Long wave asymptotics for the Euler–Korteweg system

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Abstract

The Euler–Korteweg system (EK) is a fairly general nonlinear waves model in mathematical physics that includes in particular the fluid formulation of the NonLinear Schrödinger equation (NLS). Several asymptotic regimes can be considered, regarding the length and the amplitude of waves. The first one is the free wave regime, which yields long acoustic waves of small amplitude. The other regimes describe a single wave or two counter propagating waves emerging from the wave regime. In one space dimension, it is shown that these waves are governed either by inviscid Burgers or by Korteweg-de Vries equations, depending on the spatio-temporal and amplitude scalings. In higher dimensions, those waves are found to solve Kadomtsev-Petviashvili equations. Error bounds are provided in all cases. These results extend earlier work on defocussing NLS (and more specifically the Gross–Pitaevskii equation), and sheds light on the qualitative behavior of solutions to EK, which is a highly nonlinear system of PDEs that is much less understood in general than NLS.

Key-words: Euler–Korteweg system, capillary fluids, Korteweg de Vries equation, Kadomtsev-Petviashvili equation, weakly transverse Boussinesq system.

MSC (2010): 35Q35, 35Q53.

1 Introduction

The Euler–Korteweg system is a dispersive perturbation of the Euler equations for compressible fluids. In its most general form, it reads

$$\text{(gEK)} \qquad \begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0 \\ \\ \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla(\delta \mathscr{F}[\rho]) = 0\,, \end{cases}$$

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for an isothermal or isentropic fluid whose velocity field is $\mathbf{u}$, whose energy density $\mathscr{F}$ is allowed to depend on the fluid density ρ and on its spatial gradient $\nabla\rho$, and $\delta\mathscr{F}[\rho]$ denotes the variational derivative of $\mathscr{F}$ at ρ. The standard Euler equations correspond to $\mathscr{F} = F(\rho)$ only, so that $\delta\mathscr{F}[\rho] = F'(\rho)$ (and the pressure of the fluid is $p(\rho) = \rho F'(\rho) - F(\rho)$). The most classical form of the Euler–Korteweg system corresponds to

$$\mathscr{F} = F(\rho) + \frac{1}{2}K(\rho)|\nabla\rho|^2\,,$$

where the so-called capillarity coefficient $K = K(\rho)$ can depend on the density ρ is an arbitrary way, provided that K is smooth and takes only positive values. In this case, (gEK) 'reduces' to

$$\text{(EK)}\qquad \left\{\begin{array}{l} \partial_t\rho + \nabla\cdot(\rho\mathbf{u}) = 0 \\[2ex] \partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla(g(\rho)) = \nabla\Big(K(\rho)\Delta\rho + \dfrac{1}{2}K'(\rho)|\nabla\rho|^2\Big)\,, \end{array}\right.$$

where $g \stackrel{\text{def}}{=} F'$. Would K be zero, the system (EK) would of course reduce to the standard Euler equations again, in which the sound speed is given by $\sqrt{\rho g'(\rho)}$ as long as g is a nondecreasing function of ρ. In the special case when $K = 1/(4\rho)$, the system (EK) can be derived from the (generalized) NonLinear Schrödinger equation (NLS) via the Madelung transform. An even more special case is $g(\varrho) = \varrho - 1$, which corresponds to the Gross–Pitaevskii equation. In fact, (EK) is a ubiquitous system in mathematical physics, with various choices of K and g, see for instance [3] for more details.

Associated with (gEK) is a local conservation law for the total energy $\frac{1}{2}\rho|\mathbf{u}|^2 + \mathscr{F}(\rho,\nabla\rho)$. However, the Cauchy problem for (gEK) has never been addressed for general energy densities $\mathscr{F}$. Because of analytical difficulties inherent in all systems involving high order derivatives (namely here, third order derivatives), the Cauchy problem analysis has been concentrating on (EK). The local well-posedness of (EK) is shown in [5] (one space dimension), and [4] (arbitrary space dimension). Our purpose here is to investigate the behavior of solutions of (EK) on longer times, by considering small perturbations of constant, thermodynamically stable states. By small we mean small amplitude perturbations that are significant on large space-time scales. By thermodynamically stable we mean reference densities ϱ such that $g'(\varrho)$ is positive. For any ϱ, the condition $g'(\varrho) > 0$ is equivalent to the hyperbolicity of the Euler equations at $(\varrho, 0)$ (or $(\varrho, \mathbf{u})$ for any velocity $\mathbf{u}$, by Galilean invariance) - and when applied to the fluid formulation of NLS, it corresponds to what is known as the defocussing case. This paper aims at justifying several asymptotic limits regarding small amplitude, long wave solutions to the Euler–Korteweg system (EK), thus extending a series of recent work on NLS - and similar results known for the water wave equations.

The starting point is as follows. Constant states $(\varrho, 0)$ are obviously global solutions to (EK) - and even (gEK), and small amplitude perturbations of $(\varrho, 0)$ are formally governed by the acoustic equations

$$\left\{\begin{array}{l} \partial_t\hat\rho + \varrho\nabla\cdot\hat{\mathbf{u}} = 0 \\[2ex] \partial_t\hat{\mathbf{u}} + g'(\varrho)\nabla\hat\rho = 0\,. \end{array}\right.$$

For (gEK), it suffices to replace $g'(\varrho)$ by $\frac{\partial^2 \mathscr{F}}{\partial \rho^2}(\varrho, 0)$. We are only interested here in the case when these equations are well-posed, which amounts to requiring that $g'(\varrho) > 0$. From now on, we assume that g is as smooth as necessary near $\varrho \neq 0$ - vacuum being excluded from our analysis, that $g'(\varrho) > 0$, and we denote by

$$\mathfrak{c} \stackrel{\text{def}}{=} \sqrt{\varrho g'(\varrho)} > 0$$

the sound speed at ϱ. The acoustic equations admit particular solutions that are planar traveling waves $(\hat{\rho}, \hat{\mathbf{u}}) = (\hat{\rho}, \hat{\mathbf{u}})(x - \mathfrak{c}\mathbf{n}t)$ propagating with speed $\mathfrak{c}$ in any direction $\mathbf{n}$. A natural idea is to seek genuine solutions to (EK) that are of small amplitude about $(\varrho, 0)$ and vary slowly in the frame attached to this linear wave.

In one space dimension, a prominent asymptotic regime corresponding to a weakly nonlinear limit can easily be identified by rescaling the solutions to the one D version of (EK) - or even (gEK) - as

KdVansatz (1) $$\rho(t,x) = \varrho + \varepsilon^2 \tilde{\rho}(\theta, Y)\,, \qquad u(t,x) = \varepsilon^2 \tilde{u}(\theta, Y)\,, \qquad \theta = \varepsilon^3 t\,, \quad y = \varepsilon(x - \mathfrak{c}t)\,,$$

for a small parameter $\varepsilon > 0$ (here above, the scalar, fluid velocities are denoted by u instead of the bold letter $\mathbf{u}$). Using that $\partial_t = \varepsilon^3 \partial_\theta - \varepsilon \mathfrak{c} \partial_Y$ and $\partial_x = \varepsilon \partial_Y$, we see that for (ρ, u) to solve (gEK) in one D we must have

$$\begin{cases} \partial_\theta \tilde{\rho} - \dfrac{\mathfrak{c}}{\varepsilon^2} \partial_Y \tilde{\rho} + \dfrac{1}{\varepsilon^2} \partial_Y ((\varrho + \varepsilon^2 \tilde{\rho}) \tilde{u}) = 0 \\ \\ \partial_\theta \tilde{u} - \dfrac{\mathfrak{c}}{\varepsilon^2} \partial_Y \tilde{u} + \tilde{u} \partial_Y \tilde{u} + \dfrac{1}{\varepsilon^4} (\delta \mathscr{F}[\varrho + \varepsilon^2 \tilde{\rho}]) = 0\,. \end{cases}$$

Furthermore, by Taylor expansion we have

$$\delta \mathscr{F}[\varrho + \varepsilon^2 \tilde{\rho}] = \frac{\partial \mathscr{F}}{\partial \rho}(\varrho, 0) + \varepsilon^2 \frac{\partial^2 \mathscr{F}}{\partial \rho^2}(\varrho, 0)\ \tilde{\rho} + \frac{1}{2} \varepsilon^4 \frac{\partial^3 \mathscr{F}}{\partial \rho^3}(\varrho, 0)\ \tilde{\rho}^2 - \varepsilon^4 \frac{\partial^2 \mathscr{F}}{\partial \rho_x^2}(\varrho, 0) \partial_Y^2 \tilde{\rho}\ +\ \mathcal{O}(\varepsilon^5)\,,$$

which enables us to rewrite the system above as

$$\begin{cases} \partial_\theta \tilde{\rho} - \dfrac{\mathfrak{c}}{\varepsilon^2} \partial_Y \tilde{\rho} + \dfrac{\varrho}{\varepsilon^2} \partial_Y \tilde{u} + \partial_Y (\tilde{\rho} \tilde{u}) = 0 \\ \\ \partial_\theta \tilde{u} - \dfrac{\mathfrak{c}}{\varepsilon^2} \partial_Y \tilde{u} + \tilde{u} \partial_Y \tilde{u} + \dfrac{\mathfrak{c}^2}{\varepsilon^2 \varrho} \partial_Y \tilde{\rho} + \underline{\delta} \tilde{\rho} \partial_Y \tilde{\rho} - \underline{K} \partial_Y^3 \tilde{\rho} = \mathcal{O}(\varepsilon) \end{cases}$$

with

$$\mathfrak{c}^2 = \varrho \frac{\partial^2 \mathscr{F}}{\partial \rho^2}(\varrho, 0)\,, \qquad \underline{\delta} \stackrel{\text{def}}{=} \frac{\partial^3 \mathscr{F}}{\partial \rho^3}(\varrho, 0)\,, \qquad \underline{K} \stackrel{\text{def}}{=} \frac{\partial^2 \mathscr{F}}{\partial \rho_x^2}(\varrho, 0)\,.$$

If we go on at a formal level, we find by inspecting the $\mathcal{O}(\varepsilon^{-2})$ terms that necessarily $\mathfrak{c}\tilde{\rho} \approx \varrho \tilde{u}$, and by taking a linear combination of the $\mathcal{O}(1)$ terms in the system above, we see that $w \stackrel{\text{def}}{=} \frac{1}{2}(\tilde{\rho} + \frac{\varrho}{\mathfrak{c}} \tilde{u})$ should approximately satisfy the Korteweg-de Vries equation

$$\partial_\theta w + \Gamma w \partial_Y w = \kappa \partial_Y^3 w$$

with

$$\Gamma \stackrel{\text{def}}{=} \frac{3\mathfrak{c}}{2\varrho} + \frac{\varrho\underline{\delta}}{2\mathfrak{c}}\,, \qquad\qquad \kappa \stackrel{\text{def}}{=} \frac{\varrho\underline{K}}{2\mathfrak{c}}\,.$$

When dealing with (EK), we merely have $\underline{\delta} = g''(\varrho)$ and $\underline{K} = K(\varrho)$. Of course, if $\underline{K} = 0$ we recover the well-known Burgers equation

$$\partial_\theta w + \Gamma w \partial_Y w = 0$$

as an asymptotic equation for the weakly nonlinear wave solutions to the Euler equations, in which the parameter Γ is nonzero provided that the characteristic fields are genuinely nonlinear in the neighborhood of ϱ. Indeed, both characteristic fields of the Euler equations are genuinely nonlinear in the neighborhood of ϱ if and only if $\partial_\rho(\rho\sqrt{\rho g'(\rho)})_{|\varrho} \neq 0$, and by definition of $\mathfrak{c}$ we have

$$\partial_\rho(\rho\sqrt{\rho g'(\rho)})_{|\varrho} = \mathfrak{c} + \frac{\varrho}{2\mathfrak{c}}\big(\frac{\mathfrak{c}^2}{\varrho} + \varrho g''(\varrho)\big) = \varrho\Gamma\,.$$

In fact, the dimensionless number $\varrho\Gamma/\mathfrak{c}$ is known as the Grüneisen coefficient of the fluid, which is positive in standard fluids.

