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ON THE GROUPS $J(X)$—III

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§1. INTRODUCTION

THE GENERAL OBJECT of this series of papers is to give means for computing the groups $J(X)$. A general introduction has been given at the beginning of Part I [3]. We recall that in Part II [4] we set up two further groups $J'(X)$ and $J''(X)$; here $J'(X)$ is a "lower bound" for $J(X)$, and we conjecture that $J''(X)$ is an "upper bound" for $J(X)$. The present paper, Part III, has two main objects; the first is to prove the following theorem.

THEOREM (1.1). *For each finite CW-complex X we have $J'(X) = J''(X)$.*

The precise sense in which the groups $J'(X)$ and $J''(X)$ are "equal" is the following. Both groups can be defined as quotients of $K_R(X)$, say

$$J'(X) = K_R(X)/V(X)$$
$$J''(X) = K_R(X)/W(X).$$

We shall prove that the subgroups $V(X)$ and $W(X)$ of $K_R(X)$, although differently defined, are in fact the same. Therefore the corresponding quotient groups $J'(X)$ and $J''(X)$ are (as a matter of logic) identical; that is, they are one and the same object.

Theorem (1.1) will be proved in §4. The proof is completely dependent on the existence of a certain commutative diagram (Diagram 3.1), which is established in §3. This, in turn, depends on an extension of a result of Atiyah and Hirzebruch [5], which is stated as Theorem (2.2) and proved in §2.

The proof given in §§3 and 4 may also be found in [2]. However, I would like the present account to be regarded as more complete and final; in particular, Theorem (2.2) of this paper is to be regarded as superseding Lemma (2.1) of [2].

The second main object of this paper may be explained as follows. We shall show in Part IV that the image of the stable J-homomorphism

$$J : \pi_{8m+3}(SO) \longrightarrow \pi^S_{8m+3}$$

is a direct summand. (Here π^S_r is the stable r-stem.) In other words, for the case $X = S^{8m+4}$ the group $\tilde{J}(X)$ is a direct summand in something else. It is reasonable to ask whether this is a special case of a result true for some general class of spaces X. The answer appears to be "yes", modulo some doubt as to the best way of setting up the foundations of the subject. This will be explained in §7.

Needless to say, §7 depends on §6, and §6 depends on §5. In §6 we observe that some, but not all, elements of $\prod_k (K_R(X) \otimes Q_k)$ have the form $\{\rho^k(x)\}$, and we essentially give a characterisation of the elements which have this form (Theorem (6.2)). In §5 we establish a "periodicity" property of the operations Ψ^k (Theorem (5.1)). Besides being used in §6, this property will be used in Part IV of the present series.

§2. COMPLETIONS AND CHARACTERS

In the present paper we shall sometimes have to work in the completion of a representation ring. This completion has already appeared in [4, §5]; we shall recall the details below. One calculates in such a ring by using characters; in this section we shall explain this topic, following Atiyah and Hirzebruch [5, pp. 24–27]. We shall also set up certain results needed later. Corollary (2.9) states that an element of a completed representation ring is determined by its character. Corollary (2.10) is a technical result needed in §3. These corollaries follow at once from the main result, Theorem (2.2); this is a slight extension of a result of Atiyah and Hirzebruch. Much of the proof is obtained by following these authors; but we are forced to add extra arguments, since we are interested in the real representation ing as well as the complex one.

G will be a compact connected Lie group, in which we choose a maximal torus T, whose points are given by complex coordinates $z_1, z_2, \ldots, z_n$ with $|z_r| = 1$ for each r. If $\Lambda = R$ or C, we can form the representation ring $K'_\Lambda(G)$; the notation is as in [1]. If $\theta \in K'_\Lambda(G)$, then the character $\chi(\theta)$ of θ is a finite Laurent series in the variables $z_1, z_2, \ldots, z_n$, with integer coefficients. We may identify the ring of such Laurent series with $K'_C(T)$; this amounts to identifying χ with $i^* : K'_\Lambda(G) \longrightarrow K'_C(T)$. An element of $K'_\Lambda(G)$ is determined by its character; in other words, χ (or i^*) is a monomorphism.

Let S be a subring of C; then we can form $K'_\Lambda(G) \otimes S$. If $\theta \in K'_\Lambda(G) \otimes S$, then the character $\chi(\theta)$ of θ is defined by S-linearity, and is a finite Laurent series with coefficients in S. The ring of such Laurent series may be identified with $K'_C(T) \otimes S$; this amounts to identifying χ with

$$i^* \otimes 1 : K'_\Lambda(G) \otimes S \longrightarrow K'_C(T) \otimes S.$$

Since S is torsion-free, $i^* \otimes 1$ (or χ) is again a monomorphism.

In $K'_\Lambda(G) \otimes S$ we take the ideal $I = \tilde{K}'_\Lambda(G) \otimes S$; this consists of the elements whose characters vanish at the origin. We give $K'_\Lambda(G) \otimes S$ the I-adic uniform structure and complete it; that is, we define

$$\mathrm{Comp}(K'_\Lambda(G) \otimes S) = \operatorname*{Inv\,Lim}_{q \to \infty} \frac{K'_\Lambda(G) \otimes S}{I^q}.$$

We will now define the character of an element in Comp $(K'_\Lambda(G) \otimes S)$. By substituting $z_r = 1 + \zeta_r$, any finite Laurent series in $z_1, z_2, \ldots, z_n$ yields a power-series in $\zeta_1, \zeta_2, \ldots, \zeta_n$ (convergent in a neighbourhood of the origin). An element of I yields a power-series vanishing at the origin; an element of I^q yields a power series starting with terms of the q^{th} order. Therefore an element of Comp $(K'_\Lambda(G) \otimes S)$ yields a formal power-series.

More formally, let $S[[\zeta_1, \zeta_2, \dots, \zeta_n]]$ be the ring of formal power-series in $\zeta_1, \zeta_2, \dots, \zeta_n$ with coefficients in S, and let J be the ideal consisting of power-series with zero constant term; we give $S[[\zeta_1, \zeta_2, \dots, \zeta_n]]$ the J-adic uniform structure. The substitution $z_r = 1 + \zeta_r$ yields a map

$$\chi : K'_\Lambda(G) \otimes S \longrightarrow S[[\zeta_1, \zeta_2, \dots, \zeta_n]],$$

which is uniformly continuous, since

$$I^q \subset \chi^{-1} J^q.$$

By completion we obtain a map

$$\chi : \mathrm{Comp}(K'_\Lambda(G) \otimes S) \longrightarrow S[[\zeta_1, \zeta_2, \dots, \zeta_n]],$$

since $S[[\zeta_1, \zeta_2, \dots, \zeta_n]]$ is its own completion.

This account applies, of course, to the Lie group T, and may be used to identify $S[[\zeta_1, \zeta_2, \dots, \zeta_n]]$ with Comp $(K'_C(T) \otimes S)$; cf. [5, 4.3, pp. 26, 27]. The only significant step is to check the following proposition; we include an elementary proof for completeness, but the reader who prefers to refer to [5] may omit it.

PROPOSITION (2.1). *The I-adic uniform structure on $K'_C(T) \otimes S$ coincides with that induced from the J-adic uniform structure on $S[[\zeta_1, \zeta_2, \dots, \zeta_n]]$. More precisely, we have*

$$\chi^{-1} J^q = I^q.$$

Proof. We have already seen that $I^q \subset \chi^{-1} J^q$; it remains to prove that $\chi^{-1} J^q \subset I^q$. Let

$$L = \sum_e s_e z_1^{e_1} z_2^{e_2} \dots z_n^{e_n}$$

be a finite Laurent series such that $\chi(L) \in J^q$. Without loss of generality we may suppose that all the exponents e_r which occur are positive; for otherwise we can replace L by $L z_1^{f_1} z_2^{f_2} \dots z_n^{f_n}$, since $z_1^{f_1} z_2^{f_2} \dots z_n^{f_n}$ is invertible in $K'_C(T) \otimes S$. Assuming that the e_r are positive, we may substitute $z_r = 1 + \zeta_r$ and so write L as a finite sum

$$L = \sum_g s'_g \zeta_1^{g_1} \zeta_2^{g_2} \dots \zeta_n^{g_n}.$$

By assumption, all the terms with $\sum_r g_r < q$ are zero. This displays L as an element of I^q.

The central result required for our applications is the following.

THEOREM (2.2). *If S is a subring of the ring Q of rational numbers, then the I-adic uniform structure on $K'_\Lambda(G) \otimes S$ coincides with that induced from the J-adic uniform structure on $S[[\zeta_1, \zeta_2, \dots, \zeta_n]]$. More precisely, given q there exists $r = r(G, \Lambda, q)$ independent of S such that*

$$\chi^{-1} J^r \subset I^q \subset \chi^{-1} J^q.$$

The proof is based on an argument of Atiyah and Hirzebruch [5, pp. 24–27]; it will require several lemmas.

LEMMA (2.3). *Let A be a finitely-generated commutative ring; let W be a finite group of automorphisms of A; let B be the subring of elements of A invariant under W. Then A is a finitely-generated B-module and B is a finitely-generated ring.*

This lemma is given in [5, p. 24 (iii)]. (I have altered the statement slightly, but the proof is unchanged.)

COROLLARY (2.4). *$K'_\Lambda(G)$ is a finitely-generated ring, and $K'_C(T)$ is a finitely-generated module over it.*

The case $\Lambda = C$ is due to Atiyah and Hirzebruch [5, pp. 24–27].

Proof. Since the case $\Lambda = C$ has been proved by Atiyah and Hirzebruch, and since we are mainly interested in the case $\Lambda = R$, we will give the proof for the case $\Lambda = R$.

Let us recall that $K'_C(G)$ is a free abelian group, generated by the irreducible representations of G. These may be classified into three classes. Class (c) contains each irreducible representation ρ which is distinct from its complex conjugate $\bar{\rho}$. Class (r) contains the complex forms of real representations. Class (q) contains the complex forms of quaternionic representations.

