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ON THE GROUPS $J(X)$—II

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§1. INTRODUCTION

THE GENERAL object of this series of papers is to give means for computing the groups $J(X)$. A general introduction has been given at the beginning of Part I. The object of the present paper, Part II, is to set up the groups $J'(X)$ and $J''(X)$.

The arrangement of the present paper is as follows. We reach the group $J'(X)$ in §6. Its definition depends on the "cannibalistic characteristic class" ρ^k, which is treated in §5; and this in turn depends on the Thom isomorphism in K-theory, to which we devote §4. The group $J''(X)$ is treated in §3. Here we prove Theorem (3.12), which states a formal property of J'', and is required for use in [5]. §2 is devoted to necessary number-theory about the Bernoulli numbers.

§2. NUMBER-THEORY

The work of Milnor and Kervaire [15] shows the importance of the Bernoulli numbers in studying the J-homomorphism. In what follows, we shall need certain elementary number-theoretical results about Bernoulli numbers and related topics. These results are presumably known, but for completeness, they are collected and proved in the present section.

We begin by establishing some notation. The Bernoulli numbers enter algebraic topology in various ways. One of their bridgeheads is the power-series for the function.

$$\mathrm{Log}\left(\frac{\mathrm{Sinh}\,\frac{1}{2}x}{\frac{1}{2}x}\right).$$

Here the function

$$\frac{\mathrm{Sinh}\,\frac{1}{2}x}{\frac{1}{2}x}$$

is to be interpreted as 1 for $x = 0$; it is then analytic and non-zero for $|x| < 2\pi$, and is even. Thus we have

$$\mathrm{Log}\left(\frac{\mathrm{Sinh}\,\frac{1}{2}x}{\frac{1}{2}x}\right) = \sum_{s=1}^{\infty} \alpha_{2s} \frac{x^{2s}}{(2s)!} \tag{2.1}$$

(for $|x| < 2\pi$), where this expansion defines the coefficients α_{2s}. We have

$$\frac{e^x - 1}{x} = e^{\frac{1}{2}x} \cdot \frac{\operatorname{Sinh} \frac{1}{2}x}{\frac{1}{2}x};$$

therefore

$$\operatorname{Log} \frac{e^x - 1}{x} = \sum_{t=1}^{\infty} \alpha_t \frac{x^t}{t!}, \tag{2.2}$$

where we have defined α_t for odd t by setting

$$\alpha_1 = \tfrac{1}{2}, \qquad \alpha_{2s+1} = 0 \quad \text{for} \quad s > 0.$$

It is very easy to compare (2.2) with the expansion of $x/(e^x - 1)$. Following Hardy and Wright [12, p. 90] we set

$$\frac{x}{e^x - 1} = \sum_{t=0}^{\infty} \beta_t \frac{x^t}{t!}. \tag{2.3}$$

LEMMA (2.4). *For $t > 1$ we have*

$$\alpha_t = \frac{\beta_t}{t}.$$

Proof. Differentiating (2.2), we have

$$\frac{x}{e^x - 1}\left(\frac{e^x}{x} - \frac{e^x - 1}{x^2}\right) = \sum_{t=1}^{\infty} t\alpha_t \frac{x^{t-1}}{t!}.$$

Rewriting this and using (2.3), we have

$$1 - \frac{1}{x} + \sum_{t=0}^{\infty} \beta_t \frac{x^{t-1}}{t!} = \sum_{t=1}^{\infty} t\alpha_t \frac{x^{t-1}}{t!}.$$

Equating coefficients, we find

$$\beta_t = t\alpha_t \qquad (t > 1).$$

This completes the proof.

The relation between the coefficients β_t and the classical Bernoulli numbers B_s is

$$\beta_{2s} = (-1)^{s-1} B_s \qquad (s > 0),$$

as on [12, p. 90].

The theorem of von Staudt [12, p. 91] determines the value of β_t mod 1. However, the numbers which arise in algebraic topology are not the numbers β_t themselves, but the numbers

$$\frac{\alpha_{2s}}{2} = \frac{\beta_{2s}}{4s} = (-1)^{s-1} \frac{B_s}{4s}.$$

We need to know the value of $\alpha_t/2 = \beta_t/2t$ as an element of the group of rationals mod 1. Since this group is a torsion group, it splits as the direct sum of its p-primary components. It will thus be sufficient if for each prime p we give the value mod Q'_p, where Q'_p is the additive groups of rationals with denominators prime to p.

In the next theorem, we suppose that t is even.

THEOREM (2.5). *If p is odd we have*

$$\frac{\alpha_t}{2} = \frac{\beta_t}{2t} \equiv \begin{cases} 0 & \text{mod } Q'_p \quad \textit{if} \quad t \not\equiv 0 \operatorname{mod}(p-1) \\ \dfrac{1}{2pu} & \text{mod } Q'_p \quad \textit{if} \quad t = (p-1)u. \end{cases}$$

For $p = 2$ we have

$$\frac{\alpha_t}{2} = \frac{\beta_t}{2t} \equiv \begin{cases} \frac{3}{8} & \text{mod } Q'_2 \quad \text{if} \quad t = 2 \\ \frac{1}{16} & \text{mod } Q'_2 \quad \text{if} \quad t = 4 \\ \dfrac{1}{2} + \dfrac{1}{4t} & \text{mod } Q'_2 \quad \text{if} \quad t \geqslant 6 \end{cases}$$

We defer the proof.

We now require some more notation. We write $v_p(n)$ for the exponent to which the prime p occurs in the decomposition of n into prime powers, so that

$$n = 2^{v_2(n)}3^{v_3(n)}5^{v_5(n)} \ldots .$$

We define an explicit number-theoretic function $m(t)$ as follows.

For p odd,

$$v_p(m(t)) = \begin{cases} 0 & \text{if} \quad t \not\equiv 0 \operatorname{mod}(p-1) \\ 1 + v_p(t) & \text{if} \quad t \equiv 0 \operatorname{mod}(p-1). \end{cases}$$

For $p = 2$,

$$v_2(m(t)) = \begin{cases} 1 & \text{if} \quad t \not\equiv 0 \bmod 2 \\ 2 + v_2(t) & \text{if} \quad t \equiv 0 \bmod 2. \end{cases}$$

(Note that $v_p(m(t)) = 0$ except for a finite number of p.)

For example, we have $m(2s + 1) = 2$.

THEOREM (2.6). *$m(2s)$ is the denominator of*

$$\frac{\alpha_{2s}}{2} = \frac{\beta_{2s}}{4s} = (-1)^{s-1}\frac{B_s}{4s}$$

when this fraction is expressed in its lowest terms.

This theorem is due to Milnor and Kervaire [15, Lemma (3)]. It is clear that it follows immediately from Theorem (2.5).

The function $m(t)$ also appears in a rather different situation, to which we turn next. Roughly speaking, we want to say that $m(t)$ 'is the highest common factor of the expressions

$$k^{\infty}(k^t - 1)$$

as k runs over all integers'. We proceed to make this precise.

Let f be a function which assigns to each integer k (positive, negative or zero) a non-negative integer $f(k)$. Given such a function f and a non-negative integer t, we define

$h(f, t)$ to be the highest common factor of the integers

$$k^{f(k)}(k^t - 1)$$

as k varies over all integers (positive, negative and zero).

THEOREM (2.7). *$h(f, t)$ divides $m(t)$. For each t there is a function $f(k)$ such that $h(f, t) = m(t)$.*

This result is involved in a proof by E. Dyer [11, pp. 365, 366] although it is not given a separate statement there. I owe to Dyer a suggestion for expressing the proof more elegantly.

We will now prove the results stated above. The proof of Theorem (2.5) follows the pattern of von Staudt's theorem, in that we compare a summation formula involving β_t with an independent estimate of the sum. The only difference is that the estimate holds modulo a high power of p, instead of modulo p. We begin with two well-known lemmas.

LEMMA (2.8). *If $t > 0$ we have*

$$\sum_{1 \leqslant y \leqslant q-1} y^t = \sum_{1 \leqslant v \leqslant t+1} \frac{t!}{v!(t-v+1)!} \beta_{t-v+1} q^v.$$

This follows from the identity

$$1 + e^x + e^{2x} + \ldots + e^{(q-1)x} = \frac{x}{e^x - 1} \cdot \frac{e^{qx} - 1}{x}$$

by expanding in powers of x and equating coefficients; see [12, p. 90].

We now introduce the ring J_m of residue classes mod m, and the multiplicative group G_m of units in J_m (that is, the group of residue classes of integers prime to m).

LEMMA (2.9). *If $m = p^f$ with p odd and $f \geqslant 1$ then G_m is cyclic of order $(p-1)p^{f-1}$. If $m = 2^f$ and $f \geqslant 2$ then G_m is the direct sum of the subgroup consisting of ± 1 and the subgroup of residue classes congruent to 1 mod 4; the latter subgroup is cyclic of order 2^{f-2}.*

This lemma is well-known. See [17, pp. 145, 146].

We next observe that if x lies in a given residue class mod p^a, where $a \geqslant 1$, then x^p lies in a well-determined residue class mod p^{a+1}; this follows immediately from the binomial theorem. By induction over b we see that if x lies in a given residue class mod p^a, then x^{p^b} lies in a well-determined residue class mod p^{a+b}.

For the next two lemmas, we write t in the form $t = fp^b$ with f prime to p. We also write $G_{2^a}^+$ for the subset of G_{2^a} consisting of the residue classes x for $0 < x < 2^{a-1}$.

LEMMA (2.10). *If p is odd we have*

$$\sum_{x \in G_{p^a}} x^t \equiv \begin{cases} 0 & \text{mod } p^{a+b} \quad \textit{if} \quad t \not\equiv 0 \text{ mod } (p-1) \\ (p-1)p^{a-1} & \text{mod } p^{a+b} \quad \textit{if} \quad t \equiv 0 \text{ mod } (p-1). \end{cases}$$

If $p = 2$, $a \geqslant 3$, $b \geqslant 1$ we have

$$\sum_{x \in G_{2^a}^+} x^t \equiv 2^{a-2} + 2^{a+b-1} \qquad \text{mod } 2^{a+b}.$$

Proof. We begin with the case $p = 2$. Consider the homomorphism

$$\theta : G_{2^{a+b}} \longrightarrow G_{2^{a+b}}$$

defined by $\theta(x) = x^t = x^{f2^b}$. As remarked above, the map θ factors through G_{2^a}. Since we are assuming $a \geqslant 2$, $b \geqslant 1$, Lemma (2.9) shows that each element in Im θ is the image of just two elements in G_{2^a}, namely $\pm x$ for some x. That is, the elements in Im θ are the residue classes x^t for $x \in G_{2^a}^+$. Thus we have

$$\sum_{x \in G_{2^a}^+} x^t = \sum_{y \in \operatorname{Im} \theta} y \quad \text{in} \quad J_{2^{a+b}}.$$

By Lemma (2.9), Im θ is precisely the kernel of the obvious projection

$$G_{2^{a+b}} \longrightarrow G_{2^{b+2}}.$$

That is, Im θ consists of the elements $1 + j2^{b+2} \pmod{2^{a+b}}$ for $1 \leqslant j \leqslant 2^{a-2}$. Thus we find

$$\begin{aligned}\sum_{y \in \operatorname{Im}\theta} y &\equiv \sum_{1 \leqslant j \leqslant 2^{a-2}} (1 + j2^{b+2}) \bmod 2^{a+b} \\ &\equiv 2^{a-2} + \tfrac{1}{2}.2^{a-2}(2^{a-2} + 1)2^{b+2} \bmod 2^{a+b} \\ &\equiv 2^{a-2} + 2^{a+b-1} \bmod 2^{a+b}\end{aligned}$$

(since we have assumed $a \geqslant 3$). This completes the proof for $p = 2$.

In the case p odd we consider the homomorphism

$$\theta : G_{p^{a+b}} \longrightarrow G_{p^{a+b}}$$

defined by $\theta(x) = x^t = x^{fp^b}$. The map θ factors through

$$\bar{\theta} : G_{p^a} \longrightarrow G_{p^{a+b}};$$

thus we have

$$\sum_{x \in G_{p^a}} x^t = n \sum_{y \in \operatorname{Im} \theta} y \quad \text{in} \quad J_{p^{a+b}},$$

where n is the number of elements in Ker $\bar{\theta}$. The case $t \equiv 0 \bmod (p-1)$ is now similar to the case $p = 2$.

