THE ETALE TOPOLOGY OF SCHEMES

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1. Introduction

Since Weil [56, 57] pointed out the need for invariants, analogous to topological ones, of varieties over fields of characteristic p, several proposals to define such invariants have been made, notably by Serre [47], Grothendieck [21], and Monsky and Washnitzer [38]. I would like to describe some of the recent work on one of these approaches, that of the *etale cohomology* of Grothendieck. This approach has yielded a proof of the rationality of Weil's zeta function for a variety over a finite field via the method suggested by Weil [57], and for generalized L-functions (Grothendieck [25]) [1]). The etale cohomology also provides a framework in which to state some beautiful conjectures of Tate [53] on algebraic cycles (now proved by him for divisors on abelian varieties over finite fields), and of Birch and Swinnerton-Dyer (cf. Tate [54]). Quite generally, it gives good results for coefficients prime to the characteristic p of the variety. In fact, the other proposals for a cohomology theory (Serre [47], Monsky and Washnitzer (cf. [37] or Lubkin [36]), Grothendieck's flat topology (cf. [12] or Shatz [51] for the case of a field)) all yield a cohomology with "mod p" or Witt vector coefficients, and it is not completely clear at present which of them will be the most fruitful. The problem of finding such a theory is obviously of great interest.

In this talk, I will restrict myself because of lack of time and competence to a description of some aspects of the etale theory, without going into detail on any of the applications mentioned above.

Let me begin by recalling that a morphism $X \to Y$ of schemes is called *etale* if it is flat and unramified. Those unfamiliar with the notion may get an intuitive understanding of its meaning from the fact that a map of schemes of finite type over the complex numbers is etale iff the map of associated analytic spaces is a local isomorphism.

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1) The methods also yield the functional equation and the explicit form of the zeta function as an alternating product (cf. [25]). Actually, the rationality was first proved for arbitrary varieties by Dwork [13]. For the rest, we prefer not to get involved in questions of priority. Suffice it to say that in addition, similar results have been obtained for a smooth proper variety which is a specialization from characteristic zero by Lubkin [35, 36], and that the rationality has been proved for arbitrary varieties by Monsky [38].

2. The etale topology

The first topology to be defined on an abstract variety or scheme was the *Zariski topology* (Zariski [59]). Recall that in this topology a closed set of the spectrum of a ring R is the set of zeros $V(S)$ of some subset S of R. Then Serre, in his fundamental paper FAC [45], showed that the Zariski topology could be used to define a cohomology theory of *coherent* sheaves on a variety, i.e., ones arising from modules over the coordinate rings. He also proved ([46] GAGA) that for a projective variety over the field of complex numbers, the theory thus obtained was the same as the analytic theory. These results left little doubt that the Zariski topology is a good one for the study of coherent sheaves.

For the purposes of our discussion, we may express this conclusion in a slightly different way by saying that ordinary localization in a ring R is a "sufficiently strong" process for most things in the study of modules over R. The conclusion is supported by the fact that if R is a local ring (say noetherian for simplicity), and if M, N are two finite modules over R which become isomorphic after any finitely generated faithfully flat extension of scalars $R \to R'$, then M and N are themselves isomorphic. Or, there are no twisted forms of a finite module M over a local ring R, relative to such extensions of scalars. (By descent theory (Grothendieck [23]), this can be interpreted as a generalized form of the famous Hilbert theorem 90.)

However, twisted forms of more complicated structures will usually not be locally trivial. For instance, central separable algebras over a field k are twisted forms of a full matrix algebra relative to the extension $k \to \bar{k}$ of k by its separable algebraic closure. Such examples led Serre [48] to introduce the notion of local isotriviality of a fibre space. A fibre space with given algebraic structure group G over a scheme X is called *locally isotrivial* if for every point $x \in X$ there is a Zariski open neighborhood U of x in X and a finite etale covering space U' of U such that the pull-back of the fibre space to U' is trivial. This definition yields a notion which includes the one studied by Weil [58] of fibre spaces which are locally trivial for the Zariski topology, and the one of structures over a field which become trivial after a separable algebraic field extension, studied by Lang and Tate [34] and others.