More generally, in order to find relevant asymptotic regimes, we seek solutions to (EK) of the form

waveansatz (2)
$$\rho(t,x) = \varrho + \eta\hat{\rho}(\varepsilon t, \varepsilon x)\,, \qquad\qquad \mathbf{u}(t,x) = \eta\hat{\mathbf{u}}(\varepsilon t, \varepsilon x)\,,$$

with $\eta > 0$ and $\varepsilon > 0$ some small, a priori independent parameters. The former gives an order of magnitude for the amplitude of solutions, and $1/\varepsilon$ is a spatio-temporal scale on which solutions are supposed to vary significantly.

After the linear wave regime considered in Section 3, the Korteweg-de Vries regime described above - which corresponds to the special case $\eta = \varepsilon^2$ in (2) - is fully justified in Section 4 for solutions to (EK) with well-prepared initial data, along with alternative regimes in which dispersive effects are weaker - *i.e* when $\varepsilon^2 \ll \eta$. Section 5 is devoted to more general initial data, and asymptotic regimes obtained by decoupling left-going and right-going waves. Finally, multidimensional, weakly transverse effects are taken into account in Section 6, in which we justify the so-called Kadomtsev-Petviashvili regime for (EK).

2 Preliminary material

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2.1 Statement of uniforms bounds

The ansatz (2) obviously transforms (EK) into the rescaled system

$$(\mathrm{EK}_{\varepsilon,\eta}) \qquad \begin{cases} \partial_T\hat{\rho} + \nabla_X\cdot((\varrho+\eta\hat{\rho})\hat{\mathbf{u}}) = 0 \\[2ex] \partial_T\hat{\mathbf{u}} + \eta(\hat{\mathbf{u}}\cdot\nabla_X)\hat{\mathbf{u}} + g'(\varrho+\eta\hat{\rho})\nabla_X\hat{\rho} \\ \qquad\qquad = \varepsilon^2\nabla_X\Big(K(\varrho+\eta\hat{\rho})\Delta_X\hat{\rho} + \dfrac{\eta}{2}K'(\varrho+\eta\hat{\rho})|\nabla_X\hat{\rho}|^2\Big)\,, \end{cases}$$

where $T = \varepsilon t$, $X = \varepsilon x$. Note that the acoustic equations are formally obtained by setting $\eta = 0$, $\varepsilon = 0$ in $(\mathrm{EK}_{\varepsilon,\eta})$. For $\eta > 0$, $\varepsilon > 0$, the local well-posedness of the Cauchy problem associated with $(\mathrm{EK}_{\varepsilon,\eta})$ follows from the following result.

Theorem 1 **([4])** *Let us take $s > 1 + \dfrac{d}{2}$, and $(\rho^{\mathrm{in}}, \mathbf{u}^{\mathrm{in}}) \in (\varrho, 0) + H^{s+1}(\mathbb{R}^d) \times (H^s(\mathbb{R}^d))^d$ such that ρ^{in} is positive and bounded by below in $\mathbb{R}^d$. Then, there exists a maximal time $t_* > 0$ such that the system* (EK) *possesses a unique solution*

$$(\rho, \mathbf{u}) \in (\varrho, 0) + \mathscr{C}([0, t_*], H^{s+1}(\mathbb{R}^d) \times (H^s(\mathbb{R}^d))^d) \cap \mathscr{C}^1([0, t_*], H^{s-1}(\mathbb{R}^d) \times (H^{s-2}(\mathbb{R}^d))^d)$$

such that $(\rho, \mathbf{u})(0) = (\rho^{\mathrm{in}}, \mathbf{u}^{\mathrm{in}})$. Moreover, the mapping $(\rho^{\mathrm{in}}, \mathbf{u}^{\mathrm{in}}) \mapsto (\rho, \mathbf{u})$ is continuous.

However, we need refined estimates of solutions that: 1) keep track of the parameters (η, ε); 2) take into account the nonlinear term $g'(\varrho + \eta\hat{\rho})\nabla_X \hat{\rho}$ - not as a source term as in [4] -, which will be possible thanks to the positiviy of $g'(\varrho)$. Furthermore, the following result shows that, as expected, the smaller the initial data, the longer the time of existence of the solution.

Theorem 2 *Let s be a real number greater than $1 + \dfrac{d}{2}$ and $\eta \in (0, 1]$. For $M > 0$, we consider*

$$B_\varepsilon(M) \stackrel{\text{def}}{=} \{(\hat{\rho}, \hat{\mathbf{u}}) \in H^{s+1}(\mathbb{R}^d) \times (H^s(\mathbb{R}^d))^d\, ;\ \|(\hat{\rho}, \hat{\mathbf{u}})\|_{(H^s(\mathbb{R}^d))^{d+1}} + \varepsilon \|\hat{\rho}\|_{H^{s+1}(\mathbb{R}^d)} \leqslant M\}\,.$$

If $\varrho > 0$, $g'(\varrho) > 0$, and $(\hat{\rho}^{\mathrm{in}}, \hat{\mathbf{u}}^{\mathrm{in}}) \in B_\varepsilon(M)$, then there exists $T_* > 0$, depending only on M, s and d, such that the maximal solution to $(\mathrm{EK}_{\varepsilon,\eta})$ *such that $(\hat{\rho}, \hat{\mathbf{u}})(0) = (\hat{\rho}^{\mathrm{in}}, \hat{\mathbf{u}}^{\mathrm{in}})$ exists at least on $[0, T_*/\eta]$, and $(\hat{\rho}, \hat{\mathbf{u}})(T) \in B_\varepsilon(2M)$ for all $T \in [0, T_*/\eta]$.*

A similar result is shown in [6, Theorem 1] for the hydrodynamical formulation of the Gross–Pitaevskii equation obtained with the Madelung transform. However, it is stated in terms of $\|(\hat{\mathbf{u}}, \hat{\rho})\|_{H^s \times H^{s+1}}$ instead of $\|(\hat{\rho}, \hat{\mathbf{u}})\|_{H^s} + \varepsilon \|\hat{\rho}\|_{H^{s+1}}$ (with our notations), which seems to be a slight mistake. A priori estimates rely indeed on Proposition 1 in [6], in which some quantity denoted by z is controlled in H^s, but the imaginary part of z is $2\frac{\nabla\rho}{\rho}$ with $\rho = 1 + \varepsilon\hat{\rho}$, so that only $\varepsilon\|\hat{\rho}\|_{H^{s+1}}$ is controlled. The estimate in [6, Theorem 2] should certainly be modified accordingly. Apart from this harmless correction, the main novelty here compared to [6] is twofold. First, the capillarity is arbitrary, which means in particular that it is not assumed to be proportional to $1/\rho$. As already known from [4], the a priori estimates are much trickier when $\rho K(\rho)$ is not constant. The other point is that we do not assume the vector field $\mathbf{u}$ to be potential - unlike what happens when dealing with the fluid formulation of NLS. This is again known to make a priori estimates more complicated.

Remark 1 The special case $\eta = \varepsilon^2$ is called the *Boussinesq regime*. If, in addition, the capillarity K is a positive constant and g is a convex, quadratic polynomial (*i.e.* $g' =$ constant > 0), then $(\mathrm{EK}_{\varepsilon,\varepsilon^2})$ belongs to the (a, b, c, d)-class of Boussinesq type systems as introduced in [9] and [10], with $a = b = d = 0$ and $c = -K < 0$. In this case, the existence and uniqueness of (strong) solutions on the time scale ε^{-2} has been shown by Saut and Lu [23], using hyperbolic techniques (see Theorem 1.1 in [23], case (12) in the sense of their definition 1.2). Our own result (Theorem 2 here above) applied to $\eta = \varepsilon^2$, $K =$ constant > 0, $g' =$ constant > 0, provides an alternative proof of theirs in that case.

Theorem 2 is a building block for the rigorous justification of asymptotic regimes. We need some material in order to prove it.

