}$ ***slowly increasing at infinity with all their derivatives****.*

In other terms, the functions f in the space $\mathcal{O}_M^{(n)}$ are the only smooth functions which can be generate a multiplication operator

$$M_f : \mathcal{S}_n \to \mathcal{S}_n$$

of the space $\mathcal{S}_n$ into the space $\mathcal{S}_n$ itself, (obviously) by the relation $M_f(g) = fg$. This is the motivation of the importance of these functions, and the symbol itself $\mathcal{O}_M$ depends on this fact.

Proposition. *Let* $f \in \mathcal{E}_n$ *be a smooth function. Then the following conditions are equivalent:*

1) *for all multi-index* $p \in \mathbb{N}_0^n$ *there is a polynomial* P_p *such that, for any point* $x \in \mathbb{R}^n$*, the following inequality holds*

$$|\partial^p f(x)| \leq |P_p(x)| \,;$$

2) *for all test function* $\phi \in \mathcal{S}_n$ *the product* ϕf *lies in* $\mathcal{S}_n$*;*
3) *for every multi-index* $p \in \mathbb{N}_0^n$ *and for every test function* $\phi \in \mathcal{S}_n$ *the product* $(\partial^p f)\,\phi$ *is bounded in* $\mathbb{R}^n$*.*

20.2.1 Topology

The standard topology of the space $\mathcal{O}_M^{(n)}$ is the locally convex topology defined by the family of seminorms

$$\gamma_{\phi,p}(\phi) = \sup_{x \in \mathbb{R}^n} |\phi(x)\partial^p f(x)|$$

with $\phi \in \mathcal{S}_n$ and $p \in \mathbb{N}_0^n$. This topology does not have a countable basis. Also, it can be shown that the space $\mathcal{O}_M^{(n)}$ is a complete space. A sequence (or filter) $(f_j)_{j\in\mathbb{N}}$ converges to zero in $\mathcal{O}_M^{(n)}$ if and only if for every $\phi \in \mathcal{S}_n$ and for every $p \in \mathbb{N}_0^n$, the sequence of functions $(\phi\partial^p f_j)_{j\in\mathbb{N}}$ converges to zero uniformly on $\mathbb{R}^n$; or, equivalently, if, for every test function $\phi \in \mathcal{S}_n$, the sequence $(\phi f_j)_{j\in\mathbb{N}}$ converges to zero in $\mathcal{S}_n$.

20.2.2 Bounded sets in $\mathcal{O}_M^{(n)}$

A subset B of $\mathcal{O}_M^{(n)}$ is bounded (in the topological vector space $\mathcal{O}_M^{(n)}$) if and only if, for all multi-index $p \in \mathbb{N}_0^n$, there is a polynomial P_p such that, for any function $f \in B$, the following inequality holds true

$$|\partial^p f(x)| \leq P_p(x),$$

for any point $x \in \mathbb{R}^n$.

20.2.3 Multiplications by $\mathcal{O}_M^{(n)}$ functions

The bilinear map

$$\Phi : \mathcal{O}_M^{(n)} \times \mathcal{S}_n \to \mathcal{S}_n : (\phi, f) \mapsto \phi f$$

is separately continuous with respect to the usual topologies of the spaces $\mathcal{O}_M^{(n)}$ and $\mathcal{S}_n$. It follows immediately that the multiplication operator M_f, associated with an $\mathcal{O}_M$ function f, is continuous (with respect to the standard topology on $\mathcal{S}_n$). Moreover, the transpose of the operator M_f is the operator

$${}^tM_f : \mathcal{S}'_n \to \mathcal{S}'_n$$

defined by

$$\begin{aligned} {}^tM_f(u)(g) &= u(M_f(g)) = \\ &= u(fg) = \\ &= fu(g), \end{aligned}$$

for every u in $\mathcal{S}'_n$ and for every g in $\mathcal{S}_n$; so that, the transpose of the multiplication M_f is the multiplication on $\mathcal{S}'_n$ by the function f. Indeed, the multiplication of a tempered distribution by an $\mathcal{O}_M$ function is defined by the transpose of M_f, since this last operator is self-adjoint with respect to the canonical bilinear form on $\mathcal{S}_n \times \mathcal{S}_n$; in fact, obviously, we have

$$\langle M_f(g), h \rangle = \langle g, M_f(h) \rangle,$$

for every pair (g, h) in that Cartesian product.

20.2.4 $^{\mathcal{S}}$Family of the multiplication operator M_f

We can associate to the operator $M_f : \mathcal{S}_n \to \mathcal{S}_n$ an $^{\mathcal{S}}$family v, in the canonical way, we have

$$\begin{aligned} v_p &= (M_f^{\vee})_p = \\ &= \delta_p \circ M_f = \\ &= {}^t M_f(\delta_p) = \\ &= f\delta_p = \\ &= f(p)\delta_p, \end{aligned}$$

for every p in $\mathbb{R}^n$.

20.3 Product in $\mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$ by $^{\mathcal{O}_M}$functions

The basic remark is the following.

Proposition. *Let $A \in \mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$ be a continuous linear operator and let f be a function of class $\mathcal{O}_M^{(m)}$. Then, the mapping*

$$fA : \mathcal{S}_n \to \mathcal{S}_m : \phi \mapsto fA(\phi)$$

is a linear and continuous operator too, it is indeed the composition

$$M_f \circ A,$$

where M_f is the multiplication operator by the function f.

Proof. It is absolutely straightforward. First of all we note that the product fA is well defined. In fact, we have

$$(fA)(\phi) = fA(\phi),$$

and the right-hand function lies in the space $\mathcal{S}_m$ because the function f lies in the space $\mathcal{O}_M^{(m)}$ and the function $A(\phi)$ in the space $\mathcal{S}_m$. Moreover, the bilinear application

$$\Phi : \mathcal{O}_M^{(m)} \times \mathcal{S}_m \to \mathcal{S}_m : (f, \psi) \mapsto f\psi$$

is separately continuous and since we have

$$\begin{aligned} (fA)(\phi) &= fA(\phi) = \\ &= \Phi(f, A(\phi)) = \\ &= M_f(A(\phi)), \end{aligned}$$

i.e.,

$$\begin{aligned} fA &= \Phi(f, \cdot) \circ A = \\ &= M_f \circ A, \end{aligned}$$

hence the operator fA is the composition of two linear continuous maps and then is a linear and continuous operator. ■

Definition. *Let $A \in \mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$ and $f \in \mathcal{O}_M^{(m)}$. The operator*

$$fA : \mathcal{S}_n \to \mathcal{S}_m : \phi \mapsto fA(\phi)$$

*is called **the product of the operator** A **by the function** f.*

Proposition. *Let $A, B \in \mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$ be two continuous linear operators and f, g be two functions in $\mathcal{O}_M^{(m)}$. Then, we have*

1) $(f+g)A = fA + gA$; $f(A+B) = fA + fB$; $1_{\mathbb{R}^m} A = A$, where the function $1_{\mathbb{R}^m}$ is the constant function of $\mathbb{R}^m$ into $\mathbb{K}$ with value 1;

2) the map

$$\Phi : \mathcal{O}_M^{(m)} \times \mathcal{L}(\mathcal{S}_n, \mathcal{S}_m) \to \mathcal{L}(\mathcal{S}_n, \mathcal{S}_m) : (f, A) \mapsto fA$$

is a bilinear map.

Proof. It's a straightforward computation. ■

The above bilinear application is called *multiplication of operators by $\mathcal{O}_M$ functions.*

20.3.1 The ring $\mathcal{O}_M^{(m)}$

It's easy to see that the algebraic structure $(\mathcal{O}_M^{(m)}, +, \cdot)$ is a commutative ring with identity, with respect to the usual pointwise addition and multiplications. For instance, the multiplication is the operation

$$\cdot\ : \mathcal{O}_M^{(m)} \times \mathcal{O}_M^{(m)} \to \mathcal{O}_M^{(m)} : (f, g) \mapsto fg,$$

where, obviously, if $f, g \in \mathcal{O}_M^{(m)}$, then the pointwise product fg still lies in $\mathcal{O}_M^{(m)}$. The identity of the ring is the function $1_{\mathcal{O}_M} := 1_{\mathbb{R}^m}$. Moreover, we have that subspace $\mathcal{S}_m$ of the space $\mathcal{O}_M^{(m)}$ is an ideal of the ring $\mathcal{O}_M^{(m)}$. The subring of $\mathcal{O}_M^{(m)}$ formed by the invertible elements of $\mathcal{O}_M^{(m)}$ is exactly the multiplicative subgroup of those elements f such that the multiplicative inverse f^{-1} belongs to the space $\mathcal{O}_M^{(m)}$ too.

20.3.2 The module $\mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$

Proposition. *Let $\cdot$ be the product operation defined in the above theorem. Then, the algebraic structure $(\mathcal{L}(\mathcal{S}_n, \mathcal{S}_m), +, \cdot)$ is a left module over the ring $(\mathcal{O}_M^{(m)}, +, \cdot)$.*

Proof. Recall the preceding theorem, we have to prove only the pseudo-associative law, i.e. we have to prove that for every $f, g \in \mathcal{O}_M^{(m)}$, for every $A \in \mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$, we have

$$(fg)A = f(gA).$$

In fact, for each $\phi \in \mathcal{S}_n$, we have

$$\begin{aligned} [(fg)A](\phi) &= (fg)A(\phi) = \\ &= f(gA(\phi)) = \\ &= f(gA)(\phi)) = \\ &= [f(gA)](\phi), \end{aligned}$$

as we desired. ■

20.4 Product of $^{\mathcal{S}}$families by $^{\mathcal{O}_M}$functions

The central definition of the chapter is the following.

Definition (product of families by smooth functions). *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be an $^{\mathcal{S}}$family of distributions and let $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{K})$ be a smooth function. The **product of the family** v **by the function** f is the family*

$$fv := (f(p)v_p)_{p \in \mathbb{R}^m}.$$

Theorem. *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be an $^{\mathcal{S}}$family and $f \in \mathcal{O}_M^{(m)}$. Then, the family fv lies in $\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$. Moreover, we have*

$$(fv)^\wedge = f\widehat{v}.$$

Consequently, concerning the superposition operator of the family fv, since $f\widehat{v} = M_f \circ \widehat{v}$, we have

$$^t(fv)^\wedge = {}^t\widehat{v} \circ {}^tM_f,$$

or, equivalently,

$$\int_{\mathbb{R}^m} a(fv) = \int_{\mathbb{R}^m} (fa)v,$$

for every coefficient distribution a in $\mathcal{S}'_m$.

Proof. Let $\phi \in \mathcal{S}_n$, we have

$$\begin{aligned} (fv)(\phi)(p) &= (fv)_p(\phi) = \\ &= (f(p)v_p)(\phi) = \\ &= f(p)v_p(\phi) = \\ &= f(p)\widehat{v}(\phi)(p) \end{aligned}$$

and hence the function $(fv)(\phi)$ equals $f\widehat{v}(\phi)$ which lies in $\mathcal{S}_m$. Thus, the product fv lies in the space $\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$. For any test function $\phi \in \mathcal{S}_n$, by the above consideration, we deduce

$$(fv)^\wedge(\phi) = f\widehat{v}(\phi),$$

that is,

$$(fv)^\wedge = f\widehat{v},$$

where $f\widehat{v}$, is the product of the operator $\widehat{v}$ by the function f, which belongs to $\mathcal{L}(\mathcal{S}_m, \mathcal{S}_n)$. Moreover, concerning the superposition operator of the family fv,

$$\begin{aligned} \int_{\mathbb{R}^m} a(fv) &= {}^t(fv)^\wedge(a) = \\ &= ({}^t\widehat{v} \circ {}^tM_f)(a) = \\ &= {}^t\widehat{v}({}^tM_f(a)) = \\ &= {}^t\widehat{v}(fa) = \\ &= \int_{\mathbb{R}^m} (fa)v, \end{aligned}$$

for every a in $\mathcal{S}'_m$. ■

Theorem. *Let f, g two functions in the space $\mathcal{O}_M^{(m)}$ and v, w two families in the space $\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$. Then, we have*

1) $(f + g)v = fv + gv$, $f(v + w) = fv + fw$ and $1_{\mathcal{O}_M} v = v$.

2) The map

$$\Phi : \mathcal{O}_M^{(m)} \times \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n) \to \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n) : (f, v) \mapsto fv$$

is a bilinear map.

Proof. 1) For all $p \in \mathbb{R}^m$, we have

$$\begin{aligned} [(f + g)\, v]\,(p) &= (f + g)(p)v_p = \\ &= (f(p) + g(p))v_p = \\ &= f(p)v_p + g(p)v_p = \\ &= (fv)_p + (gv)_p, \end{aligned}$$

i.e. $(f+g)v = fv + gv$. For all $p \in \mathbb{R}^m$, we have

$$\begin{aligned} [f\,(v+w)]\,(p) &= f(p)(v+w)_p = \\ &= f(p)(v_p + w_p) = \\ &= f(p)v_p + f(p)w_p = \\ &= (fv)_p + (fw)_p, \end{aligned}$$

i.e. $f(v+w) = fv + fw$. For all $p \in \mathbb{R}^m$, we have

$$(1_{\mathbb{R}^m} v)(p) = 1_{\mathbb{R}^m}(p) v_p = v_p;$$

i.e. $1_{\mathbb{R}^m} v = v$. 2) follows immediately by 1). ■

The bilinear application of the point 2) of the preceding theorem is called *multiplication of families by $\mathcal{O}_M$ functions.*

Theorem (of structure). *Let $\cdot$ the operation defined above. Then, the algebraic structure $(\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n), +, \cdot)$ is a left module over the ring $(\mathcal{O}_M^{(m)}, +, \cdot)$.*

Proof. It's analogous to the proof of the corresponding proposition for operators. ■

Theorem (of isomorphism). *The application*

$$(\cdot)^\wedge : \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n) \to \mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$$

is a module isomorphism.

Proof. It follows easily from the above theorem. ■

20.5 $^{\mathcal{O}_M}$Functions and $^{\mathcal{S}}$ basis

In this section we study some relations among a family w and its multiples fw.

Theorem. *Let $w \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ and let $f \in \mathcal{O}_M^{(m)}$. Then, the hull $^{\mathcal{S}}\mathrm{span}(w)$ contains the hull $^{\mathcal{S}}\mathrm{span}(fw)$. Moreover, if a distribution a represent the distribution u in the family fw then the distribution fa represent the distribution u in the family w.*

Proof. 1) Let u be a vector of the $^{\mathcal{S}}$linear hull $^{\mathcal{S}}\mathrm{span}(fw)$. Then, there exists a coefficient distribution $a \in \mathcal{S}'_m$ such that

$$u = \int_{\mathbb{R}^m} a(fw),$$

and this is equivalent to the equality

$$u = \int_{\mathbb{R}^m} (fa)w;$$

hence the vector u belongs also to the ${}^{\mathcal{S}}$linear hull ${}^{\mathcal{S}}\mathrm{span}(w)$. Hence the ${}^{\mathcal{S}}$linear hull ${}^{\mathcal{S}}\mathrm{span}(fw)$ is contained in the ${}^{\mathcal{S}}$linear hull ${}^{\mathcal{S}}\mathrm{span}(w)$. ■

Theorem. *Let $w \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ and let $f \in \mathcal{O}_M^{(m)}$ be a function different from* 0 *at every point. Then, the following assertions hold true:*

1) *if the family w is ${}^{\mathcal{S}}$linearly independent, the family fw is ${}^{\mathcal{S}}$linearly independent too;*

2) *the hull ${}^{\mathcal{S}}\mathrm{span}(w)$ contains the hull ${}^{\mathcal{S}}\mathrm{span}(fw)$;*

3) *if the family w is ${}^{\mathcal{S}}$linearly independent, for each vector u in the hull ${}^{\mathcal{S}}\mathrm{span}(fw)$, we have*
$$[u \mid w] = f[u \mid fw];$$

4) *if the family w is an ${}^{\mathcal{S}}$basis of a subspace V, then fw is an ${}^{\mathcal{S}}$basis of its ${}^{\mathcal{S}}$linear hull ${}^{\mathcal{S}}\mathrm{span}(fw)$ (that in general is a proper subspace of ${}^{\mathcal{S}}\mathrm{span}(w)$).*

Proof. 1) Let $a \in \mathcal{S}'_m$ be such that

$$\int_{\mathbb{R}^m} a(fw) = 0_{\mathcal{S}'_n},$$

we have

$$\begin{aligned} 0_{\mathcal{S}'_n} &= \int_{\mathbb{R}^m} a(fw) = \\ &= \int_{\mathbb{R}^m} (fa)w, \end{aligned}$$

thus, because the family w is ${}^{\mathcal{S}}$linearly independent we have $fa = 0_{\mathcal{S}'_n}$. Since f is different from 0 at every point we can conclude $a = 0_{\mathcal{S}'_n}$.

2) Let u be a vector of the span ${}^{\mathcal{S}}\mathrm{span}(fw)$. Then, there exists a coefficient distribution $a \in \mathcal{S}'_m$ such that

$$u = \int_{\mathbb{R}^m} a(fw),$$

or equivalently

$$u = \int_{\mathbb{R}^m} (fa)w,$$

and hence the vector u belongs also to ${}^{\mathcal{S}}\mathrm{span}(w)$. Hence the span ${}^{\mathcal{S}}\mathrm{span}(fw)$ is contained in the span ${}^{\mathcal{S}}\mathrm{span}(w)$.

3) If w is ${}^{\mathcal{S}}$linearly independent, from the above two equalities we deduce $(u)_{fw} = a$ and $(u)_w = fa$, from which

$$\begin{aligned} (u)_w &= fa = \\ &= f(u)_{fw}, \end{aligned}$$

as we claimed. 4) is an obvious consequence of the preceding properties. ∎

20.6 ${}^{\mathcal{O}_M}$Invertible functions and ${}^{\mathcal{S}}$basis

We recall that an invertible element of $\mathcal{O}_M^{(m)}$ is any function f everywhere different from 0 and such that its multiplicative inverse f^{-1} lives in $\mathcal{O}_M^{(m)}$ too. The set of the invertible elements of the space $\mathcal{O}_M^{(m)}$ is a group with respect to the pointwise multiplication, and we will denote (sometimes) it by $\mathcal{G}_M^{(m)}$.

Theorem. *Let $w \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ and let $f \in \mathcal{G}_M^{(m)}$ be an invertible element of the ring $\mathcal{O}_M^{(m)}$ (in particular, it must be a function different form 0 at every point). Then, the following assertions hold true:*

1) *the family w is ${}^{\mathcal{S}}$ linearly independent if and only if the family fw is ${}^{\mathcal{S}}$linearly independent;*

2) *the hull ${}^{\mathcal{S}}\mathrm{span}(w)$ coincides with the hull ${}^{\mathcal{S}}\mathrm{span}(fw)$;*

3) *if the family w is ${}^{\mathcal{S}}$linearly independent, for each vector u in the hull ${}^{\mathcal{S}}\mathrm{span}(w)$, we have*
$$[u \mid fw] = (1/f)\,[u \mid w].$$

4) *if the family w is an ${}^{\mathcal{S}}$basis of a subspace V if and only if its multiple fw is an ${}^{\mathcal{S}}$basis of the ${}^{\mathcal{S}}$linear hull ${}^{\mathcal{S}}\mathrm{span}(fw)$ (that in this case coincides with ${}^{\mathcal{S}}\mathrm{span}(w)$).*

Proof. 1) Let $a \in \mathcal{S}'_m$ be such that

$$\int_{\mathbb{R}^m} aw = 0_{\mathcal{S}'_n},$$

we have

$$\begin{aligned} 0_{\mathcal{S}'_n} &= \int_{\mathbb{R}^m} aw = \\ &= \int_{\mathbb{R}^m} (f^{-1}a)(fw), \end{aligned}$$

thus, because fw is $^{\mathcal{S}}$linearly independent we have $f^{-1}a = 0_{\mathcal{S}'_n}$. Since f^{-1} is different form 0 at every point we can conclude $a = 0_{\mathcal{S}'_n}$.