In 1958, Grothendieck found a general version of sheaf theory, which enabled him to define the notion of etale cohomology of schemes. This etale theory puts Serre's notion in a broad framework, and it provides an algebraic definition of the Betti numbers of an algebraic variety. We will describe briefly one version of the general theory. It is treated in detail in Verdier [55].

The necessary data for sheaf theory consist of the following:

(1) A category C and a collection of families of maps $\{X_i \to Y\}$ of C with common range, called *coverings* of the range Y. The following axioms are supposed to hold:

(i) Isomorphisms are coverings.

(ii) The composition of coverings is a covering, in the following sense: If $\{X_i \to Y\}$ cover Y, and $\{W_{ij} \to X_i\}_j$ is a covering of X_i for each i, then the compositions $\{W_{ij} \to Y\}$ cover Y.

(iii) A pull-back of a covering is a covering: If $\{X_i \to Y\}$ is a covering, and $Y' \to Y$ is an arbitrary map, then the fibred products $X_i \times_Y Y'$ exist in C and they form a covering of Y'.

Actually, the exact phrasing of the axioms is not very important (cf. [3, 5, 55]). We will refer to such a collection of data as a *topology*.

Given a topology, a *sheaf* is a contravariant functor F from C to (say) sets, satisfying the *sheaf axiom*.

(2) If $\{X_i \to Y\}$ is a covering, then a "section" $s \in F(Y)$ is uniquely determined by a collection of sections $s_i \in F(X_i)$ such that for each i, j the sections of $F(X_i \times_Y X_j)$ induced by s_i and s_j via the projection maps are equal.

Cohomology with values in an abelian sheaf is defined as a derived functor, as in Grothendieck [20].

The Serre notion of local isotriviality was the starting point for Grothendieck's original definition of the etale topology, but it has turned out in the meantime to be more convenient to allow localization by an arbitrary etale morphism. Thus for the *etale topology* of a prescheme X, the category C above is taken to be the category of preschemes U *etale* over X, and a covering is a family of maps which is *surjective* in the sense that the range is covered by the images.

This definition is such that the first cohomology $H^1(X, G)$ of X with values in a linear group G classifies the fibre spaces with structure group G over X having the following property: For every $x \in X$ there is an etale map $U \to X$ (not necessarily finite over a Zariski open set) whose image contains x, such that the pull-back of the fibre space to U is trivial; or, as one says, which are locally trivial for the etale topology (cf. Giraud [16] for a general treatment of H^1).

3. Relations with Galois cohomology

If X is the spectrum of a field K, then the etale schemes of finite type over X are just spectra of separable (commutative) K-algebras, i.e., products of separable field extensions, and it is not difficult to show that the resulting cohomology theory is just Tate's cohomology of the galois group $G(\bar{K}/K)$ w/here $\bar{K}$ is the separable algebraic closure of K. This theory has been treated in detail in various places (eg., Serre [50]).

The sheaf theory on an arbitrary noetherian scheme X can also be related to galois modules via the *specialization diagram* of X. We will describe it (per semplicità di discorso) only for an entire (this terminology is due to Lang [33]) normal scheme of dimension 1. This includes the case of a nonsingular algebraic curve and that of the spectrum of the ring of integers in a number field. The general case can be described in a similar way, but the specialization diagram is more complicated:

Let G be the galois group of the separable closure $\bar{K}$ of the function field K of X, and let $\bar{X}$ be the normalization of X in $\bar{K}$. For each $x \in X$, choose a point $\bar{x}$ of $\bar{X}$ above x, and let $D_x \subset G$ be the decomposition group of this point. (A change of the point $\bar{x}$ over x changes D_x by conjugation. It is an interesting feature of the etale cohomology, and one of its weaknesses, that the choices of the various points $\bar{x}$ are not important.) Let G_x be the galois group of the separable algebraic closure of the residue field $k(x)$ of X at x. Then there is a diagram of group homomorphisms

$$G \longleftarrow D_x \longrightarrow G_x$$

for each $x \in X$. The result is that the category of "constructible" abelian sheaves on X is equivalent with the category whose objects consist of

(1) (i) A G-module M and a G_x-module M_x for each $x \in X$, which are finitely generated abelian groups.

(ii) A "specialization map" Φ_x: $M_x \to M$ for each $x \in X$ which is a homomorphism of D_x-modules,

satisfying the "continuity condition" that almost all of the maps Φ_x be isomorphisms.