Let us now apply Lemma (2.3), taking A to be $K'_C(T)$, and W to be the direct product $\Gamma \times Z_2$, where Γ is the Weyl group of G and Z_2 acts by complex conjugation. The resulting ring B consists of the elements of $K'_C(G)$ invariant under complex conjugation. These constitute a free abelian group generated by the following elements: the irreducible representations of classes (r) and (q), together with the elements $\rho + \bar{\rho}$, where ρ runs over class (c). The lemma shows that B is a finitely-generated ring, and that $K'_C(T)$ is a finitely-generated module over B. Since B is finitely-generated, we may choose a set of generators consisting of a finite number of representations in the classes (r) and (q), together with a finite number of representations $\rho + \bar{\rho}$. Let the generators in class (q) be $q_1, q_2, \dots, q_m$. Let D be the subring of B generated by the remaining generators, together with the products $q_i q_j$; thus D is a finitely-generated ring. We have $D \subset K'_R(G)$, since the representations $\rho + \bar{\rho}$ and $q_i q_j$ are real. Also B is a finitely-generated D-module, generated by $1, q_1, q_2, \dots, q_m$. Since $K'_C(T)$ is a finitely-generated B-module, it follows that $K'_C(T)$ is a finitely-generated D-module, *a fortiori* a finitely-generated module over $K'_R(G)$. Finally, D is Noetherian and $K'_R(G)$ is a D-submodule of the finitely-generated D-module B; therefore $K'_R(G)$ is a finitely-generated D-module, and hence a finitely-generated ring. This completes the proof.

We will require one further consequence of the way $K'_R(G)$ is embedded in $K'_C(G)$. As in the proof above, B will be the subring of elements in $K'_C(G)$ invariant under complex conjugation. We set

$$\tilde{B} = B \cap \tilde{K}'_C(G).$$

LEMMA (2.5). *The ideals $\tilde{B}$ and $\tilde{K}_R(G) \cdot B$ define the same uniform structure on B.*

Proof. We clearly have $\tilde{K}_R(G) \cdot B \subset \tilde{B}$; we wish to argue in the opposite direction. Let $q_1, q_2, \dots, q_m$ be as in the proof above; we will show that

$$(\tilde{B})^{m+1} \subset \tilde{K}_R(G) \cdot B.$$

In fact, $\tilde{B}$ is generated as a B-module by the elements $g_r - \gamma_r$, where g_r runs over the generators of B and $\gamma_r \in Z$ is the dimension of g_r. Therefore $(\tilde{B})^{m+1}$ is generated as a

B-module by the elements

$$\prod_{1 \leqslant i \leqslant m+1} (g_{r_i} - \gamma_{r_i}).$$

If any g_{r_i} is a generator in $K_R(G)$, then the product lies in $\tilde{K}_R(G) \cdot B$. It remains only to consider the case in which every g_{r_i} is a generator q_j. In this case at least one q_j must occur twice, yielding a factor

$$(q_j - \gamma)^2 = q_j^2 - 2q_j\gamma + \gamma^2.$$

Since q_j^2 and $2q_j$ are real representations, the factor $(q_j - \gamma)^2$ lies in $\tilde{K}_R(G)$. This completes the proof.

We now require further lemmas of Atiyah and Hirzebruch.

LEMMA (2.6). *Let A, W and B be as in Lemma (2.3). Let J be an ideal of A such that $wJ = J$ for each $w \in W$; set $I = J \cap B$. Then J and $I \cdot A$ define the same uniform structure on A; more precisely, there exists an integer m such that*

$$J^m \subset I \cdot A \subset J.$$

This lemma is quoted (with minor changes of notation) from [5, p. 25 (iv)].

LEMMA (2.7). *Let A be a ring and B a subring of A such that B is Noetherian and A is a finitely-generated module over B. Let I be an ideal of B. Then the I-adic uniform structure of B coincides with that induced from the $I \cdot A$-adic uniform structure of A.*

This follows directly from [5, p. 24 (i)] by taking the modules M, N mentioned there to be the rings A, B.

COROLLARY (2.8). *Theorem (2.2) is true in the special case $S = Z$. More explicitly, let $I = \tilde{K}_\Lambda'(G)$; then the I-adic uniform structure on $K_\Lambda'(G)$ coincides with that induced from the J-adic uniform structure on $Z[[\zeta_1, \zeta_2, \ldots, \zeta_n]]$.*

The case $\Lambda = C$ is due to Atiyah and Hirzebruch [5 pp. 24–27].

Proof. Since the case $\Lambda = C$ is due to Atiyah and Hirzebruch, and since we are mainly interested in the case $\Lambda = R$, we will give the proof for the case $\Lambda = R$.

It will ease the statement of the proof if we make one convention. Let A be a ring, B a subring and J an ideal in A; then the "J-adic uniform structure on B" will mean that induced by the J-adic uniform structure on A; that is, it consists of the ideals $J^m \cap B$.

Let $J \subset Z[[\zeta_1, \zeta_2, \ldots, \zeta_n]]$ be as in the corollary. Then according to Proposition (2.1), the J-adic structure on $K_C'(T)$ coincides with the $\tilde{K}_C'(T)$-adic structure. The same is therefore true for the structures induced on any subring of $K_C'(T)$; in particular, the J-adic and $\tilde{K}_C'(T)$-adic structures on $K_r'(G)$ coincide.

We now apply Lemma (2.6), taking A to be $K_C'(T)$ and W to be $\Gamma \times Z_2$, as in the proof of Corollary (2.4). We take J to be $\tilde{K}_C'(T)$; thus I becomes $\tilde{B}$. According to Lemmas (2.6) and (2.7), the $\tilde{K}_C'(T)$-adic and $\tilde{B}$-adic structures coincide on B, and therefore on any subring of B, in particular $K_R'(G)$.

We now apply Lemma (2.7) to the ring B and the subring $K_R'(G)$. According to Lemmas (2.5) and (2.7), the $\tilde{B}$-adic and $\tilde{K}_R'(G)$-adic structures on $K_R'(G)$ coincide. This completes the proof.

Proof of Theorem (2.2). This theorem follows from the special case $S = Z$. Let us use subscripts to distinguish the ideals which occur in the two cases; thus

$$J_Z \subset Z[[\zeta_1, \zeta_2, \dots, \zeta_n]]$$

$$J_s \subset S[[\zeta_1, \zeta_2, \dots, \zeta_n]].$$

Then we have

$$J_s^r = J_Z^r \cdot S$$

and we easily check that

$$\chi^{-1}(J_S^r) = (\chi^{-1}(J_Z^r)) \cdot S.$$

The result follows.

COROLLARY (2.9). *If $S \subset Q$, the map $\chi : \mathrm{Comp}(K_\Lambda'(G) \otimes S) \longrightarrow S[[\zeta_1, \zeta_2, \dots, \zeta_n]]$ is monomorphic.*

This follows immediately from Theorem (2.2). It is this corollary which allows us to handle elements of $\mathrm{Comp}(K_\Lambda'(G) \otimes S)$ by means of their characters. The case $S = Z, \Lambda = C$ is due to Atiyah and Hirzebruch [5, pp. 26, 27].

In order to state the next corollary, suppose given two subrings $S \subset T \subset Q$. Then we have ideals

$$I_S = \tilde{K}_\Lambda'(G) \otimes S \subset K_\Lambda'(G) \otimes S$$

$$I_T = \tilde{K}_\Lambda'(G) \otimes T \subset K_\Lambda'(G) \otimes T.$$

COROLLARY (2.10). *Given q, there exists $r = r(G, \Lambda, q)$ independent of S, T such that*

$$(K_\Lambda'(G) \otimes S) \cap (I_T)^r \subset (I_S)^q.$$

This follows immediately from Theorem (2.2). It is needed for the arguments in §3.

§3. AN IDENTITY BETWEEN VIRTUAL REPRESENTATIONS

The object of this section is to prove Theorem (3.2), which is vital to the proof of Theorem (1.1). The most important part of this theorem will state the commutativity of the following diagram.

(3.1)
$$\begin{tikzcd}
\tilde{K}_{SO}(X) \arrow[r, "k^e(\Psi^k - 1)"] \arrow[d, "\theta^k"'] & \tilde{K}_{SO}(X) \arrow[d, "\rho^l"] \\
1 + \tilde{K}_{SO}(X) \arrow[r, "\delta^l"] & 1 + \tilde{K}_{SO}(X) \otimes Q_l
\end{tikzcd}$$

Here the groups are as in [4, §5]; those in the top row are additive, and those on the bottom row are multiplicative. In particular, $1 + \tilde{K}_{SO}(X)$ is the multiplicative group of elements $1 + y$ in $K_R(X)$, where $y \in \tilde{K}_{SO}(X)$; similarly for $1 + \tilde{K}_{SO}(X) \otimes Q_l$, where Q_l is the ring of rational numbers of the form a/l^b. The additive homomorphism $k^e(\Psi^k - 1)$ is defined by

$$k^e(\Psi^k - 1)x = k^e((\Psi^k x) - x);$$

thus the 1 in $\Psi_k - 1$ means the identity function. The existence of the homomorphism θ^k will be asserted as part of the theorem. The homomorphism ρ^l is as in [4, §5]. The multiplicative homomorphism δ^l is defined by

$$\delta^l(1 + y) = \frac{\Psi^l(1 + y)}{1 + y};$$

cf. [4, §6].

THEOREM (3.2). *Given integers q, k and a sufficiently large integer e* (viz. $e \geqq e_0(q, k)$) *there exists a function*

$$\theta^k = \theta^k(q, e) : \tilde{K}_{SO}(X) \longrightarrow 1 + \tilde{K}_{SO}(X)$$

defined for CW-complexes of dimension $<q$ and having the following properties.

(1) *θ^k is homomorphic from addition to multiplication, that is, exponential.*

(2) *θ^k is natural for maps of X.*

(3) *The image of $\theta^k(x)$ in $1 + \tilde{K}_{SO}(X) \otimes Q_k$ is $\rho^k(k^e x)$.*

(4) *Diagram* 3.1 *is commutative.*

We will now give a heuristic plausibility argument for Theorem (3.2). For this purpose we will abandon the real K-theory and work instead with the complex K-theory, for simplicity. We now argue that in some suitable formal setting we may hope to have

$$(\rho^l\Psi^k) \otimes \rho^k = \rho^{kl} = (\Psi^l\rho^k) \otimes \rho^l \tag{3.3}$$

In fact, all three expressions are exponential; therefore it is presumably sufficient to check the result for a complex line bundle ξ. Here we have

$$\Psi^k\xi = \xi^k, \qquad \rho^k\xi = \frac{\xi^k - 1}{\xi - 1}.$$

Therefore

$$(\rho^l\Psi^k\xi) \otimes (\rho^k\xi) = \frac{\xi^{kl} - 1}{\xi^k - 1} \cdot \frac{\xi^k - 1}{\xi - 1},$$

$$\rho^{kl}\xi = \frac{\xi^{kl} - 1}{\xi - 1},$$

$$(\Psi^l\rho^k\xi) \otimes (\rho^l\xi) = \frac{\xi^{kl} - 1}{\xi^l - 1} \cdot \frac{\xi^l - 1}{\xi - 1}.$$

(The last line uses the fact that Ψ^l preserves both addition and multiplication.) The three results are equal. We may therefore agree to suspend disbelief in (3.3). Rewriting this equation, we obtain

$$\rho^l(\Psi^k - 1) = \frac{\Psi^l\rho^k}{\rho^k}.$$

That is, if $x \in \tilde{K}_C(X)$, then $1 + y = \rho^k x$ is a formal solution of the equation

$$\rho^l(\Psi^k - 1)x = \frac{\Psi^l(1 + y)}{1 + y}.$$

Now, this formal solution involves denominators, that is, coefficients in Q_k. However, we can remove these denominators, up to dimension q, by considering

$$1 + z = (1 + y)^{k^e},$$

where e is suitably large. Raising the equation

$$\rho^l(\Psi^k - 1)x = \frac{\Psi^l(1 + y)}{1 + y}$$

to the power k^e, we obtain

$$\rho^l k^e(\Psi^k - 1)x = \frac{\Psi^l(1 + z)}{1 + z}.$$

This completes the plausibility argument.