2) Let u be in $^{\mathcal{S}}$span(w). Then, there exists an $a \in \mathcal{S}'_m$ such that

$$u = \int_{\mathbb{R}^m} aw.$$

Now, we have

$$u = \int_{\mathbb{R}^m} (f^{-1}a)\,(fw)\,,$$

so the distribution u lies in $^{\mathcal{S}}$span(fw), and hence $^{\mathcal{S}}$span(w) is contained in $^{\mathcal{S}}$span(fw). Vice versa, let u be in $^{\mathcal{S}}$span(fw). Then, there exists an $a \in \mathcal{S}'_m$ such that

$$u = \int_{\mathbb{R}^m} a(fw).$$

Now, we have

$$u = \int_{\mathbb{R}^m} (fa)w,$$

and hence u lies also in $^{\mathcal{S}}$span(w), hence $^{\mathcal{S}}$span(fw) is contained in $^{\mathcal{S}}$span(w). Concluding

$$^{\mathcal{S}}\mathrm{span}(w) = {}^{\mathcal{S}}\mathrm{span}(fw).$$

3) For any $u \in \mathcal{S}'_n$, we have

$$u = \int_{\mathbb{R}^m} [u \mid w]\, w,$$

hence

$$u = \int_{\mathbb{R}^m} (f^{-1}[u|w])\,(fw)\,,$$

as we desired. 4) It follows immediately from the above properties. ■

Theorem. *Let $e \in \mathcal{B}(\mathbb{R}^m, \mathcal{S}'_n)$ be an $^{\mathcal{S}}$basis of the space $\mathcal{S}'_n$ and let $f \in \mathcal{O}_M^{(m)}$. Then the multiple fe is an $^{\mathcal{S}}$basis of the space $\mathcal{S}'_n$ if and only if the factor f is an invertible element of the ring $\mathcal{O}_M^{(m)}$.*

Proof. We must prove that, if fe is an $^{\mathcal{S}}$basis of $\mathcal{S}'_n$, then f is an invertible element of the ring $\mathcal{O}_M^{(m)}$. First of all observe that, since fe is a basis, then fe is $^{\mathcal{S}}$linearly independent and consequently linearly independent in the ordinary algebraic sense; consequently every distribution $f(p)e_p$ must be a non zero distribution and this implies that any value $f(p)$ must be different from 0, so we can consider the multiplicative inverse f^{-1}. We now have to prove that f^{-1} lives in $\mathcal{O}_M^{(m)}$, or equivalently that, for every g in $\mathcal{S}_m$ the product $f^{-1}g$ lives in $\mathcal{S}_m$. To do this, let g be in $\mathcal{S}_m$, since fe is a basis, its associated operator from

$\mathcal{S}_n$ into $\mathcal{S}_m$ is surjective, then there is a function h in $\mathcal{S}_n$ such that $(fe)^\wedge(h) = g$, the last equality is equivalent to

$$fe(h) = g,$$

that is

$$f^{-1}g = e(h),$$

so that $f^{-1}g$ actually lives in the space $\mathcal{S}_m$. ■

Theorem. *Let $e \in \mathcal{B}(\mathbb{R}^m, V)$ be an $^{\mathcal{S}}$basis of a (weakly*) closed subspace V of the space $\mathcal{S}'_n$ and let $f \in \mathcal{O}_M^{(m)}$. Then the multiple family fe is an $^{\mathcal{S}}$basis of the subspace V if and only if the factor f is an invertible element of the ring $\mathcal{O}_M^{(m)}$.*

Proof. We must prove that, if fe is an $^{\mathcal{S}}$basis of the subspace V, then f is an invertible element of the ring $\mathcal{O}_M^{(m)}$. First of all observe that, since fe is a basis, then fe is $^{\mathcal{S}}$linearly independent and consequently linearly independent in the ordinary algebraic sense; consequently every distribution $f(p)e_p$ must be a non zero distribution and this implies that any value $f(p)$ must be different from 0. So we can consider its multiplicative inverse f^{-1} . We now have to prove that f^{-1} lives in $\mathcal{O}_M^{(m)}$, or equivalently that, for every g in $\mathcal{S}_m$ the product $f^{-1}g$ lives in $\mathcal{S}_m$. To do this, let g be in $\mathcal{S}_m$, since fe is an $^{\mathcal{S}}$ basis of a topologically closed subspace, its associated operator from $\mathcal{S}_n$ into $\mathcal{S}_m$ is surjective, then there is a function h in $\mathcal{S}_n$ such that $(fe)^\wedge(h) = g$, the last equality is equivalent to

$$fe(h) = g,$$

that is

$$f^{-1}g = e(h),$$

so that $f^{-1}g$ actually lives in the space $\mathcal{S}_m$. ■

Chapter 21

Spectral expansions

21.1 Spectral $^{\mathcal{S}}$expansions

In the following we shall use the notation $\mathcal{L}(\mathcal{S}'_n) = \mathcal{L}(\mathcal{S}'_n, \mathcal{S}'_n)$ for the space of $^{\mathcal{S}}$linear endomorphisms on the space $\mathcal{S}'_n$, it is just the space of continuous linear endomorphisms with respect to the weak* topology (or with respect to the strong* topology) on $\mathcal{S}'_n$.

Let E, F be two vector spaces and let A be a linear operator of E into F. The set of all the eigenvectors of the operator A is denoted by $\mathrm{E}(A)$. The set of all the eigenvalues of the operator A is denoted by $e(A)$; moreover the eigenspace relative to an eigenvalue $a \in \mathbb{K}$ is denoted by $|a\rangle_A$, or by $\mathrm{E}_a(A)$. For every eigenvector u of A, there is only one eigenvalue a such that $A(u) = au$, so that we can consider the projection $p : \mathrm{E}(A) \to e(A)$ associating with every eigenvector u of the operator its eigenvalue. It is clear that the set $\mathrm{E}_a^{\neq}(A)$, the eigenspace of A corresponding to the eigenvalue a without the zero vector, coincides with the fiber $p^{-}(a)$. So that we have constructed a fiber space $(\mathrm{E}(A), e(A), p)$.

Definition (of eigenfamily). *Let $A \in \mathcal{L}(\ \ \mathcal{S}'_n)$ be an $^{\mathcal{S}}$linear endomorphism on the space $\mathcal{S}'_n$, $a \in \mathcal{O}_M^{(m)}$ and $v \in \mathcal{S}(\ \mathbb{R}^m, \mathcal{S}'_n)$ be an $^{\mathcal{S}}$ family. We say that the family v is an $^{\mathcal{S}}$* ***eigenfamily of the operator** A **with respect***

to the system of eigenvalues *a if, for each index $p \in \mathbb{R}^m$, the vector v_p is an eigenvector of the operator A with respect to the eigenvalue $a(p)$. In other terms, the family v is an $^{\mathcal{S}}$eigenfamily of the operator A with respect to the system of eigenvalues a if, for each index $p \in \mathbb{R}^m$, we have*

$$A(v_p) = a(p)v_p,$$

which, in terms of families can be written as $A(v) = av$.

Now we can state and prove the principal theorem of this chapter.

Theorem (of $^{\mathcal{S}}$spectral expansion). *Let $A \in \mathcal{L}(\mathcal{S}'_n)$ be an $^{\mathcal{S}}$linear endomorphism, $a \in \mathcal{O}_M^{(m)}$ and let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be an $^{\mathcal{S}}$linearly independent eigenfamily of the operator A with respect to the system of eigenvalues a. Then, we have the spectral $^{\mathcal{S}}$ expansion*

$$A(u) = \int_{\mathbb{R}^m} a[u|v]v.$$

for each u in the $^{\mathcal{S}}$linear hull $^{\mathcal{S}}\mathrm{span}(v)$.

Proof. For each distribution u in the $^{\mathcal{S}}$linear hull $^{\mathcal{S}}\mathrm{span}(v)$, we have

$$\begin{aligned} A(u) &= A\left(\int_{\mathbb{R}^m}[u|v]v\right) = \\ &= \int_{\mathbb{R}^m}[u|v]A(v) = \\ &= \int_{\mathbb{R}^m}[u|v](av) = \\ &= \int_{\mathbb{R}^m}(a[u|v])\,v. \end{aligned}$$

In fact, the third equality holds because

$$\begin{aligned} A(v)_p &= A(v_p) = \\ &= a(p)v_p = \\ &= (av)(p), \end{aligned}$$

as we already have noted; and the fourth equality holds because

$$\begin{aligned} \left(\int_{\mathbb{R}^m}[u|v](av)\right)(\phi) &= [u|v]((av)^\wedge(\phi)) = \\ &= [u|v](a\widehat{v}(\phi)) = \\ &= (a[u|v])(\widehat{v}(\phi)) = \\ &= \int_{\mathbb{R}^m}(a[u|v])v, \end{aligned}$$

as we well know in the general case; this concludes the proof. ■

Remind. Recall the definition of superposition of an $^{\mathcal{S}}$family with respect to an operator. Let X be a subspace of the space $\mathcal{S}'_n$, $A \in \mathrm{Hom}(X, \mathcal{S}'_m)$ be a linear operator and $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be an $^{\mathcal{S}}$family of distributions. The superposition of the family v with respect to the operator A, is the operator

$$\int_{\mathbb{R}^m} Av : X \to \mathcal{S}'_n : u \mapsto \int_{\mathbb{R}^m} A(u)v.$$

Remark. So, in the conditions of the above theorem, we can write

$$A_{|X} = \int_{\mathbb{R}^n} a[.|v]v,$$

saying that the restriction of the operator A to the $^{\mathcal{S}}$ linear hull of the family v is a superposition of the family v with respect to the coordinate operator of the family.

Remark. The above theorem generalizes the Resolution of Identity theorem. Indeed, every $^{\mathcal{S}}$basis of the space $\mathcal{S}'_n$ is an $^{\mathcal{S}}$eigenfamily of the identity operator with respect to the constant unitary system $1_{\mathbb{R}^m}$, so that we have

$$(.)_{\mathcal{S}'_n} = \int_{\mathbb{R}^m} [.|v]v,$$

for every $^{\mathcal{S}}$basis v of the space $\mathcal{S}'_n$. Moreover, if j_X is the injection of the linear hull X of an $^{\mathcal{S}}$linearly independent family v, we have

$$j_X = \int_{\mathbb{R}^m} [.|v]v,$$

since j_X is just the restriction to X of the identity operator on the space $\mathcal{S}'_n$.

Remark. The above theorem holds in the particular case in which there exists a $^{\mathcal{S}}$basis of the space $\mathcal{S}'_n$ constituted by eigenvectors of the operator A. This case is the theme of the following chapter.

21.2 $^{\mathcal{S}}$Expansions and $^{\mathcal{S}}$ linear equations

Let A be an $^{\mathcal{S}}$linear operator on the space $\mathcal{S}'_n$ and let v be an $^{\mathcal{S}}$basis of the space $\mathcal{S}'_n$ such that $Av = av$, with a function of class $\mathcal{O}_M$. We desire to solve the $^{\mathcal{S}}$linear equation

$$E : A(.) = d,$$

with d in $\mathcal{S}'_n$.

Theorem. *Let A be an $^{\mathcal{S}}$linear operator on the space $\mathcal{S}'_n$ and let v be an $^{\mathcal{S}}$basis of the space $\mathcal{S}'_n$, indexed by the m-dimensional Euclidean space, such that $Av = av$, with a function of class $\mathcal{O}_M$. Then, the $^{\mathcal{S}}$linear equation*

$$E : A(.) = d,$$

with d in $\mathcal{S}'_n$, admits (at least) one solution if and only if the representation d_v, of the datum d in the $^{\mathcal{S}}$basis v, is divisible by the function a. In this case, a solution of the equation E is the representation of any quotient q, of the division of d_v by a, in the inverse basis of v, that is the superposition

$$\int_{\mathbb{R}^m} qv.$$

Proof. ($\Rightarrow$) Let u be a solution of the equation E. We have

$$A(u) = \int_{\mathbb{R}^m} a[u|v]v,$$

by the spectral $^{\mathcal{S}}$expansion theorem and

$$d = \int_{\mathbb{R}^m} [d|v]v,$$

by the definition of representation of d in the basis v. Since v is $^{\mathcal{S}}$linear independent, we obtain the eigen-representation of the equality $E(u)$, that is

$$a[u|v] = [d|v],$$

so that, if the distribution d_v is divisible by the function a, that is there exists a distribution q such that

$$aq = d_v.$$

($\Leftarrow$) It is also clear that, if the representation d_v is divisible by the function a, then any quotient q of the division of d_v by a is a solution of E. Indeed,

$$\begin{aligned} A\left(\int_{\mathbb{R}^m} qv\right) &= \int_{\mathbb{R}^m} qA(v) = \\ &= \int_{\mathbb{R}^m} q(av) = \\ &= \int_{\mathbb{R}^m} (aq)v = \\ &= \int_{\mathbb{R}^m} d_v v = \\ &= d, \end{aligned}$$

as we claimed. ■

We can see an interesting application.

Application (the Malgrange-Ehrenpreis theorem). We obtain, as a very particular case the Malgrange theorem, using the Hörmander division of a distribution by polynomials. First of all consider that the partial derivative ∂_i has the Fourier basis as an ${}^{\mathcal{S}}$ eigenfamily, indeed we have

$$\partial_i(e^{-i(p|.)}) = -ip_i e^{-i(p|.)},$$

for every positive integer i less than n. Consequently we have

$$\partial^j(e^{-i(p|.)}) = (-i)^{|j|} p^j e^{-i(p|.)},$$

for every multi-index j; thus a differential operator D with constant coefficients, say

$$D = \Sigma c_j \partial^j,$$

has the Fourier basis $v = (e^{-i(p|.)})_{p\in\mathbb{R}^m}$ as an ${}^{\mathcal{S}}$eigenbasis. If q is the quotient of the division of a distribution d by a polynomial $\Sigma(-i)^{|j|}c_j(.)^p$, the ${}^{\mathcal{S}}$linear equation

$$Du = q$$

has the solution

$$\int_{\mathbb{R}^m} qv,$$

by the above theorem, and this is exactly what the Malgrange theorem says.

21.3 Existence of Green families

Theorem. *Let $L \in \mathcal{L}(\mathcal{S}'_n)$ be an ${}^{\mathcal{S}}$linear operator. Let λ be an ${}^{\mathcal{S}}$eigenfamily of the operator L with corresponding eigenvalue system l, i.e. let the equality*

$$L(\lambda_p) = l(p)\lambda_p,$$

hold true, for every p in the index set, say I, of the family λ. Assume that

- *there is another ${}^{\mathcal{S}}$family μ such that the Dirac family of the space $\mathcal{S}'_n$ can be factorized as*

  $$\mu.\lambda = \delta,$$

  in other terms assume that the family λ has an ${}^{\mathcal{S}}$left inverse with respect to the product of families;

- *the function l is an ${}^{\mathcal{O}_M}$ function and nowhere zero and its inverse l^{-1} is of class $\mathcal{O}_M$, that is it is an element of the group of invertible elements of the ring $\mathcal{O}_M$.*

Then, the operator L has an ${}^{\mathcal{S}}$Green family, namely the family G defined by

$$G_p = \int_{\mathbb{R}^n} (1/l)\mu_p \lambda,$$

for every index p in I.

Proof. Indeed, for every index p, we have

$$\begin{aligned}
L(G_p) &= L\left(\int_{\mathbb{R}^n} (1/l)\mu_p \lambda\right) = \\
&= \int_{\mathbb{R}^n} (1/l)\mu_p L\lambda = \\
&= \int_{\mathbb{R}^n} (1/l)\mu_p L\lambda = \\
&= \int_{\mathbb{R}^n} (1/l)\mu_p l\lambda = \\
&= \int_{\mathbb{R}^n} l(1/l)\mu_p \lambda = \\
&= \int_{\mathbb{R}^n} \mu_p \lambda = \\
&= \delta_p,
\end{aligned}$$

as we claimed. ■

The above assumptions imply that the family λ is a system of ${}^{\mathcal{S}}$generators for $\mathcal{S}'_n$ and that the family μ is ${}^{\mathcal{S}}$linearly independent. In the particular case in which $\mu.\mu$ is the factorization of the Dirac basis we deduce that the family μ must be a basis too.

Let us generalize the preceding result.

Theorem. *Let $L \in \mathcal{L}(\mathcal{S}'_n)$ be an ${}^{\mathcal{S}}$linear operator with an ${}^{\mathcal{S}}$eigenfamily λ and corresponding eigenvalue system l. Assume that*

- *there is another ${}^{\mathcal{S}}$family μ such that*

$$\mu.\lambda = \delta,$$

- *any member of the family μ is divisible by the function l, that is there is a family ν of distributions such that*

$$l\nu_p = \mu_p,$$

for every index p of I (ν_p is the quotient of the division of μ_p by l).

Then,

- *the operator L has a Green family, namely the family G defined by*

$$G_p = \int_{\mathbb{R}^n} \nu_p \lambda,$$

for every index p.

- *If, moreover, the family ν is of class $\mathcal{S}$, the operator L has an $^{\mathcal{S}}$Green family, namely the family defined by the product of $^{\mathcal{S}}$families $G = \nu.\lambda$.*

Proof. Indeed, for every index p, we have

$$\begin{aligned} L(G_p) &= L\left(\int_{\mathbb{R}^n} \nu_p \lambda\right) = \\ &= \int_{\mathbb{R}^n} \nu_p L\lambda = \\ &= \int_{\mathbb{R}^n} \nu_p L\lambda = \\ &= \int_{\mathbb{R}^n} \nu_p (l\lambda) = \\ &= \int_{\mathbb{R}^n} (l\nu_p)\lambda = \\ &= \int_{\mathbb{R}^n} \mu_p \lambda = \\ &= \delta_p, \end{aligned}$$

as we claimed. ■

Open problem (the position operator). We know that, on the real line, the product of the identity mapping (.) by the Dirac distribution δ_0 is the zero distribution, i.e.

$$(.)\delta_0 = 0_{\mathcal{S}'_1};$$

then, by derivation, we deduce

$$\delta_0 + (.)\delta'_0 = 0,$$

so the Dirac distribution centered at 0 is divisible by the identity mapping (.) on the real line and the quotient of this division is the distribution $-\delta'_0$. Consider, now, the position operator on the real line $P : \mathcal{S}'_1 \to \mathcal{S}'_1$, defined by $P(u) = (.)u$. We have

$$(.)\nu_p = \delta_p,$$

where the distribution ν_p is

$$\nu_p = p^{-1}(\delta_p - h(p)\delta_0),$$

h in $\mathcal{D}'_1$ with $h(0) = 1$ and $h'(0) = 0$, for every real p different from 0, and $\nu_0 = -\delta'_0$. So we can apply the preceding result and deduce that there is a Green family of P, namely the family ν itself, since

$$G_p = \int_{\mathbb{R}^n} \nu_p \delta = \nu_p.$$

Is the family ν of class $\mathcal{S}$? Indeed, we have

$$\nu(g)(p) = \nu_p(g) = p^{-1}(g(p) - h(p)g(0)),$$

for every p different from 0, and

$$\nu(g)(0) = -\delta'_0(g) = g'(0).$$

Is the function $\nu(g)$ of class $\mathcal{S}$? So that the family G is an ${}^{\mathcal{S}}$Green family?

21.4 Superpositions in $\mathcal{O}_M$

Let $f = (f_q)_{q\in\mathbb{R}^k}$ be a family in the space $\mathcal{O}_M^{(n)}$, we say that f is an ${}^{\mathcal{S}}$family in $\mathcal{O}_M^{(n)}$ iff for every tempered distribution u in the product $\mathcal{S}_n\mathcal{S}'_n$, the function

$$f(u) : \mathbb{R}^k \to K : f(u)(q) = u(f_q),$$

is an $\mathcal{S}$ function. This is equivalent to say that the family

$$u(f) = (u(f_q))_{q\in R^k}$$

is a family of class.

In Particular, for every x in R^n, the scalar family

$$f(x) := (f_q(x))_{q\in\mathbb{R}^k} = \delta_x(f)$$

is a scalar family of class $\mathcal{S}$, canonically identified with its test function $f(\delta_x)$.

Let c be a distribution in $\mathcal{S}'_k$, we define

$$\left(\int_{\mathbb{R}^k} cf\right)(x) = \int_{\mathbb{R}^k} cf(x),$$

for every x in R^n.

Let us see that the superposition lives in $\mathcal{O}_M$. It follows form the Banach Steinhaus theorem. We have

$$\begin{aligned}
\left(\int_{\mathbb{R}^k} cf\right) g(x) &= \left(\int_{\mathbb{R}^n} cf(x)\right) g(x) = \\
&= c(f(x))g(x) = \\
&= c(g(x)f(x)) = \\
&= \left(\int_{\mathbb{R}^n} cgf\right)(x),
\end{aligned}$$

where the family of functions $gf = (g(x)f(x))_{x\in R^n}$ (a simple pointwise product) is of class S, indeed it is a family of $^{\mathcal{S}}$functions and moreover for every distribution c, the function

$$\begin{aligned}
x &\mapsto c(g(x)f(x)) \\
&= g(x)c(f(x)) = \\
&= gc(f)(x),
\end{aligned}$$

since the function $c(f)$ is $\mathcal{S}_n$.

Chapter 22

$^{\mathcal{S}}$Diagonalizable operators

22.1 $^{\mathcal{S}}$Diagonalizable operators $\mathcal{S}'_n$

Definition (of $^{\mathcal{S}}$diagonalizable operator). *Let $A \in \mathcal{L}(\mathcal{S}'_n)$ be an $^{\mathcal{S}}$linear endomorphism of the space $\mathcal{S}'_n$. The operator A is said **$^{\mathcal{S}}$diagonalizable** if there are a function $a \in \mathcal{O}_M^{(m)}$ and an $^{\mathcal{S}}$basis $\alpha \in \mathcal{B}(\mathbb{R}^m, \mathcal{S}'_n)$ such that, for every index $p \in \mathbb{R}^m$, the vector α_p is an eigenvector of the operator A with respect to the eigenvalue $a(p)$, that is if*

$$A(\alpha_p) = a(p)\alpha_p.$$

*In terms of families, we can also write $A(\alpha) = a\alpha$. Moreover, in these conditions, the basis α is said **an eigenbasis of the operator** and the function a is said **the system of eigenvalues of the operator A associated with the eigenbasis α**.*

Note that an $^{\mathcal{S}}$eigenbasis of an operator determines uniquely a system of eigenvalues.

The origin of the preceding definition and nomenclature is naturally explained by the following proposition.

Theorem. *Let α be an $^{\mathcal{S}}$ eigenbasis of an $^{\mathcal{S}}$diagonalizable operator A, and let a be the system of eigenvalues of the operator A with respect to the $^{\mathcal{S}}$basis α. Then, the $^{\mathcal{S}}$matrix representation of the operator A in the $^{\mathcal{S}}$basis α is*

$$(A)_{\alpha} = a\delta,$$

where δ is the identity $^{\mathcal{S}}$matrix, i.e. the Dirac family.

Proof. Recall that the $^{\mathcal{S}}$matrix representation of an operator L in an $^{\mathcal{S}}$basis v is the unique $^{\mathcal{S}}$ family $(L)_v$ such that the transpose product of the $^{\mathcal{S}}$ family $(L)_v$ by the coefficient distribution $(u)_v$, that is

$$(L)_v(u)_v = (L(u))_v,$$

for every distribution u in $\mathcal{S}'_n$. We have

$$\begin{aligned}(a\delta)(u)_\alpha &= a\delta(u)_\alpha = \\ &= a(u)_\alpha = \\ &= (au)_\alpha = \\ &= (Au)_\alpha,\end{aligned}$$

for every distribution u, as we desired. ■

Definition (of $^{\mathcal{S}}$diagonal matrix). *We call an $^{\mathcal{S}}$matrix (that is an $^{\mathcal{S}}$ family) $^{\mathcal{S}}$diagonal iff it is of the form $a\delta$ for some $^{\mathcal{O}_M}$function a, where δ is the Dirac basis.*

In other words, we can give the definition of $^{\mathcal{S}}$diagonalizable operator as it follows:

- *an $^{\mathcal{S}}$linear operator is said $^{\mathcal{S}}$diagonalizable if and only if there exists an $^{\mathcal{S}}$basis of the space $\mathcal{S}'_n$ in which the $^{\mathcal{S}}$matrix representation of the operator is $^{\mathcal{S}}$diagonal.*

We note, moreover, that in the definition of $^{\mathcal{S}}$diagonalizable operator is not necessary to assume the function a of any class.

