APPLICATIONS OF THE GROTHENDIECK–ATIYAH–HIRZEBRUCH FUNCTOR $K(X)$

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I shall begin by stating three results; then I shall comment on their authorship and history, and finally I shall try to show how they can be fitted into a general theory.

Let $O(n)$ be the orthogonal group; then we can consider the coset space $O(n)/O(n-k)$ and the fibering

$$O(n)/O(n-k) \to O(n)/O(n-1) = S^{n-1}. \qquad (1)$$

The classical problem about vector-fields on spheres used to ask: for what values of n and k does this fibering admit a cross-section? The answer is as follows.

THEOREM 1. *The fibering* (1) *admits a cross-section if and only if n is divisible by N_k, where N_k is the integer defined below.*

We define $N_k = 2^{a(k)}$, where $a(k)$ is the number of integers r such that $1 \leqslant r \leqslant k-1$ and $r \equiv 0$, 1, 2 or 4 mod 8.

Similarly, let $U(n)$ be the unitary group; then we can consider the coset space $U(n)/U(n-k)$ and the fibering

$$U(n)/U(n-k) \to U(n)/U(n-1) = S^{2n-1}. \qquad (2)$$

Again we ask: when is there a cross-section?

THEOREM 2. *The fibering* (2) *admits a cross-section if and only if n is divisible by M_k, where M_k is the integer defined below.*

We define $\nu_p(n)$ to be the exponent of the prime p in the decomposition of n into prime powers, so that

$$n = 2^{\nu_2(n)}\, 3^{\nu_3(n)}\, 5^{\nu_5(n)} \dots$$

We define M_k as follows:

$$\nu_p(M_k) = 0 \quad \text{if} \quad p > k,$$

$$\nu_p(M_k) = \operatorname{Sup}\,(r + \nu_p(r)) \quad \text{if} \quad p \leqslant k,$$

where the integer r runs over the range

$$1 \leqslant r \leqslant \frac{k-1}{p-1}.$$

At the last congress we heard a lecture by J. Milnor [10], which was in part about the J-homorphism of H. Hopf and G. W. Whitehead. I recall that this is a map

$$J : \pi_l(SO(n)) \to \pi_{n+l}(S^n).$$

I shall suppose that $l=4k-1$ and $n>4k$, so that we are dealing with the "stable J-homomorphism", and J is defined on a cyclic infinite group. The problem is to describe the image of J. Following Milnor and Kervaire [10], we define $m(k)$ to be the denominator of $B_k/4k$, when this fraction is expressed in its lowest terms. Here B_k is the kth Bernoulli number, so that

$$\frac{x}{e^x-1}=1-\frac{x}{2}+\sum_{k=1}^{\infty}(-1)^{k-1}B_k\frac{x^{2k}}{(2k)!}.$$

THEOREM 3. *If* $l=4k-1$ *and* $n>4k$, *then*

$$\operatorname{Im} J=\begin{cases} Z_{m(k)} & \text{if } \quad 4k\equiv 4 \bmod 8,\\ \text{either } Z_{m(k)} \quad \text{or} \quad Z_{2m(k)} & \text{if } \quad 4k\equiv 0 \bmod 8.\end{cases}$$

In Theorem 1, the existence of a cross-section when n is divisible by N_k is classical; the non-existence of a cross-section when n is not divisible by N_k is proved in [1].

The two halves of Theorem 2 were obtained in the reverse order. The non-existence of a cross-section when n is not divisible by M_k is due to Atiyah and Todd [7]; the existence of a cross-section when n is divisible by M_k is proved in [3].

In Theorem 3, the fact that the order of Im J is a multiple of $m(k)$ is the result of Milnor and Kervaire [10] as improved by Atiyah and Hirzebruch [5]. The fact that the order of Im J divides $m(k)$or $2m(k)$, as the case may be, is proved in [2].

I remark that by re-proving the result of Milnor and Kervaire, one can extract more information. One can show [2] that there is a homomorphism

$$e:\pi_{n+4k-1}(S^n)\rightarrow Z_{m(k)},$$

such that the composite eJ is an epimorphism (if $n>4k$). It follows that if $4k\equiv 4 \bmod 8$, then Im J is a direct summand in $\pi_{n+4k-1}(S^n)$. If $4k\equiv O \bmod 8$, then a similar conclusion follows except for the 2-component. The homomorphism e has other interesting properties, on which I shall not dwell.

