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THE TRANSFER MAP AND FIBER BUNDLES

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§1. INTRODUCTION

LET $p: E \to B$ be a fiber bundle whose fiber F is a compact smooth manifold, whose structure group G is a compact Lie group acting smoothly on F, and whose base B is a finite complex. Let χ denote the Euler characteristic of F. It is shown in [12] that there exists a "transfer" homomorphism $\hat{t}: H^*(E) \to H^*(B)$ with the property that the composite $\hat{t}p^*$ is multiplication by χ. The main purpose of this paper is to construct an S-map $\tau: B^+ \to E^+$ which induces the homomorphism $\hat{t}$ (+ denoting disjoint union with a base point). We call τ the transfer associated with the fiber bundle $p: E \to B$. In the case of a finite covering space τ agrees with the transfer defined by Roush [22] and by Kahn and Priddy [18].

The existence of the transfer imposes strong conditions on the projection map of a fiber bundle. Specifically, we have the following

THEOREM 5.7. *Let ξ be a fiber bundle with fiber F having Euler characteristic χ. Then*

$$p^* \otimes 1: h^*(B^+) \otimes Z[\chi^{-1}] \to h^*(E^+) \otimes Z[\chi^{-1}]$$

is a monomorphism onto a direct factor for any (reduced) cohomology theory h.

We will use the above theorem to establish a variant of the well known splitting principle for vector bundles (see Theorem 6.1). An application of this splitting principle is an alternative proof of the Adams conjecture (Quillen [21], Sullivan [25], Friedlander [11]).

The proof in outline is as follows. Theorem 6.1 asserts that if α is a $2n$-dimensional real vector bundle over a finite complex X there exists a finite complex Y and a map $\lambda: Y \to X$ such that (a) the structure group of $\lambda^*(\alpha)$ reduces to the normalizer N of a maximal torus in $0(2n)$; (b) $\lambda^*: h^*(X^+) \to h^*(Y^+)$ is a monomorphism for any cohomology theory h. By (6.1) and the result of Boardman and Vogt [7] that BF, the classifying space for spherical fibrations is an infinite loop space, one is reduced to considering vector bundles with structure group N. An argument similar to the one employed by Quillen to treat vector bundles with finite structure group is then used to treat bundles of this form.

§2. HOPF'S THEOREM

Let G denote a compact Lie group. A G-manifold F is understood to mean a compact smooth manifold together with a smooth action of G. The boundary of F will be denoted by $\dot{F}$. By a G-module V we mean a finite dimensional real G-module equipped with a G-invariant metric. The one point compactification of V will be denoted by S^V.

If $\alpha = (X_\alpha, B, p_\alpha)$ is a vector bundle we let $\bar{\alpha} = (X_{\bar{\alpha}}, B, p_{\bar{\alpha}})$ denote its fiberwise one point compactification, and we identify B with the cross section at infinity. Then for $A \subset B$ we have the Thom space

$$(B, A)\alpha = X_{\bar{\alpha}}/B \cup p_{\bar{\alpha}}^{-1}(A). \tag{2.1}$$

A theorem of Mostow [20] asserts that there exists a G-module V and a smooth equivariant embedding $F \subset V$. Let F have the induced metric, let $\omega = (X_\omega, F, p_\omega)$ denote the normal bundle, and let $X_\omega \subset V$ denote an equivariant embedding of X_ω as a tubular neighborhood of F in V.

Suppose now that F is a closed manifold. There is the associated Pontryagin–Thom map

$$c: S^V \to F^\omega \tag{2.2}$$

which is an equivariant map. Let τ denote the tangent bundle of F and let $\psi: \tau \oplus \omega \to F \times V$ denote the trivialization associated with the embedding.

Define

$$\gamma: S^V \to (F^+) \wedge S^V \tag{2.3}$$

to be

$$S^V \xrightarrow{c} F^\omega \xrightarrow{i} F^{\tau \oplus \omega} \xrightarrow{\psi} (F^+) \wedge S^V$$

where i is the inclusion.

THEOREM 2.4. *The degree of the composite*

$$S^V \xrightarrow{\gamma} (F^+) \wedge S^V \xrightarrow{\pi} S^V$$

where π is the projection, is $\chi(F)$- the Euler characteristic of F.

This is essentially [19; Theorem 1, p. 38]. However we will deduce (2.4) from Hopf's vector field theorem [13] in the form stated below.

