Final exam

3 hours

The three exercises are independent one from each other.
You can answer either in French or in English.

Exercise 1. — Let S^1 be the circle $\{z \in \mathbf{C},\ |z| = 1\}$ and let D^2 be the closed unit disk $\{z \in \mathbf{C},\ |z| \leqslant 1\}$.

Given a matrix $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in \mathrm{M}_2(\mathbf{Z})$ (with integer coefficients), we denote by f_A the following map:

$$\begin{array}{rccc} f_A : & S^1 \times S^1 & \to & S^1 \times S^1 \\ & (u, v) & \mapsto & (u^a v^c, u^b v^d) \end{array}$$

Note that for 2 matrices $A_1, A_2 \in \mathrm{M}_2(\mathbf{Z})$, we have $f_{A_1 A_2} = f_{A_1} \circ f_{A_2}$.

a) Compute the fundamental group $\pi_1(\mathrm{S}^1 \times \mathrm{S}^1; (1,1))$.

b) What is the induced group morphism

$$(f_A)_* : \pi_1(\mathrm{S}^1 \times \mathrm{S}^1; (1,1)) \to \pi_1(\mathrm{S}^1 \times \mathrm{S}^1; (1,1))?$$

Justify your answer!

c) Give a necessary and sufficient condition on A so that f_A is a homeomorphism.

d) Give a necessary and sufficient condition on A so that f_A can be extended to a (continuous) map $g_A : \mathrm{S}^1 \times \mathrm{D}^2 \to \mathrm{S}^1 \times \mathrm{S}^1$.

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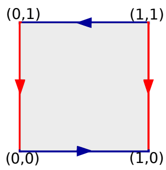
Exercise 2. — Let I be the segment $[0, 1]$ and $S = I \times I$ denote the unit square.
The Klein bottle K is defined to be the following quotient of S:



with the identifications $(0, t) \sim (1, t)$ and $(t, 0) \sim (1 - t, 1)$ for all $t \in [0, 1]$.

Let $q : S \to K$ be the canonical quotient map. We denote by ∂S the boundary of S (the union of the 4 coloured segments in the picture); by A the subspace $q(\partial S) \subset K$ and by $\overline{0}$ the base point $q(0, 0)$.

a) Show that $\pi_1(A, \overline{0})$ is a free group on two generators.

b) Use Van Kampen's theorem to prove that $\pi_1(K; \overline{0})$ is the quotient of the free group on two generators a, b by the *normal* subgroup generated by $ab^{-1}ab$.

c) Show that there are exactly 4 non isomorphic 2-fold covering spaces of K.

The aim of the rest of the exercise is to describe the universal cover of K.

Let τ and σ be the following homeomorphisms of $\mathbf{R}^2$:

$$\tau(x,y) := (x+1,y) \qquad \sigma(x,y) := (1-x,y+1).$$

We denote by $G := \langle \tau, \sigma \rangle$ the group generated by these two homeomorphisms and by $\widetilde{K} := G\backslash \mathbf{R}^2$ the space of orbits of the action of G on $\mathbf{R}^2$ (endowed with the quotient topology).

d) Show that in G we have $\tau \circ \sigma \circ \tau = \sigma$.
Deduce that any element in G can be written uniquely as $\sigma^n \tau^m$ for some integers $n, m \in \mathbf{Z}$.

e) Show that the canonical quotient map

$$\pi : \mathbf{R}^2 \to \widetilde{K}$$

is a covering space.

f) Conclude by showing that K and $\widetilde{K}$ are homeomorphic.

g) Give an example of a 2-fold covering space of K which is not trivial and describe its monodromy.

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Exercise 3. — Let $X = \mathbf{S}^1 \subset \mathbf{C}^*$ be the circle and $p : Y \to X$ a covering space on X with fiber $p^{-1}(1) = E$. For an open subset $W \subset X$ denote by $\mathfrak{S}(W)$ the set of sections $s : W \to Y$ of p defined on W.
Let $0 < \varepsilon < \frac{\pi}{4}$ and consider the connected open subsets $V = X \setminus \{1\}$, $U = \{e^{it} \mid t \in]-\varepsilon, \varepsilon[\}$, $U_- = \{e^{it} \mid t \in]-\varepsilon, 0[\}$, $U_+ = \{e^{it} \mid t \in]0, \varepsilon[\}$.

a) Explain why the map $\sigma : \mathfrak{S}(U) \to E$ defined by $\sigma(s) = s(1)$ is bijective.

b) Show that the restriction maps $\rho_U^+ : \mathfrak{S}(U) \to \mathfrak{S}(U_+)$, $\rho_U^- : \mathfrak{S}(U) \to \mathfrak{S}(U_-)$, $\rho_V^+ : \mathfrak{S}(V) \to \mathfrak{S}(U_+)$, $\rho_V^- : \mathfrak{S}(V) \to \mathfrak{S}(U_-)$ are bijective.

c) Denote by μ the composition $\sigma \circ (\rho_U^-)^{-1} \circ \rho_V^- \circ (\rho_V^+)^{-1} \circ \rho_U^+ \circ \sigma^{-1}$. Let γ be the loop $\gamma(t) = e^{2\pi t i}$. Show that for all $y \in E$ the monodromy action of $[\gamma] \in \pi_1(X,1)$ is given by $y.[\gamma] = \mu(y)$.
[**Hint:** By choosing $x_+ \in U_+$, $x_- \in U_-$ decompose γ into three parts and construct the lift of γ using suitable local sections.]

Let $f : X \to X$ be the map $f(z) = z^3$ and S be a set. Our aim is to describe $f_*(S_X)$ (where S_X denotes the sheaf of locally constant functions from X to S).
We fix $0 < \varepsilon < \frac{\pi}{4}$ and set $U_k = \{e^{(\frac{2k\pi}{3}+t)i} \mid \frac{-\varepsilon}{3} < t < \frac{\varepsilon}{3}\}$ for $k = 0, 1, 2$. Note that the U_k are the connected components of $f^{-1}(U)$.

d) Show that f_*S_X is a local system on X and calculate $(f_*S_X)_1$ as well as $f_*S_X(X)$.
Deduce that f_*S_X is not the sheaf of sections of the covering space $X \times S \to X$; $(z,s) \mapsto z^3$.

e) Identify f_*S_X to the sheaf of sections of its étalé space $p : E(f_*S_X) \to X$ and describe $f_*S_X(U)$, $f_*S_X(U_-)$, $f_*S_X(U_+)$ as well as the maps σ, ρ_U^+, ρ_U^- using the decomposition $f^{-1}(U) = U_0 \cup U_1 \cup U_2$.

f) Using your identifications from the previous question describe the map $\rho_V^- \circ (\rho_V^+)^{-1}$.
[**Hint:** for any $x \in V$ we may choose a small neighborhood U_x of x similar to $U = U_1$ such that $f^{-1}(U_x)$ has three connected components and $\rho_V^x : f_*S_X(V) \to f_*S_X(U_x)$ is bijective. Describe first $\rho_V^x \circ (\rho_V^+)^{-1}$ in the case that $U_x \cap U_+$ is not empty, then iterate the procedure to deduce $\rho_V^- \circ (\rho_V^+)^{-1}$.]

g) Describe the orbits of the monodromy action on $(f_*S_X)_1$.
Deduce the isomorphism class of the covering space associated to f_*S_X in the case that S is a set with 2 elements.

h) Deduce that $f^{-1}f_*S_X$ is a constant sheaf.