Master 2 2017

Homotopies and fundamental group

Unless otherwise stated, all spaces are topological spaces and maps are continuous maps.

I. General topology

Exercice 1. — Let X and Y be topological spaces. Let $(U_i)_{i\in I}$ be a family of open sets that covers X and $f_i : U_i \to Y$ a family of (continuous) maps such that :

$$\forall\, i, j \in I,\ f_{i|U_i\cap U_j} = f_{j|U_j\cap U_i}.$$

a) Prove that there exists a unique "global" (continuous) function $f : X \to Y$ such that :

$$\forall i \in I,\ f_{|U_i} = f_i.$$

b) Prove an analogous statement when X is covered by a finite family of closed sets $(F_i)_{i\in I}$.

Exercice 2. — **a)** Prove that a space X is connected if every continuous function f from X to the discrete space $\{0, 1\}$ is constant.

b) Let $X = [0, 1]$ be the unit interval. Show that X is connected.

c) Deduce that every path connected space is connected.

d) Give an example of a connected space that is not path connected.

e) Show that in a locally path connected space [1], each path-connected component is open and closed. Deduce that a connected space which is locally path-connected is actually path-connected.

Exercice 3. — Given a space X, we denote by $\pi_0(X)$ its set of path-connected components. For every map $f : X \to Y$, define a (set-theoretical) function $f_* : \pi_0(X) \to \pi_0(Y)$ such that

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ \pi_0(X) & \xrightarrow[f_*]{} & \pi_0(Y) \end{array}$$

commutes. Show that for composable maps $(g \circ f)_* = g_* \circ f_*$.

II. Homotopies

Exercice 4. — **[The importance of the base point]**
Let X be a topological space and $\gamma : I \to X$ be any continuous path. Show that γ is homotopic (without any condition on the endpoints) to a constant path.

Exercice 5. — Let $\mathcal{C} := S^1 \times [0, 1]$ be the cylinder and $\mathcal{S}$ be its subspace $S^1 \times \{0\}$.

a) Prove that the quotient $\mathcal{C}/\mathcal{S}$ is homeomorphic to the disk $D^2 := D(0; 1) \subset \mathbf{R}^2$.

b) Let X be a topological space and $\gamma : S^1 \to X$ be a loop in X. Prove the equivalence between :

(i) γ is homotopic (not necessarily path-homotopic) to a constant map ;

(ii) The map γ extends to a map $D^2 \to X$.

Exercice 6. — Let X, Y and Z be topological spaces. When two maps φ and ψ (with same source and target spaces) are *homotopic*, we use the notation $\varphi \simeq \psi$.

a) Show that $\simeq$ is an equivalence relation.

b) Let $f_0, f_1 : X \to Y$ and $g_0, g_1 : Y \to Z$ be some maps. Show that :

$$f_0 \simeq f_1 \implies f_0 \circ g_0 \simeq f_1 \circ g_0 \qquad \text{and} \qquad g_0 \simeq g_1 \implies f_0 \circ g_0 \simeq f_0 \circ g_1 \qquad .$$

c) Let $f : X \to Y$ and $g, h : Y \to X$ be maps such that $f \circ g \simeq \mathrm{id}_Y$ and $h \circ f \simeq \mathrm{id}_X$.
Show that f is a homotopy equivalence.
[**Indication:** Consider the composite map $h \circ f \circ g \circ f$.]

1. That is to say a space where every point has a fundamental system of open neighbourhoods which are path-connected.

Exercice 7. — **[Homotopy equivalences]**
Construct homotopy equivalences between the following pairs of spaces (you don't need to write the exact formula ; a picture may be enough) :

a) $\mathbf{R}^n \setminus \{0\}$ and the sphere $\mathbf{S}^{n-1} := \{(x_1, \dots, x_n) \in \mathbf{R}^n, \sum_1^n x_i^2 = 1\}$,

b) $\mathbf{R}^3$ minus a line and $\mathbf{R}^2 \setminus \{0\}$,

c) $\mathbf{C} \setminus]-\infty, 0]$ and $\{1\}$,

d) $\mathbf{C} \setminus \{-1, 1\}$ and the union of the two circles of radius 1 centered at -1 and 1.

III. Fundamental group

Exercice 8. — Let (X, x_0) be a pointed space. Check carefully that $\pi_1(X, x_0)$ is a group.

Exercice 9. — Let (X, x_0), (Y, y_0) be two pointed spaces. Show that the natural projection maps induce a group isomorphism :

$$p_{1*} \times p_{2*} : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

Le T be a 2-dimensional torus (the surface at the exterior of a donut). Compute its fundamental group and draw on a picture loops generating it.

Exercice 10. — Let $q : S^1 \to S^1$ be the map $z \mapsto z^2$.

a) What is the induced homomorphism $q_* : \pi_1(S^1, 1) \to \pi_1(S^1, 1)$?

b) Show that there is no continuous map $r : S^1 \to S^1$ such that

$$\forall z \in S^1,\ r(z)^2 = z.$$

c) Deduce that that there is no continuous square-root function defined on $\mathbf{C} \setminus \{0\}$.

Exercice 11. — Let G be a topological group [(2)]. Let α and β be two loops in G based at e.

a) Let $\gamma(t) = \alpha(t)\beta(t)$ (using the product in G). Show that γ is a loop based at e.

b) Show that the loops $\alpha \cdot \beta$, γ and $\beta \cdot \alpha$ are homotopic with endpoints fixed.
[**Indication:** Consider the map : $[0,1] \times [0,1] \to G$, $(t, u) \mapsto \alpha(t)\beta(u)$.]

c) Deduce that the group $\pi_1(G, \mathrm{e})$ is commutative.

Exercice 12. — **[The hairy ball theorem]**
A *vector field* on S^2 is a continuous map $V : \mathrm{S}^2 \to \mathbb{R}^3$ such that for all x in S^2 the scalar product $x.V(x) = 0$. The goal of the exercise is to prove that for every such vector field there exists a point $x_0 \in S^2$ such that $V(x_0) = 0$.
Let $\mathrm{i} : \mathrm{SO}(2) \to \mathrm{SO}(3)$ be the map $\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \mapsto \begin{bmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

a) Let α be the loop of SO(2) based at Id_2, $t \mapsto \begin{bmatrix} \cos 2\pi t & -\sin 2\pi t \\ \sin 2\pi t & \cos 2\pi t \end{bmatrix}$.
Show that the two loops of SO(3) (based at Id_3) $i_*(\alpha)$ and $i_*(\alpha^{-1})$ are homotopic.

b) Prove that there exists no continuous map $r : \mathrm{SO}(3) \to \mathrm{SO}(2)$ such that $r \circ \mathrm{i} = \mathrm{id}_{SO(2)}$.

Let assume by contradiction that there exists a nowhere-vanishing vector field V on S^2. Up to scaling, one can assume that $\forall x \in S^2, \|V(x)\| = 1$.
Let $\tilde{M} : S^2 \to SO(3)$ be the map $x \mapsto [V(x), x \times V(x), x]$. (Here x is seen as a column vector in $\mathbf{R}^3$ and $\times$ denotes the vector product usually denoted by a $\wedge$ in French).
Let $e_3 := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and set $M(x) := \tilde{M}(x)\tilde{M}(e_3)^{-1}$.

c) Prove that for the usual action of $SO(3)$ on S^2 we have

$$\forall x \in S^2,\ M(x) \cdot e_3 = x \qquad \text{and} \qquad M(e_3) = \mathrm{Id}_3.$$

d) Use question **b)** to conclude.

2. That is a topological space with a group structure where multiplication and inverse operation are continuous.