Suppose that F is connected and orientable. Let U_τ be an orientation class for τ and let U_ω be the orientation of ω determined by U_τ and ψ, that is, such that under the maps

$$F^\tau \wedge F^\omega \xleftarrow{d} F^{\tau \oplus \omega} \xrightarrow{\psi} (F^+) \wedge S^s \xrightarrow{\pi} S^s$$

we have

$$\psi^* \pi^*(v) = d^*(U_\tau \wedge U_\omega) \tag{2.5}$$

where d is the diagonal and v is the canonical generator of $\tilde{H}^s(S^s)$.

With the above data let $\mu \in \tilde{H}^n(F^+)$ denote the preimage of γ under the composite

$$\tilde{H}^n(F^+) \xrightarrow{\Phi} \tilde{H}^s(F^\omega) \xrightarrow{c^*} \tilde{H}^s(S^s)$$

where Φ is the Thom isomorphism. Next, let $h: F \to F^\tau$ denote the inclusion.

THEOREM 2.6 (Hopf [13, 23]). *We have $h^*(U_\tau) = \chi(F) \cdot \mu$ where $\chi(F)$ is the Euler characteristic of F.*

We will now prove (2.4) in the case where F is connected and orientable. We have a commutative diagram

$$\begin{array}{ccccc} S^s & \xrightarrow{c} & F^\omega & \xrightarrow{i} & F^{\tau \oplus \omega} \\ & & \Big\downarrow{\scriptstyle \rho} & & \Big\downarrow{\scriptstyle d} \\ & & (F^+) \wedge F^\omega & \xrightarrow{h \wedge 1} & F^\tau \wedge F^\omega \end{array}$$

where $\rho(v_b) = b \wedge v_b$. In view of (2.5) we must show that $c^* i^* d^*(U_\tau \wedge U_\omega) = \chi(F) \cdot v$. This follows by a simple diagram chase using the relation $h^*(U_\tau) = \chi(F) \cdot \mu$.

Suppose now that F is connected and unorientable. Let $p\colon F_o \to F$ be the orientable double cover of F. Let $F \subset R^s$ with normal bundle ω and let $F_o \subset F \times R^t$ be an embedding homotopic to p. Then the normal bundle of the composite embedding $F_o \subset F \times R^t \subset R^{s+t}$ may be identified with $p^*(\omega) \times R^t$. We have the following homotopy commutative diagrams

$$\begin{array}{ccc} & \xrightarrow{c \wedge 1} & F^\omega \wedge S^t \\ S^{s+t} & & \Big\downarrow{\scriptstyle c'} \\ & \xrightarrow{c_o} & F_o^{p^*(\omega)} \wedge S^t \end{array} \qquad \begin{array}{ccccc} F^\omega \wedge S^t & \xrightarrow{i \wedge 1} & F^{\tau \oplus \omega} \wedge S^t & \xrightarrow{\pi\psi \wedge 1} & \\ \Big\uparrow{\scriptstyle p'} & & & & S^{s+t} \\ F_o^{p^*(\omega)} \wedge S^t & \xrightarrow{i_o} & F^{p^*(\tau \oplus \omega)} \wedge S^t & \xrightarrow{\pi\psi_o} & \end{array}$$

where the triangle consists of the collapsing maps obtained from the embeddings $X_{p^*(\omega)} \times R^t \subset X_\omega \times R^t \subset R^{s+t}$, and p' is the projection.

It is well known that c' represents the transfer associated with the covering pair $(X_{p^*(\bar{\omega})}, F_o) \to (X_{\bar{\omega}}, F)$ (for a proof see [5; Appendix]). Therefore $(p'c')^*$ is multiplication by 2 in singular cohomology. It follows now that the degree of $\pi\psi_o i_o c_o$ is twice the degree of $\pi\psi ic$. Since $\chi(F_o) = 2\chi(F)$ the degree of $\pi\psi ic$ is $\chi(F)$ as desired.

Finally, if F has components $F_1, F_2, \ldots, F_m$ it is easy to see that $\pi\gamma = \sum_i \pi\gamma_i$, where $\gamma_i\colon S^s \to (F_i^+) \wedge S^s$. Since $\chi(F) = \sum \chi(F_i)$. The general case follows from the connected case.