We turn now to a rigorous version of this argument for the real case. The "suitable formal setting" for equations such as (3.3) will be the completion of a representation ring. Formulae such as

$$\frac{\xi^{kl} - 1}{\xi - 1}$$

will appear when we calculate characters.

In order to state the first lemma, let

$$\alpha_n, \beta_n \in \mathrm{Comp}(K'_\Lambda(SO(2n)) \otimes S)$$

be two exponential sequences (see [4, §5], just before Lemma (5.9)). We assume that $S \subset$ Q.

LEMMA (3.4). *If* $\chi(\alpha_1) = \chi(\beta_1)$, *then* $\alpha_n = \beta_n$ *for all* n.

Proof. Suppose that

$$\chi(\alpha_1) = \chi(\beta_1) = \phi(\zeta_1),$$

where $\phi(\zeta_1)$ is a formal power series in ζ_1. Then by the exponential law,

$$\chi(\alpha_n) = \phi(\zeta_1)\phi(\zeta_2) \ldots \phi(\zeta_n) = \chi(\beta_n).$$

The result now follows from Corollary (2.9).

We will now apply this lemma. First we observe that (with an obvious notation)

$$K'_{SO}(SO(n)) = K'_R(SO(n)).$$

In fact, any representation $\gamma : SO(n) \longrightarrow O(m)$ must map into $SO(m)$, since $SO(n)$ is connected. In particular,

$$\Psi^k - 1 \in \tilde{K}'_{SO}(SO(2n)).$$

(Here, we emphasise, 1 means the identity map of $SO(2n)$.) Thus $\rho^l \cdot (\Psi^k - 1)$ is defined, and lies in

$$\mathrm{Comp}(K'_R(SO(2n)) \otimes Q_l).$$

Again (see [4, §5]) $\Psi^l \cdot \rho^k$ is defined, and lies in

$$\mathrm{Comp}(K_R'(SO(2n)) \otimes Q_k).$$

Since ρ^k is invertible in this ring, we can define

$$\delta^l \cdot \rho^k = \frac{\Psi^l \cdot \rho^k}{\rho^k}.$$

LEMMA (3.5). *Consider the sequences of elements*

$$\alpha_n = \rho^l \cdot (\Psi^k - 1)$$
$$\beta_n = \delta^l \cdot \rho^k$$

in $\mathrm{Comp}(K_R'(SO)(2n)) \otimes Q_{kl})$. *These sequences are exponential, and satisfy* $\chi(\alpha_1) = \chi(\beta_1)$.

Proof. It is easy to check that the sequences are exponential. In fact, α_n is exponential because $(\Psi^k - 1)$ is additive and ρ^l is exponential; β_n is exponential because ρ^k is exponential and δ^l is multiplicative.

It remains to check that $\chi(\alpha_1) = \chi(\beta_1)$. The virtual representation Ψ^k of $SO(2)$ is the representation $z \longrightarrow z^k$. By definition, $\rho^l \cdot (\Psi^k - 1)$ means the element $(\rho^l \cdot \Psi^k)/\rho^l$ of $\mathrm{Comp}(K_R'(SO(2)) \otimes Q_l)$. Its character is given by

$$\chi(\alpha_1) = \frac{(z_1)^{\frac{1}{2}kl} - (z_1)^{-\frac{1}{2}kl}}{(z_1)^{\frac{1}{2}k} - (z_1)^{-\frac{1}{2}k}} \cdot \frac{(z_1)^{\frac{1}{2}} - (z_1)^{-\frac{1}{2}}}{(z_1)^{\frac{1}{2}l} - (z_1)^{-\frac{1}{2}l}}.$$

Of course, this expression is interpreted as a formal power-series in ζ_1, where $z_1 = 1 + \zeta_1$; see [4, §5].

For any virtual representation θ of G, the character of $\Psi^l \cdot \theta$ is given by

$$\chi(\Psi^l \cdot \theta)g = \chi(\theta)g^l$$

[1, Theorem (4.1) (vi).] Evidently this equation remains true when θ is replaced by an element of $K_\Lambda'(G) \otimes S$ or $\mathrm{Comp}(K_\Lambda'(G) \otimes S)$. Therefore the character of $\delta^l \cdot \rho^k = (\Psi^l \cdot \rho^k)/\rho^k$ is given by

$$\chi(\beta_1) = \frac{(z_1)^{\frac{1}{2}kl} - (z_1)^{-\frac{1}{2}kl}}{(z_1)^{\frac{1}{2}l} - (z_1)^{-\frac{1}{2}l}} \cdot \frac{(z_1)^{\frac{1}{2}} - (z_1)^{-\frac{1}{2}}}{(z_1)^{\frac{1}{2}k} - (z_1)^{-\frac{1}{2}k}}.$$

Of course, this expression also is interpreted as a formal power-series in ζ_1. We have $\chi(\alpha_1) = \chi(\beta_1)$. This completes the proof of Lemma (3.5).

PROPOSITION (3.6). *In* $\mathrm{Comp}(K_R'(SO(2n)) \otimes Q_{kl})$ *we have*

$$\rho^l \cdot (\Psi^k - 1) = \delta^l \cdot \rho^k$$

and

$$\rho^l \cdot (\Psi^k - 1) = \frac{\Psi^l(k^{-n}\rho^k)}{k^{-n}\rho^k}.$$

Proof. The first assertion follows immediately from Lemmas (3.4, 3.5). The second follows by rewriting the first.

The element $k^{-n}\rho^k$ lies in $\mathrm{Comp}(K_R'(SO(2n)) \otimes Q_k)$, and has virtual dimension 1.

LEMMA (3.7). *Given integers n, k, r and sufficiently large e (viz. $e \geqq e_0(n, k, r)$ we have*

$$(k^{-n}\rho^k)^{k^e} = 1 + x \quad \text{in} \quad (K'_R(SO(2n)) \otimes Q_k)/I^r,$$

where $x \in \tilde{K}'_R(SO(2n))$.

Proof. It is clear that if the conclusion holds for one value of e, then it holds for all larger values of e. We now proceed by induction over r. Suppose that we have found e such that

$$(k^{-n}\rho^k)^{k^e} = 1 + x \quad \text{in} \quad (K'_R(SO(2n)) \otimes Q_k)/I^r,$$

where $x \in \tilde{K}'_R(SO(2n))$. (The induction starts with $r = 1$.) Then in

$$(K'_R(SO(2n)) \otimes Q_k)/I^{2r}$$

we can write

$$(k^{-n}\rho^k)^{k^e} = 1 + x + k^{-f}y,$$

where

$$y \in K'_R(SO(2n)) \cap I^r.$$

(Here we regard $K'_R(SO(2n))$ as embedded in $K'_R(SO(2n)) \otimes Q_k$.) Now we have

$$\begin{aligned}(k^{-n}\rho^k)^{k^{e+f}} &= (1 + x + k^{-f}y)^{k^f} \\ &= (1 + x)^{k^f} + y(1 + x)^{k^f - 1} \bmod I^{2r} \\ &= 1 + z \bmod I^{2r},\end{aligned}$$

where $z \in \tilde{K}'_R(SO(2n))$. This completes the induction.

Proof of Theorem (3.2). In what follows, X will always be a CW-complex of dimension $<q$. We can thus determine $n = n(q)$ so that there is a (1–1) correspondence between homotopy classes of maps $f: X \longrightarrow BSO$ and homotopy classes of maps $f: X \longrightarrow BSO(2n)$. That is, there is a $(1 - 1)$ correspondence between the isomorphism classes of $SO(2n)$-bundles ξ over X and the elements of $\tilde{K}_{SO}(X)$; the correspondence is given by $\xi \longrightarrow \{\xi\} - 2n$.

We will now invoke Lemma (3.7), and for this purpose we define an integer r depending only on q and n. (Thus r depends ultimately only on q.) With the notation of Corollary (2.10), we set

$$\begin{aligned} r_1 &= r(SO(2n), R, q) \\ r_2 &= r(SO(2n) \times SO(2n), R, q) \\ r &= \mathrm{Max}(q, r_1, r_2). \end{aligned}$$

We now employ Lemma (3.7) to choose elements

$$\begin{aligned} (\theta^k)_n &\in K'_R(SO(2n)) \\ (\theta^k)_{2n} &\in K'_R(SO(4n)) \end{aligned}$$

such that the images of $(\theta^k)_n$, $(\theta^k)_{2n}$ in

$$\begin{aligned} &(K'_R(SO(2n)) \otimes Q_k)/I^r, \\ &(K'_R(SO(4n)) \otimes Q_k)/I^r \end{aligned}$$

are

$$(k^{-n}\rho^k)^{k^e}, \qquad (k^{-2n}\rho^k)^{k^e}.$$

We can do this for all sufficiently large e (viz. for $e \geqq e_0(q, k)$).

We shall use the element $(\theta^k)_n$ to define the function

$$\theta^k : \tilde{K}_{SO}(X) \longrightarrow 1 + \tilde{K}_{SO}(X).$$

More precisely, we define

$$\theta^k(\{\xi\} - 2n) = (\theta^k)_n \cdot \{\xi\},$$

where ξ runs over the $SO(2n)$-bundles. It is clear that this does define a function, which is natural for maps of X. Since $r \geqq q$, it also follows that the image of $\theta^k(x)$ in $1 + \tilde{K}_{SO}(X) \otimes Q_k$ is $\rho^k(k^e x)$.

