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SOME REMARKS ON ETALE HOMOTOPY THEORY AND A CONJECTURE OF ADAMS†

DANIEL G. QUILLEN

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IN [1] J. F. Adams states the following conjecture:

CONJECTURE. *If F is a real vector bundle over a finite complex X and k in an integer, then for some n the stable sphere fibration associated to the stable bundle* $k^n(\psi^k F - F)$ *is fiber homotopically trivial.*

Adams proves this in special cases such as when F is a real or complex line bundle and when F is a complex vector bundle over a sphere.

While mulling over the relation between the Adams operation ψ^p and Frobenius for vector bundles in characteristic p (number 4 below), I realized that it should be possible to give a proof of Adams' conjecture for *complex* vector bundles by using the basic comparison theorems of the etale homotopy theory for schemes recently developed by M. Artin and B. Mazur. In trying to work out the details, however, I have had to use an unproved assertion for the etale homotopy theory, namely the analogue for schemes of the fact that the complement of the zero section of a vector bundle is a sphere fibration over the base. Though this is almost certainly true its proof would be lengthy and might involve technical difficulties. The purpose of this paper is therefore to show how Adams' conjecture can be reduced to a conjecture (10) for the etale homotopy theory. I hope that it will stimulate interest in etale homotopy theory and lead to further applications in algebraic topology.

0. All vector bundles over a space will be assumed to be complex. It clearly suffices to prove Adams' conjecture when k is a prime number which will be denoted by p in the following.

1. If X is a compact space, let $\mathrm{Vect}_n(X)$ be the isomorphism classes of complex n-dimensional vector bundles on X and let $\mathrm{Vect}(X) = \bigcup_n \mathrm{Vect}_n(X)$. Let $K(X)$ be the Grothendieck group of vector bundles on X with its natural λ-ring structure and denote by $\gamma(E)$ the element of $K(X)$ belonging to a vector bundle E. If V is a prescheme then (algebraic) vector bundles over V are the same as locally free sheaves of finite rank on V, and we define $\mathrm{Vect}_n(V)$, $K(V)$, γ analogously. If V is of finite type over Spec $\underline{\underline{C}}$, then the set of complex points of V with its "classical" topology will be denoted V_{cl}. There is a canonical

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λ-ring homomorphism

$$a: K(V) \to K(V_{cl}) \tag{2}$$

given by $a(\gamma E) = \gamma(E_{cl})$.

3. If V is a prescheme of characteristic p, then to any algebraic vector bundle E is associated another algebraic vector bundle $E^{(p)} = \Phi^* E$ where $\Phi : V \to V$ is the Frobenius homomorphism. Then we have the fundamental formula

$$\psi^p \gamma(E) = \gamma(E^{(p)}). \tag{4}$$

This is essentially a formula for the representations of the general linear group $Gl(n)$ in characteristic p and may be readily checked. Another way of seeing its validity is to observe that $E \mapsto E^{(p)}$ defines an endomorphism of the functor K which coincides with ψ^p on line bundles and hence identically because of the splitting principle:

$$K(\underline{\underline{P}}E) = K(X)[T]/(\lambda_{-T}(E)) \qquad T = \gamma(\mathcal{O}(1))$$

5. We review results from etale homotopy theory [2]. Let $\underline{H}$ be the homotopy category of connected pointed CW complexes and let $V \mapsto V_{et}$ be the *etale homotopy type* functor from the category of geometrically pointed locally-connected preschemes to pro-$\underline{H}$. If V is simply-connected and locally-connected, and if V_1 and V_2 are the pointed preschemes resulting from two different geometric points of V, then $(V_1)_{et}$ and $(V_2)_{et}$ are canonically isomorphic so we may and shall forget about basepoints. Let $\underline{C}$ be the class of finite groups of order prime to p and let $\underline{CH}$ be the full subcategory of $\underline{H}$ whose objects have their homotopy groups in $\underline{C}$. Let $X \mapsto X^\wedge$ be the *$\underline{C}$-completion* functor from pro-$\underline{H}$ to pro-$\underline{CH}$. Let R be a strict localization of $\underline{\underline{Z}}$ at the prime p and choose an embedding $R \to \underline{\underline{C}}$. The residue field k of R is an algebraically closed field of characteristic p. Let V_R be a prescheme over Spec R with a given rational point Spec $R \to V_R$, and let V_C (resp. V_k) be the geometrically pointed preschemes obtained from V_R by base extension relative to the map Spec $\underline{\underline{C}} \to$ Spec R (resp. Spec $k \to$ Spec R). If V_R is proper and smooth over Spec R and if it's simply-connected, then the comparison theorems of [2] say that there are isomorphisms in pro-$\underline{CH}$