In order to prove these theorems one makes use of the "extraordinary cohomology theory" $K(X)$ of Grothendieck–Atiyah–Hirzebruch [5, 6]. I will now recall how this is constructed. Let Λ denote either the real field R or the complex field C. Let X be a "good" space, e.g. a finite connected CW-complex. If $\Lambda=R$ we take all orthogonal bundles over X; if $\Lambda=C$ we take all unitary bundles over X. In either case we divide them into isomorphism classes $\{\xi\}$. We take these classes as the generators for a free abelian group F_Λ (X). We shall define

$$K_\Lambda(X)=F_\Lambda(X)/T_\Lambda(X),$$

so that $K_\Lambda(X)$ is given by the generators $\{\xi\}$ and certain relations; we define $T_\Lambda(X)$ to be the subgroup of $F_\Lambda(X)$ generated by all elements of the form

$$\{\xi\oplus\eta\}-\{\xi\}-\{\eta\},$$

where $\xi\oplus\eta$ denotes the Whitney sum of ξ and η. The group $K_\Lambda(X)$, then, is obtained by taking the vector bundles over X and forcing them to generate an abelian group under the Whitney sum operation. The elements of $K_\Lambda(X)$ may be called "virtual bundles".

It is possible to use the groups $K_\Lambda(X)$ to prove non-existence results in just the same way that one is accustomed to use ordinary cohomology groups. Thus, if X, A is a pair and A is a retract of X, it follows that $K_\Lambda(A)$ is a direct summand in $K_\Lambda(X)$; and if we find that $K_\Lambda(A)$ is not a direct summand in $K_\Lambda(X)$, then we can conclude that A is not a retract of X. The non-existence proof in [1] is presented in this way.

However, just as in ordinary cohomology we often need to use cohomology operations, so here we need to use cohomology operations in $K_\Lambda(X)$.

The first such operation is a cup-product. We can define the tensor product of two vector-spaces over Λ; therefore we can define the tensor product $\xi \otimes \eta$ of two vector bundles over X; one shows that this defines a product in $K_\Lambda(X)$.

Similarly, we can define the dual of a vector-space over Λ; therefore we can define the dual ξ^* of a vector bundle over X; one shows that this defines an operation in $K_\Lambda(X)$.

Again, we can define the ith exterior power of a vector space over Λ; therefore we can define the ith exterior power $\lambda^i(\xi)$ of a bundle over X. It is possible to extend the definition of λ^i from bundles to virtual bundles in a unique way so as to preserve the following familiar property:

$$\lambda^i(\xi+\eta) = \sum_{j+k=i} \lambda^j(\xi) \otimes \lambda^k(\eta).$$

All this is due to Grothendieck.

Unfortunately, the formal properties of the λ^i are not very convenient. It is possible to obtain operations with better formal properties by an algebraic device. Consider

$$(x_1)^k + (x_2)^k + \ldots + (x_k)^k;$$

this is a symmetric polynomial in $x_1, x_2, \ldots, x_k$; therefore it can be written as a polynomial in the elementary symmetric functions σ_i of $x_1, x_2, \ldots, x_k$; say

$$(x_1)^k + (x_2)^k + \ldots + (x_k)^k = Q^k(\sigma_1, \sigma_2, \ldots, \sigma_k).$$

Now define

$$\Psi^k(\xi) = \begin{cases} Q^k(\lambda^1\xi, \lambda^2\xi, \ldots, \lambda^k\xi), & (k>0), \\ \Psi^{-k}(\xi^*), & (k<0). \end{cases}$$

The functions $\Psi^\cdot$ are ring homomorphisms from $K_\Lambda(X)$ to $K_\Lambda(X)$.

In order to unify the three theorems with which I started, one makes use of the groups $J(X)$ of Atiyah [4]. First I define the notion of fibre homotopy equivalence. Let ξ, η be sphere bundles over X, with total spaces E_ξ, E_η; then we say that a map $f: E_\xi \to E_\eta$ is "fibrewise" if it covers the identity map of X; we say that ξ, η are fibre homotopy equivalent if there exist fibrewise maps $f: E_\xi \to E_\eta$, $g: E_\eta \to E_\xi$ such that $gf \sim 1$ through fibrewise maps of E_ξ, and similarly for fg. We shall define

$$J_\Lambda(X) = K_\Lambda(X)/U_\Lambda(X),$$

so that $J_\Lambda(X)$ is given by the generators $\{\xi\}$ and certain relations; we define $U_\Lambda(X)$ to be the subgroup of $K_\Lambda(X)$ generated by all elements of the form

$$\{\xi\} - \{\eta\},$$

where ξ, η are fibre homotopy equivalent.