We shall use the element $(\theta^k)_{2n}$ to prove that the function θ^k is exponential. More precisely, let π, ϖ to be the projections of $SO(2n) \times SO(2n)$ onto its first and second factors. Then in

$$K'_R(SO(2n) \times SO(2n)) \otimes Q_k)/I^r$$

we have the following equations.

$$(\theta^k)_{2n} \cdot (\pi \oplus \varpi) = (k^{-2n} \rho^k)^{k^e} \cdot (\pi \oplus \varpi),$$

$$(k^{-2n} \rho^k)^{ke} \cdot (\pi \oplus \varpi) = (k^{-n} \rho^k)^{k^e} \cdot \pi \otimes (k^{-n} \rho^k)^{k^e} \cdot \varpi,$$

$$(k^{-n} \rho^k)^{k^e} \cdot \pi \otimes (k^{-n} \rho^k)^{k^e} \cdot \varpi = (\theta^k)_n \cdot \pi \otimes (\theta^k)_n \cdot \varpi.$$

We may now apply Corollary (2.10) with $G = SO(2n) \times SO(2n)$, $S = Z$, $T = Q_k$. The element

$$(\theta^k)_{2n} \cdot (\pi \oplus \varpi) - (\theta^k)_n \cdot \pi \otimes (\theta^k)_n \cdot \varpi$$

lies in

$$[K'_R(SO(2n) \times SO(2n)) \otimes S] \cap (I_T)^r.$$

By our choice of r, Corollary (2.10) applies and shows that this element lies in $(I_S)^q$. Therefore this element will annihilate any $SO(2n) \times SO(2n)$-bundle over X; for since X has dimension $<q$, we have $(\tilde{K}_R(X))^q = 0$. Let ξ, η be $SO(2n)$-bundles over X; we can apply the preceding remark to $\xi \times \eta$; we obtain the equation

$$(\theta^k)_{2n} \cdot (\xi \oplus \eta) = (\theta^k)_n \cdot \pi \otimes (\theta^k)_n \cdot \varpi$$

in $K_R(X)$. From this it follows that the function

$$\theta^k : \tilde{K}_{SO}(X) \longrightarrow 1 + \tilde{K}_{SO}(X)$$

is exponential.

We argue similarly to show that Diagram (3.1) is commutative. Proposition (3.6) states that

$$\rho^l \cdot (\Psi^k - 1) = \frac{\Psi^l \cdot (k^{-n} \rho^k)}{k^{-n} \rho^k}$$

in

$$\mathrm{Comp}(K'_R(SO(2n)) \otimes Q_{kl}).$$

Raising this equation to the power k^e, we have

$$\rho^l \cdot k^e(\Psi^k - 1) = \frac{\Psi^l (k^{-n} \rho^k)^{k^e}}{(k^{-n} \rho^k)^{k^e}}.$$

Thus

$$\rho^l \cdot k^e(\Psi^k - 1) = \frac{\Psi^l \cdot (\theta^k)_n}{(\theta^k)_n}$$

in

$$[K'_R(SO(2n)) \otimes Q_{kl}]/I^r.$$

We will now apply Corollary (2.10) with $G = SO(2n)$, $S = Q_l$, $T = Q_{kl}$. Let us take a representative

$$y \in K'_R(SO(2n)) \otimes Q_l$$

for the element

$$\rho^l \cdot k^e(\Psi^k - 1)$$

in

$$[K'_R(SO(2n)) \otimes Q_l]/(I_S)^r.$$

Then we have

$$y \in K'_R(SO(2n)) \otimes S,$$

$$\frac{\Psi^l \cdot (\theta^k)_n}{(\theta^k)_n} \in K'_R(SO(2n)),$$

$$\text{and} \quad y - \frac{\Psi^l \cdot (\theta^k)_n}{(\theta^k)_n} \in [K'_R(SO(2n)) \otimes S] \cap (I_T)^r.$$

By our choice of r, Corollary (2.10) applies and shows that this element lies in $(I_S)^q$. That is, we have

$$\rho^l \cdot k^e(\Psi^k - 1) = \frac{\Psi^l \cdot (\theta^k)_n}{(\theta^k)_n}$$

in

$$[K'_R(SO(2n)) \otimes Q_l]/(I_S)^q.$$

Arguing as above, it follows that Diagram (3.1) is commutative. This completes the proof of Theorem (3.2).

§4. PROOF OF THEOREM (1.1)

In proving Theorem (1.1), we shall have to work with square diagrams like Diagram (3.1). By a "square" S, we shall mean a commutative diagram of groups and homomorphisms which has the following form.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ C & \xrightarrow{i} & D \end{array}$$

We shall call a square "special" if it has the following property: given $b \in B$ and $c \in C$ such that $hb = ic$, there exists $a \in A$ such that $fa = b$ and $ga = c$. This is equivalent to demanding the exactness of the following sequence:

$$A \xrightarrow{(f,g)} B \oplus C \xrightarrow{(h,-i)} D.$$

By a "short exact sequence

$$0 \longrightarrow S' \longrightarrow S \longrightarrow S'' \longrightarrow 0"$$

of squares, we shall mean a commutative diagram composed of three squares S', S, S'' and four short exact sequences

$$\begin{gathered} 0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0 \\ 0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0, \text{ etc.} \end{gathered}$$

LEMMA (4.1). *Suppose that*

$$0 \longrightarrow S' \longrightarrow S \longrightarrow S'' \longrightarrow 0$$

is a short exact sequence of squares in which S' and S'' are special; then S is special.

Proof. Each square determines a sequence

$$A \xrightarrow{(f,g)} B \oplus C \xrightarrow{(h,-i)} D$$

which we may regard as a chain complex. We now have a short exact sequence of chain complexes. This yields an exact homology sequence, which leads immediately to the required result.

Alternatively, one may give a direct proof by routine diagram-chasing.

Let X be a finite CW-complex, say of dimension $<q$. The main part of the proof of Theorem (1.1) will proceed by filtering X. Let us define F_r to be the image of $\tilde{K}_R(X/X^{r-1})$ in $K_R(X)$; then for $r \geqq 2$ and sufficiently large e, Theorem (3.2) provides us with the following commutative square.

$$\begin{array}{ccc} F_r & \xrightarrow{k^e(\Psi^k - 1)} & F_r \\ \Big\downarrow{\scriptstyle \theta^k} & & \Big\downarrow{\scriptstyle \rho^l} \\ 1 + F_r & \xrightarrow{\delta^l} & 1 + F_r \otimes Q_l \end{array}$$

(Since Q_l is torsion-free, $\otimes Q_l$ is an exact functor, and the image of $\tilde{K}_R(X/X^{r-1}) \otimes Q_l$ in $\tilde{K}_R(X) \otimes Q_l$ is $F_r \otimes Q_l$.)

If we pass to a (restricted) direct sum over k and an (unrestricted) direct product over l, we obtain the following commutative square S_r (for $r \geqq 2$).

(4.2)
$$\begin{array}{ccc} \sum_k F_r & \xrightarrow{\sum_k k^{e(k)}(\Psi^k - 1)} & F_r \\ \Big\downarrow{\scriptstyle \sum_k \theta^k} & & \Big\downarrow{\scriptstyle \prod_l \rho^l} \\ 1 + F_r & \xrightarrow{\prod_l \delta^l} & \prod_l (1 + F_r \otimes Q_l) \end{array}$$

More precisely, we obtain this commutative square whenever the function $e(k)$ is sufficiently large, viz. for $e(k) \geqq e_0(q, k)$, where q is our fixed upper bound for the dimension of the complexes X considered.

THEOREM (4.3). *The square S_r displayed in Diagram* (4.2) *is special.*

Proof. We shall prove this result by downwards induction over r. For $r = q$ the result is trivial. Let us therefore assume as an inductive hypothesis that the square S_{r+1} is special, and prove that the square S_r is special. For this purpose, by Lemma (4.1), it is sufficient to construct a short exact sequence of squares

$$0 \longrightarrow S_{r+1} \longrightarrow S_r \longrightarrow S_r/S_{r+1} \longrightarrow 0$$

and prove that the square S_r/S_{r+1} is special.

Since the square S_{r+1} is embedded in S_r, it is clear that the required quotient square exists. In order to establish its structure, let us recall that $F_r \cdot F_s \subset F_{r+s}$; thus F_{r+1} is an ideal in F_r, and F_r/F_{r+1} is a ring in which the product is zero (assuming $r \geqq 1$). Besides the short exact sequence of additive groups

$$0 \longrightarrow F_{r+1} \longrightarrow F_r \longrightarrow F_r/F_{r+1} \longrightarrow 0,$$

we have also a short exact sequence of multiplicative groups:

$$1 \longrightarrow 1 + F_{r+1} \longrightarrow 1 + F_r \longrightarrow 1 + F_r/F_{r+1} \longrightarrow 1.$$

Here the product in the last group is given by

$$(1 + a)(1 + b) = 1 + (a + b).$$

Similar remarks hold for the following short exact sequence of multiplicative groups:

$$1 \longrightarrow 1 + F_{r+1} \otimes Q_l \longrightarrow 1 + F_r \otimes Q_l \longrightarrow 1 + (F_r/F_{r+1}) \otimes Q_l \longrightarrow 1$$

This shows that the square S_r/S_{r+1} has the following form.

$$\begin{array}{ccc} \sum_k F_r/F_{r+1} & \xrightarrow{\sum_k k^{e(k)}(\Psi^k - 1)} & F_r/F_{r+1} \\ \Big\downarrow{\scriptstyle \sum_k \theta^k \Sigma} & & \Big\downarrow{\scriptstyle \prod_l \rho^l} \\ 1 + (F_r/F_{r+1}) & \xrightarrow{\prod_l \delta^l} & \prod_l (1 + (F_r/F_{r+1}) \otimes Q_l) \end{array} \qquad (4.4)$$

The maps of the square S_r/S_{r+1} are induced by those of the square S_r.

Theorem (4.3) will thus be proved once we have established the following lemma.