$$V^\wedge_{\underline{\underline{C}},cl} \xrightarrow{u} V^\wedge_{\underline{\underline{C}},et} \xrightarrow{j} V^\wedge_{R,et} \xleftarrow{i} V^\wedge_{k,et} \tag{6}$$

where i and j are base-change morphisms and where u comes from the canonical map from the classical to the etale topology.

7. If X is a CW complex, let $SF_n(X)$ be the isomorphism classes of oriented $(n-1)$-sphere fibrations with base X, and let $SF(X)$ be the group of stable isomorphism classes of oriented sphere fibrations with base X. If E is an n-dimensional vector bundle over X, then the map $E - 0 \to X$ ($0 =$ zero section) defines elements of $SF_{2n}(X)$ and $SF(X)$ which we denote by $\underline{\underline{S}}E$. S induces the J homomorphism $J : K(X) \to SF(X)$. We need an analogue of J for preschemes.

Let $\underline{X}$ be an object of pro-$\underline{CH}$. Define an oriented $(n-1)$-sphere fibration with base $\underline{X}$ to be a triple (f, u, v) where $f = \{f_i : E_i \to B_i, i \in I\}$ is a pro-object in the homotopy category of maps of objects in $\underline{CH}$, where $u : \{B_i\} \to \underline{X}$ is an isomorphism, and where $v : (S^{n-1})^\flat \simeq \{Ef_i\}^\flat$. Here $Ef = E \times_B B^I \times_B \{*\}$ is the fiber of the Hurewicz fibration associated to a map f, and "$\flat$" denotes the "flat" functor of [2], §3. Then the set $SF_n(\underline{X})$ of isomorphism classes of $(n-1)$-sphere fibrations with base $\underline{X}$ is a contravariant functor of $\underline{X}$ and the fiber-join [4] operation defines a map $SF_n(\underline{X}) \times SF_m(\underline{X}) \to SF_{n+m}(\underline{X})$, permitting us to define the abelian monoid $SF(\underline{X})$ of stable oriented sphere fibrations with base $\underline{X}$ in the usual way. By [2], th.4.5, $\underline{C}$-completion defines the natural transformations

$$\wedge : SF_n(X) \to SF_n(X^\wedge)$$
$$\wedge : SF(X) \to SF(X^\wedge) \tag{8}$$

if X is a simply-connected finite complex. It is reasonable to conjecture that $SF(\underline{X})$ is in fact an abelian group and that as a functor of $\underline{X}$ it is represented by $BSF^\wedge$, however we will not need this.

9. We shall however need the following conjecture to prove Adams' conjecture. To be on the safe side we consider only those preschemes V over Spec R which are simply-connected, noetherian, and have an ample line bundle.

10. CONJECTURE. *There is a map* $\underline{\underline{S}} : \mathrm{Vect}_n(V) \to SF_{2n}(V_{et}{}^\wedge)$ *defined by taking an n-dimensional vector bundle E, choosing a basepoint of* $E - 0$, *and rigidifying in a canonical way the map* $(E-0)_{et}{}^\wedge \to V_{et}{}^\wedge$ *to obtain a* $(2n-1)$-*sphere fibration* $\underline{\underline{S}}E$ *with base* $V_{et}{}^\wedge$. *Furthermore* $\underline{\underline{S}}$ *is a natural transformation of functors of V, and when V is of finite type over* Spec $\underline{\underline{C}}$, *this* $\underline{\underline{S}}$ *for preschemes agrees with the* $\underline{S}$ *for spaces* (7) *in the sense that the following diagram is commutative:*