Thus the cohomology of a sheaf on X in the etale topology can be described in terms of the galois cohomologies of the groups G, G_x, and of the relations between them. In this way one can interpret the results of Ogg [40] and Šafarevič [44] on the cohomology of abelian varieties over function fields as calculations in the etale cohomology. Their results contain implicitly a description of the cohomology of an algebraic curve over an algebraically closed field (cf. [6], exp. IX). The formula of Ogg [40] for the Euler characteristic of a sheaf has been generalized by Grothendieck (Raynaud [42]). Similarly, the exact sequences of Tate [52] for cohomology of a number field are closely related to the etale cohomology of the ring of integers of K, but there is a slight difference in the notion of local triviality used there.

4. Cohomology with values in the multiplicative group

The sheaf of units on a scheme X for the etale topology occupies a central role. We will denote this sheaf by $\mathcal{O}^*$. It contains as subsheaf the sheaf μ of all roots of unity, and for a regular X, the inclusion

$\mu \subset \mathcal{O}^*$ induces an isomorphism on cohomology in dimensions > 2, if one ignores p-torsion for the residue characteristics p of X. The sheaf μ is clearly a locally constant torsion sheaf (ignoring p) and so its cohomology can be treated by the theory discussed in the next section. But in dimensions $\leqslant 2$, the cohomology of $\mathcal{O}^*$ gives information of an arithmetic sort not contained in μ:

It follows from descent theory [23] that

$$H^1(X, \mathcal{O}^*) = \operatorname{Pic} X$$

is the group of isomorphism classes of locally free rank one sheaves on X. On the other hand, the group $H^2(X, \mathcal{O}^*)$ contains as subgroup the "Brauer group" of sheaves of Azumaya algebras on X (cf. [26]):

$$H^2(X, \mathcal{O}^*) \supseteq \operatorname{Br} X.$$

(The notion of *Azumaya algebra*, generalizing that of central simple algebra over a field, was first introduced for rings by Azumaya [10] and Auslander and Goldman [9], and its relation to cohomology theory was studied by various authors [2, 11, 43]. The theory is discussed in detail in Grothendieck [26].)

A most interesting and apparently difficult question of Auslander and Goldman is whether or not the Brauer group is all of $H^2(X, \mathcal{O}^*)$ when X is the spectrum of a regular ring (or more generally, when X is a regular scheme). This is true when X is of dimension $\leqslant 2$ (Auslander and Goldman) or when X is a semi-local ring of an algebraic variety in characteristic zero (cf. [26]). It is generally false if X is singular (Grothendieck [26]).

To see the importance of the Brauer group, suppose that X is a complete non-singular algebraic surface over an algebraically closed field k. For any X and n prime to the characteristics of X, the *Kummer sequence*

$$0 \to \mu_n \to \mathcal{O}^* \xrightarrow{n\text{-th power}} \mathcal{O}^* \to 0 \tag{1}$$

is exact, where μ_n denotes the sheaf of n-th roots of unity. Applying (1) and the above facts to our surface X in the highest interesting dimension, we obtain an exact sequence

$$0 \to (\operatorname{Pic} X)/n \to H^2(X, \mu_n) \to (\operatorname{Br} X)_n \to 0 \tag{2}$$

where the subscript n indicates the set of elements whose order divides n. Its middle term is the cohomology of X with values in the constant sheaf $\mu_n \approx \mathbf{Z}/n$ whose rank as a $\mathbf{Z}/n$-module is, up to a bounded term, the second Betti number B_2 of X (say by definition). The term on the left yields up to a bounded term the rank of the Neron Severi group of X. Thus the Brauer group measures the algebraic analogue $\rho_0 = B_2 - \rho$ of the number of transcendental 2-cycles on a surface.

The inequality $B_2 \geqslant \rho$ was first proved in the abstract case by Igusa [31] with an ad hoc definition of B_2. His method of proof, using a pencil of curves on the surface and vanishing cycle theory, made no use of the etale cohomology, but a similar approach gives an expression for the Brauer group, and hence for the etale B_2, in terms of the pencil ([3], Tate [54]). The Brauer group is just the Šafarevič-Tate group of locally trivial principal homogeneous spaces of the Jacobian of the generic curve. This is also true for arithmetic surfaces (Tate [54]).