Since all our groups are functorial, we can write

$$K_\Lambda(X) = K_\Lambda(P) + \tilde{K}_\Lambda(X),$$

$$J_\Lambda(X) = J_\Lambda(P) + \tilde{J}_\Lambda(X);$$

here P denotes a point, and these equations are supposed to define the summands $\tilde{K}_\Lambda(X)$, $\tilde{J}_\Lambda(X)$ complementary to $K_\Lambda(P)$, $J_\Lambda(P)$. Atiyah's group $J(X)$ is the one I have called $\tilde{J}_R(X)$.

According to Atiyah [4], if you can compute $J_R(RP^{k-1})$, you can prove Theorem 1; if you can compute $J_C(CP^{k-1})$, you can prove Theorem 2; and we have

$$J_R(S^l) = J(\pi_{l-1}(SO(n))) \quad (n > l).$$

We therefore face the general problem: "compute $J_\Lambda(X)$".

Half of the problem consists in giving a lower bound for $J_\Lambda(X)$, and half of it consists in giving an upper bound for $J_\Lambda(X)$. I start with the lower bound.

It is sometimes easy to prove that two bundles ξ, η are not fibre homotopy equivalent by using the Stiefel-Whitney classes, which are fibre homotopy invariants. The reason why they are fibre homotopy invariants is that they can be defined in a particular way. Suppose given a sphere bundle ξ over B with total space E; we can embed the space E in the corresponding bundle of unit solid balls, say $\bar{E}$. Then in cohomology we have the Thom isomorphism

$$\varphi_H : H^*(B; Z_2) \to H^*(\bar{E}, E; Z_2).$$

We can consider the following diagram.

$$\begin{array}{ccc} H^*(\bar{E}, E; Z_2) & \xrightarrow{Sq = \sum_{i=0}^{\infty} Sq^i} & H^*(\bar{E}, E; Z_2) \\ \varphi_H \uparrow & & \uparrow \varphi_H \\ H^*(B; Z_2) & & H^*(B; Z_2) \end{array}$$

The total Stiefel-Witney class is given by

$$w(\xi) = \varphi_H^{-1} Sq \, \varphi_H 1$$

We can copy this procedure using the K-cohomology theory. For example, suppose that ξ is a unitary bundle, and let the other notation remain as before. The one can define a Thom isomorphism

$$\varphi_K : K_C(B) \to K_C(\bar{E}, E),$$

where the relative K groups are defined by

$$K_\Lambda(X, Y) = \tilde{K}_\Lambda(X/Y).$$

One can consider the following diagram.

$$\begin{array}{ccc} K_C(\bar{E}, E) & \xrightarrow{\Psi^k} & K_C(\bar{E}, E) \\ \varphi_K \uparrow & & \uparrow \varphi_K \\ K_C(\mathrm{B}) & & K_C(B) \end{array}$$

By analogy, we define $\varrho^k(\xi) = \varphi_K^{-1}\Psi^k\varphi_k\ 1$.

We next seek to extend the definition of ϱ^k to virtual bundles, so as to preserve the following property.

$$\varrho^k(\xi + \eta) = \varrho^k(\xi)\varrho^k(\eta).$$

The extension is possible and unique, provided we intepret $\varrho^k(\xi)$ as an element of the group

$$K_C(B) \otimes Q_k,$$

where Q_k denotes the additive group of fractions a/k^b. It is easy to see that we are forced to introduce these denominators. In fact, we have

$$\varrho_k(1) = k,$$

so

$$\varrho_k(-1) = 1/k.$$

For completeness I add that one can also adopt an intermediate approach, and consider (for example) the following diagram.

$$\begin{array}{ccc} K_C(\bar{E}, E) & \xrightarrow{ch} & H^*(\bar{E}, E; Q) \\ \varphi_K \uparrow & & \uparrow \varphi_H \\ K_C(B) & & H^*(B; Q) \end{array}$$

This method yields criteria which can be stated in terms of characteristic classes. However, it is not likely to be adequate if B has torsion; and it also fails to give best possible results for such torsion-free spaces as S^{8m+2}, CP^{4m+1}.