LEMMA (4.5). *The square displayed in Diagram* (4.4) *is special.*

Proof. The quotient group F_r/F_{r+1} of Diagram (4.4) is isomorphic to a quotient group of a subgroup of

$$\tilde{K}_R(X^r/X^{r-1}) = \tilde{K}_R(\vee S^r).$$

(Compare [4, proof of Theorem (3.11)].) Since this isomorphism is given by induced maps, the operations Ψ^k, ρ^k and θ^k on F_r/F_{r+1} are given by the same formulae that hold in S^r:

$$\Psi^k y = a(k, r)y$$
$$\rho^k y = 1 + b(k, r)y$$
$$\theta^k y = 1 + c(k, r, e)y.$$

Here the coefficients a, b and c do not depend on X or y. To give the coefficients, we have to divide the cases. Unless $r \equiv 0, 1, 2$ or $4 \bmod 8$ the group F_r/F_{r+1} is zero and there is nothing to prove. Consider first the case $r \equiv 0$ or $4 \bmod 8$; say $r = 4t$, $t > 0$. In this case we have

$$\Psi^k y = k^{2t} y$$
$$\rho^k y = 1 + \tfrac{1}{2}\alpha_{2t}(k^{2t} - 1)y$$
$$\theta^k y = 1 + \tfrac{1}{2}\alpha_{2t} k^{e(k)}(k^{2t} - 1)y.$$

Here the first result is quoted from [1, Corollary (5.2)], the second is quoted from [4, Theorem (5.18)] and the third is deduced using Theorem (3.2), part (3).

We can now check that the square (4.4) is special for $r = 4t$. Suppose given $u, v \in F_r/F_{r+1}$ such that

$$\rho^l u = \frac{\Psi^l(1 + v)}{1 + v} \qquad \text{for all } l.$$

This yields

$$\tfrac{1}{2}\alpha_{2t}(l^{2t} - 1)u = (l^{2t} - 1)v$$

in $(F_r/F_{r+1}) \otimes Q_l$ for all l. This means that for some exponent $f(l)$ we have

$$\tfrac{1}{2}\alpha_{2t} l^{f(l)}(l^{2t} - 1)u = l^{f(l)}(l^{2t} - 1)v$$

in F_r/F_{r+1} (for each l). We may write $\tfrac{1}{2}\alpha_{2t} = n(2t)/m(2t)$ where $n(2t)$, $m(2t)$ are coprime and $m(2t)$ is as in [4, §2]. According to [4, Theorem [2.7)], by taking a suitable linear combination of the equations we have just obtained, we find

$$n(2t)\, u = m(2t)\, v.$$

Since the numbers $n(2t)$, $m(2t)$ are coprime we can choose integers a, b so that

$$am(2t) + bn(2t) = 1;$$

now define

$$w = au + bv \in F_r/F_{r+1};$$

this ensures that

$$m(t)w = u, \qquad n(t)w = v.$$

Using [4, Theorem (2.7)] again, let us choose integers $c(k)$, zero except for a finite number of k, such that

$$\sum_k c(k)k^{e(k)}(k^{2t} - 1) = m(2t).$$

Let σ be the element of $\sum F_r/F_{r+1}$ with components $c(k)w$. Then

$$\begin{aligned}\sum_k k^{e(k)}(\Psi^k - 1)\sigma &= \sum_k c(k)k^{e(k)}(k^{2t} - 1)w \\ &= m(2t)w \\ &= u.\end{aligned}$$

Again,

$$\begin{aligned}\Big(\sum_k \theta^k\Big)\sigma &= 1 + \sum_k \tfrac{1}{2}\alpha_{2t}c(k)k^{e(k)}(k^{2t} - 1)w \\ &= 1 + n(t)w \\ &= 1 + v.\end{aligned}$$

Therefore the square (4.4) is special if $r \equiv 0$ or $4 \bmod 8$.

We turn now to the case $r \equiv 1$ or $2 \bmod 8$, $r \geqq 2$. In this case every element of F_r/F_{r+1} has order 2, so that

$$(F_r/F_{r+1}) \otimes Q_l \cong \begin{cases} F_r/F_{r+1} & (l \text{ odd}) \\ 0 & (l \text{ even}). \end{cases}$$

We have:

$$\Psi^k y = \begin{cases} 0 & (k \text{ even}) \\ y & (k \text{ odd}) \end{cases}$$

$$\left.\begin{aligned} \rho^k y &= 1 + y \\ \theta^k y &= 1 + y \end{aligned}\right\} \quad (k \equiv \pm 3 \bmod 8).$$

Here the second result is quoted from [4, Theorem (5.17)], and the third is deduced using Theorem (3.2), part (3). The first result is deduced from the corresponding result for RP^n[1, §7] by naturality, according to the pattern of [4, 3.5, 5.17, 6.4].

It follows from this that the map $\sum_k k^{e(k)}(\Psi^k - 1)$ of Diagram (4.4) is zero, at least for $e(k) \geqq 1$. The map $\prod_l \delta^l$ is also trivial (with image 1). The map $\sum_k \theta^k$ is epimorphic. The map $\prod_l \rho^l$ is monomorphic. It follows immediately that the square (4.4) is special.

This completes the proof of Lemma (4.5), and (therefore) of Theorem (4.3).

COROLLARY (4.6). *The following square is special (for sufficiently large functions $e(k)$.)*

(4.7)
$$\begin{array}{ccc} \sum_k \tilde{K}_{SO}(X) & \xrightarrow{\sum_k k^{e(k)}(\Psi^k - 1)} & \tilde{K}_{SO}(X) \\ \Big\downarrow{\scriptstyle \sum_k \theta^k} & & \Big\downarrow{\scriptstyle \prod_l \rho^l} \\ 1 + \tilde{K}_{SO}(X) & \xrightarrow{\prod_l \delta^l} & \prod_l 1 + \tilde{K}_{SO}(X) \otimes Q_l \end{array}$$

This follows immediately from Theorem (4.3), by setting $r = 2$.

Proof of Theorem (1.1). In proving Theorem (1.1), we may assume without loss of generality that X is connected. With the notation of §1, we require to prove that $V(X) = W(X)$.

Here we have defined

$$W(X) = \bigcap_e W(e, X),$$

where $W(e, X)$ is the subgroup of $\tilde{K}_R(X)$ generated by elements of the form

$$k^{e(k)}(\Psi^k - 1)x$$

as k runs over all integers and x runs over $K_R(X)$. I claim that for $e(k)$ sufficiently large (viz. for $e(k) \geqq e_0(q)$) we can obtain the same subgroup $W(e, X)$ by letting x run over $\tilde{K}_{SO}(X)$. In fact, any element $x \in K_R(X)$ can be written as $x = y + z$, where $y \in \tilde{K}_{SO}(X)$ and z is a linear combination of real line bundles. A real line bundle ζ can be induced by a map $f\colon X \longrightarrow RP_q$; from this we see that

$$2^{e_0(q)}(\Psi^k - 1)\zeta = 0 \qquad (k \text{ even})$$

$$(\Psi^k - 1)\zeta = 0 \qquad (k \text{ odd})$$

Thus

$$k^{e(k)}(\Psi^k - 1)z = 0$$

if $e(k) \geqq e_0(q)$, and hence

$$k^{e(k)}(\Psi^k - 1)x = k^{e(k)}(\Psi^k - 1)y.$$

We have thus shown that for each sufficiently large function $e(k)$ the group $W(e; X)$ is the image of the map $\sum_k k^{e(k)}(\Psi^k - 1)$ appearing in Diagram (4.7).

Again, we have

$$V(X) = \left(\prod_l \rho^l\right)^{-1}\left(\prod_l \delta^l\right)(1 + \tilde{K}_R(X));$$

see [4, §6]. I claim that we can obtain the same subgroup by taking

$$\left(\prod_l \rho^l\right)^{-1}\left(\prod_l \delta^l\right)(1 + \tilde{K}_{SO}(X)).$$

In fact, any element $1 + x \in 1 + \tilde{K}_R(X)$ can be written in the form $(1 + y)\zeta$, where $1 + y \in 1 + \tilde{K}_{SO}(X)$ and ζ is a real line bundle over X. By dividing the cases "l even" and "l odd", as above, we see that

$$\frac{\Psi^l\zeta}{\zeta} = 1 \qquad \text{in } 1 + \tilde{K}_R(X) \otimes Q_l$$

for all l. Hence

$$\left(\prod_l \delta^l\right)(1 + x) = \left(\prod_l \delta^l\right)(1 + y)$$

and

$$\left(\prod_l \delta^l\right)(1 + \tilde{K}_R(X)) = \left(\prod_l \delta^l\right)(1 + \tilde{K}_{SO}(X)).$$

We have thus shown that the group $V(X)$ is the group

$$\left(\prod_l \rho^l\right)^{-1}\left(\prod_l \delta^l\right)(1 + \tilde{K}_{SO}(X))$$

of Diagram (4.7).

The fact that Diagram (4.7) is commutative now shows that $W(e, X) \subset V(X)$ for sufficiently large functions $e(k)$. The fact that Diagram (4.7) is special shows that

$V(X) \subset W(e, X)$ for sufficiently large functions $e(k)$. Thus for sufficiently large functions $e(k)$ we have $W(e, X) = V(X)$; hence

$$W(X) = \bigcap_e W(e, X) = V(X).$$

This completes the proof of Theorem (1.1).

§5. A PERIODICITY THEOREM FOR THE OPERATIONS Ψ^k

In this section we shall prove Theorem (5.1), which is needed in §6 and in Part IV of the present series. Roughly speaking, it will assert that $\Psi^k(x)$ is a periodic function of k. More accurately, it will make this assertion "modulo m".

We shall suppose that X is a CW-complex such that $H_*(X)$ is finitely-generated. We make this assumption because we wish to apply the results to a Thom complex, using the devices explained in [4, §4].

THEOREM (5.1). *If $x \in K_\Lambda(X)$ and $m \in Z$, then the value of $\Psi_\Lambda^k(x)$ in $K_\Lambda(X)/mK_\Lambda(X)$ is periodic in k with period m^e. Here e depends on X and Λ, but is independent of x and m.*

In this theorem, and below, the statement "$f(k)$ is periodic in k with period m^e" means simply "$k_1 \equiv k_2 \bmod m^e$ implies $f(k_1) = f(k_2)$". It is not asserted that m^e is the smallest possible period. In particular, the theorem is true for $m = 0$ in a trivial way.

The proof will require three lemmas.

LEMMA (5.2). *Let s be a fixed positive integer. Then the binomial coefficient*

$$\frac{k(k-1)\dots(k-s+1)}{1\cdot 2\dots s+1},$$

when taken mod m, is periodic in k with period m^s.

This result is of course not "best possible", but it is sufficient for our purposes.