$$\begin{array}{ccccc}
\mathrm{Vect}_n(V_{cl}) & \xleftarrow{\quad cl \quad} & & & \mathrm{Vect}_n(V) \\
\downarrow \underline{\underline{S}} & & & & \downarrow \underline{\underline{S}} \\
SF_{2n}(V_{cl}) & \xrightarrow{\wedge} & SF_{2n}(V_{cl}{}^\wedge) & \xleftarrow{u^*} & SF_{2n}(V_{et}{}^\wedge).
\end{array}$$

From now on we shall be assuming this conjecture.

11. PROPOSITION. $\underline{\underline{S}}$ *induces a natural transformation of monoids* $J : K(V) \to SF(V_{et}{}^\wedge)$.

This will be proved in 21. Combining 6, 10, 11 we obtain the following commutative diagram if V is smooth and proper over Spec R and V is simply-connected:

$$\begin{array}{ccccccccc}
K(V_{\underline{\underline{C}},cl}) & \xleftarrow{\quad a \quad} & & & K(V_{\underline{\underline{C}}}) & \xleftarrow{j_!} & K(V) & \xrightarrow{i_!} & (KV_k) \\
\downarrow J & & & & \downarrow J & & \downarrow J & & \downarrow J \\
SF(V_{\underline{\underline{C}},cl}) & \xrightarrow{\wedge} & SF(V^\wedge_{\underline{\underline{C}},cl}) & \underset{\sim}{\xleftarrow{u^*}} & SF(V^\wedge_{\underline{\underline{C}},et}) & \underset{\sim}{\xleftarrow{j^*}} & SF(V_{et}{}^\wedge) & \underset{\sim}{\xrightarrow{i^*}} & SF(V^\wedge_{k,et}).
\end{array} \tag{12}$$

13. *Proof* of Adams' conjecture. It evidently suffices to prove Adams' conjecture when F is the canonical n-dimensional quotient bundle over the Grassmannian X of complex N planes in $n+N$ space. Let $G_{\underline{\underline{Z}}}$ be the Grassmannian scheme over Spec $\underline{\underline{Z}}$ representing the functor "isomorphism classes of n-dimensional vector bundles with $n+N$ generating sections" and let $Q_{\underline{\underline{Z}}}$ be the canonical n-dimensional bundle. Then $X = G_{\underline{\underline{C}},cl}$ and $F = Q_{\underline{\underline{C}},cl}$, where G_A, Q_A are the base extension of $G_{\underline{\underline{Z}}}$, $Q_{\underline{\underline{Z}}}$ by the map Spec $A \to$ Spec $\underline{\underline{Z}}$.

14. LEMMA. *Let V be a noetherian prescheme having an ample line bundle L. Then any* $x \in K(V)$ *may be expressed* $x = \gamma(E) - n$ *where E is an algebraic vector bundle over V.*

Proof. $K(V)$ is the free abelian group generated by $\mathrm{Vect}(V) = \bigcup_n \mathrm{Vect}_n(V)$ modulo the relation $\gamma(E) = \gamma(E') + \gamma(E'')$ for any short exact sequence

$$0 \to E' \to E \to E'' \to 0. \tag{15}$$

As $\gamma(E \oplus F) = \gamma(E) + \gamma(F)$ one may write $x = \gamma(E_1) - \gamma(E_2)$. Since L is ample there are exact sequences

$$0 \to E_3 \to n \to L^k \otimes E_2{}^* \to 0 \qquad (* = \text{dual})$$
$$0 \to E_4 \to m \to L^k \to 0$$

for suitable $k, m, n > 0$. Hence $\gamma(E_2) = n\gamma(L^k) - \gamma(L^k \otimes E_3{}^*)$ and $x = \gamma(E_1) + \gamma(L^k \otimes E_3{}^*) - n(m - \gamma(E_4)) = \gamma(E_1 \oplus L^k \otimes E_3{}^* \oplus nE_4) - nm$, proving the lemma.