5. General cohomology theory

The approach to etale cohomology has been mostly via Grothendieck's generalized sheaf theory, as we have already indicated. Actually, the first case I know of in which etale coverings were used for cohomology theory of a variety is in Kawada and Tate [32].

The results of this section (due largely to Grothendieck and myself), together with proofs, may be found in [6]. The most important single result is the following:

T h e o r e m (1) (proper base change theorem). Let $f: X \to Y$ be a proper map and F an abelian torsion sheaf on X. Let y_0 be a geometric point of Y, and X_0 the geometric fibre of f at y_0. Then the stalk at y_0 of the higher direct image $R^q f_* F$ is the cohomology $H^q(X_0, F|X_0)$ of the fibre. The assumption that F be a torsion sheaf is essential in all serious results.

For schemes of finite type over the complex numbers $\mathbf{C}$, one has

T h e o r e m (2) (comparison with the classical cohomology). Let X be a scheme of finite type over $\mathbf{C}$, and F a constructible torsion sheaf on X for the etale topology. Then F includes a sheaf on X for the classical topology, and one has isomorphisms

$$H^q(X_{\text{etale}}, F) \approx H^q(X_{\text{class}}, F).$$

The condition of constructibility is the obvious finiteness condition in this context. The proof of (2) in the general case requires resolution of singularities (Hironaka [28]).

For passing from characteristic zero to characteristic p, the following is useful (cf. [6], also Lubkin [35]):

T h e o r e m (3): (specialization theorem). Let $f: X \to Y$ be a smooth proper map, and let F be a locally constant torsion sheaf on X whose orders are prime to the residue characteristics of Y. Then the higher direct images $R^q f_* F$ are locally constant sheaves on Y (whose stalks are by (1) the cohomology of the geometric fibres of X/Y).

This result is one of a series reflecting the locally acyclic nature of smooth morphisms.

In a less definitive state are the results on cohomological dimension and finiteness of cohomology:

T h e o r e m (4) (finiteness). Let $f: X \to Y$ be a morphism of finite type, and F a constructible torsion sheaf on X. Suppose either that f is proper or that Y is an excellent (cf. [27], IV) scheme of characteristic zero. Then the higher direct images $R^q f_* F$ are again constructible. Of course, this theorem gives in particular the finiteness of the cohomology groups when $Y = \operatorname{Spec} K$ is the spectrum of a separably closed field. Assuming resolution, one can prove (4) also in the case that Y is excellent and of equal characteristics, and that F is of orders prime to the characteristics (which is a necessary assumption). But very little is known about the cohomology in the unequal characteristic case, even in low dimensions where resolution is available (Abyankhar [1]), say when X has dimension 2.

The correct upper bound for cohomological dimension of a variety over a field can be proved from the theorems on cohomological dimension for fields of Grothendieck and Tate (cf. [50]). We denote by $\mathrm{cd}_l X$ the largest integer q such that $H^q(X, F) \neq 0$ for some l-torsion sheaf F:

T h e o r e m (5): (cohomological dimension).

(i) Let X be a scheme of finite type over a separably closed field k. Then

$$\mathrm{cd}_l X \leqslant 2 \dim X.$$

If X is affine, then

$$\mathrm{cd}_l X \leqslant \dim X.$$

(ii) Let X be a scheme of finite type over the ring of integers of a number field K. Assume that either $l \neq 2$ or that K is totally imaginary. Then

$$\mathrm{cd}_l X \leqslant 2 \dim X + 1.$$

Here dim X is the Kronecker dimension. A much lower bound actually holds in (i) when l is equal to the characteristic, and for $l = 2$ in (ii) the totally imaginary number field can be replaced by $\mathbf{Q}$ if one adds to X in a formal way a "fibre at infinity" (cf. [8]). A number of other variants are treated in [6]. Again, little is known in the unequal characteristic case. Thus, for instance, the cohomology with values in $\mathbf{Z}/l$ of the scheme obtained from Spec $\mathbf{Z}_p[[t]]$ by removing the locus $\{t = 0\}$ is not known, nor is the cohomological dimension of the field of fractions of $\mathbf{Z}_p[[t]]$.