Theorem. *Let A be an $^{\mathcal{S}}$linear endomorphism of $\mathcal{S}'_n$. Assume that there are a function a and an $^{\mathcal{S}}$basis α such that $A(\alpha) = a\alpha$. Then the function a is an $^{\mathcal{O}_M}$function.*

Proof. Indeed, we have, for any test function g,

$$A(\alpha_p)(g) = a(p)\alpha_p(g),$$

that is

$$A(\alpha)^\wedge(g) = a\alpha^\wedge(g),$$

now, since the operator associated to an ${}^{\mathcal{S}}$basis is surjective, there is a test function g such that the smooth function $\alpha^\wedge(g)$ is nowhere zero, so that the function a is a pointwise quotient of two smooth functions

$$a = \frac{A(\alpha)^\wedge(g)}{\alpha^\wedge(g)},$$

and thus is a smooth function too. Moreover, since the operator $\alpha^\wedge$ is surjective and since both the functions $A(\alpha)^\wedge(g)$ and $\alpha^\wedge(g)$ are of class $\mathcal{S}$, any product ah of the smooth function a by an ${}^{\mathcal{S}}$function is an ${}^{\mathcal{S}}$function, and hence a is a function of class $\mathcal{O}_M$. ■

Note that, by the Dieudonné-Schwartz theorem, the preceding proof works also in the case in which the family of eigenvectors of the operator A is an ${}^{\mathcal{S}}$basis of a weakly* closed (or equivalently strongly closed) subspace of the space $\mathcal{S}'_n$.

22.2 ${}^{\mathcal{S}}$Diagonalizable operators on $\mathcal{S}_n$

We now pass to another characterization of the ${}^{\mathcal{S}}$ diagonalizable operators on $\mathcal{S}'$. To give it in a more complete way, we define the ${}^{\mathcal{S}}$diagonalizable operators on $\mathcal{S}_n$.

Definition (of ${}^{\mathcal{S}}$diagonalizable operator in $\mathcal{S}_n$). *Let A be an operator in $\mathcal{L}(\mathcal{S}_n)$ we say A ${}^{\mathcal{S}}$diagonalizable if there is an invertible linear continuous operator L in $\mathcal{L}(\mathcal{S}_n)$ such that*

$$LAL^{-1} = a(.)_{\mathcal{S}_n},$$

for some ${}^{\mathcal{O}_M}$function a, where we denoted the composition by the multiplicative notations.

Definition (of ${}^{\mathcal{S}}$diagonal operator in $\mathcal{S}_n$). *We say that an endomorphism A of the space $\mathcal{S}_n$ is ${}^{\mathcal{S}}$diagonal if it is of the type $a(.)_{\mathcal{S}_n}$ for some $\mathcal{O}_M$ function a.*

In other terms, recalling the definition of similitude among linear continuous operators and the definition of ${}^{\mathcal{S}}$diagonal operator, we can reformulate the definition saying that

- *an operator is diagonalizable if it is similar with a diagonal operator.*

Now we present the characterization, whose proof is trivial.

Theorem. *Let A be an operator in $\mathcal{L}(\mathcal{S}'_n)$. Then the following assertions are equivalent:*

- *the operator A is ${}^{\mathcal{S}}$ diagonalizable;*
- *the operator ${}^tA$ is ${}^{\mathcal{S}}$ diagonalizable;*
- *there are an ${}^{\mathcal{S}}$basis α and a smooth function a such that*
$$\widehat{\alpha} \circ {}^tA \circ \widehat{\alpha}^{-1} = a(.)_{\mathcal{S}_n};$$
- *there are an ${}^{\mathcal{S}}$basis α and a smooth function a such that*
$${}^t\widehat{\alpha}^{-1} \circ A \circ {}^t\widehat{\alpha} = a(.)_{\mathcal{S}'_n};$$
- *there are an ${}^{\mathcal{S}}$basis α and a smooth function a such that*
$$A \circ {}^t\widehat{\alpha} = {}^t\widehat{\alpha} \circ a(.)_{\mathcal{S}'_n};$$
- *there are an ${}^{\mathcal{S}}$basis α and a smooth function a such that*
$$[A, {}^t\widehat{\alpha}] = {}^t\widehat{\alpha} \circ (a(.)_{\mathcal{S}'_n} - A);$$
- *there are an ${}^{\mathcal{S}}$basis α and a smooth function a such that the following commutation relation does hold*
$$[A, \int_{\mathbb{R}^n} (., \alpha)] = \int_{\mathbb{R}^n} (a(.)_{\mathcal{S}'_n} - A)\alpha;$$
- *there are an invertible linear continuous operator L in $\mathcal{L}(\mathcal{S}'_n)$ and a smooth function a such that*
$$LAL^{-1} = a(.)_{\mathcal{S}'_n}.$$

22.3 Algebra of ${}^{\mathcal{S}}$diagonalizable operators

In this section we shall study the natural algebra in the space of ${}^{\mathcal{S}}$diagonalizable operators. To understand better this algebra we introduce the notion of multifiberspace.

Let E and B be two non-empty set and $p : E \to B$ be a surjective correspondence (not necessarily univocal. The triple (E, B, p) is said a linear multifiber space if, there is a vector space F such that there is a bijection

$$h : B \times F \to E$$

such that

$$p(h(b,f)) = b,$$

for each for every b in B, and f in F. In other terms,

$$p \circ h = \mathrm{pr}_1,$$

where pr_1 is the first projection of the Cartesian product $B \times F$. The non-empty set E is said the underlying set of the fiber space; the non-empty set B is called the base of the fiber space; the vector space F is said the linear space of the fiber space.

Consider an $^{\mathcal{S}}$diagonalizable operator A, then there is a pair (α, a) in the Cartesian product $\mathcal{B}(\mathcal{S}'_n) \times \mathcal{O}_M^{(n)}$ such that $A(\alpha) = a\alpha$. So we have a natural projection

$$\pi : \mathcal{B}(\mathcal{S}'_n) \times \mathcal{O}_M^{(n)} \to \mathcal{D}(\mathcal{S}'_n)$$

associating with any pair (α, a) a unique diagonalizable operator A. This correspondence is not bijective and the anti-image of the operator $A = \pi(\alpha, a)$ is the set of all pairs (α', a') such that the relation

$$a[u|\alpha] = \int_{\mathbb{R}^n} a'[u|\alpha'][\alpha'|\alpha]$$

holds true, for every tempered distribution u.

Proof.

is the set the set of basis for which there is a function satisfying that property.

Fix a basis e and consider the section

$$\pi(e, .) : \mathcal{O}_M^{(n)} \to \mathcal{D}(\mathcal{S}'_n).$$

The image of the above section is the set $\mathcal{D}_e(\mathcal{S}'_n)$ of all diagonalizable operators having e as an eigenbasis.

This section is injective.

Proof. The system E of eigenvalues is univocally determined by e. ■

So we have a bijection j_e of $\mathcal{D}_e(\mathcal{S}'_n)$ onto $\mathcal{O}_M^{(n)}$.

Proposition. *If A and B have the same eigenbasis e with eigenvalue systems a and b. Then*

- *the linear combination $cA + dB$ is in $\mathcal{D}_e(\mathcal{S}'_n)$ and*

$$(cA + dB)(e) = (ca + db)e;$$

- *the composition AB is in $\mathcal{D}_e(\mathcal{S}'_n)$ and*

$$ABe = BAe = (ab)e;$$

- *the operator A is invertible if and only if eigenvalue system a is invertible in $\mathcal{O}_M^{(n)}$ and*

$$A^{-1}e = a^{-1}e$$

- *the power A^r is in $\mathcal{D}_e(\mathcal{S}'_n)$, for every integer r, and*

$$A^r e = a^r e.$$

Let us see that the algebra $\mathcal{D}_e(\mathcal{S}'_n)$ is indeed stable under ${}^{\mathcal{S}}$linear combinations.

Reminder. If A is a family of ${}^{\mathcal{S}}$linear endomorphism of $\mathcal{S}'_n$ indexed by the k-dimensional Euclidean space, we say that A is of class $\mathcal{S}$ if, for every tempered distribution u in $\mathcal{S}'_n$, the family $A(u)$ image of u under the family A, that is the family $(A_q(u))_{q\in\mathbb{R}^k}$, is of class $\mathcal{S}$.

Moreover, if $v = (v_q)_{q\in\mathbb{R}^k}$ is a family, indexed by the k -dimensional Euclidean space, of ${}^{\mathcal{S}}$families, each of one indexed by the m-dimensional Euclidean space, we say that the family is of class $\mathcal{S}$ if the family $v(p) = (v_q(p))_{q\in\mathbb{R}^k}$ is of class $\mathcal{S}$, for every p in $\mathbb{R}^m$, where $v_q(p)$ is the p-term of the family v_q. In this conditions, if a is a distribution in $\mathcal{S}'_k$, we define the superposition

$$\int_{\mathbb{R}^k} av$$

as the family

$$\left(\int_{\mathbb{R}^k} av\right)_p = \int_{\mathbb{R}^k} av(p),$$

for every index p in $\mathbb{R}^m$.

Our aim is to prove that:

Theorem. *If $H = (H_q)_{q\in R^k}$ is an ${}^{\mathcal{S}}$family in the algebra $\mathcal{D}_e(\mathcal{S}'_n)$, then every its superposition is still in $\mathcal{D}_e(\mathcal{S}'_n)$. In other terms the algebra $\mathcal{D}_e(\mathcal{S}'_n)$ is ${}^{\mathcal{S}}$stable. In particular, if c is a coefficient distribution in $\mathcal{S}'_n$, we have*

$$\left(\int_{\mathbb{R}^k} cH\right)(e_p) = \left(\int_{\mathbb{R}^k} cE(p)\right) e_p,$$

for every p in $\mathbb{R}^m$, where the ${}^{\mathcal{O}_M}$function E_q is the eigenvalue system of the operator H_q, so that

$$H_q(e) = E_q e.$$

In terms of families, we have

$$\left(\int_{\mathbb{R}^k} cH\right)(e) = \left(\int_{\mathbb{R}^k} cE\right) e,$$

where

$$\int_{\mathbb{R}^k} cE$$

is the ${}^{\mathcal{O}_M}$*function defined by*

$$\left(\int_{\mathbb{R}^k} cE\right)(p) = \int_{\mathbb{R}^k} cE(p),$$

for every p in $\mathbb{R}^m$.

Proof. Let c in $\mathcal{S}'_k$. We have

$$\begin{aligned} \left(\int_{\mathbb{R}^k} cH\right)(e_p) &= \int_{\mathbb{R}^k} cH(e_p) = \\ &= \int_{\mathbb{R}^k} c(E(p)e_p) = \\ &= \left(\int_{\mathbb{R}^k} cE(p)\right) e_p, \end{aligned}$$

indeed observe that the family

$$H(e_p) = (H_q(e_p))_{q\in R^k},$$

that is an ${}^{\mathcal{S}}$family in $\mathcal{S}'_n$, is nothing but the family

$$E(p)e_p = (E_q(p)e_p)_{q\in R^k}.$$

We have only to prove the last equality

$$\int_{\mathbb{R}^k} c(E(p)e_p) = \left(\int_{\mathbb{R}^k} cE(p)\right) e_p,$$

where the right-hand side is the product of the number

$$\int_{\mathbb{R}^k} cE(p)$$

by the distribution e_p; namely, the above complex number is the superposition of the scalar family $E(p) = (E_q(p))_{q\in\mathbb{R}^k}$ by the coefficient distribution c. In fact, identifying canonically the family $E(p)$ with its associated test function we have

$$\begin{aligned}\int_{\mathbb{R}^k} c(E(p)e_p)(g) &= c((E(p)e_p)(g)) = \\ &= c(E(p)e_p(g)) = \\ &= c(E(p))e_p(g) = \\ &= \left(\left(\int_{\mathbb{R}^k} cE(p)\right) e_p\right)(g),\end{aligned}$$

for every g in the test function g in $\mathcal{S}_n$; note that

$$\begin{aligned}(E(p)e_p)(g)(q) &= (E_q(p)e_p)(g) = \\ &= E_q(p)e_p(g) = \\ &= (e_p(g)E(p))(q),\end{aligned}$$

for every q in $\mathbb{R}^k$, so that the proof is completely done. ■

22.4 Building some observables of QM

22.4.1 The position operator in one dimension

A particle moving on the real line can be in a state in which its position is $x \in \mathbb{R}$. It's natural to assume that this state can be represented by the distribution δ_x, so if we denote by Q the observable "position" we have $Q\delta_x = x\delta_x$, i.e., $Q\delta = \mathbb{I}_{\mathbb{R}}\delta$, applying the above theorem we have

$$\begin{aligned}Q(u) &= \int_{\mathbb{R}} \mathbb{I}_{\mathbb{R}} [u \mid \delta]\, \delta = \\ &= \int_{\mathbb{R}} (\mathbb{I}_{\mathbb{R}} u)\, \delta = \\ &= \mathbb{I}_{\mathbb{R}} u.\end{aligned}$$

This justifies the definition of the position operator, which is now possible to define, more naturally, the only observable that in the state δ_x assume the value x.

22.4.2 The position operator in three dimensions

A particle moving in the space can be in a state in which its position is the vector $x \in \mathbb{R}^3$. It's natural to assume that this state can be represented by the distribution δ_x. In this state the position has the three components x_1, x_2, x_3.

Then, if we denote by $Q = (Q_1, Q_2, Q_3)$ the triple of operators representing the observable "position" in three dimensions, we have $Q\delta_x = (x_1\delta_x, x_2\delta_x, x_3\delta_x)$, i.e., $Q\delta = (\mathbb{I}_1\delta, \mathbb{I}_2\delta, \mathbb{I}_3\delta)$. Let us apply the decomposition theorem to the i-th component, we have

$$\begin{aligned} Q_i(u) &= \int_{\mathbb{R}^3} \mathbb{I}_i [u \mid \delta] \delta = \\ &= \int_{\mathbb{R}^3} (\mathbb{I}_i u) \delta = \\ &= \mathbb{I}_i u. \end{aligned}$$

This justifies the definition of the position operator, which is now possible to define, more naturally, the only observable that in the state δ_x assume the vector-value x.

22.4.3 The momentum operator

Following De Broglie, we assume that the state of a particle moving on the real line with momentum $p \in \mathbb{R}$ be represented by the regular distribution $\left[e^{(i/\hbar)(p|\cdot)}\right]$. If we denote by P the observable "momentum", we have

$$P\left[e^{(i/\hbar)(p|\cdot)}\right] = p\left[e^{(i/\hbar)(p|\cdot)}\right].$$

Putting $f = \left(\left[e^{(i/\hbar)(p|\cdot)}\right]\right)_{p\in\mathbb{R}}$, we have thus

$$Pf = \mathbb{I}_{\mathbb{R}} f.$$

Applying the above theorem, we have

$$\begin{aligned} P(u) &= \int_{\mathbb{R}} \mathbb{I}_{\mathbb{R}} [u \mid f] f = \\ &= \left(\frac{i}{-1/\hbar}\right)^1 \left(\int_{\mathbb{R}} [u \mid f] f\right)' = \\ &= -i\hbar u'. \end{aligned}$$

22.4.4 The kinetic energy in dimension 1

Following De Broglie, we assume that the state of a particle moving on the real line with momentum $p \in \mathbb{R}$ be represented by the regular distribution

$\left[e^{(i/\hbar)(p|\cdot)}\right]$. If we denote by T the observable "Hamiltonian of a classic free particle in $\mathbb{R}$", we have

$$T\left[e^{(i/\hbar)(p|\cdot)}\right] = \frac{p^2}{2m}\left[e^{(i/\hbar)(p|\cdot)}\right].$$

Putting $f = \left(\left[e^{(i/\hbar)(p|\cdot)}\right]\right)_{p\in\mathbb{R}}$, we have

$$Tf = \frac{p^2}{2m}f.$$

Then, applying the above theorem, we have

$$\begin{aligned}
T(u) &= \int_{\mathbb{R}} \frac{(\mathbb{I}_{\mathbb{R}})^2}{2m}[u \mid f]\, f = \\
&= \frac{1}{2m}\int_{\mathbb{R}} (\mathbb{I}_{\mathbb{R}})^2 [u \mid f]\, f = \\
&= \frac{1}{2m}\left(\frac{i}{-1/\hbar}\right)^2 \left(\int_{\mathbb{R}} [u \mid f]\, f\right)'' = \\
&= -\frac{\hbar^2}{2m}u''.
\end{aligned}$$

Note that the spectrum of T is the set of non-negative real numbers and that the dimension of every eigenspace is 2.

22.5 Observables

Actually, the spectral theory treated on this chapter requires only the concept of $\mathcal{S}$-diagonalizable operator, because the spectral decomposition concerns the $\mathcal{S}$-diagonalizable operators. Nevertheless, for completeness, we give the definition of $\mathcal{S}$ -observable, that is a particular $\mathcal{S}$-diagonalizable operator

Definition (of observable with a continuous range of fundamental eigenstates). *Let $A \in \mathcal{L}(\mathcal{S}_n')$. The operator A is said to be an* ***observable with a continuous range of fundamental eigenstates (or an observable with an ${}^{\mathcal{S}}$eigenbasis or more simply an ${}^{\mathcal{S}}$observable)*** *if it is $\mathcal{S}$-diagonalizable and it is the extension of an adjointable operator on $\mathcal{S}_n$.*

For adjointable operator on $\mathcal{S}_n$, we give the following definition:

- a strongly continuous endomorphism $A : \mathcal{S}_n \to \mathcal{S}_n$ is said to be *adjointable* if there is another strongly continuous endomorphism $B : \mathcal{S}_n \to \mathcal{S}_n$ such that
$$\langle Ax|y\rangle = \langle x|By\rangle\,,$$

for every x and y in $\mathcal{S}_n$. In the above conditions the operator B is uniquely determined and it is denoted by $A^{\dagger}$. Moreover, it is possible to prove that an adjointable operator A is extendible to an $\mathcal{S}$-linear operator on $\mathcal{S}'_n$.

The most important kind of $^{\mathcal{S}}$observable is the following one. An adjointable operator $A : \mathcal{S}_n \to \mathcal{S}_n$ is said to be symmetric or Hermitian if $A^{\dagger} = A$.

- *If an $\mathcal{S}$-observable $A \in \mathcal{L}(\mathcal{S}'_n)$ is the extension of a symmetric operator it is said a real $\mathcal{S}$-observable.*

22.5.1 Observables with a singular spectrum

If we regard the constant function of value c as an observable: $M_c(u) = cu$, we have that M_c has c as unique eigenvalue. On the other hand, every $\mathcal{S}$-basis is an $\mathcal{S}$ -eigenbasis of M_c. So M_c is an observable with a continuous range of fundamental eigenstates but with a pointwise spectrum. Now, let v an arbitrary $\mathcal{S}$-basis of the space, we have

$$\begin{aligned} M_c(u) &= cu = \\ &= c\int_{\mathbb{R}^n} [u \mid v]\, v = \\ &= \int_{\mathbb{R}^n} c\,[u \mid v]\, v, \end{aligned}$$

for every tempered distribution u. The spectral decomposition then holds, note that the superposition is performed on the set indexing the $\mathcal{S}$-basis and not on the spectrum of the operator, moreover it is not an integral decomposition but an expansion via superposition.

22.5.2 The relativistic energy

Let us consider the energy of a relativistic particle moving on the real line with rest mass m_0 and momentum p:

$$E(x,p) = m_0c^2 + pc.$$

Consider its square

$$E^2(x,p) = m_0^2c^4 + p^2c^2,$$

and the corresponding operator on $\mathcal{S}'_1$

$$H^2 = M_{m_0^2c^4} + c^2\hbar^2\,(\cdot)''\,.$$

It's simple to prove that the distribution

$$f_p = \left[e^{\frac{i(p|\cdot)}{\hbar}}\right]$$

is an eigenvector of H^2 with corresponding eigenvalue $m_0^2c^4 + p^2c^2$. Consequently, being

$$f = \left(\left[e^{\frac{i(p|\cdot)}{\hbar}}\right]\right)_{p\in\mathbb{R}}$$

an $\mathcal{S}$-basis, H^2 is an ${}^{\mathcal{S}}$observable. Concerning its spectrum we have

$${}^{e}\sigma(H^2) = \left[m_0^2c^4, +\infty\right[.$$

If we consider the operators on $\mathcal{S}_1'$, defined by

$$H_-\left(f_p\right) = \left(-\sqrt{m_0^2c^4 + p^2c^2}\right)f_p,$$

and

$$H_+\left(f_p\right) = \left(\sqrt{m_0^2c^4 + p^2c^2}\right)f_p,$$

we deduce simply that

$${}^{e}\sigma(H_-) = \left]-\infty, -m_0c^2\right]$$

and

$${}^{e}\sigma(H_+) = \left[m_0c^2, +\infty\right[.$$

The operators H_- and H_+ are the Hamiltonian of a relativistic antiparticle and particle respectively.

Recall that

- *to define an ${}^{\mathcal{S}}$linear operator is enough to give an ${}^{\mathcal{S}}$image of an ${}^{\mathcal{S}}$basis.*

Chapter 23

Spectrum

23.1 Supports and vanishing-laws

We recall that, a distribution u in $\mathcal{D}'_n$ is said to vanish on an open set O contained in $\mathbb{R}^n$ if, for every function ϕ in $\mathcal{D}_n$ with support contained in O, $u(\phi) = 0$.

The set of all the functions in $\mathcal{D}_n$ with support contained in O is denoted by $\mathcal{D}_O$, so, a distribution u on $\mathbb{R}^n$ vanishes on O if, for every function ϕ in $\mathcal{D}_O$, $u(\phi) = 0$.

The set $\mathcal{D}_O$ is contained in $\mathcal{D}_n$, and we shall denote by $(\mathcal{D}_O)$ the associated topological vector subspace of $(\mathcal{D}_n)$. A distribution on an open set O is, by definition a continuous linear functional on $(\mathcal{D}_O)$.