We close this section by indicating the modifications in the above construction when F has non empty boundary $\dot{F}$. As before let $F \subset V$ be an equivalent embedding. The Pontryagin–Thom map now has the form

$$c\colon S^V \to (F, \dot{F})^\omega. \tag{2.7}$$

Let Δ denote the unit outward normal vector field on $\dot{F}$. It follows easily from the existence of an equivariant collar of F [9] that Δ can be extended to an equivariant vector field $\bar{\Delta}$ on F such that $|\bar{\Delta}(x)| \leq 1$, $x \in F$. Let

$$i\colon (F, \dot{F})^\omega \to F^{\tau \oplus \omega} \tag{2.8}$$

be defined by

$$i(v_x) = \begin{cases} (1/1 - |\bar{\Delta}(x)|)(\bar{\Delta}(x) + v_x) & |\bar{\Delta}(x)| < 1, \\ \infty, & |\bar{\Delta}(x)| = 1. \end{cases}$$

Then

$$\gamma\colon S^V \to (F^+) \wedge S^V \tag{2.9}$$

is to be the map

$$S^V \xrightarrow{c} (F, \dot{F})^\omega \xrightarrow{i} F^{\tau\oplus\omega} \xrightarrow{\psi} (F^+) \wedge S^V.$$

Theorem (2.4) remains valid for manifolds with boundary and the proof is essentially the same.

§3. THE TRANSFER

Let F denote a G-manifold as in the previous section and let $\xi = (E, B, p)$ be a fiber bundle with fiber F associated to a principal G-bundle $\tilde{\xi} = (\tilde{E}, B, \tilde{p})$, where B is a finite complex. For each such ξ we will construct an S-map

$$\tau(\xi)\colon B^+ \to E^+ \tag{3.1}$$

which we call its transfer, having the following properties.

(3.2). *If $h\colon \xi \to \xi'$ is a fiber bundle map the square*

$$\begin{array}{ccc} B^+ & \xrightarrow{\tau(\xi)} & E^+ \\ \Big\downarrow{\scriptstyle h} & & \Big\downarrow{\scriptstyle h} \\ (B')^+ & \xrightarrow{\tau(\xi')} & (E')^+ \end{array}$$

is commutative.

If X is a finite complex and ξ is a fiber bundle we let $X \times \xi$ denote the fiber bundle $(X \times E, X \times B, 1 \times p)$.

(3.3). *We have*

$$\tau(X \times \xi) = 1 \wedge \tau(\xi)\colon X^+ \wedge B^+ \to X^+ \wedge E^+.$$

For the singleton space $\{0\}$ we identify $\{0\}^+$ with $\{0\} \cup \{\infty\} = S^o$.

(3.4) If $\xi = (F, \{0\}, p)$ the composite $p\tau(\xi)\colon S^o \to S^o$ has degree $\chi(F)$.

We proceed now to construct the transfer. Recall that an ex-space of B [16], [17] is an object $X = (X, B, p, \Delta)$ consisting of maps $p\colon X \to B$ and $\Delta\colon B \to X$ such that $p\Delta = 1$. An ex-map $f\colon X \to Y$ is an ordinary map which is both fiber and cross section preserving. For example, if $\alpha = (X_\alpha, B, P_\alpha)$ is a vector bundle over B we have the ex-space $X_{\bar{\alpha}}$, the fiberwise one point compactification of X_α, by taking $\Delta\colon B \to X_{\bar{\alpha}}$ to be the cross section at infinity. As a second example, if $\tilde{p}\colon \tilde{E} \to B$ is a principal G-bundle and Y is a G-space with base point y_o fixed under the action of G, we obtain an ex-space $\tilde{E} \times_G Y$ by taking Δ: $B \to \tilde{E} \times_G Y$ to be the map $b \to [\tilde{e}, y_o]$, where $\tilde{p}(\tilde{e}) = b$.

If X and Y are ex-spaces of B we denote their fiberwise reduced join by $X \wedge_B Y$.

For the G-manifold F we have an equivariant map

$$\gamma\colon S^V \to (F^+) \wedge S^V \tag{3.5}$$

as in (2.4). We have an ex-map

$$1 \times_G \gamma\colon \tilde{E} \times_G S^V \to \tilde{E} \times_G ((F^+) \wedge S^V) \tag{3.6}$$

which we denote by γ'.