16. Using this lemma there is an algebraic vector bundle E_R over G_R with

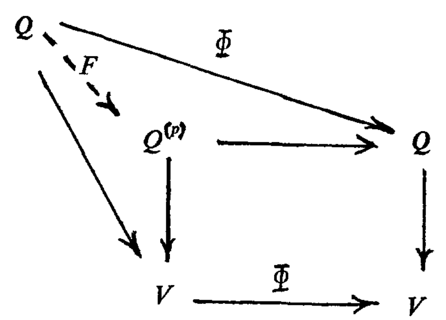
$$\psi^p(\gamma Q_R) - \gamma Q_R = \gamma(E_R) - n. \tag{17}$$

Now consider the diagram (12) with $V = G_R$ which is all right because G_R is proper and smooth over Spec R and it is simply-connected. Now

$$\begin{aligned} i^!(\gamma(E_R) - n) &= \psi^p(\gamma Q_k) - \gamma Q_k \\ &= \gamma(Q_k^{(p)}) - \gamma Q_k \end{aligned} \tag{18}$$

by 4.

If Q is a vector bundle over a variety in characteristic p there is a morphism $F : Q \to Q^{(p)}$ given by the diagram



where the square is cartesian. F is not a vector bundle morphism. In local coordinates F is given by $(X_i) \mapsto (X_i{}^p)$, hence it induces a fiber preserving map $Q - 0 \to Q^{(p)} - 0$ which is purely inseparable and therefore a "homeomorphism" for the etale topology. Therefore $(Q - 0)_{et}{}^\wedge \simeq (Q^{(p)} - 0)_{et}{}^\wedge$ over $V_{et}{}^\wedge$, so $\underline{\underline{S}}Q = \underline{\underline{S}}Q^{(p)}$ in $SF(V_{et}{}^\wedge)$.

In the situation at hand, $Ji^!(\gamma(E_R) - n) = J\gamma Q_k^{(p)} - J\gamma Q_k = 0$, so by the commutativity of 12, $J(\gamma E_{\underline{\underline{C}},cl})^\wedge = 0$. In other words the sphere fibration $E_{\underline{\underline{C}},cl} - 0 \to X$ becomes stably trivial when $\wedge$ is applied.

19. Lemma. *Let X be a finite complex and let $\xi : Y \to X$ be an n-sphere fibration. If the completed sphere fibration $\xi^\wedge : Y^\wedge \to X^\wedge$ is stably trivial, then the stable equivalence class of ξ is an element of $SF(X)$ of order a power of p.*

Proof. If $\xi^\wedge$ is stably trivial, then there is an element u in $\{Y^\wedge, (S^n)^\wedge\}$ such that $u\varepsilon^\wedge = id$. Here $\{\ ,\ \}$ denotes S-maps in pro-$\underline{\underline{CH}}$ ([2], §8) and $\varepsilon : S^n \to Y$ is the inclusion of the fiber. Consider the diagram

$$\begin{array}{ccccc} \{Y, S^n\}^\wedge & \xrightarrow{(\varepsilon^*)^\wedge} & \{S^n, S^n\}^\wedge & \xrightarrow{v^\wedge} & (Z/rZ)^\wedge \to 0 \\ \downarrow & & \downarrow & & \downarrow \\ \{Y^\wedge, (S^n)^\wedge\} & \xrightarrow{(\varepsilon^\wedge)^*} & \{(S^n)^\wedge, (S^n)^\wedge\} & \longrightarrow & 0 \end{array}$$

where v is the cokernel of ε^* and where $\wedge$ in the first row denotes $\underline{C}$-completion of an abelian group, which here is $\otimes\underline{\underline{Z}}^\wedge$. Hence the first row is exact; by [2], §8 the first two vertical maps are isomorphisms, hence $(\underline{\underline{Z}}/r\underline{\underline{Z}})^\wedge = 0$ and $r = p^i$. This means that there is an S-map $w : Y \to S^n$ such that $w\varepsilon$ is of degree p^i, so by Adams' "mod k Dold theorem" [1], ξ represents an element of order p^a in $SF(X)$. This proves the lemma.