I therefore adopt the following definition of a quotient group $J'_\Lambda(X)$ of $J_\Lambda(X)$, which will serve as a lower bound for $J_\Lambda(X)$. I define

$$J'_\Lambda(X) = K_\Lambda(X)/V,$$

where $x \in V$ if and only if there exists y in $\tilde{K}_\Lambda(X)$ such that

$$\varrho^k(x) = \frac{\Psi^k(1+y)}{1+y} \quad \text{for all } k.$$

(The experts will understand that in the case $\Lambda = R$ we impose also the conditions $w_1(x) = 0$ and $w_2(x) = 0$, in order that $\varrho^k(x)$ should be defined. Compare [8].)

The reason for adopting a definition of this form is that when one tries to prove that $\varrho^k(x)$ is a fibre homotopy invariant, one finds only that it is an invariant up to multiplication by a factor of the form

$$\frac{\Psi^k(1+y)}{1+y}.$$

I will now pass on to discuss upper bounds for $J_\Lambda(x)$. For this purpose we need a result which will prove that two sphere bundles ξ and η are fibre homotopy equivalent, although they are not isomorphic. I offer the following.

THEOREM 4. *Suppose that ξ,η are sphere bundles over a finite CW-complex and that there is a fibrewise map*

$$f: E_\xi \to E_\eta$$

of degree k on each fibre. Then there exists an integer e such that the Whitney multiples $|k^e|\xi$, $|k^e|\eta$ are fibre homotopy equivalent.

If we put $k=1$ this is a theorem of Dold [9]. Therefore one may regard this theorem as a mod k analogue of Dold's theorem. The proof of Dold may be summarised by saying that we consider the space of homotopy equivalences from S^{n-1} to S^{n-1}, and treat it seriously as a "structural group". My proof may be summarised by saying that we take the space of all maps from S^{n-1} to S^{n-1}, and treat it similarly.

By applying Theorem 4, I prove the following.

THEOREM 5. *Suppose that k is given, that $y \in K_C(X)$ (where X is a finite CW-complex) and either* (i) *y is a linear combination of complex line bundles over X, or* (ii) $X = S^{2n}$.
Then there exists an integer $e = e(k,y)$ such that

$$k^e(\Psi^k - 1)y$$

maps to zero in $J_C(X)$.

Following the hint contained in Theorem 5, I define a group $J''_\Lambda(X)$ which will act as a sort of conditional upper bound for X; that is to say, if the conclusion of Theorem 5 holds for every pair (k,y), then $J_\Lambda(X)$ will be a homomorphic image of $J''_\Lambda(X)$, no matter how large the integers $e(k,y)$ turn out to be. I define

$$J''_\Lambda(X) = K_\Lambda(X)/W,$$

where $x \in W$ if and only if for every function $e(k,y)$ there exists a function $a(k,y)$, such that

$$x = \sum_{k,\, y} a(k, y)\, k^{e(k,y)}(\Psi^k - 1)y.$$

It is understood that the functions $e(k,y)$ and $a(k,y)$ are defined for all pairs consisting of an integer k and an element $y \in K_C(X)$; the values of $e(k,y)$ are non-negative integers; $a(k,y)$ takes integer values, and is zero except for a finite number of pairs (k,y).

If we hope to estimate $J_\Lambda(X)$ by means of an upper bound and a lower bound, it is desirable to have these two bounds close together. If $X = RP^{k-1}$, CP^{k-1} or S^l, then

$$J''_R(X) = J'_R(X);$$

and this completes what I want to say about Theorems 1, 2, 3.

It would appear that we have

$$J''_R(X) = J'_R(X)$$

for any finite CW-complex X.

Problem 1. Does the conclusion of Theorem 5 hold for each finite CW-complex X and each element y in $K_R(X)$?

Problem 2. Can Theorem 4 be used to answer Problem 1?

Problem 3. Suppose given two inequivalent representations of $O(n)$, so that it acts on Euclidean spaces V, V' of the same dimension. When can one find an equivariant map $f: V \to V'$ which maps the unit sphere of V onto that of V' with degree k?

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