Let us therefore suppose that $t \not\equiv 0 \bmod (p-1)$. Consider the projection

$$G_{p^{a+b}} \longrightarrow G_p;$$

using Lemma (2.9), we see that there is an element z in Im θ such that $z \not\equiv 1 \bmod p$. As y runs over Im θ, so does zy; thus in $J_{p^{a+b}}$ we have

$$z \sum_{y \in \operatorname{Im} \theta} y = \sum_{y \in \operatorname{Im} \theta} y;$$

that is,

$$(z - 1) \sum_{y \in \operatorname{Im} \theta} y \equiv 0 \qquad \bmod p^{a+b}.$$

Since $z \not\equiv 1 \bmod p$, we have

$$\sum_{y \in \operatorname{Im} \theta} y \equiv 0 \qquad \bmod p^{a+b}.$$

This completes the proof.

LEMMA (2.11). *If p is odd we have*

$$\sum_{0<x<p^a} x^t \equiv \begin{cases} 0 & \bmod p^{a+b} \quad \textit{if} \quad t \not\equiv 0 \bmod (p-1) \\ (p-1)p^{a-1} & \bmod p^{a+b} \quad \textit{if} \quad t \equiv 0 \bmod (p-1). \end{cases}$$

If $p = 2$, $a \geqslant 3$, $b \geqslant 1$, $t \geqslant 6$ then

$$\sum_{0<x<2^{a-1}} x^t \equiv 2^{a+2} + 2^{a+b-1} \bmod 2^{a+b}.$$

Proof. Consider the case $p = 2$. We argue by induction over a; let us assume that either (i) $a = 3$, or (ii) $a > 3$ and the result is true for $a - 1$. Then we have

$$\sum_{0<x<2^{a-1}} x^t = \sum_{x \in G^+_{2^a}} x^t + 2^t \sum_{0<x<2^{a-2}} x^t.$$

Using Lemma (2.10) and, if $a > 3$, the inductive hypothesis, we have

$$\sum_{0<x<2^{a-1}} x^t \equiv 2^{a-2} + 2^{a+b-1}$$

modulo 2^{a+b} and 2^{t+a-3}. It follows from the assumption $t \geqslant 6$ that $t \geqslant b + 3$; thus the congruence holds modulo 2^{a+b}. This completes the induction.

The case in which p is odd is proved similarly, starting the induction from $a = 1$.

Proof of Theorem (2.5). We consider the case $p = 2$. Since we can evidently compute α_2 and α_4 from (2.1), we shall suppose that $t \geqslant 6$. Write $t = f2^b$ with f odd. By Lemma (2.8), with $q = 2^{a-1}$, we have

$$\sum_{0<y<2^{a-1}} y^t = \sum_{1 \leqslant v \leqslant t+1} \frac{t!}{v!(t-v+1)!} \beta_{t-v+1} 2^{v(a-1)}$$

In the terms

$$\frac{t!}{v!(t-v+1)!} \beta_{t-v+1} 2^{v(a-1)},$$

the part

$$\frac{t!}{v!(t-v+1)!} \beta_{t-v+1}$$

does not depend on a; by choosing a large enough, we can ensure that all the terms

$$\frac{t!}{v!(t-v+1)!} \beta_{t-v+1} 2^{v(a-1)}$$

with $v \geqslant 2$ are divisible by 2^{a+b}. Using Lemma (2.11), we have

$$\frac{t!}{1!t!} \beta_t 2^{a-1} \equiv 2^{a-2} + 2^{a+b-1} \bmod 2^{a+b} Q'_2.$$

(Here $2^{a+b}Q'_2$ means the additive group of rationals $2^{a+b}r$, where $r \in Q'_2$). Dividing by $2^a t = f2^{a+b}$, we find

$$\frac{\alpha_t}{2} = \frac{\beta_t}{2t} \equiv \frac{1}{4t} + \frac{1}{2f} \bmod Q'_2.$$

Since f is odd, we have

$$\frac{1}{2f} \equiv \frac{1}{2} \bmod Q_2'.$$

This completes the proof for $p = 2$. The proof for p odd is similar, by substituting $q = p^a$ in Lemma (2.8).

We now turn to the proof of Theorem (2.7). We record the essential point of the proof as a lemma, for use in Part III.

LEMMA (2.12). *For each k prime to p we have*

$$v_p(k^t - 1) \geqslant v_p(m(t)).$$

Moreover, we have

$$v_p(k^t - 1) = v_p(m(t))$$

in the following cases.

(i) *p is odd and k is a generator of G_{p^2}.*

(ii) *$p = 2$, t is even and k is a generator of $G_8/\{\pm 1\}$.*

(iii) *$p = 2$, t is odd and k is a generator of G_4.*

Proof. Consider the case $p = 2$. If t odd then $v_2(m(t)) = 1$ and the result is trivial; for if k is odd, then $k^t - 1$ is divisible by 2, i.e. $v_2(k^t - 1) \geqslant 1$; and if $k \equiv -1 \bmod 4$, then $k^t - 1 \equiv -2 \bmod 4$, i.e. $v_2(k^t - 1) = 1$.

We may therefore suppose that $t = q\,2^{v-2}$, where q is odd and $v = v_2(m(t)) \geqslant 3$. By Lemma (2.9) we have

$$k^t \equiv 1 \bmod 2^v;$$

thus

$$v_2(k^t - 1) \geqslant v.$$

Now assume that k is a generator of $G_8/\{\pm 1\}$. Then k is a generator of $G_{2^{v+1}}/\{\pm 1\}$, and by Lemma (2.9) we have

$$k^t \not\equiv 1 \bmod 2^{v+1}.$$

Thus

$$v_2(k^t - 1) = v.$$

The proof for p odd is similar.

Proof of Theorem (2.7). Suppose given a function $f(k)$. Let p^v be the highest power of p dividing the integers

$$k^{f(k)}\,(k^t - 1)$$

for all k prime to p. Thus we shall certainly have

$$v_p(h(f, t)) \leqslant v;$$

but by Lemma (2.12), we have

$$v = v_p(m(t)).$$

Since this true for each prime p, $h(f, t)$ divides $m(t)$.

Given t, we may choose f so that

$$f(k) \geqslant \operatorname*{Max}_{p|k} v_p(m(t)).$$

Then the numbers p^v considered above will also divide the integers

$$k^{f(k)}(k^t - 1)$$

when k is divisible by p. In this case therefore we shall have

$$h(f, t) = \prod_p p^v = m(t).$$

This completes the proof of Theorem (2.7).

§3. THE GROUP $J''(X)$

In this section we shall introduce the group $J''(X)$, which will serve, in favourable cases, as an upper bound for $J(X)$. After giving the definition, elementary properties and examples, we come to the result on the groups $\tilde{J}(S^{4n})$ which was announced in [2, Theorem (3); 3, Theorem (3)]; see Theorem (3.7). Finally, we establish foımal properties of the groups $J''(X)$; see especially Theorem (3.12).

In what follows, a Ψ-group will mean an abelian group Y together with given endomorphisms $\Psi^k\colon Y \to Y$ for each $k \in Z$, that is, for each integer k (positive, negative or zero). We impose no axioms on the endomorphisms Ψ^k. A Ψ-map between Ψ-groups will mean a homomorphism which commutes with the operations Ψ^k. If we speak of a Ψ-subgroup (or Ψ-quotient group) we shall mean that the injection (or projection) map is a Ψ-map.

The groups $K_\Lambda(X)$ are thus Ψ-groups, and of course this is the example of most interest to us. However, for technical reasons we sometimes have to consider other Ψ-groups, for example, Ψ-subgroups and Ψ-quotient-groups of groups $K_\Lambda(X)$.

Let Y be a Ψ-group, and let e be a function which assigns to each pair $k \in Z$, $y \in Y$ a non-negative integer $e(k, y)$. Then we define Y_e to be the subgroup of Y generated by the elements

$$k^{e(k,y)}(\Psi^k - 1)y.$$

That is, Y_e is the subgroup of linear combinations

$$\sum_{k,y} a(k, y)k^{e(k,y)}(\Psi^k - 1)y\,;$$

here the coefficients $a(k, y)$ are integers, and are zero except for a finite number of pairs (k, y). If $e_1 \geqslant e_2$, then $Y_{e_1} \subset Y_{e_2}$. We now define

$$J''(Y) = Y/\bigcap_e Y_e,$$

where the intersection runs over all functions e.

It is clear that a Ψ-map $f\colon Y_1 \to Y_2$ induces a map from $J''(Y_1)$ to $J''(Y_2)$. In fact, suppose given a function $e_2(k, y_2)$ on $Z \times Y_2$; then one defines a corresponding function e_1 by

$$e_1(k, y_1) = e_2(k, fy_1);$$

then we have

$$f(Y_1)_{e_1} \subset (Y_2)_{e_2};$$

hence

$$f \bigcap_e (Y_1)_e \subset \bigcap_{e_2} (Y_2)_{e_2}.$$

If X is a space, we define

$$J''_\Lambda(X) = J''(K_\Lambda(X)).$$

The case of most interest to us is, of course, the case $\Lambda = R$; in this case we write

$$J''(X) = J''_R(X) = J''(K_R(X)).$$

This construction is suggested, of course, by the results of Part I [4]. Let us recall conjecture 1.2 of Part I.

Conjecture (1.2) *of Part I.* If k is an integer, X is a finite CW-complex and $y \in K_R(X)$, then there exists a non-negative integer $e = e(k, y)$ such that $k^e(\Psi^k - 1)y$ maps to zero in $J(X)$.

PROPOSITION (3.1). *Suppose that for some X, Conjecture* (1.2) *of Part I holds for all k and y. Then $J''(X)$ is an upper bound for $J(X)$, in the sense of Part I.*

Proof. Take $Y = K_R(X)$, and let $T(X)$ be the kernel of the quotient map from $K_R(X)$ to $J(X)$, as in Part I. Then Conjecture 1.2 of Part I states that there is a function $e(k, y)$ such that $Y_e \subset T(X)$; *a fortiori;* $\bigcap_e Y_e \subset T(x)$. This completes the proof.

An alternative definition of $J''(Y)$, in which the functions $e(k, y)$ are replaced by functions of one variable, can be given when the abelian group Y is finitely generated (which is of course the case in our applications). In fact, we let f run over the functions $e(k, y)$ which are independent of y, so that $f(k, y) = f(k)$.

PROPOSITION (3.2). *If Y is finitely-generated then*

$$\bigcap_e Y_e = \bigcap_f Y_f,$$

so that we can write

$$J''(Y) = Y / \bigcap_f Y_f.$$

Proof. It is clear that $\bigcap_e Y_e \subset \bigcap_f Y_f$; we wish to prove the converse. Let $y_1, y_2, \dots, y_n$ generate y; for any function $e(k, y)$, define a corresponding function $f(k)$ by

$$f(k) = \operatorname*{Max}_{1 \leqslant r \leqslant n} e(k, y_r).$$

It is easy to check that $Y_f \subset Y_e$; hence $\bigcap_f Y_f \subset \bigcap_e Y_e$. This completes the proof.

In what follows we will always assume that our Ψ-groups are finitely-generated, so that Proposition (3.2) applies. Several of the results which we prove with this assumption can be proved without it, though the proofs become slightly more complicated.

PROPOSITION (3.3)(a). *Let Y_1, Y_2 be finitely-generated Ψ-groups; then*

$$J''(Y_1 \oplus Y_2) = J''(Y_1) \oplus J''(Y_2).$$

(b) *Let P be a point; then*

$$J''(P) = Z.$$

(c) *Let X be a finite connected CW-complex; then*

$$J''(X) = Z + \tilde{J}''(X),$$

where

$$\tilde{J}''(X) = J''(\tilde{K}_R(X)).$$

Proofs. (a). We have

$$(Y_1 \oplus Y_2)_f = (Y_1)_f \oplus (Y_2)_f,$$

so

$$\bigcap_f (Y_1 \oplus Y_2)_f = \bigcap_f (Y_1)_f \oplus \bigcap_f (Y_2)_f.$$

(b). $K_R(P) = Z$, and the operations are given by $(\Psi^k - 1)\,y = 0$ for all k,y.