Let u be a distribution in $\mathcal{D}'_n$, then the restriction of u to $\mathcal{D}_O$ is a distribution on O; this restriction is denoted by $u_{|O}$, and it is called the restriction of u to O.

The topological dual of the space $(\mathcal{D}_O)$, i.e., the space $(\mathcal{D}_O)'$, is denoted simply by $\mathcal{D}'_O$; this dual has the usual natural vector space structure of duals.

It is clear that, a distribution u in $\mathcal{D}'_n$ vanishes on the open O if and only if its restriction to the open O is the zero-vector of the vector space $\mathcal{D}'_O$.

In this section we shall examine the equality $fu = 0_{\mathcal{D}'_n}$, when f is a smooth function and u is a distribution in $\mathcal{D}'_n$.

Lemma. *Let K be a compact subset of $\mathbb{R}^n$ and let $g : K \to \mathbb{C}$ be a smooth function on the compact K, in the sense that there is an open neighborhood O of K and a complex smooth function g_O defined on the open O such that the restriction of g_O to K is g. Then there is a function h in $\mathcal{D}_O$ coinciding with g on K.*

Proof. Recall that for every compact K and every open neighborhood O of K, there exists a function t belonging to $\mathcal{D}_O$ equals 1 on K. With K and O as in the assumptions, define the function $h : \mathbb{R}^n \to \mathbb{K}$ by

$$h(x) := t(x)g_O(x),$$

for any x in O, and 0 elsewhere. It is clear that h is in $\mathcal{D}_O$ (the support of h is contained in the support of t) and it coincides with g on the compact K. ■

Lemma. *Let f be a smooth function, let O be the co-level 0 of f, and let T the operator from $\mathcal{D}_O$ to $\mathcal{D}_O$ defined by $T(\phi) = f\phi$. Then T is surjective.*

Proof. Let ψ be in $\mathcal{D}_O$, let K be the support of ψ. Let $g_O := 1/f_{|O}$ and let g be the restriction of g_O to K. By the preceding lemma there is a smooth function h defined on $\mathbb{R}^n$, coinciding on K with $1/f$ and with support contained in O . We have

$$T(\psi h) = f\psi h = \begin{cases} f(x)\psi(x)(1/f(x)) & \textit{for } x \in K \\ f(x) \cdot 0 \cdot h(x) & \textit{elsewhere} \end{cases},$$

hence $\psi = T(\psi h)$, and T is surjective. ■

Theorem. *Let $u \in \mathcal{D}'_n$ be a distribution and f be a smooth function. Assume that*

$$fu = 0_{\mathcal{D}'_n}.$$

Then u vanishes on the complement of the zero-level set of f.

Proof. Let O be the co-level zero of f, and let ψ be in $\mathcal{D}_O$. We have to prove that $u(\psi) = 0$. By the preceding lemma, there is a function ϕ in $\mathcal{D}_O$ such that $\psi = f\phi$, now

$$u(\psi) = u(f\phi) = (fu)(\phi) = 0,$$

and the theorem is proved. ■

Lemma. *Let $u \in \mathcal{S}'_n$ be a distribution and f be a smooth function. Assume that*

$$fu = 0_{\mathcal{S}'_n}.$$

Then u vanishes on the complement of the zero-level set of f.

Proof. Consider the complement of the zero-level of f

$$\Omega = \{p \in \mathbb{R}^n : f(p) \neq 0\} = \mathbb{R}^n \setminus f^{\leftarrow}(0).$$

We have to prove that for every test function $\phi \in \mathcal{D}(\Omega)$ is $u(\phi) = 0$. Let $\phi \in \mathcal{D}(\Omega)$, the restriction $f_{|\Omega}$ does not vanish, so the quotient $\phi/f_{|\Omega}$ is defined on Ω, it is smooth and it belongs to $\mathcal{D}(\Omega)$. Now, by definition of multiplication of a test function with a distribution, we have

$$\begin{aligned} u(\phi) &= u\left(f\phi/f_{|\Omega}\right) = \\ &= fu\left(\phi/f_{|\Omega}\right) = \\ &= 0, \end{aligned}$$

as desired. So the distribution u must be vanish in the open set

$$\{p \in \mathbb{R}^n : f(p) \neq 0\},$$

as we had to prove. ■

23.2 Structure of the eigenspectrum

We have the following results:

Theorem (on the topological structure of the eigenspectrum of an $^{\mathcal{S}}$diagonalizable operator). *Let A be an $^{\mathcal{S}}$diagonalizable operator. Then, if a is the ordered system of eigenvalues of A associated to an eigenbasis of A for $\mathcal{S}'_n$, we have*

$$\mathrm{im}a = {}^{e}\sigma(A).$$

In particular the eigenspectrum of the operator A is a connected subset of $\mathbb{C}$.

Proof of the theorem. Since the operator A is an $^{\mathcal{S}}$ diagonalizable operator, there exist a function $a \in \mathcal{O}_M(n)$ and an $^{\mathcal{S}}$basis $\alpha \in \mathcal{B}(\mathbb{R}^n, \mathcal{S}'_n)$ such that, for every point $p \in \mathbb{R}^n$, we have

$$A(\alpha_p) = a(p)\alpha_p,$$

i.e., such that $A(\alpha) = a\alpha$. We shall prove that the eigenspectrum of the operator A is the image im$a$ of the function a. Assume that e is an eigenvalue of A, then there exists a non zero vector η such that $A(\eta) = e\eta$. We then have

$$\begin{aligned} A(\eta) &= A\left(\int_{\mathbb{R}^n} [\eta \mid \alpha]\, \alpha\right) = \\ &= \int_{\mathbb{R}^n} [\eta \mid \alpha]\, A(\alpha) = \\ &= \int_{\mathbb{R}^n} [\eta \mid \alpha]\, (a\alpha) = \\ &= \int_{\mathbb{R}^n} (a\, [\eta \mid \alpha])\alpha, \end{aligned}$$

but on the other hand

$$
\begin{aligned}
A(\eta) &= e\eta = \\
&= e \int_{\mathbb{R}^n} [\eta \mid \alpha] \, \alpha = \\
&= \int_{\mathbb{R}^n} e \, [\eta \mid \alpha] \, \alpha .
\end{aligned}
$$

from the ${}^{\mathcal{S}}$linearly independence of the ${}^{\mathcal{S}}$basis α we have

$$a \, [\eta \mid \alpha] = e \, [\eta \mid \alpha] \, ,$$

then

$$(a - e) \, [\eta \mid \alpha] = 0.$$

so the distribution $[\eta \mid \alpha]$ must be vanish in the open set

$$
\begin{aligned}
\Omega_\eta &= \{ p \in \mathbb{R}^n : a(p) \neq e \} = \\
&= \mathbb{R}^n \setminus a^{\leftarrow}(e).
\end{aligned}
$$

Assume by contradiction that $e \notin \mathrm{im} a$, then there are no p such that $a(p) = e$, and then

$$\Omega_\eta = \mathbb{R}^n,$$

this implies

$$[\eta \mid \alpha] = 0_{\mathcal{S}'_n},$$

so we deduce that η is zero, and this is an absurd. We then have seen that the eigenspectrum ${}^{e}\sigma(A)$ is contained in the image $\mathrm{im} a$, the converse is true by definition of eigenbasis. Concluding *the eigenspectrum of the operator A is the image of the function a*:

$$
\begin{aligned}
{}^{e}\sigma(A) &= \mathrm{im} a = \\
&= a(\mathbb{R}^n),
\end{aligned}
$$

that is a connected set because a is continuous and $\mathbb{R}^n$ is connected. ■

Corollary. *If the eigenspectrum of an ${}^{\mathcal{S}}$ diagonalizable operator is real then it is an interval of the real line (eventually degenerate).*

23.3 Structure of the spectrum

Only a question remains open:

What about the so called residual spectrum and continuous spectrum of an ${}^{\mathcal{S}}$diagonalizable operator?

Recall that

- *1) the eigenspectrum of the operator A is the set of all the complex numbers z such that the z-characteristic operator of A, that is the operator*

$$C_z = z(.)_{\mathcal{S}'_n} - A,$$

is not injective;

- *2) the continuous spectrum is the set of all z such that the z-characteristic operator C_z is algebraically invertible (injective and surjective) but with inverse not continuous;*
- *3) the residual spectrum is the set of z such that the z-characteristic operator C_z is injective but its image is not dense in the space of tempered distribution.*
- *4) the spectrum of A is the union of the preceding ones or equivalently the set of scalars z such that the z -characteristic operator C_z is not topologically invertible.*

Theorem. *The spectrum of an $^{\mathcal{S}}$ diagonalizable operator A coincides with the union of its eigenspectrum and of its residual spectrum, since its continuous spectrum is empty.*

Proof. Since the operator A is an $^{\mathcal{S}}$diagonalizable operator, there exist a function $a \in \mathcal{O}_M(\mathbb{R}^n, \mathbb{K})$ and an $^{\mathcal{S}}$basis $\alpha \in \mathcal{B}(\mathbb{R}^n, \mathcal{S}'_n)$ such that, for every $p \in \mathbb{R}^n$, we have

$$A(\alpha_p) = a(p)\alpha_p.$$

Assume that z is not an eigenvalue of A, then the z-characteristic operator

$$C_z = z(.)_{\mathcal{S}'_n} - A$$

is injective. Moreover

$$\begin{aligned} C_z(\alpha_p) &= z(\alpha_p)_{\mathcal{S}'_n} - A(\alpha_p) = \\ &= z\alpha_p - a(p)\alpha_p = \\ &= (z - a(p))\alpha_p. \end{aligned}$$

Because z is not an eigenvalue of A, the function $z - a$ never vanishes and is of class $\mathcal{O}_M$, and then the family

$$\beta = (z - a)\,\alpha$$

is yet an $^{\mathcal{S}}$basis of the space. Indeed,

$$u = \int_{\mathbb{R}^n} (z - a)^{-1}\, u_\beta (z - a)\alpha,$$

note that the function $(z-a)^{-1}$ is smooth but in general not O_M. Moreover it is simple to prove that, if the image of an $^{\mathcal{S}}$ family by an $^{\mathcal{S}}$linear operator is an$^{\mathcal{S}}$basis than the operator is surjective. Indeed, let u be in $\mathcal{S}'_n$, then

$$\begin{aligned} u &= \int_{\mathbb{R}^n} u_\beta \beta = \\ &= \int_{\mathbb{R}^n} u_\beta C_z(\alpha) = \\ &= C_z\left(\int_{\mathbb{R}^n} u_\beta \alpha\right), \end{aligned}$$

Moreover, it's simple to prove that if the image of an $^{\mathcal{S}}$ linear operator contains an $^{\mathcal{S}}$basis of its codomain, then the operator is surjective. Consequently, the z-characteristic operator C_z is even surjective, and consequently the residual spectrum is empty. Even more, the operator C_z is $^{\mathcal{S}}$linear and then it is the transpose of a certain weakly (i.e., strongly) endomorphisms on the Fréchet space $\mathcal{S}_n$. This operator is bijective as the operator C_z is, so by the Banach inverse operator theorem it is a topological isomorphism. And even more, by the Dieudonné-Schwartz theorem, the operator C_z is a topological isomorphism too. So the continuous spectrum of an $^{\mathcal{S}}$diagonalizable operator is always empty. ■

Chapter 24

Functional Calculus

The purpose of this section is to introduce a functional calculus for the $\mathcal{S}$-diagonalizable operators. Our goal is to state and prove a theorem that allow us to define the action of a numerical function, defined on the spectrum of a certain operator A, on A itself.

If A is a linear operator, by $E(A)$ we denote the set of all the eigenvectors of A, by ${}^{e}\sigma(A)$ the set of all the eigenvalues of A, by $v_A : E(A) \to \mathbb{C}$ the mapping sending every eigenvector u of A to its unique eigenvalue. In other words, $v_A(u)$ is the unique number c such that $Au = cu$.

24.1 A vanishing lemma

First of all we need a lemma.

Lemma. *Let $M \in \mathbb{N}_0$ be a non-negative integer, O be an open subset of the field $\mathbb{K}$, $e \in O$ be a point of that open subset, $r : O \to \mathbb{K}$ be a C^M-function with all derivatives vanishing at the point e, i.e., such that $r^{(i)}(e) = 0$, for every integer $i \in \mathbb{N}_{\leq M}$. Then, for every C^M -function $a : \mathbb{R}^n \to \mathbb{K}$ such that $a(\mathbb{R}^n) \subseteq O$ we have*

$$\partial^p (r \circ a)(x) = 0,$$

for every point $x \in a^{-}(e)$ and for every multi-index $p \in \mathbb{N}_0^n$ with $|p|_1 \leq M$.

Proof. Note, first, that the composition $r \circ a$ is defined on $\mathbb{R}^n$ and is of class C^M, because it is the composition of two functions of class C^M. We shall see the proof in the case $n = 1$, the general case is wholly similar. Moreover, in this case, we shall prove a more general equality, exactly we shall prove that

$$\left(r^{(j)} \circ a\right)^{(i)}(x) = 0,$$

for every $i, j \in \mathbb{N}_0(\leq M)$ such that $i + j \leq M$, and for every x such that $a(x) = e$ (in the case $j = 0$ we obtain the statement). We shall proceed by induction on the sum $s = i + j$. If $i + j = 0$ we have necessarily $i = j = 0$, and we have to prove that $r(a(x)) = 0$, for every $x \in a^{-}(e)$, i.e., $r(e) = 0$, and this is true by assumption. If $i + j = 1$, we have to prove that $r'(a(x)) = 0$ and that $(r \circ a)'(x) = 0$. The first is equivalent to $r'(e) = 0$, that is true by assumption; for the second we have

$$\begin{aligned} (r \circ a)'(x) &= r'(a(x))a'(x) = \\ &= r'(e)a'(x) = \\ &= 0, \end{aligned}$$

that holds still by assumption (we know that $r'(e) = 0$). Now we assume (by induction) that, fixed a positive integer k strictly less than M, the equality

$$(r^{(j)} \circ a)^{(i)}(x) = 0$$

holds true, for every couple of indices $i, j \in \mathbb{N}_0(\leq M)$ such that $i + j \leq k < M$, and for every point x such that $a(x) = e$. We have to prove that the same equality

$$(r^{(j)} \circ a)^{(i)}(x) = 0$$

holds true, for every couple of indices $i, j \in \mathbb{N}_0(\leq M)$ such that $i+j = k+1 \leq M$, and for every point x such that $a(x) = e$. In fact, if $i + j = k + 1$, we have two possibilities: $i = 0$, in this case the equality becomes

$$\begin{aligned} \left(r^{(j)} \circ a\right)^{(i)}(x) &= \left(r^{(j)} \circ a\right)(x) = \\ &= r^{(j)}(a(x)) = \\ &= r^{(j)}(e) = \\ &= 0, \end{aligned}$$

and we have nothing to prove; $i > 0$, in this case- by applying the Leibniz formula, first, and using, then, the inductive assumption - we have

$$\left(r^{(j)} \circ a\right)^{(i)}(x) = \left(\left(r^{(j)} \circ a\right)'\right)^{(i-1)}(x) =$$

$$\begin{aligned}
&= \left(\left(r^{(j+1)} \circ a\right) a'\right)^{(i-1)} (x) = \\
&= \sum_{w=0}^{i-1} \binom{i-1}{w} \left(r^{(j+1)} \circ a\right)^{(w)} (x) \left(a'\right)^{(i-1-w)} (x) = \\
&= \binom{i-1}{i-1} \left(r^{(j+1)} \circ a\right)^{(i-1)} (x) a^{(i-i+1)}(x) = \\
&= \left(r^{(j+1)} \circ a\right)^{(i-1)} (x) a'(x),
\end{aligned}$$

note that the chain of equalities

$$j+1+w = k+1 = i+j$$

is equivalent to $w = i-1$. At this point, if $i = 1$ we can conclude; if, on the contrary, $i > 1$, applying yet the previous result, we have

$$\left(r^{(j)} \circ a\right)^{(i)} (x) = \left(r^{(j+2)} \circ a\right)^{(i-2)} (x) \left(a'(x)\right)^2 .$$

In general, if $i \geq q$, for some positive integer q, applying q times the previous result, we have

$$\left(r^{(j)} \circ a\right)^{(i)} (x) = \left(r^{(j+q)} \circ a\right)^{(i-q)} (x) \left(a'(x)\right)^q .$$

In particular, for $q = i$,

$$\begin{aligned}
\left(r^{(j)} \circ a\right)^{(i)} (x) &= \left(r^{(j+i)} \circ a\right) (x) \left(a'(x)\right)^i = \\
&= r^{(k+1)}(e) \left(a'(x)\right)^i = \\
&= 0,
\end{aligned}$$

as desired. ■

24.2 Transformable ${}^{\mathcal{S}}$diagonalizable operators

Definition (transformable ${}^{\mathcal{S}}$diagonalizable operator). *Let $A : \mathcal{S}'_n \to \mathcal{S}'_n$ be an ${}^{\mathcal{S}}$diagonalizable operator, let f be a real or complex smooth function defined on an open set of $\mathbb{K}$ containing the eigenspectrum ${}^{e}\sigma(A)$. The operator A is said transformable by the function f if the composition $f \circ v_A \circ \alpha$ is of class $\mathcal{O}_M$ for some eigenbasis α of A.*

Theorem (basic lemma on the functions of an ${}^{\mathcal{S}}$ diagonalizable operator). *Let A be an ${}^{\mathcal{S}}$diagonalizable operator on $\mathcal{S}'_n$, let f be a real or complex smooth function defined on an open set of the field $\mathbb{K}$ containing the* *eigenspectrum* ${}^{e}\sigma(A)$ *of* A *and such that such that the composition* $f \circ v_A \circ \alpha$ *is of class* $\mathcal{O}_M$ *for some eigenbasis* α *of* A. *Then, there is a unique* $\mathcal{S}$ *-diagonalizable operator* B *on* $\mathcal{S}'_n$ *such that, for every eigenvector* η *of* A, *the following relation holds*

$$B(\eta) = f(v_A(\eta))\eta.$$

In other words, the operator B *is such that* $v_B = f \circ v_A$. *Moreover, if* α *is* $\mathcal{S}$ *-eigenbasis of* A *and* $a = v_A \circ \alpha$ *is the ordered family of the eigenvalues associated with* α, *for every tempered distribution* u *we have*

$$B(u) = \int_{\mathbb{R}^n} (f \circ a)\, [u|\alpha]\alpha.$$

Proof. Existence. Let α be an ${}^{\mathcal{S}}$ eigenbasis of the operator A. Setting $a = v_A \circ \alpha$, consider the operator B on $\mathcal{S}'_n$ defined by

$$Bu = \int_{\mathbb{R}^n} (f \circ a)\, [u|\alpha]\alpha,$$

for every distribution u. The operator B is obviously ${}^{\mathcal{S}}$ linear. Concerning its ${}^{\mathcal{S}}$diagonalizability, we have

$$\begin{aligned}
B\alpha_p &= \int_{\mathbb{R}^n} (f \circ a)\, [\alpha_p|\alpha]\alpha = \\
&= \int_{\mathbb{R}^n} [\alpha_p|\alpha]\, (f \circ a)\, \alpha = \\
&= \int_{\mathbb{R}^n} \delta_p\, (f \circ a)\, \alpha = \\
&= (f \circ a)\, (p)\alpha_p = \\
&= f(a(p))\alpha_p = \\
&= f(v_A(\alpha_p))\alpha_p.
\end{aligned}$$

So α is an eigenbasis for B too, and then B is ${}^{\mathcal{S}}$ diagonalizable. More, the defined operator verifies the required property for the ${}^{\mathcal{S}}$basis α. Let us see that the property holds for every eigenvector. If η is an eigenvector of A,

$$\begin{aligned}
A\eta &= A\int_{\mathbb{R}^n} [\eta|\alpha]\alpha = \\
&= \int_{\mathbb{R}^n} [\eta|\alpha]A\alpha = \\
&= \int_{\mathbb{R}^n} [\eta|\alpha]a\alpha = \\
&= \int_{\mathbb{R}^n} a[\eta|\alpha]\alpha,
\end{aligned}$$

but on the other hand

$$\begin{aligned} A\eta &= v_A(\eta)\eta = \\ &= v_A(\eta)\int_{\mathbb{R}^n}[\eta|\alpha]\alpha = \\ &= \int_{\mathbb{R}^n} v_A(\eta)[\eta|\alpha]\alpha, \end{aligned}$$

from the $^{\mathcal{S}}$independence of α we have

$$a[\eta|\alpha] = v_A(\eta)[\eta|\alpha],$$

then, putting $e = v_A(\eta)$, we have

$$(a-e)\,[\eta|\alpha] = 0_{\mathcal{S}'_n}.$$

Since $[\eta \mid \alpha]$ is a tempered distribution, then it is of finite order, say of order $\leq M$. By the Taylor's formula, there is a function r such that

$$r^{(i)}(e) = 0,$$

for every $0 \leq i \leq M$, and such that

$$f(y) = \sum_{k=0}^{M} \frac{f^{(k)}(e)}{k!}(y-e)^k + r(y),$$

for every y in the domain of f, and, particularly, for y in the spectrum of A. Then, for every $x \in \mathbb{R}^n$,

$$f(a(x)) = \sum_{k=0}^{M} \frac{f^{(k)}(e)}{k!}(a(x)-e)^k + r(a(x)),$$

That is,

$$\begin{aligned} f\circ a &= \sum_{k=0}^{M} \frac{f^{(k)}(e)}{k!}(a-e)^k + r\circ a = \\ &= f(e) + \sum_{k=1}^{M} \frac{f^{(k)}(e)}{k!}(a-e)^k + r\circ a. \end{aligned}$$

Hence, multiplying by $[\eta \mid \alpha]$, and taking into account that, for $k \geq 1$,

$$(a-e)^k\,[\eta \mid \alpha] = (a-e)^{k-1}(a-e)\,[\eta \mid \alpha] = 0_{\mathcal{S}'_n},$$

we deduce

$$\begin{aligned} (f\circ a)\,[\eta \mid \alpha] &= f(e)\,[\eta \mid \alpha] + \sum_{k=1}^{M} \frac{f^{(k)}(e)}{k!}(a-e)^k\,[\eta \mid \alpha] + (r\circ a)\,[\eta \mid \alpha] = \\ &= f(e)\,[\eta \mid \alpha] + (r\circ a)\,[\eta \mid \alpha]. \end{aligned}$$

Note that (by the previous lemma) $r \circ a$ must be vanish with all its derivatives of order $\leq M$, in the closed set $a^{-}(e)$. Moreover, since $[\eta \mid \alpha]$ must vanish in the complement of this set, we have

$$\operatorname{supp}[\eta \mid \alpha] \subseteq a^{-}(e).$$

Thus $r \circ a$ vanishes on the support of $[\eta \mid \alpha]$ with all its derivatives of order $\leq M$, and then, by a classic theorem on the distributions with finite order, we have

$$(r \circ a)[\eta \mid \alpha] = 0_{\mathcal{S}'_n},$$

and consequently,

$$(f \circ a)[\eta \mid \alpha] = f(e)[\eta \mid \alpha].$$

Finally, we can conclude

$$\begin{aligned}
B\eta &= B\int_{\mathbb{R}^n}[\eta \mid \alpha]\,\alpha = \\
&= \int_{\mathbb{R}^n}[\eta \mid \alpha]\,B\alpha = \\
&= \int_{\mathbb{R}^n}[\eta \mid \alpha]\,(f \circ a)\,\alpha = \\
&= \int_{\mathbb{R}^n}(f \circ a)\,[\eta \mid \alpha]\,\alpha = \\
&= \int_{\mathbb{R}^n}f(e)\,[\eta \mid \alpha]\,\alpha = \\
&= f(e)\int_{\mathbb{R}^n}[\eta \mid \alpha]\,\alpha = \\
&= f(e)\eta.
\end{aligned}$$

Uniqueness. Two linear operators coinciding on a same $^{\mathcal{S}}$ basis are equals. ■

24.3 Functions of $^{\mathcal{S}}$diagonalizable operators

The preceding theorem allow us to give the following definition

Definition (the functions of an $^{\mathcal{S}}$ diagonalizable operator). *Let A be an $^{\mathcal{S}}$diagonalizable operator, let $E(A)$ be the set of all the eigenvectors of the operator A, let $^{e}\sigma(A)$ be the set of all the eigenvalues of A and let $v_A : E(A) \to \mathbb{C}$ be the mapping that sends every eigenvector u of A to its unique eigenvalue $v_A(u)$ (the scalar $v_A(u)$ is the unique c such that $Au = cu$). Let f be a real or complex smooth function defined on an open set of the field $\mathbb{K}$ containing the eigenspectrum $^{e}\sigma(A)$, such that the composition $f \circ v_A \circ \alpha$ is smooth and of class*

$\mathcal{O}_M$, for some eigenbasis α of A. The unique $^{\mathcal{S}}$ diagonalizable operator B such that, for every eigenvector u of A is

$$Bu = f(v_A(u))u,$$

that is, such that $v_B = f \circ v_A$ is called the image of the operator A under the function f and it is denoted by $f(A)$.