6. The fundamental group

As usual, the fundamental group classifies covering spaces: Suppose that the scheme X is connected and *locally connected* for the etale topology, i. e., that every etale scheme U over X is a disjoint union of connected components. Suppose moreover that a geometric point of X is given (X is *pointed*). Then a *pro-group* (cf. [23]) $\pi_1(X)$ is defined in such a way that it classifies etale covering spaces of X, i. e., schemes X' over X which are twisted forms for the etale topology of the "trivial covering" $\coprod_S X$ of X (S is a set). It does so in the sense that such schemes X' correspond canonically to homomorphisms from $\pi_1(X)$ to the permutation group of S. If X is geometrically unibranch ([27], IV), a connected covering space X' of this type is necessarily finite over X, and so the fundamental group is pro-finite and equal to the one introduced by Grothendieck [22]. In general, a scheme may have infinite covering spaces, and the fundamental group obtained is somewhat larger (cf. Lubkin [35], Grothendieck [24]).

Almost all of our explicit information about the fundamental group still comes from the "Riemann existence theorem". It asserts that a finite topological covering of a scheme X of finite type over $\mathbf{C}$ has a unique algebraic structure, i. e., that the etale fundamental group of X (say X is geometrically unibranch) is the profinite completion of the fundamental group of X:

$$\pi_1(X_{\text{etale}}) = \widehat{\pi_1(X_{\text{class}})}.$$

The general form of this theorem requires the results of Grauert and Remmert [18] and GAGA [46] (cf. [6]). Its importance for the etale theory is indicated for instance by the fact ([6], X) that a nonsingular variety over $\mathbf{C}$ has a Zariski open covering by $K(\pi, 1)$ spaces (ones having π_1 as only non-vanishing homotopy group).

Although Grothendieck, in his beautiful paper GFGA [22], succeeded in computing the "tame" fundamental group of an algebraic curve over an arbitrary field, the proof made use of the Riemann existence theorem in the classical case, and there is still no algebraic proof known. The difficulties which present themselves are very interesting.

Suppose for instance that X is obtained from the affine line by removing some points p_i ($i = 1, \ldots, n$), so that the tame fundamental group is free on n generators. Then the freeness can be expressed by the assertion that

$$\text{(1)} \qquad \operatorname{Hom}(\pi_1, G) \approx \prod_i \operatorname{Hom}(D_i, G)$$

where G is a finite "test group" of order prime to the characteristic, and where D_i is the decomposition subgroup of a suitably chosen point above p_i.

In fact, using Grothendieck's technique one can show by algebraic methods alone that in the above situation the ramification can be assigned arbitrarily at the points p_i, *up to inner automorphism.* By this we mean that the map

$$\text{Hex}(\pi_1, G) \longrightarrow \prod_i \text{Hex}(D_i, G) \tag{2}$$

is surjective, where Hex (A, B) is the set Hom (A, B) modulo conjugation by inner automorphisms of B. Here the choice of D_i no longer matters. This assertion is quite useful for cohomological questions, which are not very sensitive to conjugation (cf. section 3), but of course it is much weaker than (1). On the other hand, it has a chance to be true in characteristic p without restriction on the order of G.

7. Homotopy theory

In order to generalize the Riemann existence theorem to the higher homotopy groups, one needs a good way to associate something like a simplicial set to a scheme, and because of the difficulties inherent in the Čech procedure, it is not immediately clear how to do this. A way was first found by Lubkin [35]. Subsequently Verdier [55], using an idea of Cartier, found another method, and working along the lines suggested by the Cartier-Verdier approach, Quillen [41] has developed a homotopy theory for arbitrary categories.

The exact definitions are too technical to give here. Using them, one can associate to a connected and locally connected, pointed scheme X a *pro-object* in the homotopy category H of connected pointed CW-complexes, which represents the *homotopy type* of the scheme for the étale topology (cf. Lubkin [36]). Let us denote this pro-object by X_{et}. Its relation to the classical topology can be described as follows (cf. Artin and Mazur [7]):

Call a CW-complex *homotopy finite* if all of its homotopy groups are finite groups. Then one can associate to an object K in the homotopy category H a *pro-finite completion* $\hat{K}$ which is a formal inverse system of homotopy finite CW-complexes, i. e., a pro-object in the homotopy category of such complexes. The completion $\hat{K}$ is characterized by the property that any map from K to a homotopy finite complex factors through $\hat{K}$.