Part (c) follows by applying (a) and (b) to the decomposition

$$K_R(X) = K_R(P) + \tilde{K}_R(X).$$

This completes the proof.

We will now present some illustrative samples.

EXAMPLE (3.4). Take X to be real projective space RP^n; then the quotient map

$$K_R(RP^n) \to J''(RP^n)$$

is an isomorphism.

Proof. By [1, Theorem (7.4)], $\tilde{K}_R(RP^n)$ is cyclic of order 2^g, say. Let us choose f so that $f(k) \geqslant g$ for k even. Then $k^{f(k)}(\Psi^k - 1)\,y$ will be zero for k even. But for k odd $\Psi^k y = y$ in $K_R(RP^n)$ [1, Theorem (7.4)], so that $k^{f(k)}(\Psi^k - 1)\,y = 0$. Thus we have $Y_f = 0$ for this function f, and hence $\bigcap_f Y_f = 0$. This completes the proof.

EXAMPLE (3.5). Take X to be the sphere S^n with $n \equiv 1$ or $2 \bmod 8$; then the quotient map

$$K_R(S^n) \to J''(S^n)$$

is an isomorphism.

Proof. Let $f: RP^n \to S^n$ be a map of degree 1; then we have the following commutative diagram.

$$\begin{array}{ccc} K_R(S^n) & \xrightarrow{f^*} & K_R(RP^n) \\ \downarrow & & \downarrow \\ J''(S^n) & \longrightarrow & J''(RP^n) \end{array}$$

The map f^* is monomorphic, by the proof of [1, Theorem (7.4)] The right-hand column is monomorphic by Example (3.4). Therefore the left-hand column is monomorphic. This completes the proof.

EXAMPLE (3.6). Take X to be the sphere S^{4n}; then the group $\tilde{J}''(S^{4n})$ is cyclic of order $m(2n)$, where the function $m(2n)$ is in §2.

Proof. If $y \in \tilde{K}_R(S^{4n})$, we have

$$k^{f(k)}(\Psi^k - 1)\, y = k^{f(k)}(k^{2n} - 1)\, y \qquad [1, \text{Corollary } (5.2)].$$

Thus the subgroup Y_f of $\tilde{K}_R(S^{4n}) = Z$ consists of the multiples of $h(f, 2n)$, where $h(f, 2n)$ is the highest common factor of the integers

$$k^{f(k)}(k^{2n} - 1) \qquad (k \in Z).$$

The result now follows from Theorem (2.7).

THEOREM (3.7). *The image $J(\pi_{4n-1}(SO))$ of the stable J-homonomorphism—or equivalently, the group $\tilde{J}(S^{4n})$—is cyclic of order*

(i) *$m(2n)$ if $4n \equiv 4 \bmod 8$*

(ii) *either $m(2n)$ or $2m(2n)$ if $4n \equiv 0 \bmod 8$.*

This result was announced in [2, Theorem (3); 3, Theorem (3)].

Proof. The fact that the order of $\tilde{J}(S^{4n})$ is a multiple of $m(2n)$ is the result of Milnor and Kervaire [15] as improved by Atiyah and Hirzebruch [6]. We wish to argue in the opposite direction.

Suppose that $4n \equiv 4 \bmod 8$. Then the map

$$r : \tilde{K}_C(S^{4n}) \longrightarrow \tilde{K}_R(S^{4n})$$

is epimorphic; Theorem (1.4) of Part I [4] shows that Conjecture (1.2) of Part I is true for $X = S^{4n}$; the results (3.1) and (3.6) now show that the order of $\tilde{J}(S^{4n})$ divides $m(2n)$. This completes the proof in this case.

In case $4n \equiv 0 \bmod 8$ the proof is similar; we lose a factor of 2 because the image of

$$r : \tilde{K}_C(S^{4n}) \longrightarrow \tilde{K}_R(S^{4n})$$

consists of the elements divisible by 2.

We now seek to obtain formal properties of the group J''.

LEMMA (3.8). *Suppose that*

$$A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

is an exact sequence of finitely-generated Ψ-groups such that $J''(A)$ is finite. Then the sequence

$$J''(A) \xrightarrow{i_*} J''(B) \xrightarrow{j_*} J''(C) \longrightarrow 0$$

is exact.

Proof. Since $J''(B), J''(C)$ are quotients of B, C it is clear that j_* is an epimorphism; it is also clear that $j_* i_* = 0$. It remains to prove that $\operatorname{Ker} j_* \subset \operatorname{Im} i_*$. In what follows, then, we suppose given $b \in B$ such that

$$jb \in \bigcap_f C_f.$$

It is given that $J''(A)$ is finite; choose a set of representatives $\alpha_1, \alpha_2, \ldots, \alpha_q$ in A for the elements of $J''(A)$. As a first step, we will show that for each f we can find α_r such that

$$b - i\alpha_r \in B_f.$$

In fact, suppose given a function $f(k)$. Since $jb \in C_f$, we have

$$jb = \sum_k k^{f(k)}(\Psi^k - 1)c_k$$

for a suitable set of elements c_k in C, of which all but a finite number are zero. Since j is epimorphic, we can find b_k in B (of which all but a finite number are zero) such that $c_k = jb_k$. Then we have

$$j(b - \sum_k k^{f(k)}(\Psi^k - 1)b_k) = 0,$$

so by exactness there is an a in A such that

$$b = ia + \sum_k k^{f(k)}(\Psi^k - 1)b_k.$$

If α_r is the representative for the class of a in $J''(A)$ we have

$$a - \alpha_r \in A_f;$$

that is,

$$a = \alpha_r + \sum_k k^{f(k)}(\Psi^k - 1)a_k$$

for a suitable set of elements a_k in A. Hence

$$b = i\alpha_r + \sum_k k^{f(k)}(\Psi^k - 1)(b_k + ia_k);$$

that is, $b - i\alpha_r \in B_f$. This completes the first step.

We have shown that for each f there exists α_r such that $b - i\alpha_r \in B_f$. We will now show that there exists α_r such that $b - i\alpha_r \in B_f$ for all f. Suppose the contrary; then for each α_r there exists f_r such that $b - i\alpha_r \notin B_{f_r}$. Define a function f by

$$f(k) = \operatorname*{Max}_{1 \leqslant r \leqslant q} f_r(k);$$

then for each r we have $b - i\alpha_r \notin B_f$, contradicting the first step.

We have thus shown that for some α_r, $b - i\alpha_r \in \bigcap_f B_f$. That is, in $J''(B)$ we have $\{b\} = i_*\{\alpha_r\}$. This completes the proof.

LEMMA (3.9). *Suppose that a finitely-generated Ψ-group Y admits a filtration*

$$Y = Y_1 \supset Y_2 \supset \ldots \supset Y_n = 0$$

by Ψ-subgroups Y_q such that $J''(Y_q/Y_{q+1})$ is finite for each q. Then $J''(Y)$ is finite.

This is easily proved by induction over n, using Lemma (3.8) to make the inductive step.

LEMMA (3.10). *Let $\bigvee S^q$ be a finite wedge-sum of q-spheres. Let Y be a Ψ-quotient of a Ψ-subgroup of $\tilde{K}_\Lambda(\bigvee S^q)$. Then $J''(Y)$ is finite.*

Proof. If $\tilde{K}_\Lambda(S^q)$ is finite, then $\tilde{K}_\Lambda(\bigvee S^q)$ is finite, Y is finite and $J''(Y)$ is finite. It is therefore only necessary to consider the following cases:

$$\Lambda = R,\ q \equiv 0 \bmod 4\ ;\ \Lambda = C,\ q \equiv 0 \bmod 2.$$

Let us assume that $q = 2n$; then the operations Ψ^k in Y are given by

$$\Psi^k y = k^n y.$$

Arguing as in Example (3.6), we see that for each $y \in Y$ the multiple $m(n)y$ maps to zero in $J''(Y)$ (where $m(n)$ is as in §2). Since Y is finitely-generated, $J''(Y)$ must be finite.

THEOREM (3.11). *If X is a finite connected CW-complex, then $\tilde{J}''_\Lambda(X)$ is finite.*

Proof. Filter $Y = \tilde{K}_\Lambda(X)$ by taking Y_q to be the image of the map

$$j^* : K_\Lambda(X, X^{q-1}) \longrightarrow \tilde{K}_\Lambda(X),$$

where X^n is the n-skeleton of X. Then Y_q/Y_{q+1} is a Ψ-quotient of a Ψ-subgroup of $\tilde{K}_\Lambda(\bigvee S_q)$. Thus $J''(Y_q/Y_{q+1})$ is finite by Lemma (3.10) and $J''(Y)$ is finite by Lemma (3.9).

THEOREM (3.12). *Let $X \to Y \to Z$ be a cofibering of finite connected CW-complexes such that the sequence*

$$\tilde{K}_\Lambda(Z) \xrightarrow{j^*} \tilde{K}_\Lambda(Y) \xrightarrow{i^*} \tilde{K}_\Lambda(X) \longrightarrow 0$$

is exact. Then the sequence

$$\tilde{J}''_\Lambda(Z) \xrightarrow{j^*} \tilde{J}''_\Lambda(Y) \xrightarrow{i^*} \tilde{J}''_\Lambda(X) \longrightarrow 0$$

is exact.

This follows immediately from Theorem (3.11) and Lemma (3.8).

THEOREM (3.13). *Let X be a finite connected CW-complex, and let $Y = \tilde{K}_\Lambda(X)$. Then there exists a function $F(k)$ such that*

$$\bigcap_f Y_f = Y_F.$$

This theorem shows that although the definition of $J''(X)$ involves a limit over functions f (or e), the limit is actually attained.

Proof. By Theorem (3.11), $\tilde{J}''_\Lambda(X)$ is finite. Let $y_1, y_2, \ldots, y_n$ be representatives in $Y = \tilde{K}_\Lambda(X)$ for the non-zero elements of $\tilde{J}''_\Lambda(X)$. Since y_q is not in $\bigcap_f Y_f$, there is a function f_q such that y_q is not in Y_{f_q}. Define

$$F(k) = \operatorname*{Max}_{1 \leq q \leq n} f_q(k).$$

We have

$$\bigcap_f Y_f \subset Y_F,$$

so that we have a quotient map

$$\theta : Y/\bigcap_f Y_f \longrightarrow Y/Y_F.$$

By construction, y_q is not in Y_F, and this holds for each q; therefore θ is monomorphic. This proves the result.

§4. THE THOM ISOMORPHISM

In setting up the groups $J'(X)$, one should begin with a treatment of the "Thom isomorphism" in extraordinary cohomology. It is generally known that such an isomorphism can be set up. (As a matter of history, the relevant construction appears in the very sketchy sketch proof at the end of [6].) However, we have been waiting for an account which sets up this isomorphism in the best possible way, and proves that it enjoys the good properties one requires. Such an account has now been provided by Atiyah, Bott and Shapiro [18; see especially Theorem (12.3)].

In this section, I shall simply quote the result of Atiyah, Bott and Shapiro. In an earlier draft I included (for completeness and for my own security) a treatment of the Thom isomorphism, on which I based *ad hoc* proofs of certain results, especially Theorem (5.1) and (5.9) of the present paper. This treatment and these proofs are now omitted, at the referee's suggestion.

Let ξ be a vector bundle, with structural group $SO(n)$, over the finite connected CW-complex B. By the 'Thom pair $\bar{E}$, E of B', we shall mean either one of the following constructs.

(a) $\bar{E}$ is the associated bundle whose fibres are unit n-cells; E is the boundary of $\bar{E}$, so that E is the associated bundle whose fibres are unit $(n\text{-}1)$-spheres.

(b) $\bar{E}$ is the total space of the vector-bundle ξ; E is the complement of the zero cross-section in $\bar{E}$.

For cohomological purposes these two constructions are equivalent.

We recall that in ordinary cohomology we have a 'Thom isomorphism' [16]

$$\phi : H^q(B; G) \longrightarrow H^{n+q}(\bar{E}, E; G).$$

This is usually constructed as follows. We first construct a generator $u \in H^n(\bar{E}, E; Z)$. We then define

$$\phi(h) = u.(p^*h);$$

here p: $\bar{E} \rightarrow B$ is the projection map, so that p^*h lies in $H^q(\bar{E}; G)$ and the cup-product $u.(p^*h)$ lies in $H^{n+q}(\bar{E}, E; G)$.