Remark. If A has a finite spectrum, since the spectrum is connected, there is a unique eigenvalue of A, hence v_A is a constant function, so $g \circ v_A \circ \alpha$ is constant too for every smooth function g defined in an open neighborhood of the spectrum; therefore A is transformable for every such g. Moreover, for every u, if a is the unique eigenvalue of A,

$$\begin{aligned} A(u) &= \int_{\mathbb{R}^n} a\,[u \mid \alpha]\,\alpha = \\ &= a \int_{\mathbb{R}^n} [u \mid \alpha]\,\alpha = \\ &= a(0)u; \end{aligned}$$

then, the multiples of the identity are the only $\mathcal{S}$ -diagonalizable operators with finite spectrum, and, we have

$$f(A) = f(a)\mathbb{I}_{\mathcal{S}'_n}.$$

Example. Let t be a real number. Consider the function $f_t : \mathbb{R} \to \mathbb{C}$ defined by

$$f_t(x) = e^{-\frac{it}{\hbar}x}.$$

Let H be an $\mathcal{S}$-diagonalizable operator, and let η be a basis such that

$$H\eta = E\eta,$$

for some smooth real function E. Let $\psi_0 \in \mathcal{S}'_1$ and let $\psi\,(t)$ be the vector state defined by

$$\psi\,(t) = \int_{\mathbb{R}} e^{-\frac{it}{\hbar}E}\,[\psi_0 \mid \eta]\,\eta.$$

Then we have

$$\psi\,(t) = e^{-\frac{it}{\hbar}H}(\psi_0),$$

where with $e^{-\frac{it}{\hbar}H}$ we denote the operator $f_t(H)$.

24.4 Compatibility with the exponential

Theorem. *Let $A \in \mathcal{L}(\mathcal{S}'_n)$ be a continuous endomorphism on the space $\mathcal{S}'_n$, that is an $\mathcal{S}$-linear operator on $\mathcal{S}'_n$. For every tempered distribution $u \in \mathcal{S}'_n$ and for each test function $\phi \in \mathcal{S}_n$, the numerical series*

$$\sum \left(\frac{A^m (u) (\phi)}{m!} \right)_{m \in \mathbb{N}}$$

is absolutely convergent. Moreover the series in $\mathcal{S}'_n$

$$\sum \left(\frac{A^m (u)}{m!} \right)_{m \in \mathbb{N}}$$

is weakly convergent.*

Proof. Indeed, there are two positive real numbers c_A and c_u and a continuous seminorm on the topological vector space $(\mathcal{S}_n)$ such that

$$\left| \frac{A^m (u) (\phi)}{m!} \right| \leq \frac{c_A^m c_u q(\phi)}{m!},$$

moreover the numerical series

$$\sum \left(\frac{c_A^m c_u q(\phi)}{m!} \right)_{m \in \mathbb{N}},$$

is convergent (to the number $c_u q(\phi) e^{c_A}$). Consequently, from the Banach Steinhaus theorem it follows that, for every tempered distribution $u \in \mathcal{S}'_n$, there exists an other distribution $v_u \in \mathcal{S}'_n$ such that

$$v_u = \sum_{m=1}^{\infty} \frac{A^m (u)}{m!},$$

as we desire. ■

So we can define the operator

$$e^A : \mathcal{S}'_n \to \mathcal{S}'_n : u \mapsto \sum_{m=1}^{\infty} \frac{A^m (u)}{m!}.$$

Example. If $A = \mathbb{I}_{\mathcal{S}'_n}$, we have

$$e^{\mathbb{I}_{\mathcal{S}'_n}} (u) = \sum_{m=1}^{\infty} \frac{u}{m!} = eu. \ \square$$

Example. Let $A(u) = cu$ with $c \in \mathbb{C}$, we have

$$A(A(u)) = cA(u) = c(cu) = c^2 u,$$

inductively we have

$$A^n(u) = A^{n-1}(A(u)) = c^{n-1}A(u) = c^n u,$$

so

$$e^A(u) = e^c u.$$

Lemma. *If $A \in \mathcal{L}(\mathcal{S}_n')$ is an ${}^{\mathcal{S}}$linear operator, the power operator A^m is ${}^{\mathcal{S}}$linear too, that is it belongs to the space $\mathcal{L}(\mathcal{S}_n')$, for every $m \in \mathbb{N}$.*

Proof. The proof follows immediately from the fact that the composition of two continuous maps is still continuous. But we desire to prove the property using the definition of $\mathcal{S}$-linearity. Inductively, we have

$$\begin{aligned} A^m\left(\int_{\mathbb{R}^k} av\right) &= A^{m-1}\left(A\left(\int_{\mathbb{R}^k} av\right)\right) = \\ &= A^{m-1}\left(\int_{\mathbb{R}^k} aA(v)\right) = \\ &= \int_{\mathbb{R}^k} aA^{m-1}(A(v)) = \\ &= \int_{\mathbb{R}^k} aA^m(v), \end{aligned}$$

for every tempered distribution in $\mathcal{S}_k'$ and every ${}^{\mathcal{S}}$family belonging to $\mathcal{S}(\mathbb{R}^k, \mathcal{S}_n')$, as we desired. ■

Theorem. *Let $A \in \mathcal{L}(\mathcal{S}_n')$ be an ${}^{\mathcal{S}}$diagonalizable operator with the family $\alpha \in \mathcal{S}(\mathbb{R}^n, \mathcal{S}_n')$ as an $\mathcal{S}$-eigenbasis of $\mathcal{S}_n'$ with respect to the family a of eigenvalues. Then, for every distribution $u \in \mathcal{S}_n'$, the series*

$$\sum \left(\frac{A^m}{m!}(u)\right)_{m \in \mathbb{N}}$$

is $\sigma(\mathcal{S}_n')$-convergent to the superposition

$$\int_{\mathbb{R}^n} e^{\lambda}[u \mid v]\, v.$$

In other terms, if e^A is the operator $\sum_{m=1}^{\infty}(1/m!)A^m$, we have

$$e^A = \int_{\mathbb{R}^n} e^{\lambda}[\cdot \mid v]\, v,$$

in the usual (pointwise) sense, as equality among operators.

Proof. Let $\phi \in \mathcal{S}_n$, we have

$$\begin{aligned}
\frac{A^m(u)(\phi)}{m!} &= \frac{A^m\left(\int_{\mathbb{R}^n}[u \mid v]\, v\right)(\phi)}{m!} = \\
&= \frac{\left(\int_{\mathbb{R}^n}[u \mid v]\, A^m(v)\right)(\phi)}{m!} = \\
&= \frac{\left(\int_{\mathbb{R}^n}[u \mid v]\,(\lambda^m v)\right)(\phi)}{m!} = \\
&= \frac{\left(\int_{\mathbb{R}^n}\lambda^m [u \mid v]\, v\right)(\phi)}{m!} = \\
&= \left(\int_{\mathbb{R}^n}\left(\frac{\lambda^m}{m!}[u \mid v]\right) v\right)(\phi),
\end{aligned}$$

hence we have

$$\begin{aligned}
\sum_{k=1}^{\infty}\frac{A^k(u)(\phi)}{k!} &= \sum_{k=1}^{\infty}\left(\int_{\mathbb{R}^n}\left(\frac{\lambda^k}{k!}[u \mid v]\right) v\right)(\phi) = \\
&= \left(\int_{\mathbb{R}^n}\left(\sum_{k=1}^{\infty}\frac{\lambda^k}{k!}[u \mid v]\right) v\right)(\phi),
\end{aligned}$$

and since the series

$$\sum\left(\frac{\lambda^k}{k!}[u \mid v]\right)_{k\in\mathbb{N}} \overset{\sigma(\mathcal{S}'_n)}{\to} e^{\lambda}[u \mid v]$$

and since the superposition operator $\int_{\mathbb{R}^n}(\cdot, v)$ is $\sigma(\mathcal{S}'_n)$ continuous we deduce that the series

$$\sum\left(\frac{A^k(u)(\phi)}{k!}\right)_{k\in\mathbb{N}}$$

is convergent to the limit

$$\left(\int_{\mathbb{R}^n}\left(e^{\lambda}[u \mid v]\right) v\right)(\phi),$$

that is what we had to prove. ■

Remark (continuous compositions). Let $A = (A_p)_{p\in\mathbb{R}^m}$ be a family of linear continuous endomorphism on $\mathcal{S}'_n$ and assume that there is another family $H = (H_p)_{p\in\mathbb{R}^m}$ of continuous operators on $\mathcal{S}'_n$ such that $A = e^{H_p}$. We can define the composition of the (ordered!) family A as the linear continuous operator $\circ A$ defined by

$$\circ_{p\in\mathbb{R}^m} A_p := \exp\left(\int_{\mathbb{R}^m} H\right).$$

Part VII

$^{\mathcal{S}}$Linear Dynamics

Chapter 25

The Schrödinger equation

25.1 Introduction

In the present chapter we show the resolution of the Schrödinger's equation associated with an operator admitting an $^{\mathcal{S}}$ eigenbasis, that is associated with an $^{\mathcal{S}}$diagonalizable operator. We give, for such Schrödinger's equation with initial condition, a theorem of existence and uniqueness. In the first section we present the result on differentiable curves in topological vector spaces we need in the chapter. In the second one we shall prove some results for differentiable curves in the spaces $\mathcal{S}'_n$ and $\mathcal{O}_M(n)$. In the third one we shall prove results on the differentiation of curves in the space of tempered distributions. In the forth one we shall prove the main results of the chapter. In the fifth we shall discuss about applications in Quantum Mechanics.

25.2 Differentiable curve

25.2.1 Differentiable curves in topological vector spaces

First of all we begin with some background material about the differentiability of a curve in a topological vector space, this because our existence and uniqueness theorem concerns curves in the locally convex topological vector space of the tempered distributions.

Definition (differentiable curve in a complex Hausdorff topological vector space). *Let (X, τ) be a complex Hausdorff topological vector space and let $\gamma : \mathbb{R} \to X$ be a curve in X. The curve γ is said to be* $^{\tau}$***differentiable at the point*** $t \in \mathbb{R}$ *if and only if the map*

$$r_t : \mathbb{R}^{\neq} \to X : h \mapsto \frac{\gamma(t+h) - \gamma(t)}{h}$$

has $(\mathbb{R}, \tau)$-limit at 0. In this case we put

$$(\gamma)'_{\tau}(t) := {}^{(\mathbb{R},\tau)}\lim_{h \to 0} \frac{\gamma(t+h) - \gamma(t)}{h}.$$

When no confusion is possible, we shall use the notation $\gamma'(t)$ that does not emphasize the topology τ. The vector $(\gamma)'_{\tau}(t)$ is called the $^{\tau}$***derivative of the curve*** γ ***at*** t. *If γ is $^{\tau}$ differentiable at every $t \in \mathbb{R}$, γ is said to be* $^{\tau}$***differentiable*** *and the map*

$$(\gamma)'_{\tau} : \mathbb{R} \to X : t \mapsto D_{\tau}(\gamma)(t),$$

is called the $^{\tau}$***derivative of*** γ.

Theorem (continuity of differentiable curves). *Let (X, τ) be a Hausdorff topological vector space and $\gamma : \mathbb{R} \to X$ be a $^{\tau}$differentiable curve. Then γ is continuous with respect to the pair of topologies $(\tau_{\mathbb{R}}, \tau)$.*

Proof. Let $t \in \mathbb{R}$ and $h \in \mathbb{R}^{\neq}$, we have

$$\begin{aligned} \gamma(t+h) - \gamma(t) &= h\frac{\gamma(t+h) - \gamma(t)}{h} + h\gamma'(t) - h\gamma'(t) = \\ &= h\left(\frac{\gamma(t+h) - \gamma(t)}{h} - \gamma'(t)\right) + h\gamma'(t) = \\ &= h\omega_t(h) + d\gamma(t)(h), \end{aligned}$$

where we put

$$\omega_t(h) := \frac{\gamma(t+h) - \gamma(t)}{h} - \gamma'(t),$$

and

$$d\gamma(t)(h) := h\gamma'(t).$$

Obviously we have

$${}^{\tau}\lim_{h \to 0} \omega_t(h) = 0_X,$$

moreover the $^{\tau}$differential of γ at t, that is the map

$$d\gamma(t) : \mathbb{R} \to X : h \mapsto h\gamma'(t)$$

is linear and ${}^{\tau}$continuous for every t, in fact the multiplication by scalars of the space X is a ${}^{(\mathbb{C},\tau)}$continuous map and $d\gamma\,(t)$ is simply one of its sections, concluding

$$\begin{aligned} {}^{\tau}\lim_{h\to 0}\gamma\,(t+h) &= {}^{\tau}\lim_{h\to 0}(h\omega_t\,(h) + d\gamma\,(t)\,(h) + \gamma\,(t)) = \\ &= \gamma(t), \end{aligned}$$

as we desired. ■

Theorem (on the constant curves). *Let (X,τ) be a complex Hausdorff locally convex topological vector space and $\gamma : \mathbb{R} \to X$ be a ${}^{\tau}$differentiable curve. Then γ is constant if and only if*

$$\gamma'\,(t) = 0_X$$

for every $t \in \mathbb{R}$.

Proof. It follows from the mean value theorem. ■

25.2.2 Differentiable curves in $\mathcal{S}'_n$

Theorem. *Let $a : \mathbb{R} \to \mathcal{S}'_m$ be a ${}^{\sigma(\mathcal{S}'_m)}$differentiable curve and let v be an ${}^{\mathcal{S}}$family in $\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$. Then the curve*

$$\int_{\mathbb{R}^m} av : \mathbb{R} \to \mathcal{S}'_n : t \mapsto \int_{\mathbb{R}^m} a\,(t)\,v$$

is ${}^{\sigma(\mathcal{S}'_m)}$differentiable and we have

$$\left(\int_{\mathbb{R}^m} av\right)'_{\sigma(\mathcal{S}'_n)} = \int_{\mathbb{R}^m} (a)'_{\sigma(\mathcal{S}'_m)}\,v.$$

Proof. Let $t \in \mathbb{R}$ and $h \in \mathbb{R}^{\neq}$, we have

$$\frac{1}{h}\left(\int_{\mathbb{R}^m} a\,(t+h)\,v - \int_{\mathbb{R}^m} a\,(t)\,v\right) = \int_{\mathbb{R}^m} \frac{a\,(t+h) - a\,(t)}{h} v,$$

so, since the superposition operator $\int_{\mathbb{R}^m}(\cdot, v)$ of the family v is continuous with respect to the pair of topologies $(\sigma(\mathcal{S}'_m), \sigma(\mathcal{S}'_n))$, we can conclude. ■

Consequently, we have the following corollary.

Corollary. *Every differential operator* $D : \partial^1(\mathbb{R}, \mathcal{S}'_n) \to^{\mathbb{R}} \mathcal{S}'_n$ *with constant coefficients, defined on the space of all weakly* differentiable curves in the space* $\mathcal{S}'_n$*, of the form*

$$D(u) = \sum_{i=1}^{k} c_i \partial^i u,$$

is such that

$$D\left(\int_{\mathbb{R}^m} av\right) = \int_{\mathbb{R}^m} D(a)v,$$

for every coefficient curve a *in* $\partial^1(\mathbb{R}, \mathcal{S}'_n)$ *and every* ${}^{\mathcal{S}}$*family* v *in* $\mathcal{S}'_n$.

25.2.3 Differentiable curves in $\mathcal{O}_M$ (*)

Theorem (a generalized Leibniz formula). *Let* $f : \mathbb{R} \to \mathcal{O}_M(n)$ *be a* ${}^{\mathcal{O}_M}$*differentiable curve and* $u : \mathbb{R} \to \mathcal{S}'_n$ *be a* ${}^{\sigma(\mathcal{S}'_n)}$*differentiable curve. Then the curve*

$$fu : \mathbb{R} \to \mathcal{S}'_n : t \mapsto f(t)\, u(t)$$

is ${}^{\sigma(\mathcal{S}'_n)}$*differentiable and*

$$(fu)'_{\sigma(\mathcal{S}'_n)} = (f)'_{\mathcal{O}_M}\, u + f\, (u)'_{\sigma(\mathcal{S}'_n)}\, .$$

Proof. Let $t \in \mathbb{R}$ and $h \in \mathbb{R}^{\neq}$, then we have

$$\frac{(fu)(t+h) - (fu)(t)}{h} = \frac{f(t+h) - f(t)}{h} u(t+h) + f(t)\, \frac{u(t+h) - u(t)}{h},$$

now, the function

$$h \mapsto \frac{f(t+h) - f(t)}{h} u(t+h)$$

converges to $(f)'_{\mathcal{O}_M}\, u$ by hypocontinuity of the product of ${}^{\mathcal{O}_M}$functions times tempered distributions hence, by the continuity of u (it is differentiable) and by the definition of derivative, we conclude. ■

Corollary. *Let* $f : \mathbb{R} \to \mathcal{O}_M(n)$ *be a* ${}^{\mathcal{O}_M}$*differentiable curve and let* $u \in \mathcal{S}'_n$*. Then, the curve*

$$fu : \mathbb{R} \to \mathcal{S}'_n : t \mapsto f(t)\, u$$

is ${}^{\sigma(\mathcal{S}'_n)}$*differentiable and*

$$(fu)'_{\sigma(\mathcal{S}'_n)} = (f)'_{\mathcal{O}_M}\, u.$$

Proof. It follows from the Leibniz formula and because the derivative of a constant curve is the null vector at every point. ■

Corollary. *Let $f : \mathbb{R} \to \mathcal{O}_M(n)$ be a $\mathcal{O}_M$-differentiable curve, $a \in \mathcal{S}'_m$, and let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$. Then, the curve $u : \mathbb{R} \to \mathcal{S}'_n$ defined by*

$$t \mapsto \int_{\mathbb{R}^m} f\,(t)\,av$$

is $^{\sigma(\mathcal{S}'_n)}$differentiable and

$$(u)'_{\sigma(\mathcal{S}'_n)}\,(t) = \int_{\mathbb{R}^m} (f)'_{\mathcal{O}_M}\,(t)\,av,$$

or more simply

$$u'(t) = \int_{\mathbb{R}^m} f'\,(t)\,av.$$

Proof. It follows from the above results and from the continuity of superposition operators. ■

25.2.4 Pointwise differentiable curve in $\mathcal{O}_M$

We shall denote by $\mathcal{O}_M\,(n)$ the space $\mathcal{O}_M\,(\mathbb{R}^n, \mathbb{K})$.

Lemma. *Let I be a non-degenerate interval of the real line $f : I \to \mathcal{O}_M(n)$ be a pointwise differentiable curve. Assume*

- *the derivative of the curve f at any point t of I be an $^{\mathcal{O}_M}$function and let $f' : I \to \mathcal{O}_M$ be the pointwise derivative of f, defined by*

  $$f'(t)(x) = f(.)(x)'(t),$$

  for every t in I and x in $\mathbb{R}^n$;

- *assume that there exists a real function M in $\mathcal{O}_M(n)$ such that the derivative $f'(t)$ is absolutely dominated by M, i.e.*

  $$|f'(t)| \leq M,$$

  for every real t.