The natural extension of the Riemann existence theorem is the following result:

Theorem (1). Let X be a pointed geometrically unibranch scheme of finite type over the field of complex numbers. Denote by

$X_{\rm cl}$ its homotopy type for the classical topology. Then

$$X_{\rm et} = \widehat{X_{\rm cl}}.$$

It is in general not true that the homotopy groups of K are the pro-finite completions of the homotopy groups of K. However, this is true if K is simply connected and has finitely generated homotopy groups in each dimension [7]. Thus if in the above theorem $X_{\rm cl}$ is simply connected, then $\pi_q(X_{\rm et}) = \widehat{\pi_q(X_{\rm cl})}$ for each q.

The above method gives a definition of homotopy for quite general schemes X. For instance, the scheme Spec $\mathbf{Z}$ with a point at infinity added in a formal way has the homotopy type of a Moore space $K'(\mathbf{Z}/2,2)$ with the single nonvanishing homology group $\mathbf{Z}/2$ in dimension 2. However, it is likely that a good homotopy theory for Spec $\mathbf{Z}$ should allow for homotopy groups of a twisted sort, such as roots of unity.

8. Henselian rings

The study of local properties of schemes for the etale topology leads to a series of questions of an interesting sort which I want to mention briefly. The local ring of a scheme X at a geometric point x, in the etale topology, is the ring

$$R = \varinjlim_{(X', x')} \Gamma(X', O_{X'})$$

where (X', x') runs through schemes X' etale over X with chosen geometric point x' over x. The most striking property of these rings is that they are *henselian* (i. e., that Hensel's lemma holds). It seems clear that the notion of henselian ring, introduced by Azumaya [10] and studied by Nagata [39], will play an important rôle in any detailed study of local phenomena.

Let me recall the general outline, proposed by Grothendieck in the introduction to his Elements [27], for treating certain types of questions about schemes. I have rephrased it slightly for my purposes:

Step 1. One compares a global problem with the corresponding local one for the etale topology.

Step 2. By a limit argument, the local problem is reduced to a question about henselian rings R.

Step 3. One may replace the henselian ring R by its completion $\hat{R}$.

Step 4. The complete local ring is related to the artinian rings $\hat{R}/\mathfrak{m}^n$ ($\mathfrak{m} = $ rad $\hat{R}$), and the study of these rings is reduced by infinitesimal methods to a series of questions about the field $R/\mathfrak{m}$ (which are perhaps "classical"). Actually, none of these steps is under complete control, exept for 2 which is generally trivial (cf. Grothendieck [27]). Step 1 may sometimes be treated by descent theory (Grothendieck [23]), and step 4 was discussed by Serre in his Stockholm talk [49].

The somewhat novel point I want to bring out is step 3, which has not received much attention, but which seems promising. It is one aspect of the general algebraization problem of relating henselian rings to their completions. This has been studied for the divisor class group by Hironaka [29] and for algebraic extensions of rings in [45]. An interesting example is obtained from the question of the existence of a section of a map $f\colon X \to Y$ of schemes. The corresponding algebraization problem, which would handle step 3 in this case, is the following:

Suppose that X is a scheme of finite type over a henselian ring R which has a "formal section", i. e., a point with values in $\hat{R}$. What assumptions are needed to assure that one can approximate the formal section by points with values in R?

For a discrete valuation ring R, this problem was solved recently by Greenberg [19] and Raynaud. Mild restrictions on R suffice.

In a slightly different direction is the theorem of Grauert Hironaka and Rossi [17], [30] to the effect that analytic local rings with isolated singular points whose completions are isomorphic are themselves isomorphic. This theorem has an algebraic analogue in characteristic zero, which asserts that two algebraic varieties with isolated singular points whose local rings have isomorphic completions are locally isomorphic for the etale topology. (This is already a striking change from the Zariski topology in the case of simple points. For them it follows immediately from the Jacobian criterion.) The related conjecture of Grauert that any complete local ring with isolated singularity is "algebraic", i. e., is the completion of a local ring of an algebraic variety is however still open except in low dimensions [4], [29].

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