In thinking about the Thom isomorphism in extraordinary cohomology, one follows the obvious analogy, replacing H^* by K^*_Λ.

The group $K_\Lambda^*(X)$ is conveniently defined for a finite-dimensional CW-complex X by using vector-bundles over X. It will be useful to generalise the definition to more general spaces X, in order to avoid having to discuss whether our Thom pairs $\bar{E},E$ can be given the structure of CW-pairs.

We may replace X by a CW-complex Y which is weakly equivalent to X (for example, the total singular complex of X). We may now define

$$K_\Lambda^n(X) = \operatorname*{Inv\,Lim}_{q\to\infty} K_\Lambda^n(Y^q),$$

where Y^q is the q-skeleton of Y. We make the obvious definitions for pairs, maps etc.

The operations Ψ_Λ^k of [1] are defined in $K_\Lambda^n(Y^q)$; they pass to the inverse limit, and define operations Ψ_Λ^k in $K_\Lambda^n(X)$.

This use of the inverse limit is of course due to Atiyah and Hirzebruch [7] (except that they often restrict themselves to finite complexes when they could equally well allow finite-dimensional ones.)

The use of inverse limits has the disadvantage that it sacrifices exactness. However, if $H_*(X)$ is finitely-generated (which is of course the case for our Thom pairs) then the inverse limit is more apparent than real. In fact, in this case the double suspension S^2Y is simply-connected and has $H_*(S^2Y)$ finitely-generated; thus S^2Y is equivalent to a finite CW-complex Z; and we have

$$\begin{aligned}\operatorname*{Inv\,Lim}_{q\to\infty} K_\Lambda^n(Y^q) &= \operatorname*{Inv\,Lim}_{q\to\infty} K_\Lambda^{n+2}(S^2Y^q)\\ &= K_\Lambda^{n+2}(Z).\end{aligned}$$

For such spaces X, then, we do not lose exactness.

We can now describe the two cases which will concern us of the Thom isomorphism in extraordinary cohomology. In the first case, we suppose given a real vector bundle ξ over B with structural group Spin($8n$), and we obtain an isomorphism

$$\phi : K_R^*(B) \longrightarrow K_R^*(\bar{E}, E).$$

In the second case, we suppose given a complex vector bundle ξ with structural group $U(n)$, and we obtain an isomorphism

$$\phi : K_C^*(B) \longrightarrow K_C^*(\bar{E}, E).$$

In each case, φ is an isomorphism of modules over $K_\Lambda^*(B)$ (where $\Lambda = R$ or C, according to the case.) Moreover, φ is natural for bundle maps. For the definition of φ, we refer the reader to [18].

§5. THE CLASSES ρ^k.

In this section we shall study certain 'cannibalistic characteristic classes'. Following Atiyah (private communication dated 20 October 1961) we shall call them ρ^k; an independent account has been published by Bott, who calls them θ_k [8, 9].

We shall first define the class $\rho^k(\xi)$ for each bundle ξ of a suitable class, and establish certain formal properties. Then we shall give a result (Theorem (5.9)) which relates the operations ρ^k to representation-theory. After that we shall extend the definition of ρ^k from bundles ξ to virtual bundles. Finally we shall compute the values of ρ^k in RP^n and in S^n.

We begin by discussing the situation abstractly. Let K and H be extraordinary cohomology theories with products, and let $T:K \to H$ be a natural transformation (preserving products). Suppose given also some class of bundles ξ, for example, unitary bundles or Spin$(8n)$-bundles ($n = 1, 2, \dots$.) For this class of bundles, we assume, there is given a Thom isomorphism

$$\phi_K : K^*(B) \longrightarrow K^*(\bar{E}, E);$$

this is a map of modules over $K^*(B)$, and is natural for maps of bundles. Similarly for

$$\phi_H : H^*(B) \longrightarrow H^*(\bar{E}, E).$$

Under these conditions the element

$$c(T, \xi) = \phi_H^{-1} T \phi_K(1) \in H^*(B)$$

may be considered as a 'characteristic class of ξ'; in particular, it is natural for bundle maps.

We have in mind the following special cases.

(i) Let us take $K = H = H^*(\quad; Z_2)$, $T = \sum_0^\infty Sq^i$. We obtain the (total) Stiefel–Whitney class of ξ [16].

(ii) Let us take $K = K_\Lambda$, $H = H^*(\quad; Q)$. Let us write $ch_C = ch$, $ch_R = ch.c$, so that we can take $T = ch_\Lambda : K_\Lambda \to H$. We obtain characteristic classes

$$\phi_H^{-1} ch_\Lambda \phi_K(1).$$

These classes are both classical and useful in calculations, and will be discussed below.

(iii) Let us take $K = H = K_\Lambda$, and take T to be the operation Ψ_Λ^k [1]. Then we obtain a chacteristic class which we call ρ_Λ^k:

$$\rho_\Lambda^k(\xi) = \phi_K^{-1} \Psi_\Lambda^k \phi_K(1) \in K_\Lambda(B).$$

Of course the characteristic class ρ_C^k is defined for unitary bundles and the class ρ_R^k is defined for Spin$(8n)$-bundles ($n = 1, 2, \dots$).

The philosophy of characteristic classes $\phi_H^{-1} T \phi_K(1)$ has been expounded in [19, especially §§ 2.2, 2.15, 3.3; 22].

We will now discuss example (ii) above more fully. If we start from a Spin$(8n)$-bundle ξ, then the classical expression for $\phi_H^{-1}\, ch\, c\phi_K 1$ is $(\hat{A}(\xi))^{-1}$, where $\hat{A}$ is as in [6; 21 § 23]. In fact, it is by now well known that this is the way $\hat{A}$ enters the theory of characteristic classes.

I will now indicate my objection to the notation $(\hat{A}(\xi))^{-1}$. In the theory of characteristic classes we should first do all we can for general bundles; only then should we apply the theory to the tangent and normal bundles of differentiable manifolds. (In historical terms, we should follow Whitney rather than Stiefel). From this point of view the characteristic class

$$\phi_H^{-1} ch\ c\ \phi_K 1$$

is clearly fundamental, and should have its own notation; in this paper I shall use the notation

$$sh(\xi) = \phi_H^{-1} ch\ c\ \phi_K 1.$$

(The choice of notation will be explained below). One now takes a differentiable manifold, with tangent bundle τ and normal bundle ν (for some embedding in R^n). One now encounters the class

$$\hat{A}(\tau) = sh(\nu).$$

That is to say, this class 'really' arises from the normal bundle; but one introduces $\hat{A}$ in order to write it in terms of the tangent bundle.

Similar remarks apply to unitary bundles, with $(\hat{A}(\xi))^{-1}$ replaced by

$$e^{c_1(\xi)}(T(\xi))^{-1}$$

where $T(\xi)$ is the Todd class [13, §§ 1.7, 10; 21 §22]. (This expression, like the previous one, depends on the precise choice of the Thom isomorphism ϕ_K).

For later use, we require explicit formulae for the characteristic classes $\phi_H^{-1}\ ch_A\ \phi_K 1$.

Following Borel and Hirzebruch, we consider in $U(n)$ the maximal torus T which consists of diagonal matrices. We have

$$BT = CP^\infty \times CP^\infty \times \dots \times CP^\infty.$$

Let $x \in H^2(CP^\infty)$ be a generator; then the cohomology ring $H^*(BT; Q)$ is a polynomial ring on generators $x_1, x_2, \dots, x_n$ corresponding to the factors. The embedding $i : T \to U(n)$ induces a monomorphism

$$(Bi)^* : H^*(BU(n); Q) \to H^*(BT; Q)$$

whose image is the subring of symmetric polynomials.

We write bh or bh_C for the characteristic class whose image under $(Bi)^*$ is

$$\prod_{1 \leq r \leq n} \frac{e^{x_r} - 1}{x_r}.$$

The notation bh is intended to suggest 'Bernoulli'.

By means of the usual embedding $U(n) \subset SO(2n)$ we obtain a maixmal torus T in $SO(2n)$. As before, the map

$$(Bi)^* : H^*(SO(2n); Q) \to H^*(BT; Q)$$

is a monomorphism. Its image is the subring of $H^*(BT; Q)$ additively generated by symmetric polynomials in which the exponents of the variables x_r are either all even, or all odd. Using the projection $\mathrm{Spin}(2n) \to SO(2n)$, we have

$$H^*(B\,\mathrm{Spin}(2n); Q) \cong H^*(BSO(2n); Q).$$

We write sh or bh_R for the characteristic class which corresponds to

$$\prod_{1 \leqslant r \leqslant n} \frac{e^{\frac{1}{2}x_r} - e^{-\frac{1}{2}x_r}}{x_r}.$$

The notation sh is intended to suggest 'sinh'.

THEOREM (5.1). *We have*

$$\phi_H^{-1} ch_A \phi_K 1 = bh_A \xi.$$

This theorem was certainly known to previous authors; compare [19, foot of p. 149]. The proof which follows is due to Atiyah (private communication).

Proof. We shall proceed from the definition of ϕ_K given in [18], using the methods of Borel and Hirzebruch [20, 21]; compare [23, §5].

We first recall that Atiyah, Bott and Shapiro introduce a group $\mathrm{Spin}^C(n)$, defined as a subset of a certain complex Clifford algebra [18]. We will begin by obtaining the result which corresponds to Theorem (5.1) when we consider bundles with structural group $\mathrm{Spin}^C(2n)$ and take $K = K_C$. We have first to fix some notation.

Let S^1 be the subgroup of complex scalars of unit modulus in the complex Clifford algebra. Let T' be the maximal torus in $\mathrm{Spin}^R(2n) \subset \mathrm{Spin}^C(2n)$. Then $S^1 \cap T' = Z_2$, consisting of ± 1; and $S^1 \times_{Z_2} T'$ is a maximal torus T'' in $\mathrm{Spin}^C(2n)$. This torus is a double cover of $(S^1/Z_2) \times T$, where T is the maximal torus in $SO(2n)$. We take the coordinate in S^1/Z_2 as x_0 mod 1; thus the coordinate in S^1 is $\frac{1}{2}x_0$ mod 1. Similarly, we write $x_1, \ldots, x_n$ for the coordinates in T.

We begin by considering the case $n = 1$. Let $\bar{E}, E$ be the universal Thom pair with structural group $\mathrm{Spin}^C(2)$; and consider the induced homomorphisms

$$K_C(\bar{E}, E) \xrightarrow{j^*} K_C(\bar{E}) \underset{\cong}{\xleftarrow{p^*}} K_C(B).$$

Because of the 'difference bundle construction' employed by Atiyah, Bott and Shapiro, $j^*\phi_K 1$ can actually be written in the form $p^*\eta - p^*\zeta$, where η and ζ are bundles obtained from the universal bundle by known complex representations. We have to calculate the characters of these representations; it is sufficient to calculate their restrictions to S^1 and $\mathrm{Spin}^R(2)$. The representations are one-dimensional, and the complex scalars in the Clifford algebra act as complex scalars; therefore the restriction of either character to S^1 is $e^{\frac{1}{2}x_0 \cdot 2\pi i}$. We turn to $\mathrm{Spin}^R(2)$, which is the subset of elements

$$\mathrm{Cos}\, \tfrac{1}{2}x_1 + e_1 e_2 \,\mathrm{Sin}\, \tfrac{1}{2}x_1$$

in the Clifford algebra. By definition, the 'positive' basic representation is the one which represents $e_1 e_2$ as $+i$; the 'negative' basic representation is the one which represents $e_1 e_2$ as $-i$. Therefore the restriction of the characters to $\mathrm{Spin}^R(2)$ are $e^{\frac{1}{2}x_1 \cdot 2\pi i}$, $e^{-\frac{1}{2}x_2 \cdot 2\pi i}$. Thus the characters are $e^{\frac{1}{2}(x_0 + x_1) \cdot 2\pi i}$ for the 'positive' representation, $e^{\frac{1}{2}(x_0 - x_1) \cdot 2\pi i}$ for the 'negative' representation. This leads immediately to the formula

$$\begin{aligned} ch(p^*)^{-1} j^* \phi_K 1 &= ch(\eta - \zeta) \\ &= e^{\frac{1}{2}x_0}\,(e^{\frac{1}{2}x_1} - e^{-\frac{1}{2}x_1}). \end{aligned}$$

Now by a standard result, we have

$$\phi_H^{-1} y = \frac{1}{x_1} (p^*)^{-1} j^* y.$$

This yields

$$\phi_H^{-1}\, ch\, \phi_k\, 1 = e^{\frac{1}{2}x_0} \left(\frac{e^{\frac{1}{2}x_1} - e^{-\frac{1}{2}x_1}}{x_1} \right).$$

This completes the calculation for $\mathrm{Spin}^C(2)$.