Then, for every tempered distribution a, the curve $f \otimes a$ in the space $\mathcal{S}'_n$, defined by

$$f \otimes a : t \mapsto f(t)a,$$

is weakly star differentiable (differentiable with respect to the topology $\sigma(\mathcal{S}'_n)$) and its derivative is the curve

$$(f \otimes a)'_{\sigma(\mathcal{S}'_n)} = f' \otimes a,$$

that is the curve in the space $\mathcal{S}'_n$ defined by

$$f' \otimes a : t \mapsto f'(t)a,$$

for every t in I.

Proof. We have to prove that the difference quotient

$$\frac{f(t+h) - f(t)}{h} a : \mathbb{R}_{\neq} \to \mathcal{S}'_n$$

converges in the weak* topology $\sigma(\mathcal{S}'_n)$ to the distribution $f'(t)a$ at 0. This is equivalent to prove that the Fourier transform

$$\mathcal{F}\left(\frac{f(t+h) - f(t)}{h} a\right)$$

converges in the weak* topology $\sigma(\mathcal{S}'_n)$ to the Fourier transform

$$\mathcal{F}(f'(t)a),$$

that is

$$\mathcal{F}(\left[\frac{f(t+h) - f(t)}{h}\right]) * \mathcal{F}(a) \to_{\sigma(\mathcal{S}'_n)} \mathcal{F}([f'(t)]) * \mathcal{F}(a),$$

that is true if we prove that

$$\mathcal{F}(\left[\frac{f(t+h) - f(t)}{h}\right]) \to_{\sigma(\mathcal{S}'_n)} \mathcal{F}([f'(t)]),$$

pointwise (since the convolution is separately continuous). We have

$$\begin{aligned} \mathcal{F}(\left[\frac{f(t+h) - f(t)}{h}\right])(g) &= \left[\frac{f(t+h) - f(t)}{h}\right](\mathcal{S}g) = \\ &= \int_{\mathbb{R}^n} \frac{f(t+h) - f(t)}{h} \mathcal{S}g \, \mu, \end{aligned}$$

this last integral tends to

$$\int_{\mathbb{R}^n} f'(t)\mathcal{S}(g) \, \mu = [f'(t)](\mathcal{S}g),$$

as h tends to 0, by the Lebesgue's dominated convergence theorem. Indeed, let $h < 1$, for every x in $\mathbb{R}^n$, by the mean value theorem applied to the function $f(.)(x)$, there is a point t_x in the unit interval $[0, 1]$ such that

$$\frac{f(t+h)(x) - f(t)(x)}{h} \mathcal{S}(g) = f'(t_x)(x)\mathcal{S}(g),$$

hence we have

$$\begin{aligned}\left|\frac{f(t+h)(x)-f(t)(x)}{h}\mathcal{S}(g)\right| &= \left|f'(t_x)(x)\mathcal{S}(g)\right| \leq \\ &\leq \left|M(x)\mathcal{S}(g)\right|,\end{aligned}$$

so that the family of test functions

$$\left(\frac{f(t+h)-f(t)}{h}\mathcal{S}(g)\right)_{h\in[0,1]}$$

is locally absolutely bounded at the index 0 by the summable function $|M\mathcal{S}(g)|$. The theorem is so proved. ■

25.3 The Schrödinger's equation in $\mathcal{S}'_n$

25.3.1 Solutions of the eigen-representation

Theorem. *Let $a_0 \in \mathcal{S}'_n$ be a tempered distribution and let $E \in \mathcal{O}_M(n)$ be a real function. Then, the curve $a : \mathbb{R} \to \mathcal{S}'_n$ in the space $\mathcal{S}'_n$ defined by*

$$t \mapsto e^{-(i/\hbar)tE}a_0$$

is ${}^{\sigma(\mathcal{S}'_n)}$differentiable and it is the only one verifying

- *the differential equality*
$$i\hbar a' = Ea;$$
- *the initial condition $a(0) = a_0$.*

Proof. Existence. The curve $a : \mathbb{R} \to \mathcal{S}'_n$, defined by

$$t \mapsto e^{-(i/\hbar)tE}u_0,$$

is differentiable by the lemma on pointwise differentiable curves in the space $\mathcal{O}_M(n)$. Indeed it is immediate that the curve $t \mapsto e^{-(i/\hbar)tE}$, in $\mathcal{O}_M$, verifies all the requirements of the lemma. Consequently, the curve a is such that

$$\begin{aligned}a'(t) &= -(i/\hbar)Ee^{-(i/\hbar)tE}a_0 = \\ &= -(i/\hbar)Ea(t),\end{aligned}$$

and hence it resolves the equation

$$i\hbar x'(t) = Ex(t).$$

Moreover, we have

$$\begin{aligned} a(0) &= e^{-(i/\hbar)0E} a_0 = \\ &= a_0. \end{aligned}$$

Uniqueness. Let $x : \mathbb{R} \to \mathcal{S}'_n$ be a ${}^{\sigma(\mathcal{S}'_n)}$differentiable curve such that

$$i\hbar x' = Ex,$$

and $x(0) = a_0$. We have

$$i\hbar e^{(i/\hbar)tE} x'(t) = E e^{(i/\hbar)tE} x(t),$$

that is

$$i\hbar e^{(i/\hbar)tE} x'(t) - E e^{(i/\hbar)tE} x(t) = 0.$$

The above equality is equivalent to

$$\left[i\hbar e^{(i/\hbar)tE} x \right]'(t) = 0,$$

and the preceding one holds true if and only if, there exists a $c \in \mathcal{S}'_1$, such that, for every $t \in \mathbb{R}$, we have

$$i\hbar e^{(i/\hbar)tE} x(t) = c.$$

Moreover, it's obvious that

$$\begin{aligned} c &= i\hbar e^{(i/\hbar)0E} x(0) = \\ &= i\hbar u_0, \end{aligned}$$

thus we have

$$\begin{aligned} e^{(i/\hbar)tE} x(t) &= \frac{c}{i\hbar} \\ &= \frac{i\hbar u_0}{i\hbar} \\ &= u_0, \end{aligned}$$

and hence

$$x(t) = e^{-(i/\hbar)tE} u_0,$$

as we desire. ■

25.3.2 The Schrödinger's equation in $\mathcal{S}'_n$

Recall the following definition.

Definition. *Let $A \in \mathcal{L}(\mathcal{S}'_n)$ be a continuous linear endomorphism on the space $\mathcal{S}'_n$. We say that the operator A* ***has an ${}^{\mathcal{S}}$eigenbasis of the space***

indexed by $\mathbb{R}^m$ *if there exists an* $^{\mathcal{S}}$*basis* $\alpha \in \mathcal{B}(\mathbb{R}^m, \mathcal{S}'_n)$ *and a smooth function* $a \in C^\infty(\mathbb{R}^m, \mathbb{K})$ *such that* $A(\alpha) = a\alpha$*. In this case the smooth function* a *is called* ***the system of eigenvalues of the operator*** A ***in the eigenbasis*** α *and we say also that the operator* A *is* $^{\mathcal{S}}$ *diagonalizable.*

Theorem (on the abstract Schrödinger equation). *Let* $H \in \mathcal{L}(\mathcal{S}'_n)$ *be an* $^{\mathcal{S}}$*diagonalizable operator with real eigenvalues. Then, for every vector* $\psi_0 \in \mathcal{S}'_n$*, there exists a unique differentiable curve* $\psi : \mathbb{R} \to \mathcal{S}'_n$*, verifying the differential equality*

$$i\hbar\psi'(t) = H(\psi(t)),$$

and the initial condition $\psi(0) = \psi_0$*. Explicitly, the unique solution is given by*

$$\psi(t) = e^{-(i/\hbar)tH}(\psi_0),$$

for every t*.*

Proof. Let $\eta \in \mathcal{B}(\mathbb{R}^n, \mathcal{S}'_n)$ be an $^{\mathcal{S}}$eigenbasis of the operator H for the space $\mathcal{S}'_n$ and let E be the system of the eigenvalues of the operator H in the $^{\mathcal{S}}$basis η, i.e., the smooth function such that

$$H(\eta) = E\eta.$$

We recall that

$$e^{-(i/\hbar)tH}(\psi_0) := \int_{\mathbb{R}^n} e^{-(i/\hbar)tE}\,[\psi_0 \mid \eta]\,\eta.$$

Existence. Let $\psi : \mathbb{R} \to \mathcal{S}'_1$ be the curve defined by

$$\psi(t) = \int_{\mathbb{R}^n} e^{-(i/\hbar)tE}\,[\psi_0 \mid \eta]\,\eta.$$

We have by derivation

$$\begin{aligned} i\hbar\psi'(t) &= \int_{\mathbb{R}^n} Ee^{-(i/\hbar)tE}\,[\psi_0 \mid \eta]\,\eta = \\ &= \int_{\mathbb{R}^n} e^{-(i/\hbar)tE}\,[\psi_0 \mid \eta]\,(E\eta). \end{aligned}$$

On the other hand

$$\begin{aligned} H(\psi(t)) &= H\Big(\int_{\mathbb{R}^n} e^{-(i/\hbar)tE}\,[\psi_0 \mid \eta]\,\eta\Big) = \\ &= \int_{\mathbb{R}^n} e^{-(i/\hbar)tE}\,[\psi_0 \mid \eta]\,H(\eta) = \\ &= \int_{\mathbb{R}^n} e^{-(i/\hbar)tE}\,[\psi_0 \mid \eta]\,(E\eta), \end{aligned}$$

and hence, recalling the expression of $i\hbar\psi'$, we deduce

$$i\hbar\psi'(t) = H(\psi(t)).$$

Uniqueness. Let ψ be a solution of the Schrödinger equation we have

$$\begin{aligned} 0_{\mathcal{S}_1'} &= i\hbar\psi'(t) - H(\psi(t)) = \\ &= i\hbar\int_{\mathbb{R}^n}[\psi \mid \eta]'(t)\,\eta - H\left(\int_{\mathbb{R}^n}[\psi(t) \mid \eta]\,\eta\right) = \\ &= \int_{\mathbb{R}^n} i\hbar\,[\psi \mid \eta]'\,\eta - \int_{\mathbb{R}^n}[\psi \mid \eta]\,H(\eta) = \\ &= \int_{\mathbb{R}^n}\left(i\hbar\,[\psi \mid \eta]' - E\,[\psi \mid \eta]\right)\eta, \end{aligned}$$

now, because η is linearly independent we have

$$i\hbar\,[\psi \mid \eta]'(t) = E\,[\psi \mid \eta](t),$$

i.e., set $a = [\psi \mid \eta]$ we have

$$i\hbar a'(t) = Ea(t).$$

Now, if $\psi(0) = \psi_0$, then

$$\begin{aligned} a(0) &= [\psi(0) \mid \eta] = \\ &= [\psi_0 \mid \eta], \end{aligned}$$

so, applying the above result we have

$$a(t) = e^{-(i/\hbar)tE}a(0),$$

Concluding

$$\begin{aligned} \psi(t) &= \int_{\mathbb{R}^n}[\psi(t) \mid \eta]\,\eta = \\ &= \int_{\mathbb{R}^n} a(t)\,\eta = \\ &= \int_{\mathbb{R}^n} e^{-(i/\hbar)tE}a(0)\,\eta = \\ &= \int_{\mathbb{R}^n} e^{-(i/\hbar)tE}\,[\psi(0) \mid \eta]\,\eta, \end{aligned}$$

as we desired. ■

25.3.3 The evolution group

Note that, for every tempered distribution ψ,

$$\begin{aligned}
e^{-(i/\hbar)tH}(e^{-(i/\hbar)hH}(\psi)) &= e^{-(i/\hbar)tH}\left(\int_{\mathbb{R}^n} e^{-(i/\hbar)hE}\,[\psi|\eta]\,\eta\right) = \\
&= \int_{\mathbb{R}^n} e^{-(i/\hbar)hE}\,[\psi|\eta]\,e^{-(i/\hbar)tH}\,(\eta) = \\
&= \int_{\mathbb{R}^n} e^{-(i/\hbar)hE}\,[\psi|\eta]\,(e^{-(i/\hbar)tE}\eta) = \\
&= \int_{\mathbb{R}^n} (e^{-(i/\hbar)tE}e^{-(i/\hbar)hE})\,[\psi\,(0)\,|\eta]\,\eta = \\
&= \int_{\mathbb{R}^n} e^{-(i/\hbar)(t+h)E}\,[\psi\,(0)\,|\eta]\,\eta = \\
&= e^{-(i/\hbar)(t+h)H}(\psi).
\end{aligned}$$

So the curve $U : \mathbb{R} \to \mathcal{L}(\mathcal{S}'_n)$ defined by

$$U(t) = e^{-(i/\hbar)tH},$$

for every real t, is a one-parameter group of endomorphisms.

25.3.4 The abstract Heat equation on $\mathcal{S}'_n$

Theorem (the Heat equation). *Let $A \in \mathcal{L}(\mathcal{S}'_n)$ be an $^{\mathcal{S}}$ diagonalizable operator with real and positive eigenvalues. Then, for every $\psi_0 \in \mathcal{S}'_n$, there exists a unique differentiable curve $\psi : \mathbb{R}_{\geq} \to \mathcal{S}'_n$, such that*

$$\psi'\,(t) = -A\,(\psi\,(t))\,,$$

and $\psi\,(0) = \psi_0$. Explicitly, the unique solution is given by

$$\psi\,(t) = e^{-tA}(\psi_0),$$

for every $t \geq 0$.

Chapter 26

$^{\mathcal{S}}$Diagonalizable equations

26.1 Superpositions with respect to curves

Let $b : I \to \mathcal{S}'_n$ be a curve in the space $\mathcal{S}'_n$, and let v be an $^{\mathcal{S}}$family in $\mathcal{S}'_n$. The superposition of the $^{\mathcal{S}}$family v with respect to the curve b (in $^I\mathcal{S}'_n$) is the curve

$$\int_{\mathbb{R}^n} bv : I \to \mathcal{S}'_n,$$

defined by

$$\int_{\mathbb{R}^n} bv : t \mapsto \int_{\mathbb{R}^n} b(t)v,$$

for every t in I.in $^I\mathcal{S}'_n$;

26.2 Solution of the eigenrepresentations

Theorem. *Let $H : \mathcal{S}'_n \to \mathcal{S}'_n$ be an ${}^{\mathcal{S}}$ diagonalizable operator, let (E, e) be an ${}^{\mathcal{S}}$ eigensolution of H, that is a pair (E, e) in the Cartesian product of the space $\mathcal{O}_n^{(n)}$ times the space $\mathcal{S}(\mathbb{R}^n, \mathcal{S}'_n)$ such that $H(e) = Ee$. Let X be a subspace of the space of curves ${}^I\mathcal{S}'_n$ and let*

$$D : X \to {}^I\mathcal{S}'_n$$

be an operator defined on that subspace X with values in the space of curves ${}^I\mathcal{S}'_n$. Assume that

1) *there exists a curve a in X such that the superposition*

$$\int_{\mathbb{R}^n} ae$$

belongs to the subspace X and such that

$$D\left(\int_{\mathbb{R}^n} ae\right) = \int_{\mathbb{R}^n} (Da)e,$$

where the superposition of an ${}^{\mathcal{S}}$family v with respect to a curve b in ${}^I\mathcal{S}'_n$ is the curve

$$\int_{\mathbb{R}^n} bv : t \mapsto \int_{\mathbb{R}^n} b(t)v,$$

in ${}^I\mathcal{S}'_n$;

2) *the curve a satisfies the equation*

$$D(a)(t) = Ea(t),$$

for every point t in I.

Then the curve

$$u = \int_{\mathbb{R}^n} ae$$

is a solution of the equation

$$D(u)(t) = H(u(t)),$$

for every t in the interval I. Moreover, if the property 1) of action of the operator D under the superposition sign, is true for every curve a in the domain of the operator, then a curve

$$u = \int_{\mathbb{R}^n} be$$

is a solution of the equation if and only if the coefficient curve b satisfies the equation

$$D(b)(t) = Eb(t),$$

for every point t in I.

Proof. **First part.** We have, for any t in I,

$$\begin{aligned}
D\left(\int_{\mathbb{R}^n} ae\right)(t) &= \left(\int_{\mathbb{R}^n} D(a)e\right)(t) = \\
&= \int_{\mathbb{R}^n} D(a)(t)e = \\
&= \int_{\mathbb{R}^n} Ea(t)e = \\
&= \int_{\mathbb{R}^n} a(t)(Ee) = \\
&= \int_{\mathbb{R}^n} a(t)H(e) = \\
&= H\left(\int_{\mathbb{R}^n} a(t)e\right),
\end{aligned}$$

as we claimed. **Second part.** We have to prove that if a curve

$$u = \int_{\mathbb{R}^n} be$$

satisfies the equation

$$D(u)(t) = H(u(t)),$$

for every t then the coefficient b satisfies the equation

$$D(b)(t) = Eb(t).$$

Indeed, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} D(b)(t)e &= \left(\int_{\mathbb{R}^n} D(b)e\right)(t) = \\
&= D\left(\int_{\mathbb{R}^n} be\right)(t) = \\
&= H\left(\int_{\mathbb{R}^n} b(t)e\right) = \\
&= \int_{\mathbb{R}^n} b(t)H(e) = \\
&= \int_{\mathbb{R}^n} b(t)Ee = \\
&= \int_{\mathbb{R}^n} Eb(t)e,
\end{aligned}$$

for every t, and, since the family e is linearly independent, we deduce that the coefficient distribution $D(b)(t)$ must be equal to the coefficient distribution $Eb(t)$. ■

Remark. It is clear that any differential operator

$$D : \partial^1(I, \mathcal{S}'_n) \to {}^I\mathcal{S}'_n$$

with constant coefficient - with $\partial^1(I, \mathcal{S}'_n)$ we denote the space of all differentiable curves defined on the non-degenerate interval I and with value in the space $\mathcal{S}'_n$ - verifies the equality

$$D\left(\int_{\mathbb{R}^n} av\right) = \int_{\mathbb{R}^n} (Da)v,$$

for any coefficient curve a in $\mathcal{S}'_n$ and every ${}^{\mathcal{S}}$family in $\mathcal{S}'_n$. So that the above resolution of the Schrödinger equation can be considered a particular case of the above theorem.

26.3 The eigenrepresentations

Definition (of eigenrepresentation of an equation). *Let $H : \mathcal{S}'_n \to \mathcal{S}'_n$ be an ${}^{\mathcal{S}}$diagonalizable operator with ${}^{\mathcal{S}}$ eigensolution a pair (E, e). Let X be a subspace of the space of curves ${}^I\mathcal{S}'_n$ and let*

$$D : X \to {}^I\mathcal{S}'_n$$

be an operator defined on that subspace X with values in the space of curves ${}^I\mathcal{S}'_n$. We say that the equation

$$D(b) = Eb,$$

where Eb is the curve $t \mapsto Eb(t)$, is the representation in the basis e of the equation

$$D(u) = H \circ u.$$

Since the curve b is the representation of the curve

$$\int_{\mathbb{R}^n} be,$$

in the basis e, we can conclude the following corollary.

Corollary. *In the conditions of the above theorem, a curve u is a solution of the equation*

$$D(u) = H \circ u,$$

if and only if the representation of u is a solution of the eigenrepresentation of the equation itself.

Remark. Let us observe that the eigenrepresentation $D(b) = Eb$ is indeed an eigenvalue equation but not in a vector space, it is an eigen vector equation in a module. Indeed, the space of curves ${}^I\mathcal{S}'_n$ is a module with respect to the multiplication by $\mathcal{O}_M$ functions, that defined by

$$\mathcal{O}_M(n) \times^I \mathcal{S}'_n \to^I \mathcal{S}'_n : (f, b) \mapsto fb$$

with

$$fb : I \to {}^I\mathcal{S}'_n : t \mapsto Eb(t),$$

note that in the equation $D(b) = Eb$, the function E is one eigenvalue (belonging to the ring of the module) and the curve b is one eigenvector corresponding to the eigenvalue E. From a physical point of view D can be see as an observable but the states are curves (dynamical evolutions) and the observed physical measures are functions.

26.4 Solutions in functional form

Theorem. *Assume, in the conditions of the above theorem that the curve coefficient distribution a have the form*

$$a(t) = (f_t \circ E)a_0,$$

for every t in I, where we put $a_0 = a(0)$. Then, the solution of the equation is the curve defined by

$$t \mapsto f_t(H)(u_0),$$

where by f_t we mean a complex function defined on the eigenspectrum of the operator H, for every t in the interval I, such that the composition $f_t \circ E$ is function of class $\mathcal{O}_M$.

Proof. Note first that

$$\begin{aligned} u(0) &= \left(\int_{\mathbb{R}^n} ae\right)(0) = \\ &= \int_{\mathbb{R}^n} a(0)e = \\ &= \int_{\mathbb{R}^n} a_0 e, \end{aligned}$$

so that $a_0 = [u_0|e]$. Moreover, for every t in I, we have

$$\begin{aligned}\left(\int_{\mathbb{R}^n} ae\right)(t) &= \left(\int_{\mathbb{R}^n} ae\right)(t) = \\ &= \int_{\mathbb{R}^n} a(t)e = \\ &= \int_{\mathbb{R}^n} (f_t \circ E)a_0 e = \\ &= \int_{\mathbb{R}^n} (f_t \circ E)[u_0|e]e = \\ &= f_t(H)(u_0),\end{aligned}$$

as we claimed. ■

Chapter 27

Feynman propagators

27.1 Propagators as family-valued functions

Definition (of Feynman propagator). *We call a function*

$$G : \mathbb{R}^2 \to \mathcal{S}(\mathbb{R}^n, \mathcal{S}'_n)$$

a Feynman (or group) $\mathcal{S}$-propagator on the space $\mathcal{S}'_n$ if

- *1) for every real t, $G(t,t) = \delta$;*
- *2) for every pair of reals t_0 and t, the family $G(t_0,t)$ is invertible and*

$$G(t_0,t) = G(t,t_0)^{-1};$$

- *3) for every triple of times t_0, t_1 and t_2, we have*

$$G(t_0,t_2) = G(t_0,t_1) \cdot G(t_1,t_2).$$

Remark. Note that a group $\mathcal{S}$-propagator G is a $\mathcal{S}$-family-valued function, indeed

$$G(t_0, t) = (G(t_0, t)_y)_{y \in \mathbb{R}^n}.$$

Remark. Note that the property 2 of the definition derives from properties 1 and 3. In fact, from property 3 we have, for $t_2 = t_0$,

$$G(t_0, t_0) = G(t_0, t_1) \cdot G(t_1, t_0),$$

and then, applying 1 we obtain 2.