Let η denote the vector bundle with fiber B associated to $\tilde{\xi}$ and let $\zeta = (X_\zeta, B, p_\zeta)$ be a complimentary bundle with trivialization $\phi\colon \eta \oplus \zeta \to B \times R^s$. Now we have

$$\gamma' \wedge_B 1\colon (\tilde{E} \times_G S^V) \wedge_B X_\zeta \to \tilde{E} \times_G ((F^+) \wedge S^V) \wedge_B X_\zeta. \tag{3.7}$$

If we identify B to a point on each side the resulting quotient space on the left is $B^{\eta \oplus \zeta}$ whereas the one on the right is $E^{p^*(\eta \oplus \zeta)}$. Let

$$\sigma\colon B^{\eta \oplus \zeta} \to E^{p^*(\eta \oplus \zeta)} \tag{3.8}$$

denote the induced map. Now we define $\tau(\xi)$ in (3.1) to be the S-map represented by

$$(B^+) \wedge S^s \xrightarrow{\phi^{-1}} B^{\eta \oplus \zeta} \xrightarrow{\sigma} E^{p^*(\eta \oplus \zeta)} \xrightarrow{p^*(\phi)} (E^+) \wedge S^s.$$

This construction of the transfer, by applying standard bundle techniques to the G-map $\gamma\colon S^V \to (F^+) \wedge S^V$, is parallel to Boardman's construction [6] of the "umkehr" map from the Pontryagin–Thom map $S^V \to F^\omega$ (see §4).

Suppose that $e\colon F \to V$ and $e'\colon F \to V'$ are equivariant embeddings yielding γ and γ' respectively as in (2.4). The equivariant isotopy $H\colon F \times I \to V \oplus V'$ by $H(y, t) = (1 - t)e(y) \oplus t\, e'(y)$ yields, by a standard argument, an equivariant homotopy

$$K\colon S^{V \oplus V'} \times I \to (F^+) \wedge S^{V \oplus V'}$$

such that $K_o = \gamma \wedge 1$ and K_1 is the composite

$$S^{V \oplus V'} \xrightarrow{1 \wedge \gamma'} S^V \wedge F^+ \wedge S^{V'} \longrightarrow F^+ \wedge S^{V \oplus V'}$$

(identifying $S^{V \oplus V'}$ with $S^V \wedge S^{V'}$). Using K it is easy to show that a transfer constructed from the embedding e is stably homotopic to one constructed from the embedding e'. Therefore the transfer is well defined, i.e. independent of the choices involved.

Properties (3.2) and (3.3) of the transfer now follow immediately from its definition. Property (3.4) is simply a restatement of Theorem 2.4.

§4. THE UMKEHR MAP

In this section we will make explicit the relation between the transfer and the classical umkehr map. Let ξ be a fiber bundle with fiber F a smooth n-dimensional G-manifold without boundary. Let $\tilde{\xi}$ be the underlying principal bundle of ξ. Retaining the notation of §2 and §3, the bundle α of tangents along the fiber is given by

$$\tilde{E} \times_G X_\tau \xrightarrow{1 \times_G p_\tau} \tilde{E} \times_G F = E.$$

Let β denote the bundle

$$\tilde{E} \times_G X_\omega \xrightarrow{1 \times_G p_\omega} \tilde{E} \times_G F = E.$$

The trivialization $\psi\colon \tau \oplus \omega \to F \times V$ yields an equivalence $\hat{\psi}\colon \alpha \oplus \beta \to p^*(\eta)$ and we have

$$\alpha \oplus \beta \oplus p^*(\gamma) \xrightarrow{\hat{\psi} \oplus 1} p^*(\eta) \oplus p^*(\zeta) \xrightarrow{p^*(\phi)} E \times R^s.$$

Let $\alpha' = \beta \oplus p^*(\zeta)$ and let $\theta\colon \alpha \oplus \alpha' \to E \times R^s$ denote the above trivialization.

The Pontryagin–Thom map $c\colon S^V \to F^\omega$ yields

$$(E \times_G S^V) \wedge_B X_\zeta \xrightarrow{(1 \times Gc) \wedge 1} (E \times_G F^\omega) \wedge_B X_\zeta.$$

Identifying B to a point on each side we have a map $t'\colon B^{\eta \oplus \zeta} \to E^{\alpha'}$. The *umkehr* map

$$t\colon (B^+) \wedge S^s \to E^{\alpha'} \tag{4.1}$$

is the composite

$$(B^+) \wedge S^s \xrightarrow{\phi^{-1}} B^{\eta \oplus \zeta} \xrightarrow{t'} E^{\alpha'}.$$

This construction of t is due to Boardman [6].