We now consider $\mathrm{Spin}^C(2n)$. We observe that our characteristic classes $c(T, \xi)$ are exponential, in the sense that

$$c(T, \xi \oplus \eta) = c(T, \xi) \, . \, c(T, \eta).$$

In fact, this follows from the "product formulae" for ϕ_H and ϕ_K in $\xi \oplus \eta$; that for ϕ_H is classical, while that for ϕ_K is one of the main results of Atiyah, Bott and Shapiro [18, Proposition (11.1)]. We also observe that the homomorphism

$$\mathrm{Spin}^C(2) \times \mathrm{Spin}^C(2) \times \ldots \times \mathrm{Spin}^C(2) \to \mathrm{Spin}^C(2n)$$

induces a monomorphism in rational cohomology of the classifying spaces. Therefore the result for $\mathrm{Spin}^C(2n)$ follows immediately from the result for $\mathrm{Spin}^C(2)$; we obtain the formula

$$e^{\frac{1}{2}x_0} \prod_{1 \leqslant r \leqslant n} \frac{e^{\frac{1}{2}x_r} - e^{-\frac{1}{2}x_r}}{x_r}.$$

Finally, we deduce the two parts of Theorem (5.1) by naturality. For a bundle with structural group $\mathrm{Spin}^R(8n)$ the constructions of [18] lead to

$$c\phi_R 1 = \phi_C 1$$

(with an obvious notation.) This yields the formula

$$\prod_{1 \leqslant r \leqslant 4n} \frac{e^{\frac{1}{2}x_r} - e^{-\frac{1}{2}x_r}}{x_r}$$

in $B\mathrm{Spin}^R(8n)$. For a $U(n)$-bundle one has to employ the homomorphism

$$U(n) \longrightarrow \mathrm{Spin}^C(2n)$$

given in [18, end of §3]. This homomorphism sends x_0 into $\sum_1^n x_r$, and sends x_r into x_r for $r \geq 1$. This yields the formula

$$\prod_{1 \leqslant r \leqslant n} \frac{e^{x_r} - 1}{x_r}$$

in $BU(n)$.

Alternatively, this last formula can be deduced from the construction in terms of exterior algebras, given in [18, Proposition (11.6)].

This completes the proof of Theorem (5.1).

PROPOSITION (5.2). *We have*

$$\mathrm{Log}\, bh\xi = \sum_{t=1}^{\infty} \alpha_t ch_t \xi$$

$$\mathrm{Log}\, sh\eta = \sum_{s=1}^{\infty} \tfrac{1}{2}\alpha_{2s} ch_{2s} c\eta.$$

In these formulae, we define Log $(1 + x)$ for $x \in \sum_{t>0} H^{2t}(X; Q)$ by means of the usual power-series expansion. The coefficients α_t are as in §2. We write ch_t for the component of ch in dimension $2t$.

This proposition follows from the definitions by standard methods and obvious manipulations.

We now return to the assumptions made at the beginning of this section, so that K, H are extraordinary cohomology theories and $T : K \to H$ is a natural transformation. We take up the study of the characteristic classes $\phi_H^{-1} T \phi_K(1)$.

LEMMA (5.3). *Suppose given the following commutative diagrams.*

$$\begin{array}{ccc}
K^*(\bar{E}_1, E_1) & \xrightarrow{\alpha} & H^*(\bar{E}_2, E_2) \\
\phi_1 \uparrow \cong & & \phi_2 \uparrow \cong \\
K^*(B_1) & & H^*(B_1)
\end{array}$$

$$\begin{array}{ccc}
K^*(\bar{E}_1) & \xrightarrow{\beta} & H^*(\bar{E}_2) \\
p_1^* \uparrow & & p_2^* \uparrow \\
K^*(B_1) & \xrightarrow{\gamma} & H^*(B_2)
\end{array}$$

Suppose that $\alpha(xy) = (\alpha x)(\beta y)$ for $x \in K^*(\bar{E}_1, E_1)$, $y \in K^*(\bar{E}_1)$. *Then we have*

$$\phi_2^{-1}\alpha\phi_1 x = (\phi_2^{-1}\alpha\phi_1 1)(Yx).$$

The proof is purely formal, and is obvious.

COROLLARY (5.4). *Taking* $\xi_1 = \xi_2 = \xi$, $\alpha = \beta = \gamma = T$ *we have*

$$\phi_H^{-1} T \phi_K(x) = c(T, \xi) . T(x).$$

In particular, we have

$$\phi_H^{-1} ch_A \phi_K(x) = bh_A(\xi) . ch_A(x)$$

$$\phi_K^{-1} \Psi_A^k \phi_K(x) = \rho_A^k(\xi) . \Psi_A^k(x).$$

PROPOSITION (5.5).

$$(\rho_A^k \xi) . (\Psi_A^k \rho_A^l \xi) = \rho_A^{kl} \xi.$$

This result has also been found by Bott [8, 9].

Proof. In [1] it is shown that $\Psi_\Lambda^k \Psi_\Lambda^l = \Psi_\Lambda^{kl}$. Take the equation

$$(\phi_K^{-1} \Psi_A^k \phi_K)(\phi_K^{-1} \Psi_A^l \phi_K)1 = (\phi_K^{-1} \Psi_A^{kl} \phi_K)1$$

and evaluate each side. We find

$$(\phi_K^{-1}\,\Psi_A^k\,\phi_K)(\rho_A^l\,\xi) = \rho_A^{kl}\,\xi,$$

or using Corollary (5.4),

$$(\rho_A^k\,\xi)\,.\,(\Psi_A^k\,\rho_A^l\,\xi) = \rho_A^{kl}\,\xi.$$

This completes the proof.

For our next proposition, we define

$$\Psi_H^k : \sum_{s\geqslant 0} H^{2s}(X; Q) \longrightarrow \sum_{s\geqslant 0} H^{2s}(X; Q)$$

by

$$\Psi_H^k(x) = k^s x \qquad \text{if} \qquad x \in H^{2s}(X; Q).$$

The point of this definition is that

$$ch_A\,\Psi_A^k = \Psi_H^k\,ch_A;$$

see [1]. If ξ is a vector bundle whose dimension over the reals is $2n$, we have

$$\phi_H^{-1}\,\Psi_H^k\,\phi_H(x) = k^n\,\Psi_H^k(x).$$

PROPOSITION (5.6).

$$(bh_A\,\xi).(ch_A\,\rho_A^k\,\xi) = k^n\,(\Psi_H^k\,bh_A\,\xi).$$

Proof. Take the equation

$$(\phi_H^{-1}\,ch_A\,\phi_K)(\phi_K^{-1}\,\Psi_A^k\,\phi_K)1 = (\phi_H^{-1}\,\Psi_H^k\,\phi_H)(\phi_H^{-1}\,ch_A\,\phi_K)1$$

and evaluate both sides. We find

$$(\phi_H^{-1}\,ch_A\,\phi_K)\,(\rho_A^k\,\xi) = (\phi_H^{-1}\,\Psi_H^k\,\phi_H)\,(bh_A\,\xi);$$

using Corollary (5.4) and the remark above, we have

$$(bh_A\,\xi)\,.\,(ch_A\,\rho_A^k\,\xi) = k^n\,(\Psi_H^k\,bh_A\,\xi).$$

This completes the proof.

We will next carry out the analogue, for our context, of the proof that Stiefel–Whitney classes are fibre-homotopy invariants. We suppose given a commutative diagram of the following form; it is not assumed that it arises from a bundle map.

$$\begin{tikzcd} \dot{E}_1, E_1 \arrow[r, "g"] \arrow[d, "p_1"'] & \dot{E}_2, E_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, "f"] & B_2 \end{tikzcd}$$

We define

$$k = \phi_{K,1}^{-1}\,g^*\,\phi_{K,2}\,(1) \in K^*(B_1)$$
$$h = \phi_{H,1}^{-1}\,g^*\,\phi_{H,2}\,(1) \in H^*(B_1).$$

PROPOSITION (5.7). *We have*

$$h\,.f^*c(T, \xi_2) = c(T, \xi_1)\,.\,(Tk).$$

Proof. We have $g^*T = Tg^*$. Consider the equation

$$(\phi^{-1}g^*\phi)(\phi^{-1}T\phi)1 = (\phi^{-1}T\phi)(\phi^{-1}g^*\phi)1,$$

where $1 \in K^*(B_2')$, and the suffix for each φ can be determined from the context.

The equation yields

$$(\phi^{-1}g^*\phi)\, c(T, \xi_2) = (\phi^{-1}T\phi)k,$$

or using Lemma (5.3),

$$h \,.\, f^*c(T, \xi_2) = c(T, \xi_1)\,.\,(Tk).$$

This completes the proof.

COROLLARY (5.8). *Let ξ_1, ξ_2 be unitary bundles over B in case $\Lambda = C$, or $Spin(8n)$-bundles ($n = 1, 2, \dots$) in case $\Lambda = R$. If the sphere-bundles associated with ξ_1, ξ_2 are fibre-homotopy equivalent, then there exists an element $x_A \in \tilde{K}_A(B)$ such that*

$$bh_A\, \xi_2 = bh_A\, \xi_1 \,.\, ch_A(1 + x_A)$$

$$\rho_A^l\, \xi_2 = \rho_A^l\, \xi_1 \,.\, \frac{\Psi_A^l(1 + x_A)}{1 + x_A}.$$

Note that in the second equation x_A is independent of l.

Proof. If the sphere-bundles associated with ξ_1, ξ_2 are fibre-homotopy equivalent, then there is a diagram

$$\begin{tikzcd} \bar{E}_1, E_1 \arrow[rr, "g"] \arrow[dr, "p_1"'] & & \bar{E}_2, E_2 \arrow[dl, "p_2"] \\ & B & \end{tikzcd}$$

in which g has degree ± 1 on each fibre. We may therefore apply Proposition (5.7) with $f = 1$. The result involves $k = \phi_{K,1}^{-1}\, g^*\phi_{K,2}(1)$. We may determine the virtual dimension of k over each component of B by restricting on a single fibre; we find that this virtual dimension is ± 1 (according to the degree of g). Let ε be a trivial virtual bundle with the same virtual dimension as k on each component of B; then we have $\varepsilon k = 1 + x_A$ for some $x_A \in \tilde{K}_A(B)$.

Consider now the case of ρ_A^l. Since $h = k$, the result of Proposition (5.7) is

$$k\,.\,(\rho_A^l\, \xi_2) = (\rho_A^l\, \xi_1)\,.\,(\Psi_A^l\, k).$$

Multiplying by the equation $\varepsilon = \Psi_A^l$ or ε, we find

$$(1 + x_A)\,.\,(\rho_A^l\, \xi_2) = (\rho_A^l\, \xi_1)\,.\,\Psi_A^l\,(1 + x_A).$$

Now, $1 + x_A$ is invertible. (If B is finite this follows from the usual power-series for $(1 + x_A)^{-1}$; but in any case, if we take an equivalence γ inverse to g, we obtain an element $\phi_{K,2}^{-1}\, \gamma^*\, \phi_{K,1}(1)$ inverse to k.) We thus obtain

$$\rho_A^l\, \xi_2 = \rho_A^l\, \xi_1 \quad \frac{\Psi_A^l(1 + x_A)}{1 + x_A}$$

The case of bh_A is closely similar. We have $(ch_A\,\varepsilon)\ h = 1$ in $H^0(B)$. The result of Proposition (5.7) is

$$h\,.\,bh_A\,\xi_2 = bh_A\,\xi_1\,.\,ch_A\,k.$$

Multiplying by $ch_A\,\varepsilon$, we find

$$bh_A\,\xi_2 = bh_A\,\xi_1\,.\,ch_A(1 + x_A).$$

This completes the proof.

We now turn to a result useful for calculating ρ^k_Λ.