2.2 Basic tools for the proof of uniform bounds

As in [4], we shall derive uniform Sobolev bounds through an extended formulation of the system (EK). The idea is to introduce the complex-valued unknown $\mathbf{z} = \mathbf{u} + i\mathbf{w}$ which is naturally involved in the global energy

$$\mathscr{E} = \int_{\mathbb{R}^d} \left(\frac{1}{2}\rho|\mathbf{u}|^2 + F(\rho) + \frac{1}{2}K(\rho)|\nabla\rho|^2 \right) \, dx \, .$$

This integral is indeed well defined provided that we redefine $F(\rho) \stackrel{\text{def}}{=} \displaystyle\int_{\varrho}^{\rho} g$, and conserved along (smooth) solutions $(\rho, \mathbf{u})$ to (EK) that tend to $(\varrho, 0)$ sufficiently fast at infinity. Now, we have

$$\frac{1}{2}\rho|\mathbf{u}|^2 + \frac{1}{2}K(\rho)|\nabla\rho|^2 = \frac{1}{2}\rho|\mathbf{z}|^2 \quad \text{with } \mathbf{z} = \mathbf{u} + i\mathbf{w}\, , \; \mathbf{w} \stackrel{\text{def}}{=} \sqrt{\frac{K(\rho)}{\rho}}\nabla\rho \, .$$

Then, if we also introduce

$$a(\rho) \stackrel{\text{def}}{=} \sqrt{\rho K(\rho)}\, , \qquad\qquad b(\rho) \stackrel{\text{def}}{=} \frac{\rho g'(\rho)}{a(\rho)}\, ,$$

by differentiating the first equation in (EK) we obtain the following, equivalent system for $(\rho, \mathbf{z})$,

$$\text{(ES)} \qquad \left\{ \begin{array}{l} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0 \\[2ex] \partial_t \mathbf{z} + (\mathbf{u} \cdot \nabla)\mathbf{z} + i(\nabla \mathbf{z})\mathbf{w} + b(\rho)\mathbf{w} + i\nabla(a(\rho)\nabla \cdot \mathbf{z}) = 0 \, , \end{array} \right.$$

in which the notation $(\nabla\mathbf{z})\mathbf{w}$ stands for the standard product of the matrix-valued function $\nabla\mathbf{z} = (\partial_j \mathbf{z}_k)_{1 \leqslant j,k \leqslant d}$ and the vector field $\mathbf{w}$, so that

$$((\nabla\mathbf{z})\mathbf{w})_j = \sum_{k=1}^{d} (\partial_j \mathbf{z}_k)\mathbf{w}_k = (\partial_j \mathbf{z}) \cdot \mathbf{w} \, .$$

The scaling in (2) urges us to define

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$$\hat{\mathbf{w}} \stackrel{\text{def}}{=} \varepsilon\sqrt{\frac{K(\rho)}{\rho}}\nabla_X\hat{\rho}\,, \qquad \hat{\mathbf{z}} \stackrel{\text{def}}{=} \hat{\mathbf{u}} + i\hat{\mathbf{w}}\,, \tag{3}$$

so that $\mathbf{z}(t,x) = \eta(\hat{\mathbf{u}} + i\hat{\mathbf{w}})(T,X) = \eta\hat{\mathbf{z}}(T,X)$, and (ES) equivalently reads

$$(\mathrm{ES}_{\varepsilon,\eta}) \qquad \begin{cases} \partial_T\rho + \eta\nabla_X\cdot(\rho\hat{\mathbf{u}}) = 0 \\ \\ \partial_T\hat{\mathbf{z}} + \eta(\hat{\mathbf{u}}\cdot\nabla_X)\hat{\mathbf{z}} + i\eta(\nabla_X\hat{\mathbf{z}})\hat{\mathbf{w}} + \dfrac{1}{\varepsilon}\,b(\rho)\hat{\mathbf{w}} + i\varepsilon\nabla_X(a(\rho)\nabla_X\cdot\hat{\mathbf{z}}) = 0\,. \end{cases}$$

Our main purpose here is to derive some a priori estimates for solutions to $(\mathrm{ES}_{\varepsilon,\eta})$ that are valid uniformly in (ε,η). In this respect, we are going to use a modified version of the energy

$$\mathscr{E} = \frac{\eta^2}{2\varepsilon^d}\int_{\mathbb{R}^d}\left(\rho|\hat{\mathbf{z}}|^2 + \frac{2}{\eta^2}(F(\varrho+\eta\hat{\rho}) - F(\varrho))\right)\,dy\,,$$

obtained by expanding F about ϱ and by omitting the linear term in $\hat{\rho}$. Indeed, the latter does not contribute to - at least the lowest order - a priori estimates since $\hat{\rho}$ is conserved. Forgetting also the harmless factor $\eta^2\varepsilon^{-d}$, the modified energy reads

$$E_0[\hat{\rho},\hat{\mathbf{z}}] \stackrel{\text{def}}{=} \frac{1}{2}\int_{\mathbb{R}^d}\rho|\hat{\mathbf{z}}|^2 + g'(\rho)\hat{\rho}^2\,dy\,, \qquad \rho = \varrho + \eta\hat{\rho}\,.$$

Clearly, even though E_0 depends on η through ρ, the assumption $g'(\varrho) > 0$ ensures that $\sqrt{E_0[\hat{\rho},\hat{\mathbf{z}}]}$ is equivalent to the L^2 norm of $(\hat{\rho},\hat{\mathbf{z}})$ as long as ρ and $g'(\rho)$ remain bounded and bounded away from zero. Moreover, going back to (3), we may see E_0 as a functional applied to $(\hat{\rho},\hat{\mathbf{u}})$, and, as such, $E_0[\hat{\rho},\hat{\mathbf{u}}]$ enjoys the following estimates.

degrezero **Proposition 1** *Let $r \in (0,\varrho/2]$ be such that $g'(\rho) > 0$ and $K(\rho) > 0$ if $|\rho-\varrho| \leqslant r$. Then for all $(\hat{\rho},\hat{\mathbf{u}}) \in H^1\times L^2$ such that $\|\hat{\rho}\|_{L^\infty} \leqslant r$, for all $\eta \in (0,1]$, for all $\varepsilon > 0$,*

$$c_0(\|\hat{\mathbf{u}}\|_{L^2}^2 + \|\hat{\rho}\|_{L^2}^2 + \varepsilon^2\|\hat{\rho}\|_{H^1}^2) \leqslant E_0[\hat{\rho},\hat{\mathbf{u}}] \leqslant C_0(\|\hat{\mathbf{u}}\|_{L^2}^2 + \|\hat{\rho}\|_{L^2}^2 + \varepsilon^2\|\hat{\rho}\|_{H^1}^2)\,,$$

where $c_0 > 0$ and $C_0 > 0$ depend only on r (and the functions g, K).

Proof. We obviously have these inequalities with, explicitly,

$$c_0 \stackrel{\text{def}}{=} \frac{1}{2}\min_{|\rho-\varrho|\leqslant r}\min\left(\rho, g'(\rho), \sqrt{K(\rho)/\rho}\right)\,, \qquad C_0 \stackrel{\text{def}}{=} \frac{1}{2}\max_{|\rho-\varrho|\leqslant r}\max\left(\rho, g'(\rho), \sqrt{K(\rho)/\rho}\right)\,. \qquad \square$$

Now, the following, zero-th order a priori estimate is reminiscent of the fact that the exact energy $\mathscr{E}$ is conserved along solutions of (EK).

Proposition 2 *Let $\eta \in (0,1]$. Assume that $(\hat\rho, \hat{\mathbf{u}}) \in \mathscr{C}([0,t_*], H^{s+1}(\mathbb{R}^d) \times (H^s(\mathbb{R}^d))^d) \cap \mathscr{C}^1([0,t_*], H^{s-1}(\mathbb{R}^d) \times (H^{s-2}(\mathbb{R}^d))^d)$ is a solution of* $(\mathrm{EK}_{\varepsilon,\eta})$ *for some $s > 1 + d/2$, such that $\|\hat\rho\|_{L^\infty} \leqslant r$, where r is as in Proposition 1. Then there exists $C > 0$ depending only on r such that*

$$\frac{d}{dT} E_0[\hat\rho, \hat{\mathbf{u}}] \leqslant C\eta \|(\nabla_X \hat\rho, \nabla_X \mathbf{u})\|_{L^\infty} E_0[\hat\rho, \hat{\mathbf{u}}] \,.$$

Proof. Of course, we are going to use that $(\hat\rho, \hat{\mathbf{z}} = \hat{\mathbf{u}} + i\hat{\mathbf{w}})$ solves $(\mathrm{ES}_{\varepsilon,\eta})$ if $\rho\hat{\mathbf{w}} = \varepsilon a(\rho) \nabla_X \hat\rho$ - this equality just being a different way of writing (3). Recall also that $a(\rho)b(\rho) = \rho g'(\rho)$. The notation $\langle \cdot, \cdot \rangle$ will stand everywhere for real-valued inner products, and more precisely $\langle \mathbf{z}, \boldsymbol{\zeta} \rangle = \frac{1}{2} \sum_{j=1}^d (\overline{\mathbf{z}}_j \boldsymbol{\zeta}_j + \mathbf{z}_j \overline{\boldsymbol{\zeta}}_j)$ for all $\mathbf{z}$, $\boldsymbol{\zeta} \in \mathbb{C}^d$ (whatever d, including $d = 1$). Using $(\mathrm{ES}_{\varepsilon,\eta})$, we find by straightforward differentiation that

$$\begin{aligned} 2\frac{d}{dT} E_0[\hat\rho, \hat{\mathbf{u}}] = &- \eta \int \nabla_X \cdot (\rho\hat{\mathbf{u}}) |\hat{\mathbf{z}}|^2 - \eta \int \rho\hat{\mathbf{u}} \cdot \nabla_X |\hat{\mathbf{z}}|^2 - 2\eta \int \rho \langle i(\nabla_X \hat{\mathbf{z}})\hat{\mathbf{w}}, \hat{\mathbf{z}} \rangle \\ &- \frac{2}{\varepsilon} \int \rho b(\rho) \langle \hat{\mathbf{w}}, \hat{\mathbf{u}} \rangle - 2\varepsilon \int \rho \langle i \nabla_X (a(\rho) \nabla_X \cdot \hat{\mathbf{z}}), \hat{\mathbf{z}} \rangle \\ &+ \int \hat{\mathbf{u}} \cdot \nabla_X (2\rho g'(\rho)\hat\rho + \eta(\rho g''(\rho) - g'(\rho))\hat\rho^2) \,. \end{aligned}$$

By the relations recalled above and an integration by part, this reduces to

$$\begin{aligned} 2\frac{d}{dT} E_0[\hat\rho, \hat{\mathbf{u}}] = &- 2\varepsilon \int a(\rho) \langle i(\nabla_X \hat{\mathbf{z}}) \cdot \nabla_X \rho, \hat{\mathbf{z}} \rangle \\ &- 2\int \rho g'(\rho)(\hat{\mathbf{u}} \cdot \nabla_X \hat\rho) + 2\varepsilon \int a(\rho) \langle i \nabla_X \cdot \hat{\mathbf{z}}, \hat{\mathbf{z}} \cdot \nabla_X \rho \rangle \\ &+ \int \hat{\mathbf{u}} \cdot \nabla_X (2\rho g'(\rho)\hat\rho + \eta(\rho g''(\rho) - g'(\rho))\hat\rho^2) \,. \end{aligned}$$

Now, using that $a(\rho)\nabla\rho$ is potential, we see that the ε-terms cancel out, and simplifying/integrating by parts the remaining terms we obtain

$$2\frac{d}{dT} E_0[\hat\rho, \hat{\mathbf{u}}] = 2\eta \int \partial_\rho(\rho g'(\rho)) \ \hat\rho \ \hat{\mathbf{u}} \cdot \nabla_X \hat\rho + \eta \int (\rho g''(\rho) - g'(\rho)) \ \hat\rho^2 \ \nabla_X \hat{\mathbf{u}} \,.$$

The claimed inequality thus holds true with

$$C = \frac{1}{c_0} \max_{|\rho - \varrho| \leqslant r} \Big(|\partial_\rho(\rho g'(\rho))| + |\rho g''(\rho) - g'(\rho)| \Big) \,. \qquad \square$$