Let $\mathbf{M}$ be a ring spectrum [26]. We will say that ξ is $\mathbf{M}$-orientable if its bundle α of tangents along the fiber is $\mathbf{M}$-orientable in the usual sense. In this case let $U \in \mathbf{M}^n(E^\alpha)$ be an orientation class for α, let $\chi_\alpha \in \mathbf{M}^n(E^+)$ be its Euler class and let $U' \in \mathbf{M}^{s-n}(E^{\alpha'})$ be the orientation of α' determined by U and the trivialization θ. With this data we obtain from t a homomorphism (depending on U)

$$p_\#\colon \mathbf{M}^k(E^+) \to \mathbf{M}^{k-n}(B^+) \tag{4.2}$$

by

$$\mathbf{M}^k(E^+) \xrightarrow{\Phi'} \mathbf{M}^{k+s-n}(E^{\alpha'}) \xrightarrow{t'} \mathbf{M}^{k+s-n}((B^+) \wedge S^s) \xrightarrow{\sigma} \mathbf{M}^{k-n}(B^+),$$

where σ denotes suspension and Φ' is the Thom isomorphism associated with U'.

THEOREM 4.3. *If ξ is M-orientable the transfer*

$$\tau^*\colon M^k(E^+) \longrightarrow M^k(B^+)$$

is given by $\tau^*(x) = p_\#(x \cup \chi_\alpha)$.

Proof. We may easily check that τ is the composite

$$(B^+) \wedge S^s \xrightarrow{t} E^{\alpha'} \xrightarrow{i} E^{\alpha \oplus \alpha'} \xrightarrow{\theta} (E^+) \wedge S^s \tag{4.4}$$

where i is the inclusion. The result now follows from the commutativity of the following diagram.

$$\begin{array}{ccccccc}
\mathbf{M}^k(E^+) & & \xrightarrow{-\cup \chi_\alpha} & & \mathbf{M}^{k+n}(E^+) & \xrightarrow{p_\#} & \mathbf{M}^k(B^+) \\
\uparrow \sigma & & & & \downarrow \Phi' & & \uparrow \sigma \\
\mathbf{M}^{k+s}((E^+) \wedge S^s) & \xrightarrow{\theta^*} & \mathbf{M}^{k+s}(E^{\alpha \oplus \alpha'}) & \xrightarrow{i^*} & \mathbf{M}^{k+s}(E^{\alpha'}) & \xrightarrow{t^*} & \mathbf{M}^{k+s}((B^+) \wedge S^s)
\end{array}$$

Our object now is to point out that our transfer agrees with that given by F. Roush [22] and D. Kahn and S. Priddy [18] in the case of a finite covering. If $p\colon E \to B$ is an n-fold covering we regard it as a fiber bundle with fiber $\{1, 2, \ldots, n\}$ and structure group the symmetric group $\mathscr{S}_n$ in the usual way. Thus the transfer constructed above yields a transfer for any n-fold covering. In this case the bundle α of tangents along the fiber is 0 and the map i in (4.4) is the identity. Hence we have

$$\begin{array}{ccc} & & E^{\alpha'} \\ & \overset{t}{\nearrow} & \big\downarrow \theta \\ (B^+) \wedge S^s & \underset{\tau}{\searrow} & \\ & & (E^+) \wedge S^s \end{array}$$

so that (modulo the identification θ) the transfer τ is the same as the umkehr map t. In [5; Appendix] a direct proof is given that the transfer of Roush and Kahn–Priddy is the same as θt. Hence it is the same as τ.

§5. MULTIPLICATIVE PROPERTIES

If ξ is a fiber bundle we have a commutative diagram

$$\begin{array}{ccccc} E & \xrightarrow{d} & E \times E & \xrightarrow{p \times 1} & B \times E \\ \big\downarrow p & & & & \big\downarrow 1 \times p \\ B & & \xrightarrow{\qquad d \qquad} & & B \times B \end{array} \tag{5.1}$$

where d in each case denotes the diagonal map.