THEOREM (5.9). *If ξ is a $U(n)$-bundle then $\rho^k_C\,\xi$ is induced from ξ by the virtual representation whose character is*

$$\prod_{1 \leqslant r \leqslant n} \frac{z_r^k - 1}{z_r - 1} = \prod_{1 \leqslant r \leqslant n} (z_r^{k-1} + z_r^{k-2} + \dots + z_r + 1).$$

If ξ is a Spin(8*n*)*-bundle then $\rho^k_R\,\xi$ is induced from ξ by the real virtual representation whose character is*

$$\prod_{1 \leqslant r \leqslant 4n} \frac{z_r^{\frac{1}{2}k} - z_r^{-\frac{1}{2}k}}{z_r^{\frac{1}{2}} - z_r^{-\frac{1}{2}}} = \prod_{1 \leqslant r \leqslant 4n} (z_r^{\frac{1}{2}(k-1)} + z_r^{\frac{1}{2}(k-3)} + \dots + z_r^{-\frac{1}{2}(k-1)}).$$

This theorem was first published by Bott [8, 9]. It follows fairly easily from the definition of the Thom isomorphism ϕ used in [8, 9]. However, it is shown in [18] that this definition coincides with the definition given in [18].

In the above, we have defined $\rho^k_\Lambda(\xi)$ for suitable bundles ξ. Next we shall seek to extend the definition of ρ^k_Λ from bundles to virtual bundles.

We shall see that on bundles, ρ^k is 'homomorphic from addition to multiplication', or more shortly, 'exponential'. Also if τ is the trivial bundle of dimension $2n$ over the reals, we have $\rho^k(\tau) = k^n$. Therefore we are forced to define $\rho^k(-\tau) = k^{-n}$. This indicates that we can define ρ^k on virtual bundles only at the price of introducing denominators. We shall therefore define Q_k to be the additive group of fractions of the form p/k^q, where p and q are integers. If k is a virtual bundle over X, we shall seek to define $\rho^k(k)$ as an element of $K_A(X) \otimes Q_k$. More generally, we may be willing to consider $K_A(X) \otimes S$, where S is a suitable subring of C.

We face a similar situation if we try to define the composite $\rho^k\,\theta$, where θ is a virtual representation. (In Part III we shall be forced to consider such composites.) In this case we are forced, not only to introduce denominators, but also to introduce the completion of the representation ring. Let G be a compact connected Lie group, let $\Lambda = R$ or C, and let S be a subring of C. Let $K'_\Lambda(G)$ be the representation ring of G; then we can form $K'_\Lambda(G) \otimes S$. In $K'_\Lambda(G) \otimes S$ we take the ideal I consisting of elements of virtual dimension zero; these are the elements whose characters vanish at the identity of G. We can now form

$$\mathrm{Comp}(K'_\Lambda(G) \otimes S) = \operatorname*{Inv\,Lim}_{m \to \infty} \frac{K'_\Lambda(G) \otimes S}{I^m}.$$

If θ is a virtual representation of G, we shall seek to define $\rho^k\theta$ as an element of $\mathrm{Comp}(K'_\Lambda(G) \otimes Q_k)$.

If X is a finite CW-complex, one may complete $K_\Lambda(X) \otimes S$ in a similar way. However, $\mathrm{Comp}(K_\Lambda(X) \otimes S)$ can be identified with $K_\Lambda(X) \otimes S$. For let I be the ideal of elements of virtual dimension zero in $K_\Lambda(X) \otimes S$, and let q be the dimension of X; then, as in [7], we have $(\tilde{K}_\Lambda(X))^{q+1} = 0$, so that $I^{q+1} = 0$, and

$$\frac{K_\Lambda(X) \otimes S}{I^m} = K_\Lambda(X) \otimes S$$

for $m \geqslant q + 1$.

We shall require the following lemma on completions. Here the letter K stands for an augmented ring, which in the applications becomes $K_\Lambda(X)$ or $K'_\Lambda(G)$.

LEMMA (5.10)(a). *An element of* $\mathrm{Comp}(K \otimes S)$ *whose virtual dimension is invertible in* S *is invertible in* $\mathrm{Comp}(K \otimes S)$.

(b) *An element* $\hat{c}$ *of* $\mathrm{Comp}(K \otimes S)$ *has in* $\mathrm{Comp}(K \otimes S)$ *a square root, unique up to sign, provided that the virtual dimension of* $\hat{c}$ *has a square root* s *in* S *and* $2s$ *is invertible in* S.

Proof of (b). Suppose given s, as in the data. Let c_m be the component of $\hat{c}$ in $(K \otimes S)/I^m$. In $(K \otimes S)/I^m$ we seek an element of the form $s + i_m$, where $i_m \in I/I^m$, such that

$$(s + i_m)^2 = c_m .$$

For $m = 1$ such an element exists and is unique. Suppose, as an inductive hypothesis, that such an element exists and is unique for some value of m. Choose in $(K \otimes S)/I^{m+1}$ a trial element $s + j_m$ mapping to $s + i_m$ in $(K \otimes S)/I^m$; then we have

$$(s + j_m)^2 = c_m - \varepsilon_m ,$$

where $\varepsilon_m \in I^m/I^{m+1}$. If $(K \otimes S)/I^{m+1}$ contains a square root $s + i_{m+1}$ for c_{m+1} at all, we can write this square root in the form $s + j_m + \delta_m$; and since the square root $s + i_m$ in $(K \otimes S)/I^m$ is unique, we must have $\delta_m \in I^m/I^{m+1}$. If we assume that $\delta_m \in I^m/I^{m+1}$, the equation

$$(s + j_m + \delta_m)^2 = c_{m+1} \quad \text{in} \quad (K \otimes S)/I^{m+1}$$

is equivalent to

$$2s\delta_m = \varepsilon_m \quad \text{in} \quad I^m/I^{m+1}.$$

By assumption, this equation has a unique solution for δ_m. This completes the induction. We have shown that for each m, c_m has a unique square root of virtual dimension s in $(K \otimes S)/I^m$. This yields the result stated.

Part (a) may be proved similarly, or by using the power-series for $(1 + x)^{-1}$.

The work to be done in defining $\rho^k(\kappa)$ is very similar to that in [1, pp. 606–609]. We follow [1] and adopt a convention. The letters f, g will denote maps of complexes such as X. The letters ξ, η will denote bundles; the letters κ, λ will denote elements of $K_\Lambda(X)$; the letters μ, ν will denote elements of $K_\Lambda(X) \otimes S$. The letters α, β will denote representations; the letters θ, ϕ will denote elements of $K'_\Lambda(G)$; the letters ψ, ρ will denote elements of $K_\Lambda'(G) \otimes S$; the letters $\hat{\psi}$, $\hat{\rho}$ will denote elements of $\mathrm{Comp}(K_\Lambda'(G) \otimes S)$. Initially, as on [1, p. 606], we have composites

$$\beta . \alpha, \qquad \alpha . \xi, \qquad \xi . f, \qquad f . g.$$

By linearising over the first factor, as on [1, p. 607], we obtain composites

$$\phi . \alpha, \qquad \theta . \xi, \qquad \kappa . f$$

By S-linearity over the first factor we obtain composites

$$\rho . \alpha, \qquad \psi . \xi, \qquad \mu . f$$

lying in appropriate groups $K \otimes S$. Since composition with a factor on the right preserves virtual dimensions ahd tensor-products, we obtain composites

$$\hat{\rho} . \alpha, \qquad \hat{\psi} . \xi.$$

These lie in the appropriate groups

$$\mathrm{Comp}(K_\Lambda'(G) \otimes S), \mathrm{Comp}(K_\Lambda(X) \otimes S) = K_\Lambda(X) \otimes S.$$

We have remarked earlier that ρ^k is 'homomorphic from addition to multiplication', that is, 'exponential'. We will now make this notion more precise. Let $G(n)$ be one of the series of groups $U(dn)$, $SO(dn)$ or $\mathrm{Spin}(dn)$ (for some integer d). Let

$$\pi : G(n) \times G(m) \longrightarrow G(n),$$

$$\varpi : G(n) \times G(m) \longrightarrow G(m)$$

be the projections of $G(n) \times G(m)$ onto its two factors; thus

$$\pi \oplus \varpi : G(n) \times G(m) \longrightarrow G(n+m)$$

is the 'universal Whitney sum map'. Here the universal Whitney sum map

$$\pi \oplus \varpi : \mathrm{Spin}(dn) \times \mathrm{Spin}(dm) \longrightarrow \mathrm{Spin}(d(n+m))$$

is constructed by lifting the map

$$\pi \oplus \varpi : SO(dn) \times SO(dm) \longrightarrow SO(d(n+m)).$$

For each n, let ρ_n be an element of $K_\Lambda'(G(n)) \otimes S$. We will say that the sequence $\rho = (\rho_n)$ is 'exponential' if we have

$$\rho_{n+m} . (\pi \oplus \varpi) = (\rho_n . \pi) \otimes (\rho_m . \varpi)$$

for all n, m. The sides of this equation lie in $K_\Lambda'(G(n) \times G(m)) \otimes S$; cf [1, p. 607, 609]. Similarly for a sequence

$$\hat{\rho}_n \in \mathrm{Comp}(K_\Lambda'(G(n)) \otimes S).$$

LEMMA (5.11). *If the sequence $\rho = (\rho_n)$ is exponential, then for any two representations $\alpha : H \to G(n)$, $\beta : H \to G(m)$ we have*

$$\rho_{n+m} . (\alpha \oplus \beta) = (\rho_n . \alpha) \otimes (\rho_m . \beta).$$

Moreover, for any two bundles ξ, η with groups $G(n)$, $G(m)$ we have

$$\rho_{n+m} . (\xi \oplus \eta) = (\rho_n . \xi) \otimes (\rho_n . \eta).$$

Similarly for a sequence

$$\hat{\rho} = (\hat{\rho}_n).$$

This lemma is strictly analogous to those of [1, pp. 607–609], and so is its proof.

For the next lemma, we introduce the Grothendieck groups $K_G(X), K'_G(H)$. Here $K_G(X)$ is defined in the obvious fashion using bundles over X with group $G(n)$ for $n = 1, 2, \ldots$; similarly, $K'_G(H)$ is defined using representations $\alpha : H \to G(n)$ of the group H. If $G(n)$ is the sequence of groups $SO(dn)$ or $\mathrm{Spin}(dn)$, we write $K_{SO(d)}$ or $K_{\mathrm{Spin}(d)}$ for the resulting Grothendieck groups K_G. Thus (for example) $K_{\mathrm{Spin}(8)}(X)$ is defined in terms of bundles with structural group $\mathrm{Spin}(8n)$ for $n = 1, 2, \ldots$. If $d = 1$, we write $K_{SO}(X)$ for $K_{SO(1)}(X)$. The group $K_{SO}(X)$ is monomorphically embedded in $K_R(X)$ as the subgroup of classes κ such that $w_1(\kappa) = 0$. Under the decomposition

$$K_R(X) = Z + \tilde{K}_R(X),$$

we have

$$K_{SO(d)}(X) = dZ + \tilde{K}_{SO}(X).$$

We suppose given an exponential sequence $\hat{\rho} = (\hat{\rho}_n)$, where $\hat{\rho}_n \in \mathrm{Comp}(K'_\Lambda(G(n)) \otimes S)$. We assume that the virtual dimension of $\hat{\rho}_1$ is invertible in S.

LEMMA (5.12). *If $\theta \in K'_G(H)$, $\kappa \in K_G(X)$ it is possible to form composites*

$$\hat{\rho} . \theta \in \mathrm{Comp}(K'_\Lambda(H) \otimes S)$$
$$\hat{\rho} . \kappa \in \mathrm{Comp}(K_\Lambda(X) \otimes S) = K_\Lambda(X) \otimes S$$

so that these have the following properties.

(i) *$\hat{\rho}$ is exponential (in the obvious sense).*

(ii) *If we replace θ by α or κ by ξ, then these composites reduce to those considered above.*

Proof. If the virtual dimension of $\hat{\rho}_1$ is s, then the virtual dimension of $\hat{\rho}_n$ is s^n (since $\hat{\rho}$ is exponential). If s is invertible in S, so is s^n. Lemma (5.10)(a) now shows that every element $\hat{\rho}_n \cdot \alpha$ or $\hat{\rho}_n \cdot \xi$ is invertible. Therefore $\hat{\rho}$ can be defined, so as to be exponential, on the free abelian group F generated by the isomorphism classes of such α or ξ. It remains to show that $\hat{\rho}$ passes to the quotient, so that it is defined on $K'_G(H)$ or $K_G(X)$. This follows from the fact that $(\hat{\rho}_n)$ is exponential, using Lemma (5.11). This completes the proof.