20. By the lemma $0 = p^a J(\gamma E_{\underline{\underline{C}},cl})$ which equals $p^a J(\psi^p\gamma F - \gamma F)$ by 17 and 13, thus proving Adams' conjecture.

21. *Proof* of 11. It is necessary to show that the image of $\underline{\underline{S}} : \mathrm{Vect}(V) \to SF(V_{et}{}^\wedge)$ lies in the subgroup of invertible elements of the monoid $SF(V_{et}{}^\wedge)$ and that if 15 is an exact sequence of vector bundles over V, then $\underline{\underline{S}}E = \underline{\underline{S}}E' + \underline{\underline{S}}E''$. If V is proper and smooth over Spec R and V is simply-connected, then by 10 there is a commutative diagram

$$\begin{array}{ccccccc} K(V_{\underline{\underline{C}},cl}) & \xleftarrow{a} & K(V_{\underline{\underline{C}}}) & \longleftarrow & K(V) & \xleftarrow{\gamma} & \mathrm{Vect}\ V \\ \downarrow J & & & & & & \downarrow \underline{\underline{S}} \\ SF(V_{\underline{\underline{C}},cl}) & \xrightarrow{\wedge} & SF(V^\wedge_{\underline{\underline{C}},cl}) & \xleftarrow{\sim} & SF(V^\wedge_{\underline{\underline{C}},et}) & \xleftarrow{\sim} & SF(V_{et}{}^\wedge). \end{array}$$

This reduces 11 for V to the topological situation where it is clear, so 11 holds for such a V

We shall call an algebraic vector bundle *full* if it is generated by its global sections. Let $D_{\underline{\underline{Z}}} = D_{\underline{\underline{Z}}}^{Nij}$ be the drapeaux scheme over Spec $\underline{\underline{Z}}$ which classifies diagrams

$$N \to E_1 \to E_2$$

of surjective maps of algebraic vector bundles where $\dim E_1 = i$ and $\dim E_2 = j$. Given an exact sequence 15 on V with E full then for a suitable choice of N, i, j there is a map from V to D_R inducing 15 from an exact sequence on D_R. As D_R is proper and smooth over Spec R and simply-connected, we know 11 is true for it, hence by naturality of $\underline{\underline{S}}$ (10), $\underline{\underline{S}}E = \underline{\underline{S}}E' + \underline{\underline{S}}E''$ and $\underline{\underline{S}}E, \underline{\underline{S}}E', \underline{\underline{S}}E''$ are invertible in $SF(V_{et}{}^\wedge)$. Now given any vector bundle F on V we may write F^* as a quotient of nL^{-k} where L is an ample line bundle on V, and hence F is isomorphic to an E' occurring in an exact sequence 15 with E full. Thus $\underline{\underline{S}}F$ is invertible for any F on V. Finally given any exact sequence 15 on V there is a commutative diagram

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \uparrow & & \uparrow & & \\ & & & & F_3 & = & F_3 & & \\ & & & & \uparrow & & \uparrow & & \\ 0 & \to & E' & \to & F_1 & \to & F_2 & \to & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \to & E' & \to & E & \to & E'' & \to & 0 \\ & & & & \uparrow & & \uparrow & & \\ & & & & 0 & & 0 & & \end{array}$$

with F_1, hence F_2 full. Then $\underline{\underline{S}}E = \underline{\underline{S}}F_1 - \underline{\underline{S}}F_3 = (\underline{\underline{S}}E' + \underline{\underline{S}}F_2) - (\underline{\underline{S}}F_2 - \underline{\underline{S}}E'') = \underline{\underline{S}}E' + \underline{\underline{S}}E''$, completing the proof of 11.

22. *Remark.* There is the following evidence for conjecture 10: In [3] Exposé XVI, remark 3.10, there is a statement which may be used to prove the existence of a Gysin sequence

$$\cdots \to H^{q-2d}(V) \to H^q(V) \to H^q(E-0) \to H^{q-2d+1}(V) \to \cdots$$

when E is a vector bundle of dimension d over V.

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Massachusetts Institute of Technology