Since it involves the $W^{1,\infty}$ norm of the solution $(\hat\rho, \hat{\mathbf{u}})$, the estimate in (2) is clearly not sufficient to get a priori estimates without loss of derivatives. In order to close the estimates, we need higher order ones. If s is a large enough integer, we may use

$$E_s[\hat\rho, \hat{\mathbf{z}}] \stackrel{\text{def}}{=} \sum_{\sigma=0}^{s} \dot{E}_\sigma(\hat\rho, \hat{\mathbf{z}}) \,,$$

$$\dot{E}_\sigma[\hat{\rho}, \hat{\mathbf{z}}] \stackrel{\text{def}}{=} \sum_{\substack{\alpha \in \mathbb{N}_0^d, \\ |\alpha| = \sigma}} \frac{\sigma!}{\alpha!} \int_{\mathbb{R}^d} \frac{1}{2} a(\rho)^\sigma \left(\rho |\partial^\alpha \hat{\mathbf{z}}|^2 + g'(\rho)(\partial^\alpha \hat{\rho})^2\right) \, dX \,, \quad \rho = \varrho + \eta \hat{\rho} \,,$$

where ∂^α stands for $\partial^{|\alpha|}/\partial X_1^{\alpha_1} ... \partial X_d^{\alpha_d}$. The coefficients $\frac{\sigma!}{\alpha!}$ here above, as well as the weights a^σ, are chosen so as to eliminate bad terms in our a priori estimates, as we shall see. The usefulness of these estimates will be based on the following, in which E_s is viewed as a functional applied to $(\hat{\rho}, \hat{\mathbf{u}})$, by using (3) as for E_0.

degresept **Proposition 3** *Let s be a positive integer. Let $r \in (0, \varrho/2]$ be such that $g'(\rho) > 0$ and $K(\rho) > 0$ if $|\rho - \varrho| \leqslant r$. Then for all $(\hat{\rho}, \hat{\mathbf{u}}) \in H^{s+1}(\mathbb{R}^d) \times (H^s(\mathbb{R}^d))^d$ such that $\|\hat{\rho}\|_{W^{1,\infty}} \leqslant r$, for all $\eta \in (0,1]$, for all $\varepsilon > 0$,*

$$c(\|\hat{\mathbf{u}}\|_{H^s}^2 + \|\hat{\rho}\|_{H^s}^2 + \varepsilon^2 \|\hat{\rho}\|_{H^{s+1}}^2) \leqslant E_s[\hat{\rho}, \hat{\mathbf{u}}] \leqslant C(\|\hat{\mathbf{u}}\|_{H^s}^2 + \|\hat{\rho}\|_{H^s}^2 + \varepsilon^2 \|\hat{\rho}\|_{H^{s+1}}^2) \,,$$

where $c > 0$ and $C > 0$ depend only on r, s, d (and the functions g, K).

Proof. As in the proof of Proposition 1, we readily see that

$$c_\sigma(\|\partial^\alpha \hat{\mathbf{z}}\|_{L^2}^2 + \|\partial^\alpha \hat{\rho}\|_{L^2}^2) \leqslant \int_{\mathbb{R}^d} a(\rho)^\sigma \left(\rho |\partial^\alpha \hat{\mathbf{z}}|^2 + g'(\rho)(\partial^\alpha \hat{\rho})^2\right) \, dX \leqslant C_\sigma(\|\partial^\alpha \hat{\mathbf{z}}\|_{L^2}^2 + \|\partial^\alpha \hat{\rho}\|_{L^2}^2) \,,$$

with

$$c_\sigma \stackrel{\text{def}}{=} \min_{|\rho - \varrho| \leqslant r} \left(a(\rho)^\sigma \min(\rho, g'(\rho), \sqrt{K(\rho)/\rho})) \right) , C_\sigma \stackrel{\text{def}}{=} \max_{|\rho - \varrho| \leqslant r} \left(a(\rho)^\sigma \max(\rho, g'(\rho), \sqrt{K(\rho)/\rho})) \right) .$$

By summation we thus find $c_s > 0$ and $C_s > 0$ such that

$$c_s(\|\hat{\mathbf{z}}\|_{H^s}^2 + \|\hat{\rho}\|_{H^s}^2) \leqslant E_s[\hat{\rho}, \hat{\mathbf{u}}] \leqslant C_s(\|\hat{\mathbf{z}}\|_{H^s}^2 + \|\hat{\rho}\|_{H^s}^2) \,.$$

So the only point is to check that $\|\hat{\mathbf{z}}\|_{H^s}^2 + \|\hat{\rho}\|_{H^s}^2$ is equivalent to $\|\hat{\mathbf{u}}\|_{H^s}^2 + \|\hat{\rho}\|_{H^s}^2 + \varepsilon^2 \|\hat{\rho}\|_{H^{s+1}}^2$ when $\hat{\mathbf{z}} = \hat{\mathbf{u}} + i \hat{\mathbf{w}}$, $\hat{\mathbf{w}} = \varepsilon c(\varrho + \eta \hat{\rho}) \nabla_X \hat{\rho}$ for some smooth function c - here $c(\rho) = \sqrt{K(\rho)/\rho}$. This comparison relies on Proposition A.1, which gives that

$$\begin{aligned} \|c(\varrho + \eta \hat{\rho}) \nabla_X \hat{\rho}\|_{H^s} &\leqslant c(\varrho) \|\nabla_X \hat{\rho}\|_{H^s} + \gamma \|c'\|_{W^{s,\infty}([\varrho - r, \varrho + r])} \, (1 + \|\hat{\rho}\|_{L^\infty})^\sigma \, \|\nabla_X \hat{\rho}\|_{L^\infty} \, \|\hat{\rho}\|_{H^s} \\ &\qquad + 2\gamma \|c\|_{L^\infty([\varrho - r, \varrho + r])} \|\nabla_X \hat{\rho}\|_{H^s} \\ &\leqslant C(\|\hat{\rho}\|_{W^{1,\infty}}) \, \|\hat{\rho}\|_{H^{s+1}} \,, \end{aligned}$$

and in a similar way, using the notation d for $1/c$,

$$\begin{aligned} \varepsilon \|\nabla_X \hat{\rho}\|_{H^s} &\leqslant d(\varrho) \|\hat{\mathbf{w}}\|_{H^s} + \gamma \|d'\|_{W^{s,\infty}([\varrho - r, \varrho + r])} \, (1 + \|\hat{\rho}\|_{L^\infty})^\sigma \, \|\hat{\mathbf{w}}\|_{L^\infty} \, \|\hat{\rho}\|_{H^s} \\ &\qquad + 2\gamma \|c\|_{L^\infty([\varrho - r, \varrho + r])} \|\hat{\mathbf{w}}\|_{H^s} \\ &\leqslant C(\|(\hat{\rho}, \hat{\mathbf{w}})\|_{L^\infty}) \, \|\hat{\rho}\|_{H^s} \, + \, C(r) \, \|\hat{\mathbf{w}}\|_{H^s} \,. \end{aligned}$$

Here above, γ stands for a 'universal' constant (depending only on s and d), and $C(q)$ stands for a positive number depending only on q, whatever the quantity q. We can thus conclude that

$$\|\hat{\mathbf{z}}\|_{H^s}^2 + \|\hat{\rho}\|_{H^s}^2 \leqslant \max(1, C(\|\hat{\rho}\|_{W^{1,\infty}})^2) \, (\|\hat{\mathbf{u}}\|_{H^s}^2 + \|\hat{\rho}\|_{H^s}^2 + \varepsilon^2 \|\hat{\rho}\|_{H^{s+1}}^2) \,,$$

$$\|\hat{\mathbf{u}}\|_{H^s}^2 + \|\hat{\rho}\|_{H^s}^2 + \varepsilon^2 \|\hat{\rho}\|_{H^{s+1}}^2 \leqslant \max(1 + 2C(\|(\hat{\rho}, \hat{\mathbf{w}})\|_{L^\infty})^2, C(r)^2) \, (\|\mathbf{z}\|_{H^s}^2 + \|\hat{\rho}\|_{H^s}^2) \,.$$

$\square$

2.3 Proof of uniform bounds in the potential case

In this section, we are going to show that for any integer $s > 1 + d/2$, E_s enjoys an a priori estimate that is similar to the one in Proposition 2 for E_0, at least when the velocity vector field $\mathbf{u}$ is potential. We start with this simpler case for the sake of clarity - all computations below are detailed enough to be readable without any pencil. As was noticed in [4], the fact that $\mathbf{u}$ is potential or, equivalently, that $\mathbf{u}$ is curl-free is preserved along (smooth) solutions. So it will be sufficient to assume that the initial velocity field is curl-free.

Proposition 4 *Assume that* $(\hat{\rho}, \hat{\mathbf{u}}) \in \mathscr{C}([0,t_*], H^{s+1}(\mathbb{R}^d)\times(H^s(\mathbb{R}^d))^d)\cap\mathscr{C}^1([0,t_*], H^{s-1}(\mathbb{R}^d)\times(H^{s-2}(\mathbb{R}^d))^d)$ *is a solution of* $(\mathrm{EK}_{\varepsilon,\eta})$ *for some integer* $s > 1 + d/2$*, such that* $\|\hat{\rho}\|_{L^\infty} \leqslant r$*, where* r *is as in Proposition 1. Assume moreover that* $\hat{\mathbf{u}}(0)$ *is curl-free. Then there exists* $C > 0$ *depending only on* r*,* s *and* d *such that*

$$\frac{d}{dT}E_s[\hat{\rho},\hat{\mathbf{u}}] \leqslant C\eta\Big(\|(\nabla_X\hat{\rho},\nabla_X\hat{\mathbf{u}})\|_{L^\infty} + \varepsilon\|D^2_X\hat{\rho}\|_{L^\infty}\Big)(1+\eta\varepsilon\|\nabla_X\hat{\rho}\|_{L^\infty})E_s[\hat{\rho},\hat{\mathbf{u}}]\,.$$

Proof. Let $0 \leqslant \sigma \leqslant s$ be given and $\alpha \in \mathbb{N}_0^d$ such that $|\alpha| = \sigma$. We work in the X variable only, and use the simplified notations $\partial_j = \partial_{X_j}$, $\nabla = \nabla_X$. We recall that when $\hat{\mathbf{z}}$ is related to $(\hat{\rho},\hat{\mathbf{u}})$ through (3), if the latter satisfies $(\mathrm{EK}_{\varepsilon,\eta})$ then $(\rho = \varrho + \eta\hat{\rho}, \hat{\mathbf{z}})$ satisfies $(\mathrm{ES}_{\varepsilon,\eta})$. Applying ∂^α to the second equation in $(\mathrm{ES}_{\varepsilon,\eta})$, we obtain

$$\begin{aligned}(4)\qquad \partial_T\partial^\alpha\hat{\mathbf{z}} &+ \eta(\hat{\mathbf{u}}\cdot\nabla)\partial^\alpha\hat{\mathbf{z}} + i\eta(\nabla\partial^\alpha\hat{\mathbf{z}})\hat{\mathbf{w}} + \frac{1}{\varepsilon}\,b(\rho)\partial^\alpha\hat{\mathbf{w}} + i\varepsilon\partial^\alpha\nabla(a(\rho)\nabla\cdot\hat{\mathbf{z}})\\ &= \eta[\hat{\mathbf{u}}\cdot\nabla,\partial^\alpha]\hat{\mathbf{z}} + i\eta\Big((\nabla\partial^\alpha\hat{\mathbf{z}})\hat{\mathbf{w}} - \partial^\alpha((\nabla\hat{\mathbf{z}})\hat{\mathbf{w}})\Big) + \frac{1}{\varepsilon}\,[b(\rho),\partial^\alpha]\hat{\mathbf{w}} \stackrel{\text{def}}{=} \mathbf{R}\,.\end{aligned}$$