Proof. We proceed by induction over s. The result is certainly true for $s = 1$; assume it true for s. Consider the summation formula

$$\frac{k(k-1)\dots(k-s)}{1\cdot 2\dots s+1} = \sum_{0\leqslant l\leqslant k-1} \frac{l(l-1)\dots(l-s+1)}{1\cdot 2\dots s}.$$

By the inductive hypothesis, the summand

$$\frac{l(l-1)\dots(l-s+1)}{1\cdot 2\dots s+1},$$

when taken mod m, is periodic in l with period m^s. Let the sum over m^s consecutive terms be σ; then the sum over m^{s+1} consecutive terms is $m\sigma$, that is, 0 mod m. Therefore the sum is periodic with period m^{s+1}. This completes the induction, and proves the lemma.

LEMMA (5.3). *Let ξ be a real line bundle over X. Then the value of $\Psi_R^k(\xi)$ in $K_R(X)/mK_R(X)$ is periodic in k with period m.*

Proof. For a real line bundle ξ we have

$$\Psi^k\xi = \begin{cases} 1 & (k \text{ even}) \\ \xi & (k \text{ odd}) \end{cases}.$$

Thus $\Psi^k\xi$ is periodic with period 2. If m is even, this is all that is required. On the other hand, $\xi - 1$ has order 2^t for some t, and so if m is odd $\xi - 1$ is divisible by m; in this case, therefore, the mod m value of $\Psi^k\xi$ is constant. This completes the proof.

LEMMA (5.4). *Let ξ be an $SO(2n)$-bundle (if $\Lambda = R$) or a $U(n)$-bundle (if $\Lambda = C$) over X. Then there exists an integer $e = e(X, \Lambda, n)$ such that the value of $\Psi_\Lambda^k\xi$ in $K_\Lambda(X)/mK_\Lambda(X)$ is periodic in k with period m^e.*

Proof. Since $H_*(X)$ is finitely generated, there exists $q = q(X, \Lambda)$ such that the filtration subgroup F_q of $K_\Lambda(X)$ is zero. We now apply Theorem (2.2), taking $S = Z$ and $G = SO(2n)$ or $G = U(n)$ according to the case. We obtain $r = r(n, \Lambda, q)$ such that

$$\chi^{-1}(J^r) \subset I^q.$$

That is, if a virtual representation θ of G is such that its character $\chi(\theta)$ is small of the r^{th} order at the identity of G, then we shall have $\theta\xi = 0$ in $K_\Lambda(X)$.

We now introduce virtual representations Φ^k of G, for $k \geqq 0$, by the following equation.

$$\Phi_\Lambda^k = \Psi_\Lambda^k - k\Psi_\Lambda^{k-1} + \frac{k(k-1)}{1\cdot 2}\Psi_\Lambda^{k-2} - \cdots + (-1)^k\Psi_\Lambda^0.$$

In the case $G = U(n)$, $\Lambda = C$ the character of Ψ_C^k is $\sum_{1 \leqslant t \leqslant n} (z_t)^k$, and therefore the character of Φ_C^k is

$$\sum_{1 \leqslant t \leqslant n} (z_t - 1)^k = \sum_{1 \leqslant t \leqslant n} (\zeta_t)^k.$$

Thus

$$\Phi_C^k \in \chi^{-1}(J^k).$$

Since $c\Phi_R^k = \Phi_C^k c$, we also have

$$\Phi_R^k \in \chi^{-1}(J^k).$$

We can invert the definition and write

$$\Psi_\Lambda^k = \Phi_\Lambda^0 + k\Phi_\Lambda^1 + \frac{k(k-1)}{1\cdot 2}\Phi_\Lambda^2 + \cdots.$$

Here we only have to take r terms if we wish to work modulo $\chi^{-1}(J^r)$. With this interpretation, the formula is true whether k is positive, negative or zero; in the case $\Lambda = C$ this can be checked at once by taking characters; the case $\Lambda = R$ follows since the Φ's and Ψ's commute with c.

We now have

$$\Psi_\Lambda^k\xi = \sum_{0 \leqslant s \leqslant r-1} \frac{k(k-1)\ldots(k-s+1)}{1\cdot 2\cdot \ldots s}\Phi_\Lambda^s\xi.$$

According to Lemma (5.2), the mod m value of each summand is periodic in k with period m^r. This proves the lemma.

Proof of Theorem (5.1). We recall that since X may be infinite-dimensional, $K_\Lambda(X)$ is defined as an inverse limit. Consider first the case $\Lambda = C$; then an element of this inverse limit may be represented by a map $X \longrightarrow Z \times BU$. Since X is cohomologically finite-dimensional and $BU(n)$ is simply-connected, this map can be compressed into $Z \times BU(n)$ for some $n = n(X)$. That is, any element of $K_C(X)$ can be represented in the form $h + \xi$, where $h \in H^0(X; Z)$ and ξ is a bundle with structural group $U(n)$. Since $\Psi^k(h) = h$, the result now follows from Lemma (5.4).

Similarly, in the case $\Lambda = R$, every element of $K_R(X)$ can be represented in the form $h + \xi + \eta$, where ξ is a real line bundle and η is an $SO(2n)$-bundle. The result now follows from Lemmas (5.3, 5.4). This completes the proof.

§6. CHARACTERISATION OF THE POSSIBLE VALUES $\{\rho^k\}$

Let X be a finite CW-complex, and let $x \in \tilde{K}_{\mathrm{Spin}}(X)$, $y \in \tilde{K}_R(X)$ be two elements. Then the equation

$$v^k = \rho^k(x)\frac{\Psi^k(1+y)}{1+y} \qquad \text{(all } k) \tag{6.1}$$

defines an element

$$\{v^k\} \in \prod_{k \neq 0} (1 + \tilde{K}_R(X) \otimes Q_k).$$

It is the object of this section to characterise the elements which arise in this way.

THEOREM (6.2). *An element*

$$\{v^k\} \in \prod_{k \neq 0} (1 + \tilde{K}_R(X) \otimes Q_k)$$

can be written in the form (6.1) *if and only if the following conditions hold.*

(a) *Let* $i : X^2 \longrightarrow X$ *be the inclusion map of the* 2*-skeleton; then* $i^*v^k \in Q_k$.

(b) $v^{-1} = 1$.

(c) $v^k \cdot \Psi^k v^l = v^{kl}$ in $1 + \tilde{K}_R(X) \otimes Q^{kl}$.

(d) *For each prime power* p^f *there exists* p^g *such that the mod* p^f *value of* v^k *is periodic in* k *with period* p^g.

These conditions call for a few comments. Condition (a) states in effect that v^k "is of filtration 3". For the purposes of this theorem we could equally well have written "$i^*v^k = 1$" instead of "$i^*v^k \in Q^k$"; the condition is written in the form given so that it can be applied to more general sequences v^k. Conditions (a) and (b) are of course fairly trivial. Condition (c) has been stressed by Bott, who calls it the "cocycle condition" [7]. Personally I do not see the point of stressing (c) without (d), since in the presence of torsion both are certainly needed to make any realistic algebraic model of the topological situation.

It remains to explain what we mean by "periodic" in condition (d); for on the face of it, the mod p^f value of v^k lies in a group which is dependent on k. More precisely, let us write K for $K_R(X)$; then the mod p^f value of v^k lies in the ring

$$(K \otimes Q_k)/p^f(K \otimes Q_k).$$

However, we have canonical isomorphisms

$$(K \otimes Q_k)/p^f(K \otimes Q_k) \cong (K/p^f K) \otimes Q_k$$

$$\cong \begin{cases} K/p^f K & (k \not\equiv 0 \bmod p) \\ 0 & (k \equiv 0 \bmod p). \end{cases}$$

This allows us to identify the rings

$$(K \otimes Q_k)/p^f(K \otimes Q_k)$$

$$(K \otimes Q_l)/p^f(K \otimes Q_l)$$

whenever $k \equiv l \bmod p$.

The proof of Theorem (6.2) will require three lemmas.

LEMMA (6.3). *In Theorem* (6.2), *the conditions* (a), (b), (c) *and* (d) *are necessary.*

This is the easy half of Theorem (6.2).

Proof. We note that if two sequences v^k, w^k satisfy these conditions, so does their product $v^k w^k$ and the inverse of v^k. It is now sufficient to check the conditions for $\rho^k x$ when x is a Spin($8n$)-bundle, and for

$$\frac{\Psi^k(1+y)}{1+y}.$$

Let x be a Spin($8n$)-bundle ξ; we check condition (a). Since Spin($8n$) is 1-connected, the bundle $i^*\xi$ over X^2 is trivial; thus

$$i^*\rho^k\xi = \rho^k i^*\xi = k^{4m}.$$

Condition (c) is given by [4, Proposition (5.5)]. Now consider the equation

$$\rho^k\xi = \phi^{-1}\Psi^k\phi 1.$$

This makes condition (b) obvious, since Ψ^{-1} is the identity. Also the mod p^f value of $\Psi^k\phi 1$ is periodic in k with period p^{ef}, by Theorem (5.1) applied in the Thom space X^ξ; thus $\rho^k\xi$ satisfies condition (d).

We turn to the sequence

$$\frac{\Psi^k(1+y)}{1+y}$$

and check condition (a). We note (from the spectral sequence) that $K_R(X^2)$ is generated by real line bundles and elements of filtration 2; for each of these we have $\Psi^k z = z$ so long as k is odd; and for k even we have

$$\tilde{K}_R(X^2) \otimes Q_k = 0,$$

since $\tilde{K}_R(X^2)$ is 2-primary. Thus

$$\frac{\Psi^k(1+y)}{1+y} = 1 \quad \text{in} \quad 1 + \tilde{K}_R(X^2) \otimes Q_k,$$

proving condition (a). It is easy to check that

$$\frac{\Psi^k(1+y)}{1+y}$$

satisfies conditions (b) and (c); condition (d) is given by Theorem (5.1). This completes the proof of Lemma (6.3).

The proof of Theorem (6.2) will be by filtering $K_R(X)$ and using induction over the dimension. To make the inductive step we shall require Lemma (6.4).

We shall assume that Y is a finitely-generated abelian group, on which operations Ψ^k and ρ^k are defined by the same formulae that hold in the particular case $Y = \tilde{K}_R(\vee S^r)$ (compare the proof of Lemma (4.5)). If $r \equiv 1$ or $2 \bmod 8$, we assume that evey element of Y has order 2.

LEMMA (6.4). *Suppose that an element*

$$\{v^k\} \in \prod_{k \neq 0} (1 + Y \otimes Q_k)$$

satisfies conditions (b), (c) *and* (d) *of Theorem* (6.2). *Then it can be written in the form* (6.1) *for some* x, y *in* Y.