Finally, we introduce one further generalised composite. Suppose that $\theta = (\theta_n)$ is an additive sequence of virtual representations with $\theta_n \in K'_\Lambda(G(n))$. Then by S-linearity over the second factor we can define composites

$$\theta . \mu, \qquad \theta . \rho$$

lying in the appropriate groups $K \otimes S$. If (moreover) θ is multiplicative and maps elements of virtual degree zero into elements of virtual degree zero, then we can define composites

$$\theta . \hat{\rho}$$

lying in the appropriate group $\mathrm{Comp}(K \otimes S)$. In practice this situation arises when θ is the sequence Ψ^k [1, §4].

As in [1, pp. 606–609], the appropriate associative laws continue to hold for all the composites we have discussed.

We shall now give some examples of exponential sequences.

EXAMPLE (5.13). Let ρ_n^k be the element in $K_C'(U(n))$ with character

$$\prod_{1 \leqslant r \leqslant n} \frac{z_r^k - 1}{z_r - 1}.$$

The sequence (ρ_n^k) is exponential, as one verifies immediately by checking characters. The virtual dimension of ρ_1^k is k (since $(z_1)^{k-1} + \ldots + z_1 + 1$ takes the value 1 at $z_1 = 1$). If $k = 0$, then $\rho_n^k = 0$; otherwise k is invertible in Q_k. The foregoing theory applies; if $\kappa \in K_C(X)$, we can define $\rho^k.\kappa$ as an element of $K_C(X) \otimes Q_k$. If κ is represented by a $U(n)$-bundle ξ, then $\rho^k.\kappa = \rho_C^k(\xi)$, according to Theorem (5.9).

We now turn to the "real" case. We have already defined $\rho_R^k(\xi)$ when ξ is a Spin$(8n)$-bundle over X. It would therefore be plausible to define $\rho^k(\kappa)$ when $\kappa \in K_{\mathrm{Spin}(8)}(X)$. Actually we shall do more; we shall define $\rho^k(\kappa)$ when $\kappa \in K_{SO(2)}(X)$. (That is, κ may be a linear combination of $SO(2n)$-bundles for $n = 1, 2, \ldots$). For this purpose we need to distinguish the cases 'k odd' and 'k even'.

EXAMPLE (5.14). Assume that k is odd. Consider the formula

$$\prod_{1 \leqslant r \leqslant n} \frac{z_r^{\frac{1}{2}k} - z_r^{-\frac{1}{2}k}}{z_r^{\frac{1}{2}} - z_r^{-\frac{1}{2}}} = \prod_{1 \leqslant r \leqslant n} (z_r^{\frac{1}{2}(k-1)} + z_r^{\frac{1}{2}(k-3)} + \ldots + z_r^{-\frac{1}{2}(k-1)}).$$

It represents a polynomial in which the z_r occur to integral powers. It is also invariant under the Weyl group of $SO(2n)$; therefore it is the character of some virtual representation ρ_n^k of $SO(2n)$. We will show that this virtual representation is real. It must be real if n is even, because every virtual representation of $SO(4m)$ is real. It is also clear that the restriction of ρ_{2m}^k to $SO(4m\text{-}2)$ is $k\rho_{2m-1}^k$. In order to prove that ρ_{2m-1}^k is real, it is sufficient to recall the general fact that if $k\theta$ is real and k is odd, then θ is real. In fact, irreducible representations can be divided into those which coincide with their complex conjugate and those which do not; and the former can be divided into real and quaternionic representations. In order that a representation be real it is necessary and sufficient that it contain each quaternionic irreducible representation an even number of times, and each irreducible representation the same number of times as its complex conjugate when that is a distinct representation. If this condition holds for $k\theta$, it holds for θ.

Alternatively, assuming a little more representation-theory, we can argue that the given formula is invariant under the Weyl group of $O(2n)$; thus ρ_n^k can be written as a polynomial in the exterior powers, so it is real.

We have therefore established the existence of a sequence of virtual representations

$$\rho_n^k \in K_R'(SO(2n))$$

with the characters given. This sequence is exponential, as one verifies immediately by checking characters. The virtual dimension of ρ_1^k is k. The foregoing theory applies; if $\kappa \in K_{SO(2)}(X)$, we can define $\rho^k\kappa$ as an element of $K_R(X) \otimes Q_k$. If κ is represented by a

Spin($8n$)-bundle ξ, then $\rho^k_\cdot \kappa = \rho^k_R(\xi)$, according to Theorem (5.9). This completes the case 'k odd'.

EXAMPLE (5.15). Assume that k is even. We will now construct a sequence of elements

$$\rho^k_n \in \mathrm{Comp}(K'_R(SO(2n)) \otimes Q_k).$$

First, let θ^k_n be the virtual representation of Spin($2n$) with character

$$\prod_{1 \leq r \leq n} \frac{z_r^{\frac{1}{2}k} - z_r^{-\frac{1}{2}k}}{z_r^{\frac{1}{2}} - z_r^{-\frac{1}{2}}}.$$

Since this character is real, it follows that $2\theta^k_n$ is a real virtual representation. Since k is even, $\frac{1}{2} \in Q_k$ and we have

$$\theta^k_n \in K'_R(\mathrm{Spin}(2n)) \otimes Q_k.$$

We now remark that if $S \subset C$, the character of an element of $K'_\Lambda(G) \otimes S$ is defined, and is a finite Laurent series in the z_r with coefficients in S. The elements of $K'_\Lambda(G) \otimes S$ are distinguished by their characters. Therefore we can prove that the sequence θ^k_n is exponential, by checking characters in the obvious way.

Next, consider the map

$$1 \oplus 1 : SO(2n) \longrightarrow SO(4n).$$

This sends a matrix M into

$$\begin{bmatrix} M & O \\ O & M \end{bmatrix}$$

It can be lifted to a unique homomorphism

$$\tau : SO(2n) \longrightarrow \mathrm{Spin}(4n).$$

We will now define

$$\rho^k_n = (\theta^k_{2n} \cdot \tau)^{\frac{1}{2}} \in \mathrm{Comp}(K'_R(SO(2n)) \otimes Q_k).$$

Here the square root exists by Lemma (5.10)(b); we choose its sign so that the virtual dimension of ρ^k_n is k^n.

It is now easy to check that the sequence ρ^k_n is exponential, using the fact that θ^k_n is exponential and the fact that square roots are unique (Lemma (5.10)(b)).

The virtual dimension of ρ^k_1 is k. The foregoing theory applies; if $\kappa \in K_{SO(2)}(X)$, we can define $\rho^k_\cdot \kappa$ as an element of $K_R(X) \otimes Q_k$.

Using Theorem (5.9), it is easy to check that if κ is represented by a Spin($8n$)-bundle ξ, then $\rho^k_\cdot \kappa = \rho^k_R(\xi)$. This completes the case 'k even'.

We add one note. The character of an element

$$\hat{\rho} \in \mathrm{Comp}(K'_\Lambda(G) \otimes S)$$

can be defined, and is a formal power-series in the variables $\zeta_r = z_r - 1$ with coefficients in

S. Cf [7]; we shall give further details in Part III. One can then check that the character of ρ_n^k is

$$\prod_{1 \leqslant r \leqslant n} \frac{z_r^{\frac{1}{2}k} - z_r^{-\frac{1}{2}k}}{z_r^{\frac{1}{2}} - z_r^{-\frac{1}{2}}},$$

where this formula is interpreted using the expansion of $(z_r)^m$ as a binomial series in ζ_r.

We turn next to the calculation of the operations ρ_R^k for the space $X = RP^n$. We recall the structure of $\tilde{K}_R(RP^n)$ from [1, Theorem (7.4)]. Let ξ be the canonical line bundle over RP^n; then $\xi^2 = 1$, and $\lambda = \xi - 1$ is a generator in $K_R(RP^n)$, which is cyclic of order 2^f, where f is the number of integers s such that $0 < s \leqslant n$ and $s \equiv 0, 1, 2$ or $4 \bmod 8$. In particular, $\tilde{K}_R(RP^n) \otimes Q_k = 0$ if k is even. The only case of interest is therefore that in which k is odd. The operation ρ^k is defined on $K_{SO(2)}(X)$, so that ρ^k is defined on all multiples of 2λ in $\tilde{K}_R(RP^n)$. The value $\rho^k(2l\lambda)$ will lie in the multiplicative group $1 + \tilde{K}_R(RP^n) \otimes Q_k \cong 1 + \tilde{K}_R(RP^n)$ of elements of virtual dimension 1. In order to make the structure of this group more transparent, let $J_{2^{f+1}}$ be the ring of residue classes mod 2^{f+1}, and let $G_{2^{f+1}}$ be the multiplicative group of odd residue classes mod 2^{f+1}. Then we have $J_{2^{f+1}} \otimes Q_k \cong J_{2^{f+1}}$ for k odd; and we can define a ring homomorphism

$$\alpha : K_R(RP^n) \otimes Q_k \longrightarrow J_{2^{f+1}} \otimes Q_k$$

by setting $\alpha(\xi) = -1$, or equivalently $\alpha(\lambda) = -2$. The map α induces an isomorphism of $1 + \tilde{K}_R(RP^n) \otimes Q_k$ onto $G_{2^{f+1}}$. The subgroup $1 + \tilde{K}_{SO}(RP^n) \otimes Q_k$ (defined in term of orientable bundles) maps by α onto the group of residue classes congruent to 1 mod 4.

THEOREM (5.16). *The operations ρ^k on $\tilde{K}_{SO}(RP^n)$ are given by*

$$\rho^k(2l\lambda) = 1 + \frac{k^l - \varepsilon^l}{2k^l}\lambda$$

$$\text{where} \qquad \varepsilon = \begin{cases} 1 & \text{if } k \equiv 1 \bmod 4 \\ -1 & \text{if } k \equiv 3 \bmod 4. \end{cases}$$

Equivalently, they are given by

$$\alpha\rho^k(2l\lambda) = \left(\frac{\varepsilon}{k}\right)^l.$$

Remark (*a*). For l divisible by 4 this result has been found by Bott [8, 9]. Similar calculations have been made by Atiyah (private communication).

Remark (*b*). The values of ρ^k lie in $1 + \tilde{K}_{SO}(RP^n) \otimes Q_k$. This must necessarily happen, since all the representations we have used map into $SO(m)$.

Remark (*c*). One can choose an odd number k so that k (and hence ε/k) has the maximum possible order in $G_{2^{f+1}}$; then ε/k will have no square root in $G_{2^{f+1}}$; this proves that it is impossible to define ρ^k on λ so as to preserve the exponential property.

Proof. We have $2l\xi = i\xi$, where $i : O(1) \to SO(2l)$ maps ± 1 into ± 1. We will compute the representation $\rho_l^k i$ of $O(1)$, where ρ_l^k is as in (5.14). The value of the character of ρ_l^k at 1 is k^l. If we substitute $z_r = -1$, the value of

$$z_r^{\frac{1}{2}(k-1)} + z_r^{\frac{1}{2}(k-3)} + \ldots + z_r^{-\frac{1}{2}(k-1)}$$

is ε, where

$$\varepsilon = \begin{cases} 1 & \text{if} \quad k \equiv 1 \bmod 4 \\ -1 & \text{if} \quad k \equiv 3 \bmod 4 \end{cases}$$

Therefore the value of the character at -1 is ε^l. Let λ^1 be the identity representation of $O(1)$; we conclude that

$$\rho_l^k \, i = a + b\lambda^1$$

where

$$a + b = k^l$$
$$a - b = \varepsilon^l.$$

Therefore

$$\rho_l^k (2l\xi) = \rho_l^k \, i \, \xi = a + b\xi,$$
$$\alpha \, \rho_l^k (2l\xi) = a - b = \varepsilon^l,$$
$$\alpha \, \rho_l^k (2l\lambda) = \frac{\varepsilon^l}{k^l}.$$

The same method yields the first part of the theorem. This completes the proof.