Here above, the notation $[\cdot,\cdot]$ stands for a commutator, that is,

$$[\partial^\alpha,\hat{\mathbf{u}}\cdot\nabla]\hat{\mathbf{z}} \stackrel{\text{def}}{=} \partial^\alpha((\hat{\mathbf{u}}\cdot\nabla)\hat{\mathbf{z}}) - (\hat{\mathbf{u}}\cdot\nabla)(\partial^\alpha\hat{\mathbf{z}})\,,\quad [\partial^\alpha,b(\rho)]\hat{\mathbf{w}} \stackrel{\text{def}}{=} \partial^\alpha(b(\rho)\hat{\mathbf{w}}) - b(\rho)\partial^\alpha\hat{\mathbf{w}}\,.$$

All three commutators in the right-hand side $\mathbf{R}$ of (4) can be estimated by using the inequality (A.3) recalled in the appendix, and by noting in addition that $[\partial^\alpha, b(\rho)] = [\partial^\alpha, b(\rho) - b(\varrho)]$ (since ϱ is constant), and, by definition of $\hat{\mathbf{w}}$, that

$$\|\hat{\mathbf{w}}\|_{H^{s-1}} \leqslant C(r)\varepsilon\|\nabla\hat{\rho}\|_{H^{s-1}} \leqslant C(r)\varepsilon\|\hat{\rho}\|_{H^s} \leqslant C(r)\varepsilon\sqrt{E_s[\hat{\rho},\hat{\mathbf{z}}]}$$

(by definition of E_s). We then infer that

$$\begin{aligned}\|\mathbf{R}\|_{L^2} &\leqslant C(r,s,d)\eta\Big(\|\nabla\hat{\mathbf{z}}\|_{L^\infty}\|\hat{\mathbf{z}}\|_{H^s} + \frac{1}{\varepsilon}\|\hat{\rho}\|_{H^s}\|\hat{\mathbf{w}}\|_{L^\infty} + \frac{1}{\varepsilon}\|\nabla\hat{\rho}\|_{L^\infty}\|\hat{\mathbf{w}}\|_{H^{s-1}}\Big)\\ &\leqslant C(r,s,d)\eta\|(\nabla\hat{\mathbf{z}},\nabla\hat{\rho})\|_{L^\infty}\sqrt{E_s[\hat{\rho},\hat{\mathbf{z}}]}\,.\end{aligned}$$

Here above and in what follows, $C(q)$ stands for a positive number depending only on q, whatever the quantity q. For convenience, the actual value of $C(q)$ may change from line to line. Therefore, using that

$$\frac{\partial}{\partial T}(a^\sigma(\rho)\rho) + \eta\hat{\mathbf{u}}\cdot\nabla(a^\sigma(\rho)\rho) + \eta\rho\partial_\rho(\rho a^\sigma(\rho))\nabla\cdot\hat{\mathbf{u}} = 0$$

by the first equation in ($\mathrm{ES}_{\varepsilon,\eta}$), we obtain after integrations by parts that

$$
\frac{d}{dT}\int_{\mathbb{R}^d}\rho a^\sigma(\rho)|\partial^\alpha\hat{\mathbf{z}}|^2\,dX \leqslant \sum_{k=1}^{6}\mathcal{I}_k + C(r,s,d)\eta\|(\nabla\hat{\mathbf{z}},\nabla\hat\rho)\|_{L^\infty}E_s[\hat\rho,\hat{\mathbf{z}}]\,, \tag{5}
$$

$$
\begin{aligned}
\mathcal{I}_1 &\stackrel{\text{def}}{=} -\eta\int_{\mathbb{R}^d}(\hat{\mathbf{u}}\cdot\nabla(\rho a^\sigma(\rho)))|\partial^\alpha\hat{\mathbf{z}}|^2\,dX\,,\\
\mathcal{I}_2 &\stackrel{\text{def}}{=} -\eta\int_{\mathbb{R}^d}\rho\partial_\rho(\rho a^\sigma(\rho))(\nabla\cdot\hat{\mathbf{u}})|\partial^\alpha\hat{\mathbf{z}}|^2\,dX\,,\\
\mathcal{I}_3 &\stackrel{\text{def}}{=} \eta\int_{\mathbb{R}^d}(\nabla\cdot(\rho a^\sigma(\rho)\hat{\mathbf{u}}))|\partial^\alpha\hat{\mathbf{z}}|^2\,dX\,,\\
\mathcal{I}_4 &\stackrel{\text{def}}{=} -2\eta\int_{\mathbb{R}^d}\rho a^\sigma(\rho)\langle i(\nabla(\partial^\alpha\hat{\mathbf{z}}))\hat{\mathbf{w}},\partial^\alpha\hat{\mathbf{z}}\rangle\,dX\,,\\
\mathcal{I}_5 &\stackrel{\text{def}}{=} -\frac{2}{\varepsilon}\int_{\mathbb{R}^d}\rho a^\sigma(\rho)b(\rho)\langle\partial^\alpha\hat{\mathbf{w}},\partial^\alpha\hat{\mathbf{u}}\rangle\,dX\,,\\
\mathcal{I}_6 &\stackrel{\text{def}}{=} 2\varepsilon\int_{\mathbb{R}^d}\rho a^\sigma(\rho)\langle i\partial^\alpha\hat{\mathbf{z}},\partial^\alpha(\nabla(a(\rho)\nabla\cdot\hat{\mathbf{z}}))\rangle\,dX\,.
\end{aligned}
$$

We can expand the divergence in $\mathcal{I}_3$ and notice that the term involving $\hat{\mathbf{u}}\cdot\nabla(\rho a^\sigma(\rho))$ cancels out with $\mathcal{I}_1$. As a consequence,

$$
\begin{aligned}
\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3 &= \eta\int_{\mathbb{R}^d}\Big(\rho a^\sigma(\rho)-\rho\partial_\rho(\rho a^\sigma(\rho))\Big)(\nabla\cdot\hat{\mathbf{u}})|\partial^\alpha\hat{\mathbf{z}}|^2\,dX\\
&= -\eta\int_{\mathbb{R}^d}\rho^2\partial_\rho(a^\sigma(\rho))(\nabla\cdot\hat{\mathbf{u}})|\partial^\alpha\hat{\mathbf{z}}|^2\,dX \leqslant C(r,s,d)\eta\|\nabla\cdot\hat{\mathbf{u}}\|_{L^\infty}E_s[\hat\rho,\hat{\mathbf{z}}]\,.
\end{aligned}
$$

Concerning $\mathcal{I}_6$, an integration by parts and the Leibniz formula give

$$
\begin{aligned}
\mathcal{I}_6 &= -2\varepsilon\int_{\mathbb{R}^d}\langle i\nabla\cdot(\rho a^\sigma(\rho)\partial^\alpha\hat{\mathbf{z}}),\partial^\alpha(a(\rho)\nabla\cdot\hat{\mathbf{z}})\rangle\,dX\\
&= -2\varepsilon\int_{\mathbb{R}^d}\Big\langle i\rho a^\sigma(\rho)\nabla\cdot(\partial^\alpha\hat{\mathbf{z}})+i(\nabla(\rho a^\sigma(\rho)))\cdot\partial^\alpha\hat{\mathbf{z}},\\
&\qquad\qquad a(\rho)\nabla\cdot(\partial^\alpha\hat{\mathbf{z}})+\sum_{\substack{\beta\leqslant\alpha,\\|\beta|=\sigma-1}}\binom{\alpha}{\beta}\partial^{\alpha-\beta}(a(\rho))\nabla\cdot(\partial^\beta\hat{\mathbf{z}})+\mathbf{L}\Big\rangle\,dX\,,
\end{aligned}
$$

where the lower order terms in $\mathbf{L}$ are such that

$$
\begin{aligned}
\varepsilon\|\nabla\mathbf{L}\|_{L^2} &\leqslant C(r,s,d)\eta\Big(\|\hat{\mathbf{z}}\|_{H^s}\|\varepsilon D^2\hat\rho\|_{L^\infty}+\varepsilon\|\hat\rho\|_{H^{s+1}}\|\nabla\hat{\mathbf{z}}\|_{L^\infty}\Big)\\
&\leqslant C(r,s,d)\eta\|(\nabla\hat\rho,\nabla\hat{\mathbf{z}})\|_{L^\infty}\sqrt{E_s[\hat\rho,\hat{\mathbf{z}}]}\,.
\end{aligned} \tag{6}
$$

We now expand the big inner product involved in $\mathcal{I}_6$, and notice that:

- the term $\langle i\rho a^\sigma(\rho)\nabla\cdot(\partial^\alpha\hat{\mathbf{z}}),a(\rho)\nabla\cdot(\partial^\alpha\hat{\mathbf{z}})\rangle$ vanishes point wise (recall that $\langle\cdot,\cdot\rangle$ stands for a for real-valued inner product);

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- by (6) and an integration by parts, the contribution of $\mathbf{L}$ to $\mathcal{I}_6$ is bounded by
$$\varepsilon C(r,s,d)\eta\|(\nabla\hat\rho,\nabla\hat{\mathbf{z}})\|_{L^\infty}(1+\eta\|\nabla\hat\rho\|_{L^\infty})E_s[\hat\rho,\hat{\mathbf{z}}]\,;$$
- the contribution of derivatives of $\hat{\mathbf{z}}$ of order σ, coming from the inner product of the second term in the left factor and the sum on β in the right factor of the integrand, is bounded by $\varepsilon C(r,s,d)\eta^2\|\nabla\hat\rho\|^2_{L^\infty}\|\hat{\mathbf{z}}\|^2_{H^s}$ by the Cauchy-Schwarz inequality.