Since $(p \times 1)d$ is a bundle map we obtain from (3.2) and (3.3) the following commutative diagram of S-maps.

$$\begin{array}{ccccc} E^+ & \xrightarrow{d} & E^+ \wedge E^+ & \xrightarrow{p \wedge 1} & B^+ \wedge E^+ \\ \big\uparrow \tau & & & & \big\uparrow 1 \wedge \tau \\ B^+ & & \xrightarrow{\qquad d \qquad} & & B^+ \wedge B^+ \end{array} \tag{5.2}$$

Now suppose that $\mathbf{M}$ is a ring spectrum and $\mathbf{N}$ is an $\mathbf{M}$ module [26]. The commutativity of (5.2) together with elementary properties of the cup and cap product imply that the transfer satisfies the following basic relations.

$$\tau^*(p^*(x) \cup y) = x \cup \tau^*(y), \qquad x \in \mathbf{M}^s(B^+), \qquad y \in \mathbf{N}^t(E^+). \tag{5.3}$$

$$p^*(\tau^*(x) \cap y) = x \cap \tau^*(y), \qquad x \in \mathbf{N}_s(B^+), \qquad y \in \mathbf{M}^t(E^+). \tag{5.4}$$

Let $\tilde{H}(\ ;\Lambda)$ denote reduced singular theory with coefficients in Λ.

THEOREM 5.5. *Let ξ be a fiber bundle with fiber F. The composite*

$$\tilde{H}^*(B^+;\Lambda) \xrightarrow{p^*} \tilde{H}^*(E^+;\Lambda) \xrightarrow{\tau^*} \tilde{H}^*(B^+;\Lambda)$$

is multiplication by $\chi(F)$.

Proof. Let $b \in B$ and let $i_b\colon F \to E$ be a bundle map covering $j_b\colon \{0\} \to \{b\}$. By (3.2) and (3.4)

$$j_b{}^*\colon \tilde{H}^o(B^+;Z) \to \tilde{H}^o(S^o;Z)$$

sends $\tau^*p^*(1)$ to $\chi(F)\cdot 1$. It follows now that $\tau^*(1) = \tau^*p^*(1) = \chi(F)\cdot 1$. Now if $x \in \tilde{H}^s(B^+;\Lambda)$ we have by (5.3),

$$\tau^*p^*(x) = \tau^*(p^*(x) \cup 1) = x \cup \tau^*(1) = \chi(F)\cdot x.$$

A dual result for singular homology follows from (5.4).

Let $\chi = \chi(F)$ and let $Z[\chi^{-1}]$ denote the ring of integers with χ^{-1} adjoined if $\chi \neq 0$ and let $Z[\chi^{-1}] = 0$ if $\chi = 0$. Let h be a (reduced) cohomology theory on the category of finite CW-complexes and consider the cohomology theory $h \otimes Z[\chi^{-1}]$. The S-map

$$p\tau\colon B^+ \to B^+$$

induces

$$(p\tau)^* \otimes 1\colon h^*(B^+) \otimes Z[\chi^{-1}] \to h^*(B^+) \otimes Z[\chi^{-1}]. \tag{5.6}$$

Applying the Atiyah–Hirzebruch spectral sequence [10], we have on the E_2-level

$$(p\tau)^*\colon \tilde{H}^*(B^+; h^*(S^o) \otimes Z[\chi^{-1}]) \to \tilde{H}^*(B^+; h^*(S^o) \otimes Z[\chi^{-1}])$$

and $(p\tau)^*$, being multiplication by χ, is an isomorphism. Therefore, by the comparison theorem, $(p\tau)^* \otimes 1$ in (5.6) is also an isomorphism. We now have the following generalization of a result of Borel [8].

THEOREM 5.7. *Let ξ be a fiber bundle with fiber F having Euler characteristic χ. Then*

$$p^* \otimes 1\colon h^*(B^+) \otimes Z[\chi^{-1}] \to h^*(E^+) \otimes Z[\chi^{-1}]$$

is a monomorphism onto a direct factor, for any cohomology theory h.

In particular, if $\chi = 1$, $p^*\colon h^*(B^+) \to h^*(E^+)$ is a monomorphism onto a direct factor.

§6. VECTOR BUNDLES

Let T denote a maximal torus of the compact Lie group G and let $N(T)$ be the normalizer of T in G. If G is connected, a theorem of Hopf and Samelson [14] states that $\chi(G/T) = |N(T)/T|$, the order of the Weyl group $N(T)/T$. Now we have a finite covering space

$$N(T)/T \to G/T \to G/N(T)$$

so that $\chi(G/T) = \chi(G/N(T)) \cdot |N(T)/T|$. Comparing this with the preceding formula we see that $\chi(G/N(T)) = 1$.