We will now consider the case $X = S^n$, where $n \equiv 1$ or $2 \bmod 8$. In this case the group $\tilde{K}_R(S^n)$ is Z_2, and we have $\tilde{K}_R(S^n) \otimes Q_k = 0$ for k even. The only case of interest is therefore again that in which k is odd. The operation ρ^k is defined on $\tilde{K}_R(S^n)$ for $n \geqslant 2$.

THEOREM (5.17). *If $n \equiv 1$ or $2 \bmod 8$ and $n \geqslant 2$, then the operations ρ^k on $\tilde{K}_R(S^n)$ are given by*

$$\rho^k x = \begin{cases} 1 & \text{if} \quad k \equiv \pm 1 \bmod 8 \\ 1 + x & \text{if} \quad k \equiv \pm 3 \bmod 8. \end{cases}$$

Proof. Consider a map $g : RP^n \to S^n$ of degree 1, as in 3.5. The map

$$g^* : \tilde{K}_R(S^n) \longrightarrow \tilde{K}_R(RP^n)$$

is monomorphic, by the proof of [1, Theorem (7.4)]. We can now compute $\rho^k x$ by naturality. If $x = 0$ the result is trivial, so we may assume $x \neq 0$; then $g^* x = 2^{f-1}\lambda$, where 2^f is the order of λ. By Lemma (2.9), the group $G_{2^{f+1}}$ is $Z_2 + Z_{2^{f-1}}$. The element ε/k (see Theorem (5.16)) has multiplicative order dividing 2^{f-2} if $k \equiv \pm 1 \bmod 8$, or 2^{f-1} if $k \equiv \pm 3 \bmod 8$. Thus

$$\alpha \, \rho^k (2^{f-1}\lambda) = (\varepsilon/k)^{2^{f-2}}$$

is equal to 1 if $k \equiv \pm 1 \bmod 8$, and otherwise not equal to 1. Thus $\rho^k x$ is equal to 1 or not according to the case; but if it is not 1, it can only be $1 + x$. This completes the proof.

It remains to consider the case $X = S^{4n}$.

THEOREM (5.18). *If $x \in \tilde{K}_R(S^{4n})$* then

$$\rho^k(x) = 1 + \tfrac{1}{2}(k^{2n} - 1) \, \alpha_{2n} \, x,$$

where α_{2n} is as in §2.

Remark. The coefficient $\frac{1}{2}(k^{2n}-1)\alpha_{2n}$ lies in Q_k, according to Theorem (2.7).

Proof. We may suppose that x is a linear combination of Spin($8m$)-bundles for various values of m. By applying Proposition (5.6) to each bundle, we find

$$(shx).(ch\, c\, \rho^k x) = \Psi_H^k\, sh\, x.$$

By Corollary (5.2) we have

$$sh\, x = 1 + \tfrac{1}{2}\alpha_{2n}\, ch_{2n}\, cx,$$

so that

$$\Psi_H^k\, sh\, x = 1 + \tfrac{1}{2}k^{2n}\, \alpha_{2n}\, ch_{2n}\, cx$$

and

$$ch\, c\, \rho^k\, x = 1 + \tfrac{1}{2}(k^{2n}-1)\, \alpha_{2n}\, ch_{2n}\, cx.$$

Hence

$$\rho^k x = 1 + \tfrac{1}{2}(k^{2n}-1)\, \alpha_{2n}\, x.$$

This completes the proof.

§6. THE GROUP $J'(X)$.

In this section we shall introduce the group $J'(X)$. We shall prove (Theorem (6.1)) that $J'(X)$ is a lower bound for $J(X)$. We also compute the groups $J'(X)$ when $X = RP^n$ or S^n.

We will now give the definition of J'. First recall that $K_{SO(2)}(X)$ is monomorphically embedded in $K_R(X)$ as the subgroup of elements x such that (i) the first Stiefel–Whitney class $w_1(x)$ is zero, and (ii) the virtual dimension of x is even. We define $V(X)$ to be the subgroup of elements $x \in K_{SO(2)}(X)$ which satisfy the following condition: there exists $y \in \tilde{K}_R(X)$ such that

$$\rho^k x = \frac{\Psi^k(1+y)}{1+y} \quad \text{in} \quad K_R(X) \otimes Q_k$$

for all $k \neq 0$. We now define

$$J'(X) = K_R(X)/V(X).$$

It is necessary to check that $V(X)$ is a subgroup. We first note that any x which satisfies the condition given has virtual dimension zero. Let $1 + \tilde{K}_R(X) \otimes Q_k$ be the multiplicative group of elements of virtual dimension 1 in $K_R(X) \otimes Q_k$. Let Π be the multiplicative group

$$\prod_{k \neq 0} (1 + \tilde{K}_R(X) \otimes Q_k).$$

Let us define a function

$$\delta : 1 + \tilde{K}_R(X) \longrightarrow \prod$$

by

$$\delta(1+y) = \left\{\frac{\Psi^k(1+y)}{1+y}\right\};$$

then δ is a homomorphism, because Ψ^k is multiplicative for each k. Similarly, the function

$$\rho : \tilde{K}_{SO}(X) \longrightarrow \prod$$

defined by

$$\rho(x) = \{\rho^k(x)\}$$

is a homomorphism. Therefore the set

$$V(X) = \rho^{-1}\delta(1 + \tilde{K}_R(X))$$

is a subgroup.

THEOREM (6.1). *$J'(X)$ is a lower bound for $J(X)$, in the sense of Part I* [4].

We recall that in Part I we defined

$$J(X) = K_R/T(X);$$

here $T(X)$ is the subgroup of $K_R(X)$ generated by elements of the form $\{\xi\} - \{\eta\}$, where ξ and η are orthogonal bundles whose associated sphere-bundles are fibre homotopy equivalent. The theorem states that $T(X) \subset V(X)$, so that the quotient map $K_R(X) \to J'(X)$ factors through $J(X)$.

Proof. Suppose given a finite connected CW-complex X. I claim that $T(X)$ is generated by elements $\{\xi'\} - \{\eta'\}$, where ξ' and η' are orthogonal bundles whose associated sphere bundles are fibre homotopy equivalent, and η' is trivial of dimension divisible by 8. In fact, let ξ, η be orthogonal bundles over X whose associated sphere-bundles are fibre homotopy equivalent. Then the same is true for $\xi \oplus \zeta$ and $\eta \oplus \zeta$, whatever the bundle ζ. We have

$$\{\xi \oplus \zeta\} - \{\eta \oplus \zeta\} = \{\xi\} - \{\eta\}.$$

We can choose ζ so that $\eta \oplus \zeta$ is a trivial bundle of dimension divisible by 8.

Let ξ', η' be as above. Then the Stiefel–Whitney classes of ξ' are zero, since the Stiefel–Whitney classes are fibre homotopy invariants. Thus we can lift ξ' to a Spin($8n$)-bundle. Corollary (5.8) applies, and shows that there exists $y \in \tilde{K}_R(X)$ such that

$$\rho^k(\xi') = \rho^k(\eta') . \frac{\Psi^k(1+y)}{1+y}$$

for all k. That is,

$$\rho^k(\{\xi'\} - \{\eta'\}) = \frac{\Psi^k(1+y)}{1+y}$$

in $K_R(X) \otimes Q_k$. This shows that $\{\xi'\} - \{\eta'\} \in V(X)$; thus $T(X) \subset V(X)$. This completes the proof.

By way of illustration, we will now calculate the groups J' for the examples considered in §3.

EXAMPLE (6.3). Take $X = RP^n$. Then the quotient map

$$K_R(RP^n) \longrightarrow J'(RP^n)$$

is an isomorphism, and consequently the quotient map

$$K_R(RP^n) \longrightarrow J(RP^n)$$

is an isomorphism.

The fact that the results of [1] can be rephrased in this way is an observation of ATIYAH (private communication) and of BOTT [8, 9].

Proof. As observed above, the group $\tilde{K}_R(RP^n)$ is of order 2^f, and therefore $\tilde{K}_R(RP^n) \otimes Q_k$ is zero for k even. It is therefore sufficient to consider odd values of k, for which

$$\tilde{K}_R(RP^n) \otimes Q_k \cong \tilde{K}_R(RP^n).$$

According to [1, Theorem (7.4), p. 625], for k odd and $y \in \tilde{K}_R(RP^n)$ we have

$$\frac{\Psi^k(1+y)}{1+y} = 1.$$

The elements $v \in V(RP^n)$ have therefore to satisfy the conditions

(i) $w_1(v) = 0$

(ii) $\rho^k(v) = 1$ for k odd.

The first condition ensures that $v = 2l\lambda$, and disposes entirely of the low-dimensional case $n = 1$. By Theorem (5.16), if $k \equiv \pm 3 \bmod 8$ the element $\rho^k(2\lambda)$ has order 2^{f-1} in the multiplicative group $1 + \tilde{K}_R(RP^n)$. (Here the integer f is as in §5. The same application of Theorem (5.16) was made in the proof of Theorem (5.17).) Therefore the condition

$$\rho^k(2l\lambda) = 1, \qquad \text{all odd } k$$

implies that l is divisible by 2^{f-1}, i.e. that $2l\lambda = 0$ in $\tilde{K}_R(RP^n)$. Thus $V_R(RP^n) = 0$. This completes the proof.

EXAMPLE (6.4). Take $X = S^n$ with $n \equiv 1$ or $2 \bmod 8$. Then the quotient map

$$K_R(S^n) \longrightarrow J'(S^n)$$

is an isomorphism, and consequently the quotient map

$$K_R(S^n) \longrightarrow J(S^n)$$

is an isomorphism. Equivalently, the image $J(\pi_{n-1}(SO))$ of the stable J-homomorphism is Z_2 for $n \equiv 1$ or $2 \bmod 8$.

Proof. As in proving Theorem (5.17), we consider a map $f : RP^n \to S^n$ of degree 1, so that the induced map

$$f^* : \tilde{K}_R(S^n) \longrightarrow \tilde{K}_R(RP^n)$$

is monomorphic. Consider the following commutative diagram.

$$\begin{array}{ccc} K_R(S^n) & \xrightarrow{K(f)} & K_R(RP^n) \\ \downarrow{\scriptstyle q_S} & & \downarrow{\scriptstyle q_P} \\ J'(S^n) & \xrightarrow{J'(f)} & J'(RP^n) \end{array}$$

Since $K(f)$ is monomorphic and q_P is an isomorphism, q_S must be monomorphic. This completes the proof.

EXAMPLE (6.5). Take $X = S^{4n}$. Then $J'(S^{4n})$ is cyclic of order $m(2n)$.

This is essentially the theorem of MILNOR and KERVAIRE [15], as improved by ATIYAH and HIRZEBRUCH [6]; it states that the image $J(\pi_{4n-1}(SO))$ of the stable J-homomorphism has an order divisible by the denominator of $B_n/4n$.

Proof. Suppose $x \in V(S^{4n})$; that is, suppose that

$$\rho^k x = \frac{\Psi^k(1+y)}{1+y}$$

for all k. Using Theorem (5.18), this becomes

(6.6) $$1 + \tfrac{1}{2}(k^{2n}-1)\alpha_{2n}x = 1 + (k^{2n}-1)y$$

where α_{2n} is as in §2; by Theorem (2.6) we have

$$\tfrac{1}{2}\alpha_{2n} = \frac{d(2n)}{m(2n)},$$

where $d(2n)$ and $m(2n)$ are coprime. Equation (6.6) holds in $1 + \tilde{K}_R(S^{4n}) \otimes Q_k$; in $\tilde{K}_R(S^{4n})$ we have

(6.7) $$k^{f(k)}(k^{2n}-1)\frac{d(2n)}{m(2n)}x = k^{f(k)}(k^{2n}-1)y,$$

for some exponent $f(k)$. According to Theorem (2.7) the highest common factor of the integers $k^{f(k)}(k^{2n}-1)$ divides $m(2n)$. Therefore by taking a linear combination of the equations (6.7), we can show

$$d(2n)x = m(2n)y.$$

That is, x is divisible by $m(2n)$ in $\tilde{K}_R(S^{4n})$.

Conversely, if x is divisible by $m(2n)$ in $\tilde{K}_R(S^{4n})$, then we can solve the equation

$$d(2n)x = m(2n)y$$

for y, and we can calculate that

$$\rho_x^k = \frac{\Psi^k(1+y)}{1+y}$$

for all k. Thus $x \in V(S^{4n})$.

This determines the subgroup $V(S^{4n})$, and proves the result stated.

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