This in turn gives
$$\begin{aligned}\mathcal{I}_6\leqslant&-2\varepsilon\int_{\mathbb{R}^d}\Big\langle i\rho a^\sigma(\rho)\nabla\cdot(\partial^\alpha\hat{\mathbf{z}}),\sum_{\substack{\beta\leqslant\alpha,\\|\beta|=\sigma-1}}\binom{\alpha}{\beta}\partial^{\alpha-\beta}(a(\rho))\nabla\cdot(\partial^\beta\hat{\mathbf{z}})\Big\rangle\,dX\\&+\mathcal{S}+C(r,s,d)\eta\varepsilon\|(\nabla\hat\rho,\nabla\hat{\mathbf{z}})\|_{L^\infty}(1+\eta\|\nabla\hat\rho\|_{L^\infty})E_s[\hat\rho,\hat{\mathbf{z}}]\,,\\\mathcal{S}&\stackrel{\text{def}}{=}-2\varepsilon\int_{\mathbb{R}^d}\langle i(\nabla(\rho a^\sigma(\rho)))\cdot\partial^\alpha\hat{\mathbf{z}},a(\rho)\nabla\cdot(\partial^\alpha\hat{\mathbf{z}})\rangle\,dX\,.\end{aligned}$$

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By (3) we readily have that $\nabla(\rho a^\sigma(\rho))=\dfrac{\eta\rho}{\varepsilon a(\rho)}\partial_\rho(\rho a^\sigma(\rho))\hat{\mathbf{w}}$, and integrating by parts once more we see that
$$\mathcal{S}\leqslant 2\eta\int_{\mathbb{R}^d}\rho\partial_\rho(\rho a^\sigma(\rho))\langle i\nabla(\partial^\alpha\hat{\mathbf{z}})\hat{\mathbf{w}},\partial^\alpha\hat{\mathbf{z}}\rangle\,dX+2\eta\|\nabla(\rho\partial_\rho(\rho a^\sigma(\rho))\hat{\mathbf{w}})\|_{L^\infty}E_s[\hat\rho,\hat{\mathbf{z}}]\,.$$
We now use that
$$\begin{aligned}\eta\|\nabla(\rho\partial_\rho(a^\sigma(\rho)\rho)\hat{\mathbf{w}})\|_{L^\infty}&\leqslant C(r,s,d)\eta\Big(\eta\|\hat{\mathbf{w}}\|_{L^\infty}\|\nabla\hat\rho\|_{L^\infty}+\|\nabla\hat{\mathbf{w}}\|_{L^\infty}\Big)\\&\leqslant C(r,s,d)\varepsilon\eta\Big(\eta\|\nabla\hat\rho\|^2_{L^\infty}+\|D^2\hat\rho\|_{L^\infty}\Big)\end{aligned}$$
to infer
$$\mathcal{S}\leqslant 2\eta\int_{\mathbb{R}^d}\rho\partial_\rho(\rho a^\sigma(\rho))\langle i\nabla(\partial^\alpha\hat{\mathbf{z}})\hat{\mathbf{w}},\partial^\alpha\hat{\mathbf{z}}\rangle\,dX+C(r,s,d)\varepsilon\eta\Big(\eta\|\nabla\hat\rho\|^2_{L^\infty}+\|D^2\hat\rho\|_{L^\infty}\Big)E_s[\hat\rho,\hat{\mathbf{z}}]\,.$$
Since $\partial_\rho(\rho a^\sigma(\rho))=a^\sigma(\rho)+\rho\partial_\rho(a^\sigma(\rho))$, the addition of
$$\mathcal{I}_4=-2\eta\int_{\mathbb{R}^d}a^\sigma(\rho)\rho\langle i(\nabla(\partial^\alpha\hat{\mathbf{z}}))\hat{\mathbf{w}},\partial^\alpha\hat{\mathbf{z}}\rangle\,dX$$
to $\mathcal{I}_6$ cancels out the term involving $a^\sigma(\rho)$ in the bound found above for $\mathcal{S}$, so that

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$$\begin{aligned}\mathcal{I}_4+\mathcal{I}_6\leqslant&\, C(r,s,d)\varepsilon\eta\Big(\|(\nabla\hat\rho,\nabla\hat{\mathbf{z}},D^2\hat\rho)\|_{L^\infty}+\eta\|(\nabla\hat\rho,\nabla\hat{\mathbf{z}})\|_{L^\infty}\|\nabla\hat\rho\|_{L^\infty}\Big)E_s[\hat\rho,\hat{\mathbf{z}}]\\&+\mathcal{K}+\sum_{\substack{\beta\leqslant\alpha,\\|\beta|=\sigma-1}}\mathcal{J}_\beta\,,\\\mathcal{J}_\beta&\stackrel{\text{def}}{=}-2\varepsilon\int_{\mathbb{R}^d}\rho a^\sigma(\rho)\Big\langle i\nabla\cdot(\partial^\alpha\hat{\mathbf{z}}),\binom{\alpha}{\beta}\partial^{\alpha-\beta}(a(\rho))\nabla\cdot(\partial^\beta\hat{\mathbf{z}})\Big\rangle\,dX\,,\\\mathcal{K}&\stackrel{\text{def}}{=}2\eta\int_{\mathbb{R}^d}\rho^2\partial_\rho(a^\sigma(\rho))\langle i(\nabla\partial^\alpha\hat{\mathbf{z}})\hat{\mathbf{w}},\partial^\alpha\hat{\mathbf{z}}\rangle\,dX\,.\end{aligned}\tag{7}$$

If, for any smooth enough mapping $\mathbf{Z} : \mathbb{R}^d \to \mathbb{C}^d$ we denote by $\mathrm{curl}\mathbf{Z}$ the matrix-valued function defined by

$$(\mathrm{curl}\mathbf{Z})_{jk} = \partial_j \mathbf{Z}_k - \partial_k \mathbf{Z}_j \,,$$

we see that for $\mathbf{W}$, $\mathbf{Y}$ any other smooth enough mappings $\mathbb{R}^d \to \mathbb{C}^d$,

$$\langle (\nabla \mathbf{Z})\mathbf{W}, \mathbf{Y}\rangle = \langle (\mathbf{W}\cdot\nabla)\mathbf{Z}, \mathbf{Y}\rangle \;+\; \langle (\mathrm{curl}\mathbf{Z})\mathbf{W}, \mathbf{Y}\rangle \,.$$

In particular, we can write

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$$\begin{aligned} \mathcal{K} = 2\sigma\eta \int_{\mathbb{R}^d} & \rho^2 a^{\sigma-1}(\rho) a'(\rho) \langle i\hat{\mathbf{w}}\cdot\nabla\partial^\alpha \hat{\mathbf{z}}, \partial^\alpha\hat{\mathbf{z}}\rangle \; dX \\ &+ 2\sigma\eta \int_{\mathbb{R}^d} \rho^2 a^{\sigma-1}(\rho) a'(\rho) \langle i(\partial^\alpha \mathrm{curl}\,\hat{\mathbf{z}})\hat{\mathbf{w}}, \partial^\alpha\hat{\mathbf{z}}\rangle \; dX \,. \end{aligned} \tag{8}$$

On the other hand, using that $\partial^{\alpha-\beta}(a(\rho)) = a'(\rho)\partial^{\alpha-\beta}\rho$ when $\alpha-\beta$ has length one, we have

$$\mathcal{J}_\beta = -2\varepsilon \int_{\mathbb{R}^d} \rho a^\sigma(\rho) a'(\rho) (\partial^{\alpha-\beta}\rho) \langle i\nabla\cdot(\partial^\alpha\hat{\mathbf{z}}), \nabla\cdot(\partial^\beta\hat{\mathbf{z}})\rangle \; dX \,,$$

which gives, after integrating by parts and using (A.3) tamecomm,

$$\begin{aligned} \mathcal{J}_\beta \leqslant 2\varepsilon \int_{\mathbb{R}^d} & \rho a^\sigma(\rho) a'(\rho) (\partial^{\alpha-\beta}\rho) \langle i\partial^\alpha\hat{\mathbf{z}}, \nabla(\nabla\cdot(\partial^\beta\hat{\mathbf{z}}))\rangle \; dX \\ &+ C(r,s,d)\varepsilon \|\nabla(\rho a^\sigma(\rho)a'(\rho)\partial^{\alpha-\beta}\rho)\|_{L^\infty} E_s[\hat{\rho},\hat{\mathbf{z}}] \,. \end{aligned}$$

Now, observing that for any smooth enough mappings $\mathbf{Z}, \mathbf{Y} : \mathbb{R}^d \to \mathbb{C}^d$,

$$\langle \mathbf{Z}, \nabla(\nabla\cdot\mathbf{Y})\rangle = \langle \mathbf{Z}, \Delta\mathbf{Y}\rangle \;+\; \langle \mathbf{Z}, \nabla\cdot(\mathrm{curl}\mathbf{Y})\rangle \,,$$

(where we have used the notation $\nabla\cdot\mathbf{M}$ for the vector field defined by $(\nabla\cdot\mathbf{M})_j = \sum_{k=1}^d \partial_k \mathbf{M}_{jk}$, associated with the matrix-valued function $\mathbf{M} = \mathrm{curl}\mathbf{Y}$), we find that

$$\begin{aligned} \mathcal{J}_\beta \leqslant & C(r,s,d)\varepsilon\eta\Big(\eta\|\nabla\hat{\rho}\|_{L^\infty}^2 + \|D^2\hat{\rho}\|_{L^\infty}\Big) E_s[\hat{\rho},\hat{\mathbf{z}}] + 2\varepsilon \int_{\mathbb{R}^d} \rho a^\sigma(\rho) a'(\rho) (\partial^{\alpha-\beta}\rho) \langle i\partial^\alpha\hat{\mathbf{z}}, \Delta\partial^\beta\hat{\mathbf{z}}\rangle \; dX \\ &+ 2\varepsilon \int_{\mathbb{R}^d} \rho a^\sigma(\rho) a'(\rho) (\partial^{\alpha-\beta}\rho) \langle i\partial^\alpha\hat{\mathbf{z}}, \nabla\cdot(\partial^\beta \mathrm{curl}\hat{\mathbf{z}})\rangle \; dX \,. \end{aligned}$$

To finish with the estimate of $\mathcal{J}_\beta$, we integrate by parts again, and arrive at

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$$\begin{aligned} \mathcal{J}_\beta \leqslant & C(r,s,d)\varepsilon\eta\Big(\eta\|\nabla\hat{\rho}\|_{L^\infty}^2 + \|D^2\hat{\rho}\|_{L^\infty}\Big) E_s[\hat{\rho},\hat{\mathbf{z}}] \\ &- 2\varepsilon \int_{\mathbb{R}^d} \rho a^\sigma(\rho) a'(\rho) (\partial^{\alpha-\beta}\rho) \sum_{j=1}^d \langle i\partial^{\alpha-\beta}\partial_j\partial^\beta\hat{\mathbf{z}}, \partial_j\partial^\beta\hat{\mathbf{z}}\rangle \; dX \\ &- 2\varepsilon \int_{\mathbb{R}^d} \rho a^\sigma(\rho) a'(\rho) (\partial^{\alpha-\beta}\rho) \Big\langle i\partial^\alpha D\hat{\mathbf{z}}, \partial^\beta \mathrm{curl}\,\hat{\mathbf{z}}\Big\rangle_{\mathcal{M}_d(\mathbb{C})} \; dX \,, \end{aligned} \tag{9}$$

where $\langle A, B\rangle_{\mathcal{M}_d(\mathbb{C})} \stackrel{\text{def}}{=} \mathrm{Re}(\mathrm{Tr}(AB^*))$ is the usual real inner product on $\mathcal{M}_d(\mathbb{C})$, and $D\hat{\mathbf{z}} \stackrel{\text{def}}{=} (\nabla\hat{\mathbf{z}})^{\mathrm{T}}$.