Definition (propagator of a process). *Let $u : \mathbb{R} \to \mathcal{S}'_n$ be a dynamic process. We say that a propagator*

$$G : \mathbb{R}^2 \to \mathcal{S}(\mathbb{R}^n, \mathcal{S}'_n)$$

is an $\mathcal{S}$-Green function, or an $^{\mathcal{S}}$propagator, for the process u if

$$u(t) = \int_{\mathbb{R}^n} u(t_0) G(t_0, t),$$

for every t_0 and t in T.

Remark. Hence $G(t_0, t)$ is a family such that the state of u at the time t is the superposition of the family $G(t_0, t)$ with respect to the system of coefficients coinciding with the state of u at t_0.

If a process u admits a Feynman propagator, then it is a *strongly causal* and *reversible* process. In fact, by definition, the state of the process at every time t_0, determines the state of the process u at every other time t. Moreover, the process u is reversible, since, if

$$u(t) = \int_{\mathbb{R}^n} u(t_0) G(t_0, t),$$

then

$$u(t_0) = \int_{\mathbb{R}^n} u(t) G(t_0, t)^{-1}.$$

27.2 Evolution operators

Definition (evolution operator). *An evolution operator*

$$E : \mathbb{R} \times \mathcal{S}'_n \to C^0(\mathbb{R}, \mathcal{S}'_n),$$

is an operator that sends every initial condition (t_0, u_0) belonging to $\mathbb{R} \times \mathcal{S}'_n$ into a continuous dynamical process

$$E_{(t_0,u_0)} : \mathbb{R} \to \mathcal{S}'_n$$

such that

- *1)* $E_{(t_0,u_0)}(t_0) = u_0$*;*
- *2) if* $E_{(t,u)}(t_0) = u_0$ *then* $E_{(t_0,u_0)}(t) = u$*;*
- *3) if* $E_{(t_0,u_0)}(t_1) = u_1$ *and* $E_{(t_1,u_1)}(t_2) = u_2$ *then* $E_{(t_0,u_0)}(t_2) = u_2$*.*

Definition (the propagator of an evolution). *Let E be an evolution operator. We say that a function*

$$G : \mathbb{R}^2 \to \mathcal{S}(\mathbb{R}^n, \mathcal{S}'_n)$$

is an $\mathcal{S}$-Green function, or an ${}^{\mathcal{S}}$propagator, for the evolution operator E if

$$E_{(t_0,u_0)}(t) = \int_{\mathbb{R}^n} u_0 G(t_0, t),$$

for every u_0 in $\mathcal{S}'_n$ and for every t_0 and t in $\mathbb{R}$.

Theorem (characterization of evolution operators). *Let $E : \mathbb{R}\times\mathcal{S}'_n \to C^0(\mathbb{R}, \mathcal{S}'_n)$, be an operator that sends every initial condition (t_0, u_0) belonging to $\mathbb{R} \times \mathcal{S}'_n$ into a continuous dynamical process $E_{(t_0,u_0)} : \mathbb{R} \to \mathcal{S}'_n$, and let*

$$G : \mathbb{R}^2 \to \mathcal{S}(\mathbb{R}^n, \mathcal{S}'_n),$$

be a family-valued function such that

$$E_{(t_0,u_0)}(t) = \int_{\mathbb{R}^n} u_0 G(t_0, t),$$

for every state u_0 in $\mathcal{S}'_n$ and for every t_0 and t in $\mathbb{R}$. Then, E is an evolution operator if and only if G is a Feynman propagator.

Proof. Assume E be an evolution operator, we must verify the properties of the Feynman propagator.

1) Let t be a real. for every u, we have

$$u = E_{(t,u)}(t) = \int_{\mathbb{R}^n} uG(t, t);$$

thus $G(t, t) = \delta$.

2) Consider two instant of time t_0 and t. For every u_0 in $\mathcal{S}'_n$ set

$$u := E_{(t_0,u_0)}(t) = \int_{\mathbb{R}^n} u_0 G(t_0, t);$$

by axiom 2, we have

$$u_0 = E_{(t,u)}(t_0) = \int_{\mathbb{R}^n} uG(t, t_0),$$

then, consequently

$$\begin{aligned} u_0 &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} u_0 G(t_0, t) \right) G(t, t_0) = \\ &= \int_{\mathbb{R}^n} u_0 \int_{\mathbb{R}^n} G(t_0, t) G(t, t_0). \end{aligned}$$

This is equivalent to

$$\int_{\mathbb{R}^n} G(t_0, t) G(t, t_0) = \delta,$$

and then

$$G(t_0, t) = G(t, t_0)^{-1}.$$

3) Consider three instants of time t_0, t_1 and t_2, we have

$$G(t_0, t_2) = G(t_0, t_1) \cdot G(t_1, t_2).$$

In fact, if $E_{(t_0,u_0)}(t_1) = u_1$ and $E_{(t_1,u_1)}(t_2) = u_2$ then $E_{(t_0,u_0)}(t_2) = u_2$. Now $E_{(t_0,u_0)}(t_1) = u_1$ is equivalent to

$$\int_{\mathbb{R}^n} u_0 G(t_0, t_1) = u_1,$$

and $E_{(t_1,u_1)}(t_2) = u_2$ is equivalent to

$$\int_{\mathbb{R}^n} u_1 G(t_1, t_2) = u_2.$$

Hence

$$\begin{aligned} u_2 &= \int_{\mathbb{R}^n} u_1 G(t_1, t_2) = \\ &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} u_0 G(t_0, t_1) \right) G(t_1, t_2) = \\ &= \int_{\mathbb{R}^n} u_0 \int_{\mathbb{R}^n} G(t_0, t_1) G(t_1, t_2). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} u_2 &= E_{(t_0,u_0)}(t_2) = \\ &= \int_{\mathbb{R}^n} u_0 G(t_0, t_2), \end{aligned}$$

and so

$$\int_{\mathbb{R}^n} u_0 G(t_0, t_2) = \int_{\mathbb{R}^n} u_0 \left(\int_{\mathbb{R}^n} G(t_0, t_1) G(t_1, t_2) \right),$$

thus

$$G(t_0, t_2) = G(t_0, t_1) \cdot G(t_1, t_2).$$

The vice versa is a simple calculation. ■

27.3 Operatorial propagators

Definition (operatorial propagator). *An operatorial propagator*

$$S : \mathbb{R}^2 \to \mathcal{L}(\mathcal{S}'_n),$$

is an operator-valued function verifying the following properties:

- *1)* $S(t_0, t_0) = (\cdot)_{\mathcal{S}'_n}$*;*
- *2)* $S(t_0, t)^{-1} = S(t, t_0)$*;*
- *3)* $S(t_0, t_1) \circ S(t_1, t_2) = S(t_0, t_2)$*.*

The following evident theorem shows the relation among operatorial and Feynman propagators.

Theorem. *Let* $S : \mathbb{R}^2 \to \mathcal{L}(\mathcal{S}'_n)$ *be an operator-valued function, and let*

$$G : \mathbb{R}^2 \to \mathcal{S}(\mathbb{R}^n, \mathcal{S}'_n),$$

be a family-valued function such that

$$G(t_0, t)_y = S(t_0, t)\delta_y.$$

Then, S *is an operatorial propagator if and only if* G *is a Feynman propagator.*

Let $u : \mathbb{R} \to \mathcal{S}'_n$ be a process. u is said generated by an operator-valued function

$$S : \mathbb{R}^2 \to \mathcal{L}(\mathcal{S}'_n),$$

if, for every pair of times t and t_0, we have

$$u(t) = S(t_0, t)u(t_0).$$

Theorem. *Let* $S : \mathbb{R}^2 \to \mathcal{L}(\mathcal{S}'_n)$ *be an operatorial propagator, and let* $u : \mathbb{R} \to \mathcal{S}'_n$ *be a process generated by* S*. Then, the function*

$$G : \mathbb{R}^2 \to \mathcal{S}(\mathbb{R}^n, \mathcal{S}'_n),$$

defined by

$$G(t_0, t)_y = S(t_0, t)\delta_y,$$

for every y *in* $\mathbb{R}^n$*, i.e., by*

$$G(t_0, t) = S(t_0, t)\delta,$$

is a Green function for u*.*

Proof. In fact, by the $\mathcal{S}$-linearity of $S(t_0, t)$ we have

$$\begin{aligned} u(t) &= S(t_0,t)\int_{\mathbb{R}^n} u(t_0)\delta = \\ &= \int_{\mathbb{R}^n} u(t_0)S(t_0,t)\delta, \end{aligned}$$

as we desired. ■

Theorem. *Let the operator valued function $S : T^2 \to \mathcal{L}(\mathcal{S}'_n)$ be of the form*

$$S(t_0, t) = \exp(-i(t - t_0)H),$$

for a certain $\mathcal{S}$-linear operator H. Then S is an operatorial propagator.

Proof. It's enough to prove that the Green function of S is a Feynman propagator. For every times t_0 and t, we have - by definition of Green function, the exponential assumption, expanding in the Dirac basis and applying the $\mathcal{S}$-linearity of the operator $\exp(-i(t - t_1)H)$ -

$$\begin{aligned} G(t_0,t) &= S(t_0,t)\delta = \\ &= \exp(-i(t-t_0)H)\delta = \\ &= \exp(-i(t-t_1+t_1-t_0)H)\delta = \\ &= \exp(-i(t-t_1)H)\circ\exp(-i(t_1-t_0)H)(\delta) = \\ &= \exp(-i(t-t_1)H)(G(t_0,t_1)) = \\ &= \exp(-i(t-t_1)H)(\int_{\mathbb{R}^n} G(t_0,t_1)\delta) = \\ &= \int_{\mathbb{R}^n} G(t_0,t_1)\exp(-i(t-t_1)H)\delta = \\ &= \int_{\mathbb{R}^n} G(t_0,t_1)G(t_1,t) = \\ &= G(t_0,t_1)\cdot G(t_1,t), \end{aligned}$$

as we desired. ■

27.4 Feynman propagator of a free particle

Let us evaluate the Green function of the evolution of a free particle. The operatorial propagator, in this case, is of the form

$$S(t_0, t) = \exp(-\frac{i}{\hbar}(t - t_0)H),$$

where H is the $\mathcal{S}$-linear operator defined by

$$H = \frac{1}{2m} P^2,$$

where

$$P : \mathcal{S}_1' \to \mathcal{S}_1' : u \mapsto -i\hbar u'$$

is the momentum operator on $\mathcal{S}_1'$. Moreover, let φ be the Dirac-orthonormal standard eigenbasis of P, that is the following family of regular tempered distributions

$$\varphi = \left(\frac{1}{\sqrt{2\pi\hbar}} \left[e^{-\frac{i(p|\cdot)}{\hbar}} \right] \right)_{p\in\mathbb{R}}.$$

It's obvious that $P\varphi_p = p\varphi_p$, for every real p. We have

$$\begin{aligned}
G(0,t)_q &= \exp(-itH)(\delta_q) = \\
&= \exp(-itH)(\int_{\mathbb{R}^n} (\delta_q \mid \varphi)\varphi) = \\
&= \int_{\mathbb{R}^n} (\delta_q \mid \varphi) \exp(-itH)(\varphi) = \\
&= \int_{\mathbb{R}^n} [\delta_q \mid \varphi] \exp(-it\frac{(\cdot)^2}{2m})\varphi = \\
&= \int_{\mathbb{R}^n} \exp(-it\frac{(\cdot)^2}{2m}) \left(\int_{\mathbb{R}^n} \delta_q \varphi^{-1} \right) \varphi = \\
&= \int_{\mathbb{R}^n} \exp(-it\frac{(\cdot)^2}{2m}) \left(\int_{\mathbb{R}^n} \delta_q \overline{\varphi} \right) \varphi = \\
&= \int_{\mathbb{R}^n} \exp(-it\frac{(\cdot)^2}{2m}) \overline{\varphi_q} \varphi.
\end{aligned}$$

The family φ is a regular family, and denoted by f_q the $\mathcal{S}$-function generating the regular tempered distribution φ_q, the function

$$\exp(-it\frac{(\cdot)^2}{2m})\overline{f_q}$$

is an $\mathcal{S}$-function (it the product of a bounded function by an $\mathcal{S}$-function). Hence the superposition

$$\int_{\mathbb{R}^n} \exp(-it\frac{(\cdot)^2}{2m}) \overline{\varphi_q} \varphi$$

is a regular distribution of class $\mathcal{S}$; say g_q the generating $\mathcal{S}$-function, it's simple to see that

$$g_q(q') = \int_{\mathbb{R}^n} \exp(-it\frac{p^2}{2m}) \overline{f_q}(p) f_p(q^{'}) dp,$$

in fact, the superposition

$$[g_q] = \int_{\mathbb{R}^n} \exp(-it\frac{(\cdot)^2}{2m}) \overline{\varphi_q} \varphi$$

is the Fourier transform of the tempered distribution

$$\exp(-it\frac{(\cdot)^2}{2m})\overline{\varphi_q},$$

and so g_p is the Fourier transform of the function

$$p \mapsto \exp(-it\frac{p^2}{2m})\overline{f_q}(p).$$

Substituting the expression of f, we have

$$\begin{aligned}
g_q(q') &= \int_{\mathbb{R}^n} \exp(-it\frac{p^2}{2m})\overline{f_q}(p) f_p(q')dp = \\
&= \frac{1}{2\pi\hbar}\int_{\mathbb{R}^n} \exp(-it\frac{p^2}{2m})\exp(\frac{ipq}{\hbar})\exp(-\frac{ipq'}{\hbar})dp = \\
&= \frac{1}{2\pi\hbar}\int_{\mathbb{R}^n} \exp(-it\frac{p^2}{2m} + \frac{ipq}{\hbar} - \frac{ipq'}{\hbar})dp.
\end{aligned}$$

The last integral is classic, after the standard calculation, we, at last, conclude

$$G(0,t)_q = [g_q] = \left[\left(\frac{m}{2\pi it}\right)^{1/2} e^{im((\cdot)-q)^2/2t}\right].$$

Chapter 28

Evolutions in the space $\mathcal{S}'_n$

28.1 Introduction

We start from the abstract Schrödinger equation

$$i\hbar u'(t) = H(t)u(t),$$

where: $H(t)$ is a linear operator on some topological vector space (X, σ), for every real time t; u is a σ-differentiable curve in that space; and u' is the derivative of u with respect to the topology σ. When X is an Hilbert space, physicists solve formally this equation using propagators, more precisely, for every time t and for every initial condition $(t_0, u_0) \in \mathbb{R} \times X$, they find a unique differentiable solution u starting from the initial condition defined by

$$u(t) = S(t_0, t)u_0,$$

for every time t, where the operator $S(t_0, t)$ is given by the *Dyson formula*

$$S(t_0, t) = \exp\left(\frac{1}{i\hbar}\int_{t_0}^{t} H(\tau)d\tau\right).$$

Unfortunately, it is not true that in Hilbert spaces the abstract Schrödinger equations has always a solution (because the operators $H(t)$, that are of interest in Quantum Mechanics, are always unbounded-linear operators and not-everywhere defined) and moreover in those spaces the Dyson formula has not a precise mathematical sense. In this chapter we solve the problem in the space of tempered distributions (where the operators of Quantum Mechanics are always continuous and everywhere defined) we will find a unique solution, for every initial condition, in the propagator-form desired by physicists, and we shall give a meaning to the Dyson formula in this context. We solve in this space the abstract heat evolution equation too, in view of financial applications.

28.2 Integral of function from $\mathbb{R}^m$ to $\mathcal{S}'_n$

Let us define the integral of function from $\mathbb{R}^m$ to $\mathcal{S}'_n$.

Definition. *Let u be a function from $\mathbb{R}^m$ into the distributions space $\mathcal{S}'_n$ and let ϕ be a test function in $\mathcal{S}_n$. We define image of ϕ by u the function*

$$u(\phi) : \mathbb{R}^m \to \mathbb{C} : u(\phi)(p) = u(p)(\phi).$$

The images by u of the test functions give informations on the entire function u, as, for instance, states the following.

Theorem. *Let u be a function from $\mathbb{R}^m$ into the space $\mathcal{S}'_n$. Then, u is continuous with respect to the topology $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$, if and only if the function*

$$u(\phi) : \mathbb{R}^m \to \mathbb{C} : u(\phi)(t) = u(t)(\phi),$$

is continuous, for every test function ϕ in $\mathcal{S}_n$.

Proof. In fact, u is continuous at t_0, with respect to the topology $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$, if and only if, for every test function ϕ (recall that $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$ is induced by the family of seminorms $|\langle \cdot, \phi \rangle|$), the semi-distance

$$|\langle u(t) - u(t_0), \phi \rangle|$$

vanishes, as $t \to t_0$; on the other hand

$$\lim_{t \to t_0} |(u(t) - u(t_0))(\phi)| = \lim_{t \to t_0} |u(\phi)(t) - u(\phi)(t_0)|,$$

so u is $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$-continuous at t_0 if and only if for every test function ϕ, the complex function $u(\phi)$ is continuous at t_0. ■

The preceding theorem allow us to associate with the function u, in an extremely natural way, an operator from $\mathcal{S}_n$ into the space of continuous complex functions $C^0(\mathbb{R}^m, \mathbb{C})$, which in the following we shall denote by C^0_m.

Definition. *Let u be a function from $\mathbb{R}^m$ into the space $\mathcal{S}'_n$, continuous in the topology $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$. The operator $\widehat{u} : \mathcal{S}_n \to C^0_m$ defined by $\widehat{u}(\phi) = u\,(\phi)$, for every test function ϕ, is called the operator induced by u.*

The operator associated with a continuous function in the topology $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$ is automatically continuous in the natural topologies of the two spaces $\mathcal{S}_n$ and C^0_m.

Theorem. *Let u be a function from $\mathbb{R}^m$ into the space $\mathcal{S}'_n$ continuous in the topology $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$ and let a be a Radon-measure on $\mathbb{R}^m$ with compact support. Then, the composition $a \circ \widehat{u}$ is $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$-continuous for every a in $C^{0\prime}_m$ and, consequently, the operator $\widehat{u}$ is continuous in the topologies $\sigma(\mathcal{S}_n, \mathcal{S}'_n)$ and $\sigma(C^0_m, C^{0\prime}_m)$.*

Proof. Denote by $C^{0\prime}_m$ the dual of C^0_m with respect to its standard locally convex topology. The dual $C^{0\prime}_m$ is, by definition, the space of Radon-measures on $\mathbb{R}^m$ with compact support. Note, that the linear operator $\widehat{u}$ is continuous with respect to the topologies $\sigma(\mathcal{S}_n, \mathcal{S}'_n)$ and $\sigma(C^0_m, C^{0\prime}_m)$ if and only if for every a in $C^{0\prime}_m$, i.e., for every Radon-measure on $\mathbb{R}^m$ with compact support, the functional $a \circ \widehat{u}$ is $\sigma(\mathcal{S}_n, \mathcal{S}'_n)$-continuous (see, for example, [5],[8]). Note, moreover, that this is true for a sequentially dense subset of $C^{0\prime}_m$, the linear subspace spanned by the delta-measures on $\mathbb{R}^m$, so by the Banach-Steinhaus theorem $a \circ \widehat{u}$ is $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$ -continuous, being $\mathcal{S}_n$ a Fréchet space and thus a Baire-space (see Banach-Steinhaus on Laurent Schwartz's *Functional Analysis*). ■

Definition (integral of a function with respect to a measure). *In the conditions of the above theorem the composition $a \circ \widehat{u}$ is said the integral on $\mathbb{R}^m$ of the function u with respect to the measure a and it is denoted by $\int_{\mathbb{R}^m} u\,(a)$ or by $a(u)$.*

Our aim is, generalizing the above definition, to integrate u on every bounded Borel-subset of $\mathbb{R}^m$, with respect to a Radon-measure on $\mathbb{R}^m$ eventually not with compact support. To this end we have the following theorem.

Theorem. *Let a be a Radon-measure on $\mathbb{R}^m$, and let B be a bounded Borel subset of $\mathbb{R}^m$. Consider the functional $a_B : C^0_m \to \mathbb{C}$ defined*

$$f \mapsto \int_B f(\mu_a) = \int_{\mathbb{R}^m} \chi_B f(\mu_a),$$

where μ_a is the Borel-measure associated canonically to a by the Riesz theorem. Then, a_B is a Radon-measure with compact support on $\mathbb{R}^m$.

Proof. We must prove that a_B is continuous in the standard topology of C^0_m (that is the topology of compact convergence), it is the locally convex topology induced by the family of seminorms $(q_K)_{K\in\mathcal{K}(\mathbb{R}^m)}$ indexed by the set of compact subset of $\mathbb{R}^m$ and defined by

$$q_K(f) := \max_K |f| \, .$$

Remember that a linear functional T on C^0_m is continuous in this topology if there exists a positive real number M and a seminorm q_K such that, for every function f in C^0_m is

$$|T(f)| \leq M q_K(f).$$

Indeed, we have

$$\begin{aligned} |a_B(f)| &= \left| \int_B f d\mu_a \right| \leq \\ &\leq \sup_B |f| \cdot |\mu_a| (B) \leq \\ &\leq |\mu_a| (B) \cdot \max_{\overline{B}} |f| \, , \end{aligned}$$

so, since the closure of B is compact, a_B is continuous with respect to the topology of compact convergence, and then it is a compact-support Radon-measure on $\mathbb{R}^m$. ■

We call the functional a_B of the preceding theorem restriction of a to B. We can give, so, the following definition.