Now consider the orthogonal group $0(2n)$ and let $T = \overset{n}{\underset{1}{\times}}\ SO(2)$ denote the standard maximal torus. Then T is also a maximal torus of $SO(2n)$ and if $N_o(T)$ denotes the normalizer of T in $SO(2n)$ we have, by the above remarks, that $\chi(SO(2n)/N_o(T)) = 1$. Observe that $O(2n)/N(T) = SO(2n)/N_o(T)$ and therefore $\chi(O(2n)/N(T)) = 1$.

Let $\alpha = (E, B, p)$ be a $2n$-plane bundle over a finite complex B and let $\tilde{\alpha} = (\tilde{E}, B, \tilde{p})$ be its associated principal $O(2n)$ bundle so that

$$E = \tilde{E} \times_{O(2n)} R^{2n}.$$

Let $X = \tilde{E}/N(T)$ and let $\lambda\colon X \to B$ be the natural map. It is the projection of a fiber bundle whose fiber F is the space of left cosets of $N(T)$ in$O(2n)$. Since F is diffeomorphic to $O(2n)/N(T)$ we have $\chi(F) = 1$. According to Theorem 5.7,

$$\lambda^*\colon h^*(B^+) \to h^*(X^+)$$

is a monomorphism for any cohomology theory h.

By the *standard* $N(T)$-module W we mean R^{2n} together with the action of $N(T)$ obtained by restricting the usual action of $O(2n)$. Let ζ denote the principal $N(T)$-bundle $\tilde{E} \to \tilde{E}/N(T) = X$. In view of the commutative square

$$\begin{array}{ccc} \tilde{E} \times_{N(T)} R^{2n} & \xrightarrow{\tilde{\lambda}} & \tilde{E} \times_{O(2n)} R^{2n} \\ \downarrow & & \downarrow \\ X & \xrightarrow{\lambda} & B \end{array}$$

where $\tilde{\lambda}$ is the quotient map, we see that $\lambda^*(\alpha)$ is equivalent to the vector bundle with fiber W associated to ζ.

Recall that the wreath product $\mathscr{S}_n \int H$ of the symmetric group with a group H is the semi-direct produce $\mathscr{S}_n \times_\theta \left(\overset{n}{\underset{1}{\times}} H\right)$ where $\theta\colon \mathscr{S}_n \to \mathrm{Aut}\left(\overset{n}{\underset{1}{\times}} H\right)$ is the obvious map. We then have $N(T) = \mathscr{S}_n \int O(2)$ (cf. [3], [15]). Summarizing, we have the following result.

THEOREM 6.1. *Let α be a $2n$-plane bundle over a finite complex B. There exists a finite complex X, a map $\lambda\colon X \to B$, and a principal $\mathscr{S}_n \int O(2)$-bundle ζ over X such that*

(1) $\lambda^*(\alpha)$ *is the vector bundle associated to ζ having fiber the standard $\mathscr{S}_n \int O(2)$-module W.*

(2) $\lambda^*\colon h^*(B^+) \to h^*(X^+)$ *is a monomorphism for any cohomology theory h.*

The space constructed above has the homotopy type of a finite CW-complex by [24; Proposition 0]. We take X in the statement of the theorem to be a finite complex homotopy equivalent to the original X.

§7. THE ADAMS CONJECTURE

In this section we will show how the transfer can be used to prove the following.

THEOREM 7.1 (Quillen [21], Sullivan [25], Friedlander [11]). *Let B be a finite complex, let k be an integer and let $x \in KO(B)$. Then there is an integer n such that $k^n J(\psi^k(x) - x) = 0$.*

This was proved by Adams [2] for vector bundles of dimension 1 and 2. The group $\mathrm{Sph}(B)$ is the group of stable equivalence classes of spherical fibrations over B and

$$J\colon KO(B) \to \mathrm{Sph}(B)$$

is the extension of the map which assigns to each vector bundle its underlying sphere bundle.

First observe that it is sufficient to prove (7.1) in the case where $x = [\alpha]$ with α a $2n$-dimensional vector bundle. With $\lambda: X \to B$ as in Theorem 6.1 we have the following commutative diagram:

$$\begin{array}{ccc} KO(X) & \xrightarrow{J} & \mathrm{Sph}(X) \\ \downarrow{\scriptstyle \lambda^*} & & \downarrow{\scriptstyle \lambda^*} \\ KO(B) & \xrightarrow{J} & \mathrm{Sph}(B) \end{array} \tag{7.2}$$

Let F_n denote the space of base point preserving homotopy equivalences of S^n; let $F = \text{inj lim}_n F_n$; and let BF denote the classifying space for F. It follows from a result of Stasheff [24] that there is a natural equivalence

$$\mathrm{Sph}(B) \to [B^+; BF]. \tag{7.3}$$

(Here [;] denotes base point preserving maps.)