The remaining term $\mathcal{I}_5$ will turn out to cancel out, up to a remainder term, with the time derivative of $\int_{\mathbb{R}^d} g'(\rho)a^\sigma(\rho)(\partial^\alpha\hat{\rho})^2\,dX$. In order to see this, we differentiate the first equation in $(\mathrm{ES}_{\varepsilon,\eta})$ and obtain

$$\partial_T\partial^\alpha\hat{\rho} + \eta(\hat{\mathbf{u}}\cdot\nabla)\partial^\alpha\hat{\rho} + \rho\nabla\cdot\partial^\alpha\hat{\mathbf{u}} = -\eta[\partial^\alpha, \hat{\mathbf{u}}\cdot\nabla]\hat{\rho} - [\partial^\alpha, \rho\nabla\cdot]\hat{\mathbf{u}}\,.$$

By (A.3), the commutators in the right-hand side here above have an L^2 norm bounded by

$$C(r,s,d)\,\eta\|(\nabla\hat{\rho},\nabla\hat{\mathbf{u}})\|_{L^\infty}\sqrt{E_s[\hat{\rho},\hat{\mathbf{z}}]}\,.$$

Furthermore, by the first equation in $(\mathrm{ES}_{\varepsilon,\eta})$ again, we have

$$\partial_T(g'(\rho)a^\sigma(\rho)) + \eta\hat{\mathbf{u}}\cdot\nabla(g'(\rho)a^\sigma(\rho)) + \eta\rho\partial_\rho(g'(\rho)a^\sigma(\rho))\nabla\cdot\hat{\mathbf{u}} = 0\,.$$

Arguing as for $\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3$, we thus find that

$$\begin{aligned}\frac{d}{dT}\int_{\mathbb{R}^d} g'(\rho)a^\sigma(\rho)(\partial^\alpha\hat{\rho})^2\,dX \leqslant{}& C(r,s,d)\,\eta\|(\nabla\hat{\rho},\nabla\hat{\mathbf{u}})\|_{L^\infty}E_s[\hat{\rho},\hat{\mathbf{z}}]\\ &-\int_{\mathbb{R}^d} 2\rho g'(\rho)a^\sigma(\rho)\partial^\alpha\hat{\rho}\nabla\cdot(\partial^\alpha\hat{\mathbf{u}})\,dX\,.\end{aligned}$$

Integrating by parts, using again that

$$\rho\hat{\mathbf{w}} = \varepsilon a(\rho)\nabla\hat{\rho}\,,\quad \|\hat{\mathbf{w}}\|_{H^{s-1}} \leqslant C_{s,d}\varepsilon\sqrt{E_s[\hat{\rho},\hat{\mathbf{z}}]}$$

and combining this with (A.3), we arrive at

$$\begin{aligned}\frac{d}{dT}\int_{\mathbb{R}^d} g'(\rho)a^\sigma(\rho)(\partial^\alpha\hat{\rho})^2\,dX \leqslant{}& C(r,s,d)\,\eta\|(\nabla\hat{\rho},\nabla\hat{\mathbf{u}})\|_{L^\infty}E_s[\hat{\rho},\hat{\mathbf{z}}]\\ &+\frac{2}{\varepsilon}\int_{\mathbb{R}^d}\rho\frac{\rho g'(\rho)}{a(\rho)}a^\sigma(\rho)\langle\partial^\alpha\hat{\mathbf{w}},\partial^\alpha\hat{\mathbf{u}}\rangle\,dX\,.\end{aligned}\tag{10}$$

Since $a(\rho)b(\rho) = \rho g'(\rho)$, the integral in the right-hand side of (10) here above cancels out with the integral $\mathcal{I}_5$ in (5). Therefore, using (8) and (9) in (7), and combining (10) with (5), we obtain

Calypso
$$
\begin{aligned}
&\frac{d}{dT}\int_{\mathbb{R}^d} \rho a^\sigma(\rho)|\partial^\alpha \hat{\mathbf{z}}|^2 + g'(\rho)a^\sigma(\rho)(\partial^\alpha\hat\rho)^2 \; dX \\
&\leqslant C(r,s,d)\eta\Big(\|(\nabla\hat\rho,\nabla\hat{\mathbf{z}})\|_{L^\infty}(1+\eta\varepsilon\|\nabla\hat\rho\|_{L^\infty}) + \varepsilon\|D^2\hat\rho\|_{L^\infty}\Big)E_s[\hat\rho,\hat{\mathbf{z}}] \\
&\quad + 2\sigma\eta\int_{\mathbb{R}^d}\rho^2 a^{\sigma-1}(\rho)a'(\rho)\langle i\hat{\mathbf{w}}\cdot\nabla\partial^\alpha\hat{\mathbf{z}},\partial^\alpha\hat{\mathbf{z}}\rangle \; dX \\
&\quad + 2\sigma\eta\int_{\mathbb{R}^d}\rho^2 a^{\sigma-1}(\rho)a'(\rho)\langle i(\partial^\alpha \operatorname{curl}\hat{\mathbf{z}})\hat{\mathbf{w}},\partial^\alpha\hat{\mathbf{z}}\rangle \; dX \\
&\quad - 2\sum_{\substack{\beta\leqslant\alpha,\\ |\beta|=\sigma-1}} \varepsilon\binom{\alpha}{\beta}\int_{\mathbb{R}^d}\rho a^\sigma(\rho)a'(\rho)(\partial^{\alpha-\beta}\rho)\sum_{j=1}^d\langle i\partial^{\alpha-\beta}\partial_j\partial^\beta\hat{\mathbf{z}},\partial_j\partial^\beta\hat{\mathbf{z}}\rangle \; dX \\
&\quad - 2\sum_{\substack{\beta\leqslant\alpha,\\ |\beta|=\sigma-1}} \varepsilon\binom{\alpha}{\beta}\int_{\mathbb{R}^d}\rho a^\sigma(\rho)a'(\rho)(\partial^{\alpha-\beta}\rho)\Big\langle i\partial^\alpha D\hat{\mathbf{z}},\partial^\beta\operatorname{curl}\hat{\mathbf{z}}\Big\rangle_{\mathcal{M}_d(\mathbb{C})} \; dX\,.
\end{aligned}
\tag{11}
$$

At this stage, we use the hypothesis that $\hat{\mathbf{z}}$ is a gradient vector field, so that the two terms involving the curl operator in (Calypso 11) cancel out. Summing over α with $|\alpha|=\sigma$ then gives

Cousteau
$$
\begin{aligned}
&\frac{d}{dT}\dot E_\sigma(\hat\rho,\hat{\mathbf{z}}) \leqslant C(r,s,d)\eta\|(\nabla\hat\rho,\nabla\hat{\mathbf{z}})\|_{L^\infty}(1+\eta\varepsilon\|\nabla\hat\rho\|_{L^\infty})E_s[\hat\rho,\hat{\mathbf{z}}] \\
&\qquad + 2\eta\sigma!\sum_{|\alpha|=\sigma}\frac{\sigma}{\alpha!}\int_{\mathbb{R}^d}\rho^2 a^{\sigma-1}(\rho)a'(\rho)\langle i\hat{\mathbf{w}}\cdot\nabla\partial^\alpha\hat{\mathbf{z}},\partial^\alpha\hat{\mathbf{z}}\rangle \; dX \\
&\qquad - 2\sigma!\sum_{|\alpha|=\sigma}\sum_{\substack{\beta\leqslant\alpha,\\ |\beta|=\sigma-1}}\frac{\varepsilon}{\alpha!}\binom{\alpha}{\beta}\int_{\mathbb{R}^d}\rho a^\sigma(\rho)a'(\rho)(\partial^{\alpha-\beta}\rho)\sum_{j=1}^d\langle i\partial^{\alpha-\beta}\partial_j\partial^\beta\hat{\mathbf{z}},\partial_j\partial^\beta\hat{\mathbf{z}}\rangle \; dX\,.
\end{aligned}
\tag{12}
$$

In the double sum, there holds $\dfrac{1}{\alpha!}\dbinom{\alpha}{\beta} = \dfrac{1}{\beta!}$, since $\alpha-\beta$ has length one. Exchanging the order of summation on α and β, then summing at fixed $\alpha'=\beta+e_j$, and using again that

$\varepsilon a(\rho)\partial_k\rho = \eta\rho\hat{\mathbf{w}}_k$, we can rewrite this double sum as

$$\begin{aligned}
&\varepsilon \sum_{|\beta|=\sigma-1} \frac{1}{\beta!} \sum_{k=1}^{d} \int_{\mathbb{R}^d} \rho a^{\sigma}(\rho) a'(\rho) \partial_k\rho \sum_{j=1}^{d} \langle i\partial_k\partial_j\partial^{\beta}\hat{\mathbf{z}}, \partial_j\partial^{\beta}\hat{\mathbf{z}}\rangle \; dX \\
&= \eta \sum_{|\beta|=\sigma-1} \frac{1}{\beta!} \int_{\mathbb{R}^d} \rho^2 a^{\sigma-1}(\rho) a'(\rho) \sum_{j=1}^{d} \langle i\hat{\mathbf{w}}\cdot\nabla\partial_j\partial^{\beta}\hat{\mathbf{z}}, \partial_j\partial^{\beta}\hat{\mathbf{z}}\rangle \; dX \\
&= \eta \sum_{|\alpha'|=\sigma} \sum_{j=1}^{d} \frac{\alpha'_j}{(\alpha')!} \int_{\mathbb{R}^d} \rho^2 a^{\sigma-1}(\rho) a'(\rho) \langle i\hat{\mathbf{w}}\cdot\nabla\partial^{\alpha'}\hat{\mathbf{z}}, \partial^{\alpha'}\hat{\mathbf{z}}\rangle \; dX \\
&= \eta\sigma \sum_{|\alpha'|=\sigma} \frac{1}{(\alpha')!} \int_{\mathbb{R}^d} \rho^2 a^{\sigma-1}(\rho) a'(\rho) \langle i\hat{\mathbf{w}}\cdot\nabla\partial^{\alpha'}\hat{\mathbf{z}}, \partial^{\alpha'}\hat{\mathbf{z}}\rangle \; dX
\end{aligned}$$

since the integral does not depend on j and $\sum_j \alpha'_j = \sigma$. Therefore, the two sums in (12) cancel out (this is due to the coefficients $1/\alpha!$ in the definition of $\dot{E}_\sigma$). The conclusion then follows by summation over σ. $\square$

2.4 Proof of uniform bounds in the general case

In this section, s is any real number greater than $1 + d/2$. Our aim is to prove Theorem 2 in the general case. As we have seen in the a priori estimates above, there remain some 'bad' terms when the velocity field $\mathbf{u}$ is not potential. This is why, as in [4], the solenoidal part of $\mathbf{u}$ requires a different weight than the potential part. In fact, our proof of Theorem 2 will parallel very closely the proof of Proposition 3.4 in [4], except that we pay attention to the parameters (η, ε), and insert the contribution of the nonlinear function $g'(\rho)$.