Proof. Since Y is a finitely-generated abelian group, we may express it as a direct sum of cyclic groups Z and Z_{p^f}. It is now easy to see that it is sufficient to prove the result for these summands. This reduces the proof to three cases.

We consider first the case $r \equiv 0$ or $4 \bmod 8$ (say $r = 4t$) and $Y = Z$. Let us write

$$v^k = 1 + w^k,$$

where

$$w^k \in Y \otimes Q_k \cong Q_k.$$

In this case condition (c) yields

$$v^k \cdot \Psi^k v^l = v^l \cdot \Psi^l v^k,$$

which gives

$$w^k + k^{2t} w^l = w^l + l^{2t} w^k,$$

that is,

$$(k^{2t} - 1)w^l = (l^{2t} - 1)w^k.$$

Therefore there is a rational number c such that

$$w^k = (k^{2t} - 1)c.$$

Now there exists $f(k)$ such that $k^{f(k)} w^k$ is integral; and by [4, Theorem (2.7)], the highest common factor of the expressions $k^{f(k)}(k^{2t} - 1)$ divides $m(2t)$; therefore $m(2t)c$ is integral. That is, we may write

$$w^k = \frac{(k^{2t} - 1)d}{m(2t)}$$

where d is integral.

Now let us write $\frac{1}{2}\alpha_{2t} = n(2t)/m(2t)$, where $n(2t)$ and $m(2t)$ are coprime. (See [4, §2]; compare the proof of Lemma (4.5)). We may now set

$$am(2t) + bn(2t) = 1.$$

Setting $x = bd$, $y = ad$ we easily calculate that

$$\rho^k(x) \frac{\Psi^k(1 + y)}{1 + y} = v^k.$$

This completes this case.

Secondly, we consider the case $r \equiv 0$ or $4 \bmod 8$ (say $r = 4t$), and $Y = Z_{p^f}$. In this case

$$Y \otimes Q_k \cong \begin{cases} Y & \text{if } k \not\equiv 0 \bmod p \\ 0 & \text{if } k \equiv 0 \bmod p. \end{cases}$$

We may therefore restrict attention entirely to those values of k prime to p.

When we apply condition (d), we shall of course take p^f to be the order of Y. Thus condition (d) asserts that the actual value of v^k is periodic with period p^g. According to Lemma (6.3) (for the space $X = S^r$), we can suppose g chosen so large that if x is a generator of $\tilde{K}_R(S^r)$, then the mod p^f value of $\rho^k(x)$ is periodic in k, with period p^g, and similarly for

$$\frac{\Psi^k(1+y)}{1+y}.$$

(Of course this is also easy to check using the explicit formulae for ρ^k and Ψ^k in S^r.) Therefore the same periodicity statement will be true for every expression

$$\rho^k(x)\frac{\Psi^k(1+y)}{1+y}$$

with x, y in Y. In Y, of course, the periodicity statement asserts that the actual value of this expression is periodic with period p^g.

We now fix attention on a particular value of k. If p is odd, we choose k to be a generator for the multiplicative group G_{p^g} of residue classes prime to p modulo p^g. If $p = 2$, we choose k to be a generator of the quotient group $G_{2^g}/\{\pm 1\}$, which is again cyclic.

By [4, Lemma (2.12)], when the fraction

$$\frac{k^{2t}-1}{m(2t)}$$

is written in its lowest terms, both numerator and denominator are prime to p. Therefore we can solve the equation

$$v^k = 1 + \frac{k^{2t}-1}{m(2t)}z$$

for a solution $z \in Y \otimes Q_k \cong Y$. As before, we set $\frac{1}{2}\alpha_{2t} = n(2t)/m(2t)$, $am(2t) + bn(2t) = 1$, $x = bz$ and $y = az$; we easily calculate that

$$\rho^k(x)\frac{\Psi^k(1+y)}{1+y} = v^k$$

for this particular value of k.

Thus the two expressions

$$\rho^l(x)\frac{\Psi^l(1+y)}{1+y}, \qquad v^l$$

agree for $l = k$. But both expressions satisfy condition (c), which allows us to calculate v^{k^r} in terms of v^k; thus we see that the two expressions agree for $l = k^r$. In fact, using condition (b) also, we see that they agree for $l = \pm k^r$. But by the choice of k, the integers $\pm k^r$ give all residue classes prime to p modulo p^g; hence by periodicity, the two expressions agree for all l prime to p. This completes this case.

Finally, we consider the case $r \equiv 1$ or $2 \bmod 8$, $Y = Z_2$. In this case

$$Y \otimes Q_k \cong \begin{cases} Z_2 & \text{if k is odd} \\ 0 & \text{if } k \text{ is even.} \end{cases}$$

We may therefore restrict attention entirely to odd values of k. By condition (d), the value of v^k is periodic in k with period 2^g.

If k and l are odd, condition (c) gives

$$v^k \cdot v^l = v^{kl}.$$

Therefore the function v^k of k gives a homomorphism from the multiplicative group of odd residue classes mod 2^g to the multiplicative group $1 + Y \cong Z_2$. By condition (b), this homomorphism factors through $G_{2^g}/\{\pm 1\}$. But this group is cyclic; so there are only two possible homomorphisms. We must have

$$v^k = \begin{cases} 1 & \text{for } k \equiv \pm 1 \bmod 8 \\ 1 + x & \text{for } k \equiv \pm 3 \bmod 8 \end{cases}$$

where x is one of the two elements in Y. According to [4, Theorem (5.17)], this shows that

$$v^k = \rho^k x.$$

This completes the proof of Lemma (6.4).

We need one more lemma. As in §4, we define F_r to be the image of $\tilde{K}_R(X/X^{r-1})$ in $K = K_R(X)$.

LEMMA (6.5). *Suppose given a sequence*

$$\{v^k\} \in \prod_{k \neq 0} (1 + F_r \otimes Q_k)$$

and suppose that for each prime power p^f there exists p^g such that the value of v^k in

$$(K \otimes Q_k)/p^f(K \otimes Q_k)$$

is periodic with period p^g. Then for each prime power p^f there exists p^h such that the value of v^k in

$$1 + (F_r \otimes Q_k)/p^f(F_r \otimes Q_k)$$

is periodic with period p^h.

Proof. Suppose given p^f. Consider the subgroup S_t of elements x in $K_R(X)$ such that

$$p^{ft} x \in F_r.$$

This is an increasing sequence of Z-submodules in the finitely-generated Z-module $\tilde{K}_R(X)$, therefore convergent. That is, there is a t such that $x \in \tilde{K}_R(X)$ and $p^{f(t+1)} x \in F_r$ imply $p^{ft} x \in F_r$. According to the data, there is now an h such that the value of v^k in $K \otimes Q_k/p^{f(t+1)}(K \otimes Q_k)$ is periodic in k with period p^h. Suppose then that $k \equiv l \bmod p^h$ and k, l are prime to p. We have

$$v^k = 1 + k^{-a} w^k$$

$$v^l = 1 + l^{-b} w^l$$

for some w^k, $w^l \in F_r$. The periodicity statement gives

$$l^b w^k - k^a w^l = p^{f(t+1)} x$$

for some $x \in \tilde{K}_R(X)$ with $p^{f(t+1)} x \in F_r$. By our choice of t, this gives $p^{ft} x \in F_r$; that is,

$$l^b w^k - k^a w^l = p^f y$$

with $y \in F_r$. Since k and l are prime to p, this gives

$$k^{-a} w^k = l^{-b} w^l$$

in $F_r/p^f F_r$. This proves the lemma.

Proof of Theorem (6.2). We will prove by downwards induction over r that if an element

$$\{v^k\} \in \prod_{k \neq 0} (1 + F_r \otimes Q_k)$$

satisfies conditions (b), (c) and (d) of Theorem (6.2), then it can be written in the form (6.1), provided $r \geqq 3$. Here it does not matter whether the periodicity condition (d) is interpreted as an equation in $K_R(X)/p^f K_R(X)$ or as an equation in $F_r/p^f F_r$, according to Lemma (6.5). The inductive hypothesis is trivial for r greater than the dimension of X; we assume it true for $r + 1$, where $r \geqq 3$. Let

$$\{v^k\} \in \prod_k (1 + F_r \otimes Q_k)$$

be an element satisfying conditions (b), (c) and (d) of Theorem (6.2). Then the image of $\{v^k\}$ in

$$\prod_k (1 + (F_r/F_{r+1}) \otimes Q_k)$$

satisfies the conditions of Lemma (6.4). Therefore there are elements x, y in F_r such that

$$v^k \left[\rho^k(x) \frac{\Psi^k(1+y)}{1+y} \right]^{-1} \in 1 + F_{r+1} \otimes Q_k$$

for all k. (Note that $x \in \tilde{K}_{\text{Spin}}(X)$, since $r \geqq 3$.) By Lemma (6.3) the element

$$\left\{ \rho^k(x) \frac{\Psi^k(1+y)}{1+y} \right\}$$

satisfies the conditions (b), (c) and (d); therefore the inductive hypothesis applies to the sequence

$$v^k \left[\rho^k(x) \frac{\Psi^k(1+y)}{1+y} \right]^{-1},$$

which can accordingly be written

$$\rho^k(x') \frac{\Psi^k(1+y')}{1+y'}.$$

Thus v^k can be written in the form (6.1) (with x replaced by $x + x'$, and similarly for y). This completes the induction. Therefore the inductive hypothesis is true for $r = 3$; this proves that the conditions (a), (b), (c) and (d) are sufficient. The proof of Theorem (6.2) is complete.

§7. POSSIBLE GENERALISATIONS

The trend of this section can be explained by considering the special case $X = S^{8m+4}$. Let π_r^S be the stable r-stem. We shall show in Part IV that the image of the stable J-homomorphism

$$J : \pi_{8m+3}(SO) \longrightarrow \pi^S_{8n+3}$$

is a direct summand. The reason is essentially that the quotient map

$$\tilde{J}(S^{8m+4}) \longrightarrow \tilde{J}'(S^{8m+4})$$

can be extended over the whole of π^S_{8m+3}.

It is reasonable to seek for a generalisation of this phenomenon. The generalisation should state that for a suitable class of spaces X, $J(X)$ is a natural direct summand in $L(X)$, where L is a functor such that $\tilde{L}(S^n) = \pi^S_{n-1}$. Unfortunately, I do not feel certain as to the best way of arranging the details of the construction of the functor L. For this reason, the present section is written as a tentative and heuristic explanation of the phenomena involved.

The construction of the functor L should secure the following properties.