Now Boardman and Vogt [7; Theorems A and B] have shown that BF is an infinite loop space. That is, there is an Ω-spectrum M such that $M_o = BF$. We then have natural equivalences

$$\mathrm{Sph}(B) \to [B^+, BF] \to M^o(B^+). \tag{7.4}$$

It follows now from Theorem 6.1 that

$$\lambda^*: \mathrm{Sph}(B) \to \mathrm{Sph}(X)$$

is a monomorphism. Then by the commutativity of (7.2) we see that (7.1) is true for α if it is true for $\lambda^*(\alpha)$.

Let $G = \mathscr{S}_n \int O(2)$. It remains to prove (7.1) for vector bundles such as $\lambda^*(\alpha)$ which have the form

$$\eta: E \times_G W \xrightarrow{p} X, \tag{7.5}$$

where $p: E \to X$ is a principal G-bundle. The argument here is similar to the one employed by Quillen in treating vector bundles with finite structure group. The group G consists of elements $(\rho, T_1, \ldots, T_n)$ where $\rho \in \mathscr{S}_n$ and $T_i \in O(2)$, $1 < i < n$. The multiplication is given by

$$(\rho, T_1, \ldots, T_n)(\sigma, S_1, \ldots, S_n) = (\rho\sigma, T_{\sigma(1)} S_1, \ldots, T_{\sigma(n)} S_n).$$

Let H be the subgroup of G consisting of elements $(\rho, T_1, \ldots, T_n)$ such that $\rho(1) = 1$, and define a homomorphism $\phi: H \to O(2)$ by $\phi(\rho, T_1, \ldots, T_n) = T_1$. This defines a 2-dimensional H-module which we shall denote by V.

Now H has finite index n in G so we have the induced G-module $i(V)$ defined as follows: let $\sigma_1 H, \ldots, \sigma_n H$ be a complete set of left cosets of H in G and let

$$i(V) = \{\sigma_1\} \times V \oplus \ldots \oplus \{\sigma_n\} \times V.$$

For $g \in G$ let $g\sigma_i = \sigma_k h$, $h \in H$. The action of G on $i(V)$ is defined by

$$g \cdot (\sigma_i \times V) = \sigma_k \times hv.$$

Now by a direct calculation we see that

$$i(V) = W. \tag{7.6}$$

We have the finite covering space

$$\tilde{E}/H \longrightarrow \tilde{E}/G = X$$

and the vector bundle

$$\zeta : \tilde{E} \times_H V \xrightarrow{p} \tilde{E}/H.$$

Since ζ is 2-dimensional (7.1) is true for ζ as shown by Adams [2]. We have the transfer

$$\tau^* : KO(\tilde{E}/H) \to KO(X)$$

associated with the above covering space. The proof of (7.1) for η is now a consequence of the following two facts (see [21]):

(7.7). $\tau^*(\zeta) = \eta$.

(7.8). *If ζ is a 2-dimensional bundle over $\tilde{E}/H$,* (7.1) *is true for* $\tau^*(\zeta)$.

It is known [18], [22] that τ^* agrees with the geometrically defined transfer as described by Atiyah [4]. Using the geometric description it is easy to see that τ^* sends the vector bundle with fiber the H-module V associated with $E \to E/H$ to the vector bundle with fiber the G-module $i(V)$ associated with $E \to E/G$. Since $i(V) = W$ this yields $\tau^*(\zeta) = \eta$.

The proof of (7.8) is given by Quillen in the case where k in (7.1) is an odd prime. The proof for k an odd integer or for k even and ζ orientable is identical. Finally, suppose that k is even and ζ is non-orientable. Let γ be the line bundle classified by the first Stiefel–Whitney class $w_1(\zeta)$. Since $\zeta \otimes \gamma$ is orientable (7.1) is true for $\tau^*(\zeta \otimes \gamma)$. Since $[\gamma] - 1$ has order a power of 2 [1], we have $[\zeta \otimes \gamma] = [\zeta]$ modulo 2-torsion and therefore

$$\tau^*[\zeta \otimes \gamma] = \tau^*[\zeta]$$

modulo 2-torsion. Now since k is even it is easy to see that (7.1) also holds for $\tau^*(\zeta)$.

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