Definition (integral on a bounded Borel set with respect to a measure). *In the above conditions the composition $a_B \circ \widehat{u}$ is said the integral on B of the function u with respect to the measure a, it is denoted by $\int_B uda$. In other terms we put*

$$\int_B uda := \int_{\mathbb{R}^m} uda_B.$$

Remark. Let B be a bounded Borel-subset of $\mathbb{R}^m$, and a_B be the restriction of a Radon-measure a to B. Then, a_B is the Radon-measure on $\mathbb{R}^m$ associated with the Borel-measure defined for every Borel-set E by $(\mu_a)_B (E) = \mu_a(E \cap B)$. In fact, we have

$$\begin{aligned} T(f) &= \int_{\mathbb{R}^m} f \ (\mu_a)_B = \\ &= \int_B f \ \mu_a = \\ &= a_B(f). \end{aligned}$$

28.3 Integral of functions of $\mathbb{R}^m$ into $\mathcal{L}(\mathcal{S}'_n)$

In this section we give the definition of integral of an operator-valued function H, namely of functions from $\mathbb{R}^m$ into $\mathcal{L}(\mathcal{S}'_n)$ (the space of $\mathcal{S}$-endomorphisms on the space $\mathcal{S}'_n$, see [1] or [3]) in a Radon-measure on $\mathbb{R}^m$. Analogously to the preceding section it is possible to prove the following proposition.

Proposition. *A function $H : \mathbb{R}^m \to \mathcal{L}(\mathcal{S}'_n)$ is continuous with respect to the pointwise topology induced by the topology $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$ if and only if the function $H(u) : \mathbb{R}^m \to \mathcal{S}'_n$ defined by*

$$H(u)(p) := H(p)(u),$$

for every p, is continuous in the topology $\sigma(\quad \mathcal{S}'_n)$, for every tempered distribution u.

Definition. *Let $H : \mathbb{R}^m \to \mathcal{L}(\mathcal{S}'_n)$ be a continuous function. We define integral of H in a Radon-measure a the operator*

$$a(H) = \int_{\mathbb{R}^m} H da : \mathcal{S}'_n \to \mathcal{S}'_n : u \mapsto \int_{\mathbb{R}} H(u)\, da.$$

The integral of H in the Lebesgue-measure is denoted by $\int_{\mathbb{R}^m} H$.

Theorem. *Let $H : \mathbb{R}^m \to \mathcal{L}(\mathcal{S}'_n)$ be a continuous function. Then, the integral of H in a Radon-measure a, is continuous in the topology $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$ and consequently $\mathcal{S}$-linear.*

Proof. Recall that an linear operator T from a seminormed space (E, p) to another seminormed space (F, q) is continuous if for every seminorm q_j of the family q there is a seminorm p_i of the family p and a positive real M such that $q_j(T(x)) \leq M p_i(x)$. Let ϕ be a test function, we have, then, to prove that there are another test function ψ and a positive real M such that

$$\left| \left\langle \phi, \int_{\mathbb{R}^m} H(u) da \right\rangle \right| \leq M \left| \langle \psi, u \rangle \right|.$$

Concluding we have, if K is the compact support of a, that there exist an m-index p_0 (by the Weierstrass theorem) and a test function ψ with a real M (by continuity of the operator H_{p_0}) such that

$$\begin{aligned}
\left| \left\langle \phi, \int_{\mathbb{R}^m} H(u) da \right\rangle \right| &= \left| \int_{\mathbb{R}^m} H(u)(\phi)\, da \right| \leq \\
&\leq |a|(K) \sup_K |H(u)(\phi)| = \\
&= |a|(K) \left| H_{p_0}(u)(\phi) \right| = \\
&= |a|(K) \left| \langle \phi, H_{p_0}(u) \rangle \right| \leq \\
&\leq (|a|(K) M) \left| \langle \psi, u \rangle \right|.
\end{aligned}$$

So the integral of H, by a Radon-measure with compact support, is a continuous operator in the weak* topology, and consequently (D. Carfi, *Topological characterizations of* $\mathcal{S}$ *-linearity*, preprint) is $\mathcal{S}$-linear. ■

The last definition concerns integral on bounded Borel-set.

Definition (integral on a bounded Borel set with respect to a measure). *Let a be a Radon-measure on $\mathbb{R}^m$, B be a bounded Borel-subset of $\mathbb{R}^m$, a_B be the restriction of a to B and let $H : \mathbb{R}^m \to \mathcal{L}(\mathcal{S}'_n)$ be a continuous function. We define*

$$\int_B H da := \int_{\mathbb{R}^m} H da_B.$$

The integral of H in the Lebesgue measure is denoted by $\int_B H$.

28.4 Curves of $\mathcal{S}$-linear operators

From this section to the end we shall use intensively the $\mathcal{S}$ -linear Algebra, that will be assumed known. For the fundamentals of $\mathcal{S}$-linear Algebra and for the functional calculus of $\mathcal{S}$ -diagonalizable endomorphisms see [1], [3]. We observe only that if A is an $\mathcal{S}$-diagonalizable operator and if α is one of its $\mathcal{S}$-eigenbasis for the space, then there is only a function a associating to each n-tuple p the unique complex eigenvalues a_p of A on the eigenvector α_p. We call a the system of eigenvalues of A on the $\mathcal{S}$-basis α. We remark that this function a is necessarily a smooth function of class $\mathcal{O}_M$. In fact, from $A\alpha_p = a_p\alpha_p$ we have

$$A\,(\alpha)^\wedge\,(\phi)(p) = a(p)\alpha^\wedge(\phi)(p);$$

the functions $A\,(\alpha)^\wedge\,(\phi)$ and $\alpha^\wedge(\phi)$ are, by definition of $\mathcal{S}$-family, of class $\mathcal{S}$; since α is an $\mathcal{S}$-eigenbasis we have that the operator $\alpha^\wedge$ is surjective and injective. Let us prove that a is smooth at every p_0, let ϕ be a test function such that $\alpha^\wedge(\phi)(p_0)$ is different from 0 (it certainly exists because $\alpha^\wedge$ is surjective), it follows that there is a neighborhood of p_0 in which $\alpha^\wedge(\phi)$ is different from 0, for each p in this neighborhood we have

$$a(p) = \frac{A\,(\alpha)^\wedge\,(\phi)(p)}{\alpha^\wedge(\phi)(p)},$$

then a is smooth at p_0. Moreover, since

$$A\,(\alpha)^\wedge\,(\phi) = a\alpha^\wedge(\phi)$$

and since $\alpha^\wedge$ is surjective, the product of the function a with all the functions of class $\mathcal{S}$ is yet of class $\mathcal{S}$, and then a is of class $\mathcal{O}_M$.

Now, let $A : \mathbb{R} \to \mathcal{L}(\mathcal{S}'_n)$ be a curve of $\mathcal{S}$-diagonalizable endomorphisms with a common $\mathcal{S}$-eigenbasis α, the family $a = (a(t))_{t\in\mathbb{R}}$ of the systems of eigenvalues of the operators $A(t)$ on α is called the system of eigenvalues of the curve A. For every p, we define the function $a_p : \mathbb{R} \to \mathbb{C}$ associating to t the value of the function $a(t)$ on the n-tuple p: $a_p(t) := a(t)(p)$. If A is a continuous curve of $\mathcal{S}$-diagonalizable endomorphisms with a common $\mathcal{S}$ -eigenbasis α, the functions a_p are continuous. In fact, from the equality

$$A(t)(\alpha_p) = a_p(t)\alpha_p,$$

if ϕ does not belong to the kernel of α_p, we deduce

$$a_p(t) = \frac{A(\alpha_p)(t)(\phi)}{\alpha_p(\phi)},$$

for every t, so a_p is proportional to the continuous function $A(\alpha_p)^\wedge(\phi)$ and then it is continuous.

Theorem. *Let A be a differentiable curve of $\mathcal{S}$-diagonalizable endomorphisms on $\mathcal{S}_n$, in the pointwise topology induced by $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$. Assume that the endomorphisms of the curve have a common $\mathcal{S}$-eigenbasis for the space and real eigenvalues. Then the curve of states*

$$B : t \mapsto \exp\left(iA\left(t\right)\right)$$

is differentiable in the topology $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$, and

$$B'(t) = iA'\left(t\right) \exp iA\left(t\right),$$

where we used the multiplicative notation for the composition of endomorphisms. As a consequence, for every state ψ the curve u defined by

$$u(t) := \exp\left(iA\left(t\right)\right)\left(\psi\right)$$

satisfies the differential equality

$$u'\left(t\right) = iA'\left(t\right)\left(u\left(t\right)\right).$$

Proof. Let t_0 be a real time, we have

$$B'\left(t_0\right)\left(\psi\right) = {}^{\sigma(\mathcal{S}'_n)}\lim_{t\to t_0} \frac{\exp iA\left(t\right)\left(\psi\right) - \exp iA\left(t_0\right)\left(\psi\right)}{t - t_0},$$

if the right-hand limit exists. We prove the pointwise existence of $B'(t_0)$ for an eigenbasis α of A. To this aim $a_p : \mathbb{R} \to \mathbb{C}$ will be, for every n-tuple p the unique function such that

$$A(t)(\alpha_p) = a_p(t)\alpha_p,$$

it is continuous by the above argumentations; we have

$$\begin{aligned}
B'(t_0)(\alpha_p) &= \lim_{t\to t_0}\frac{\exp iA(t)(\alpha_p)-\exp iA(t_0)(\alpha_p)}{t-t_0}= \\
&= \lim_{t\to t_0}\frac{e^{ia_p(t)}\alpha_p-e^{ia_p(t_0)}\alpha_p}{t-t_0}= \\
&= \left(\lim_{t\to t_0}\frac{e^{ia_p(t)}-e^{ia_p(t_0)}}{t-t_0}\right)\alpha_p= \\
&= ia'_p(t_0)\left(e^{ia_p(t_0)}\alpha_p\right)= \\
&= ia'_p(t_0)\exp iA(t_0)(\alpha_p)= \\
&= iA'(t_0)(\exp iA(t_0)(\alpha_p)).
\end{aligned}$$

Now, let z be a tempered distribution, for the difference-quotient we have

$$\begin{aligned}
Q_B(t_0,t)(z) &= \frac{\exp A(t)(z)-\exp A(t_0)(z)}{t-t_0}= \\
&= \frac{1}{t-t_0}\left[\exp A(t)\left(\int_{\mathbb{R}^n}(z)_\alpha\alpha\right)-\exp A(t_0)\left(\int_{\mathbb{R}^n}(z)_\alpha\alpha\right)\right]= \\
&= \frac{1}{t-t_0}\left[\int_{\mathbb{R}^n}(z)_\alpha\exp A(t)(\alpha)-\int_{\mathbb{R}^n}(z)_\alpha\exp A(t_0)(\alpha)\right]= \\
&= \int_{\mathbb{R}^n}(z)_\alpha\frac{\exp A(t)(\alpha)-\exp A(t_0)(\alpha)}{t-t_0},
\end{aligned}$$

passing to limit

$$\begin{aligned}
B'(t_0)(z) &= \lim_{t\to t_0}\int_{\mathbb{R}^n}(z)_\alpha\frac{\exp iA(t)(\alpha)-\exp iA(t_0)(\alpha)}{t-t_0}= \\
&= \int_{\mathbb{R}^n}(z)_\alpha iA'(t_0)\circ\exp iA(t_0)(\alpha)= \\
&= iA'(t_0)\circ\exp iA(t_0)\int_{\mathbb{R}^n}(z)_\alpha\alpha= \\
&= iA'(t_0)\circ\exp iA(t_0)(z),
\end{aligned}$$

so

$$B'(t)=iA'(t)\exp iA(t).$$

Applying the preceding result, we have

$$\begin{aligned}
u'(t) &= iA'(t)(\exp iA(t)(\psi))= \\
&= iA'(t)(u(t)),
\end{aligned}$$

and the theorem is proved. ■

The preceding theorem can be generalized.

Theorem. *Let A be a curve of $\mathcal{S}$ -diagonalizable operators with a same $\mathcal{S}$-basis of the space, namely α, and let, for every n-tuple p, a_p the complex function defined on the real line such that*

$$A(t)(\alpha_p) = a_p(t)\alpha_p,$$

for every real time t. Then, the curve A is differentiable with respect to the pointwise topology τ induced by the weak topology $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$ on the space $\mathcal{L}(\mathcal{S}'_n)$ at a real time t_0 if and only if a_p is differentiable in t_0 for every n-tuple p. Moreover, in these conditions the operator $A'(t_0)$ is $\mathcal{S}$-diagonalizable with the same $\mathcal{S}$-basis α and its system of eigenvalues is the system of derivatives

$$p \mapsto a'_p\,(t_0)\,.$$

Proof. For every n-tuple p, assume the function a_p be differentiable at a time t_0, we have to prove that the pointwise-limit

$$^{\tau}\lim_{t\to t_0} \frac{A\,(t) - A\,(t_0)}{t - t_0}$$

there exists, i.e., that for every u in $\mathcal{S}'_n$ the $\sigma(\mathcal{S}'_n, \mathcal{S}_n)$-limit

$$^{\sigma(\mathcal{S}'_n)}\lim_{t\to t_0} \frac{A\,(t) - A\,(t_0)}{t - t_0}(u)$$

there exists; we begin with the basis α:

$$\begin{aligned}
A'\,(t_0)\,(\alpha_p) &= {}^{\sigma(\mathcal{S}'_n)}\lim_{t\to t_0} \frac{A\,(t) - A\,(t_0)}{t - t_0}(\alpha_p) = \\
&= {}^{\sigma(\mathcal{S}'_n)}\lim_{t\to t_0} \frac{A\,(t)\,(\alpha_p) - A\,(t_0)\,(\alpha_p)}{t - t_0} = \\
&= \lim_{t\to t_0} \frac{a_p\,(t)\,\alpha_p - a_p\,(t_0)\,\alpha_p}{t - t_0} = \\
&= \lim_{t\to t_0} \frac{a_p\,(t) - a_p\,(t_0)}{t - t_0}(\alpha_p) = \\
&= a'_p\,(t_0)\,\alpha_p.
\end{aligned}$$

Now, let u be a tempered distribution, for the difference-quotient, we have

$$\begin{aligned}
\frac{A\,(t) - A\,(t_0)}{t - t_0}(u) &= \frac{A\,(t) - A\,(t_0)}{t - t_0}(\int_{\mathbb{R}^n} (u)_\alpha\,\alpha) = \\
&= \frac{1}{t - t_0}\left(A\,(t)\int_{\mathbb{R}^n} (u)_\alpha\,\alpha - A\,(t_0)\int_{\mathbb{R}^n} (u)_\alpha\,\alpha\right) = \\
&= \frac{1}{t - t_0}\left(\int_{\mathbb{R}^n} (u)_\alpha\ A\,(t)\,\alpha - \int_{\mathbb{R}^n} (u)_\alpha\ A\,(t_0)\,\alpha\right) = \\
&= \frac{1}{t - t_0}\int_{\mathbb{R}^n} (u)_\alpha\ (A\,(t) - A\,(t_0))\alpha = \\
&= \int_{\mathbb{R}^n} (u)_\alpha\ (\frac{A\,(t) - A\,(t_0)}{t - t_0})\alpha,
\end{aligned}$$

so by $\sigma(\mathcal{S}'_n)$-continuity of superpositions, the limit

$$^{\sigma(\mathcal{S}'_n)}\lim_{t\to t_0}\frac{A(t)-A(t_0)}{t-t_0}(u)$$

exists and its value is the superposition

$$\int_{\mathbb{R}^n}(u)_\alpha\,(a'_p(t_0)\,\alpha_p)_{p\in\mathbb{R}^n},$$

as we desired. ■

28.5 The Dyson Formula

First of all we generalize a fact of elementary Linear-Algebra. If A and B are two diagonalizable endomorphisms on a finite-dimensional vector space with the same eigenbasis v, then every linear combination $aA+bB$ has the same eigenbasis and

$$\mathrm{ev}_{aA+bB}(v_i)=a\mathrm{ev}_A(v_i)+b\mathrm{ev}_B(v_i),$$

for every vector v_i of v, where ev_L is the mapping associating to every eigenvector of the linear operator L the corresponding (unique) eigenvalues.

Theorem. *Let H be a continuous curve of $\mathcal{S}$-diagonalizable operators with the same $\mathcal{S}$ -eigenbasis and let μ be a Radon measure on $\mathbb{R}$. Let $\eta=(\eta_p)_{p\in\mathbb{R}^n}$ be the common eigenbasis of the operators of the curve, and let, for every n-tuple p , $E_p:\mathbb{R}\to\mathbb{C}$ be the unique function such that*

$$H(t)(\eta_p)=E_p(t)\eta_p,$$

for every time t. Then, for every n-tuple p,

$$(\int_{t_0}^t Hd\mu)(\eta_p)=(\int_{t_0}^t E_p(\mu))(\eta_p).$$

Proof. It's straightforward,

$$\begin{aligned}(\int_{t_0}^t H\,(\mu))(\eta_p) &= \int_{t_0}^t H(\eta_p)(\mu)=\\ &= \int_{t_0}^t E_p\eta_p\,(\mu)=\\ &= (\int_{t_0}^t E_p\,(\mu))(\eta_p),\end{aligned}$$

as we desired. ■

Theorem. *Let H be a continuous curve of $\mathcal{S}$-diagonalizable operators with the same $\mathcal{S}$ -eigenbasis. Then, for every time t,*

$$\left(\int_{t_0}^{(\cdot)} H d\lambda\right)'(t) = H\left(t\right).$$

Proof. Let η be the common eigenbasis of the operators of the curve, and let $E_p : \mathbb{R} \to \mathbb{C}$ the unique function such that

$$H(t)(\eta_p) = E_p(t)\eta_p,$$

for every n-tuple p and for every time t. We have, for every n-tuple p,

$$\left(\int_{t_0}^{t} H\left(\lambda\right)\right)(\eta_p) = \left(\int_{t_0}^{t} E_p\left(\lambda\right)\right)(\eta_p).$$

Let us compute the difference-quotient at a time t_1

$$\begin{aligned}\frac{1}{t-t_1}\left(\int_{t_0}^{t} H - \int_{t_0}^{t_1} H\right)(\eta_p) &= \frac{1}{t-t_1}\left(\left(\int_{t_0}^{t} H\right)(\eta_p) - \left(\int_{t_0}^{t_1} H\right)(\eta_p)\right) = \\ &= \frac{1}{t-t_1}\left(\int_{t_0}^{t} E_p - \int_{t_0}^{t_1} E_p\right)\eta_p,\end{aligned}$$

passing to limit we have

$$\begin{aligned}\left(\int_{t_0}^{(\cdot)} H d\lambda\right)'(t_1)\left(\eta_p\right) &= E_p(t_1)(\eta_p) = \\ &= H\left(t_1\right)\left(\eta_p\right).\end{aligned}$$

For every u, we have

$$\begin{aligned}\frac{1}{t-t_1}\left(\int_{t_0}^{t} H - \int_{t_0}^{t_1} H\right)(u) &= \frac{1}{t-t_1}\left(\int_{t_0}^{t} H - \int_{t_0}^{t_1} H\right)\left(\int_{\mathbb{R}^n} (u)_\eta\, \eta\right) = \\ &= \int_{\mathbb{R}^n} (u)_\eta \frac{1}{t-t_1}\left(\int_{t_0}^{t} H - \int_{t_0}^{t_1} H\right)(\eta),\end{aligned}$$

when $t \to t_1$, by the previous step, the right hand side tend to

$$\int_{\mathbb{R}^n} (u)_\eta\, H(t_1)\eta = H(t_1)(u),$$

and the theorem is proved. ■

We finally can conclude with the main result.

Theorem. *Let H be a continuous curve of $\mathcal{S}$-diagonalizable operators with a same $\mathcal{S}$ -eigenbasis for the space $\mathcal{S}'_n$, let (t_0, u_0) be an initial condition in $\mathbb{R}\times\mathcal{S}'_n$ and let u be the curve in $\mathcal{S}'_n$ defined by*

$$u : t \mapsto \exp\left(\int_{t_0}^{t} H d\lambda\right)(u_0),$$

for every real time t. Then u is $\sigma(\ \mathcal{S}'_n)$-differentiable and it is such that

$$u'(t) = H(t)(u(t)),$$

for every real time t, and $u(t_0) = u_0$. Moreover, for every pair of times s and t we have

$$u(t) = S(s,t)u(s),$$

where $S : \mathbb{R}^2 \to \mathcal{L}(\mathcal{S}'_n)$ is the propagator defined by

$$S(s,t) = \exp\left(\int_{s}^{t} H d\lambda\right).$$

Proof. Applying the preceding theorem we obtain

$$\begin{aligned} u'(t) &= \left(\int_{t_0}^{(\cdot)} H d\lambda\right)'(t)\left(\exp\left(\int_{t_0}^{t} H d\lambda\right)(u_0)\right) = \\ &= \left(\int_{t_0}^{(\cdot)} H d\lambda\right)'(t)(u(t)) = \\ &= H(t)(u(t)). \end{aligned}$$

Concerning the propagator we have

$$\begin{aligned} u(t) &= S(t_0,s)S(s,t)u(t_0) = \\ &= S(s,t)S(t_0,s)u(t_0) = \\ &= S(s,t)u(s), \end{aligned}$$

as we desired. ■

Analogously, for the Abstract Heat equation, fundamental in the study of financial evolution, we have the following.

Theorem (on the abstract Heat equation). *Let $A : \mathbb{R}^{\geq} \to \mathcal{L}(\mathcal{S}'_n)$ be a continuous curve of ${}^{\mathcal{S}}$diagonalizable operators with a common ${}^{\mathcal{S}}$eigenbasis for $\mathcal{S}'_n$ and with real and positive eigenvalues. Then, for every initial condition $(0, u_0) \in \mathbb{R}^{\geq} \times \mathcal{S}'_n$, the curve $\psi : \mathbb{R}^{\geq} \to \quad \mathcal{S}'_n$, given by*

$$\psi(t) = \exp\left(-\int_{t_0}^{t} A(\lambda)\right)(u_0),$$

for every $t \geq 0$, is such that

$$\psi'(t) = -A(\psi(t)),$$

and $\psi(0) = \psi_0$.

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