(i) $L(X)$ should be a Grothendieck group generated by equivalence classes of "fiberings", in some weak sense.

The senses of the word "fibering" which might be considered include (for example) the following.

(a) Fiberings which are locally fibre homotopy equivalent to products $B \times S^m$; see [8].

(b) Hurewicz fiberings in which the fibres are homotopy-equivalent to spheres; see [9, 10, 11].

(c) Suitable CSS-fiberings; see [6].

As stated above, I do not feel certain as to the best choice of details.

(ii) The fiberings considered should admit suitable "Whitney sums" and "induced fiberings" in order that Grothendieck's construction should apply.

(iii) Sphere-bundles should qualify as "fiberings" in the sense considered; moreover the Whitney sums and induced bundles which we have for sphere-bundles should qualify as "Whitney sums" and "induced bundles" in the generalised sense. We require this in order to obtain a natural transformation

$$K_R(X) \longrightarrow L(X).$$

(iv) The image of $K_R(X)$ in $L(X)$ should be $J(X)$, up to natural isomorphism.

For practical purposes, this means that the equivalence relation used in constructing $L(X)$ must be fibre homotopy equivalence.

(v) The functor $L(X)$ should be a representable contravarient functor; moreover, the representing space should be essentially that constructed as follows.

Let H_n be the space of homotopy-equivalences from S^n to S^n, considered as a monoid under composition. Let BH_n be its classifying space (under some interpretation of this phrase). By suspension we define an embedding $H_n \longrightarrow H_{n+1}$, whence a map $BH_n B \longrightarrow H_{n+i}$. Set $BH = \lim_{n \to \infty} BH_n$ and take the space $Z \times BH$.

(vi) The "fiberings", "Whitney sums" and "induced fiberings" employed in the construction of $L(X)$ should retain sufficient of the cohomological properties of their classical counterparts, in respect both of ordinary and of extraordinary cohomology theories.

This is necessary in order that we should be able to define cohomological invariants of these fiberings, according to the pattern introduced in Part II.

Let us assume that with some arrangement of the details, we can secure what has been indicated above. The whole of the rest of this section will depend on this assumption, although I will not bother to write every sentence in a conditional form. However, I will not call the results "theorems", since the underlying assumptions have not been stated precisely enough. Of course it would not be hard to drag every assumption out into the open and give it the status of an axiom; but the result might be somewhat tedious to read.

We will now consider the set of "fiberings" whose Thom pairs can be oriented over the cohomology theory K_Λ.

EXAMPLE (6.1). *If $H^*(X; Z)$ is torsion-free, then every "fibering" ξ over X is orientable over the cohomology theory K_C.*

For the Thom space X^ξ of ξ will also be torsion-free; hence the spectral sequence

$$H^*(X^\xi; K_C^*(P)) \Rightarrow K_C^*(X^\xi)$$

will be trivial; therefore the required orientation will exist.

A similar argument will work if $H^*(X; Z)$ is even-dimensional.

The condition of "orientability over K_Λ" is invariant under fibre homotopy equivalence, and even under stable fibre homotopy equivalence; it is inherited by induced fiberings; if two fiberings admit orientations, we can put the product orientation on their Whitney sum. (This assumes that the details chosen for "Whitney sums" and "induced fiberings" allow one to draw the usual diagrams. (See (vi) above.))

We now introduce the subgroup $L_{OR}(X)$ of $L(X)$, generated by fiberings orientable over K_R and of dimension divisible by 8. Our cohomological invariants, such as ρ^k, are defined on such fiberings. More precisely, given an orientation u of ξ over K_R, this determines a Thom isomorphism ϕ in K_R-cohomology. It thus determines

$$\rho^k(\xi) = \phi^{-1}\Psi^k\phi 1.$$

However, if $y \in \tilde{K}_R(X)$, then we can replace u by $\pm(p^*(1 + y))u$; this changes $\rho^k(\xi)$, multiplying it by

$$\frac{\Psi^k(1 + y)}{1 + y}.$$

The classes $\rho^k\xi$ satisfy formulae such as

$$\rho^k(\xi \oplus \eta) = \rho^k\xi \cdot \rho^k\eta$$

$$\rho^k(f^*\xi) = f^*(\rho^k\xi)$$

up to the relevant indeterminacy

$$\frac{\Psi^k(1+y)}{1+y}.$$

We can therefore define our invariant $\rho = \prod_k \rho^k$ on $L_{OR}(X)$ so as to be exponential. More precisely, let $k^* + \tilde{K}_R(X) \otimes Q_k$ be the set of elements in $K_R(X) \otimes Q_k$ of the form $k^f + x$, where $f \in Z$ and $x \in \tilde{K}_R(X) \otimes Q^k$. This set is a multiplicative group. Let us define a map

$$\delta : 1 + \tilde{K}_R(X) \longrightarrow \prod_k (k^* + \tilde{K}_R(X) \otimes Q_k)$$

by

$$\delta(1+y) = \left\{\frac{\Psi^k(1+y)}{1+y}\right\},$$

as above. Then we can define ρ so that its values lie in the cokernel of δ.

We will next prove that the image of the map ρ is no larger than it would have been if we had used only classical bundles, instead of more general fiberings. The proof uses Theorem (6.2), which was introduced for this purpose. More precisely, suppose given an extraordinary fibering ξ of dimension $8n$ over X, and an orientation u of the Thom space X^ξ. We will show that the sequence

$$\rho^k\xi = \phi^{-1}\Psi^k\phi\, 1$$

satisfies conditions (a,), (b), (c) and (d) of Theorem (6.2).

In fact, the proofs that conditions (b), (c) and (d) are necessary (see Lemma (6.3) and [4, Proposition (5.5)]) are purely cohomological; therefore these proofs remain valid when we consider extraordinary fiberings. Thus the sequence $\rho^k\xi$ satisfies conditions (b), (c) and (d).

It remains to give a new argument for condition (a). We observe that if ξ has an orientation u over K_R, then this yields an orientation h over $H^*(\quad; Z)$ such that $Sq^1 h = 0$ and $Sq^2 h = 0$. (The second assertion follows by considering the spectral sequence

$$H^*(X^\xi; K_R^*(P)) \Rightarrow K_R^*(X^\xi),$$

since Sq^2 is a differential in this spectral sequence.) It follows that the Stiefel–Whitney classes $w_1(\xi)$ and $w_2(\xi)$ are zero. Now we observe that the embedding $BO \longrightarrow BH$ induces an isomorphism of π_1 and π_2 (both groups being Z_2). Therefore extraordinary fiberings over 2-dimensional complexes are classified by w_1 and w_2, as in the classical case. Thus if $i : X^2 \longrightarrow X$ is the inclusion map, $i^*\xi$ is fibre homotopy trivial. It follows that

$$\rho^k(i^*\xi) = k^{4n}\frac{\Psi^k(1+y)}{1+y}.$$

But in proving Lemma (6.3), we have shown that

$$\frac{\Psi^k(1+y)}{1+y} = 1 \quad \text{in} \quad 1 + \tilde{K}_R(X^2) \otimes Q_k.$$

Thus $i^*\rho^k\xi = k^{4n}$, which establishes condition (a).

Theorem (6.2) now shows that if $x \in \tilde{L}_{OR}(X)$, then there exists $x' \in \tilde{K}_{\mathrm{Spin}}(X)$ such that

$$\rho(x) = \rho(x').$$

This completes our argument about the image of ρ.

We will now show that the phenomenon which we have observed for S^{8m+4} generalises to any space X such that (i) $J(X) = J'(X)$, and (ii) every extraordinary fibering over X is orientable over K_R.

In fact, if condition (ii) holds we have the following commutative diagram.

$$\begin{tikzcd} \tilde{K}_R(X) \arrow[rr] \arrow[dr] & & \tilde{L}(X) \arrow[dl] \\ & \tilde{J}'(X) = \operatorname{Im} \rho & \end{tikzcd}$$

The image of $\tilde{K}_R(X)$ in $\tilde{L}(X)$ is $\tilde{J}(X)$. If condition (i) holds, then $\tilde{J}(X)$ is a natural direct summand in $\tilde{L}(X)$.

This certainly succeeds in providing an acceptable generalisation of the case $X = S^{8m+4}$. However, it must be pointed out that there are many spaces X which do not satisfy condition (ii).

For suppose we confine attention to spaces X satisfying condition (i) and (ii). Then $J(X)$ is a natural direct summand in $L(X)$, and $L(X)$ is an exact functor; therefore $J(X)$ is exact. But this is not true for general spaces X, as may be shown by the following example.

Consider the cofibering

$$S^4 \xrightarrow{f} S^4 \longrightarrow S^4 \cup_f e^5,$$

where f is a map of degree 24. Applying the functors $\tilde{K}$, $\tilde{J}$ we obtain

$$Z \xleftarrow{24} Z \longleftarrow 0,$$

$$Z_{24} \xleftarrow{0} Z_{24} \longleftarrow 0.$$

The last sequence is not exact.

In this example, both spaces satisfy condition (i) and S^4 satisfies condition (ii); therefore $S^4 \cup_f e^5$ does not satisfy condition (ii). We note that for the space $X = S^4 \cup_f e^5$, $H^*(X; Z)$ is neither torsion-free nor even-dimensional (see (6.1)).

An alternative line of argument is to point out that extraordinary fiberings (such as must be used to construct $L(X)$) have greater freedom to be non-orientable over K_R than sphere-bundles have. This may be shown by the following example.

We know that every sphere-bundle over S^m is orientable over K_R, provided $m > 2$. If we translate this into terms of π_{m-1}^S, it states that every map $f: S^{n+m-1} \longrightarrow S^n$ which lies in the image of J induces the zero homomorphism of $\tilde{K}^*_R$ (provided $m > 2$). On the other hand, for $m = 3$ the map $\eta\eta$ (which is not in the image of J) induces a non-zero homomorphism of $\tilde{K}^*_R$; and we shall see in Part IV that the same thing is true whenever $m \equiv 2$ or 3 mod 8 and $m > 2$.

This second example tends to indicate that the question, "When is an extraordinary fibering non-orientable over K_R?" can sometimes be answered by quite simple invariants defined using K_R. Indeed there are some grounds for supposing it to be a less subtle question than those with which we have mainly been concerned.

In both examples it happens to be true that $J(X)$ is a direct summand in $L(X)$ (though not by a natural map or for a general reason). It would perhaps be interesting to look for an example in which $J(X)$ is not a direct summand in $L(X)$.

This completes our discussion of the extent to which we can hope to generalise the case $X = S^{8m+4}$.

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