As in the proof of Proposition 4, ∇ stands for ∇_X in what follows. As a preliminary step, we rewrite the second equation in $(\mathrm{ES}_{\varepsilon,\eta})$ as an equation for $\hat{\mathbf{Z}} \stackrel{\text{def}}{=} \sqrt{\rho}\hat{\mathbf{z}}$ instead of $\hat{\mathbf{z}}$. Using that $\rho\eta\hat{\mathbf{w}} = \varepsilon a(\rho)\nabla\rho$ (which is just a reformulation of (3)), the first order term $i\eta(\nabla\hat{\mathbf{z}})\hat{\mathbf{w}}$ can combined with the second order one $i\varepsilon\nabla(a(\rho)\nabla\cdot\hat{\mathbf{z}})$ to obtain

$$\begin{aligned}
\partial_T\hat{\mathbf{Z}} + \eta(\hat{\mathbf{u}}\cdot\nabla)\hat{\mathbf{Z}} + \frac{1}{\varepsilon} b(\rho)\hat{\mathbf{w}} + i\varepsilon\nabla(a(\rho)\nabla\cdot\hat{\mathbf{Z}}) + i\varepsilon a(\rho)(\nabla_0\hat{\mathbf{Z}})\nabla\log\sqrt{\rho} = \\
-\tfrac{1}{2}\eta(\nabla\cdot\hat{\mathbf{u}})\hat{\mathbf{Z}} + i\varepsilon\nabla(a(\rho)\nabla\sqrt{\rho})\frac{\hat{\mathbf{Z}}}{\sqrt{\rho}}\ ,
\end{aligned}$$

where the operator ∇_0 is defined by

$$(\nabla_0\mathbf{Z})_{jk} = \partial_j Z_k - (\nabla\cdot\mathbf{Z})\ \delta_{jk}\,, \quad \text{or equivalently}\,,\ \nabla_0\mathbf{Z} \stackrel{\text{def}}{=} \nabla\mathbf{Z} - (\nabla\cdot\mathbf{Z})\ \mathbf{I}\,.$$

The advantage of this formulation is that it trivializes the proof of zeroth order estimates (Proposition 2), since

$$\int_{\mathbb{R}^d} \langle i(\nabla_0\hat{\mathbf{Z}})\mathbf{W}, \hat{\mathbf{Z}}\rangle = 0$$

for all potential vector fields $\mathbf{W}$, and in particular for $\mathbf{W} = a(\rho)\nabla \log \sqrt{\rho}$. The idea is to keep this nice structure for higher order derivatives, which means writing equations for $\hat{\mathbf{Z}}^s := \sqrt{\rho}\Lambda^s \hat{\mathbf{z}}$ instead of $\Lambda^s \hat{\mathbf{z}}$, where Λ^s denotes the Fourier multiplier operator

$$\Lambda^s \stackrel{\text{def}}{=} (1-\Delta)^{s/2}\,.$$

However, we have to cope with a 'bad' commutator, namely in $\nabla[a(\rho), \Lambda^s]\nabla\cdot$, which already appears in the equation for $\Lambda^s\hat{\mathbf{z}}$. Pointing out its principal part, we can write as in [4]

$$\begin{aligned}\nabla[a(\rho), \Lambda^s]\nabla \cdot \hat{\mathbf{z}} &\stackrel{\text{def}}{=} \mathbf{R}_0 + s\nabla(\nabla a(\rho) \cdot \Lambda^{s-2}\nabla(\nabla \cdot \hat{\mathbf{z}})) \\ &\stackrel{\text{def}}{=} \mathbf{R}_0 + \mathbf{R}_{00} - s\nabla a(\rho) \cdot \nabla(\mathcal{Q}\Lambda^s \hat{\mathbf{z}})\,,\end{aligned}$$

where

$$\begin{aligned}\|\mathbf{R}_0\|_{L^2} &\lesssim \|D^2 a(\rho)\|_{H^{s-1}}\,\|\nabla \cdot \hat{\mathbf{z}}\|_{L^\infty} + \|D^2 a(\rho)\|_{L^\infty}\,\|\nabla \cdot \hat{\mathbf{z}}\|_{H^{s-1}}\,, \\ \|\mathbf{R}_{00}\|_{L^2} &\lesssim \|\nabla a(\rho)\|_{W^{1,\infty}} \|\hat{\mathbf{z}}\|_{H^s}\,,\end{aligned}$$

and $\mathcal{Q}$ is the L^2-orthogonal projector onto potential vector fields. Consequently, by applying Λ^s to the second equation in ($\mathrm{ES}_{\varepsilon,\eta}$), multiplying by $\sqrt{\rho}$, and using also the first equation in ($\mathrm{ES}_{\varepsilon,\eta}$), we see that

$$\begin{aligned}\partial_T \hat{\mathbf{Z}}^s + \eta(\hat{\mathbf{u}}\cdot\nabla)\hat{\mathbf{Z}}^s + \frac{1}{\varepsilon}\, b(\rho)\sqrt{\rho}\Lambda^s \hat{\mathbf{w}} + i\varepsilon\nabla(a(\rho)\nabla\cdot\hat{\mathbf{Z}}^s) + \\ + i\varepsilon a(\rho)(\nabla_0 \hat{\mathbf{Z}}^s)\nabla \log\sqrt{\rho} + i\varepsilon s\sqrt{\rho}\nabla a(\rho)\cdot\nabla(\mathcal{Q}\Lambda^s\hat{\mathbf{z}}) = \\ -\tfrac{1}{2}\eta(\nabla\cdot\hat{\mathbf{u}})\hat{\mathbf{Z}}^s + i\varepsilon\nabla(a(\rho)\nabla\sqrt{\rho})\frac{\hat{\mathbf{Z}}^s}{\sqrt{\rho}} + i\varepsilon\sqrt{\rho}\,(\mathbf{R}_0 + \mathbf{R}_{00}) + \sqrt{\rho}\mathbf{R}\end{aligned}$$

with

$$\mathbf{R} \stackrel{\text{def}}{=} \eta[\hat{\mathbf{u}}\cdot\nabla, \Lambda^s]\hat{\mathbf{z}} + i\eta\Big((\nabla\Lambda^s\hat{\mathbf{z}})\hat{\mathbf{w}} - \Lambda^s((\nabla\hat{\mathbf{z}})\hat{\mathbf{w}})\Big) + \frac{1}{\varepsilon}\,[b(\rho), \Lambda^s]\hat{\mathbf{w}}$$

being bounded as in the proof of Proposition 4 by

$$\|\mathbf{R}\|_{L^2} \leqslant C(r,s,d)\eta\|(\nabla\hat{\mathbf{z}}, \nabla\hat{\rho})\|_{L^\infty}\|(\hat{\rho},\hat{\mathbf{z}})\|_{H^s}\,,$$

and also

$$\|i\varepsilon(\mathbf{R}_0 + \mathbf{R}_{00})\|_{L^2} \leqslant C(r,s,d)\eta\|(\nabla\hat{\mathbf{z}}, \nabla\hat{\rho})\|_{L^\infty}\|(\hat{\rho},\hat{\mathbf{z}})\|_{H^s}$$

by the estimates mentioned above and the fact that $\varepsilon\|\hat{\rho}\|_{H^{s+1}} \leqslant C(r,s,d)\|(\hat{\rho},\hat{\mathbf{z}})\|_{H^s}$. Therefore, apart form the term $\varepsilon^{-1}\, b(\rho)\sqrt{\rho}\Lambda^s\hat{\mathbf{w}}$ that we will deal with afterwards, the only troublesome term regarding the time derivative of $\|\hat{\mathbf{Z}}^s\|_{L^2}$ is the one involving $\nabla(\mathcal{Q}\Lambda^s\hat{\mathbf{z}})$, which corresponds to derivatives of order $s+1$. This is where the use of an appropriate weight comes into play. In fact, whatever the positive-valued weight (or gauge) $\psi = \psi(\rho)$, the equation above for $\hat{\mathbf{Z}}^s$ and the first equation in $(\mathrm{ES}_{\varepsilon,\eta})$ give, after some manipulations,

$$
\begin{aligned}
\partial_T\hat{\mathbf{Y}}^s+\eta(\hat{\mathbf{u}}\cdot\nabla)\hat{\mathbf{Y}}^s+\frac{1}{\varepsilon}\,b(\rho)\sqrt{\rho}\psi(\rho)\Lambda^s\hat{\mathbf{w}}+i\varepsilon\nabla(a(\rho)\nabla\cdot\hat{\mathbf{Y}}^s)+&\\
+i\varepsilon a(\rho)(\nabla_0\hat{\mathbf{Y}}^s)\nabla\log\left(\frac{\sqrt{\rho}a^s(\rho)}{\psi(\rho)}\right)&\\
-i\varepsilon s\sqrt{\rho}\psi(\rho)(\nabla\mathcal{P}\Lambda^s\hat{\mathbf{z}})\nabla a(\rho)+i\varepsilon a(\rho)(\nabla\cdot\hat{\mathbf{Y}}^s)\nabla\log\left(\frac{a^s(\rho)}{\psi^2(\rho)}\right)=&\\
-\tfrac12\eta(\nabla\cdot\hat{\mathbf{u}})(1+\rho\partial_\rho\log(\psi^2(\rho)))\hat{\mathbf{Y}}^s&\\
+i\varepsilon\left(\frac{\nabla(a(\rho)\nabla\sqrt{\rho})}{\sqrt{\rho}}+\frac{\nabla(a(\rho)\nabla\psi(\rho))}{\psi(\rho)}+a(\rho)(\nabla\log\psi(\rho))\otimes\nabla\log\left(\frac{a^s(\rho)}{\psi(\rho)}\right)\right)\hat{\mathbf{Y}}^s&\\
+i\varepsilon\sqrt{\rho}\,\psi(\rho)\,(\mathbf{R}_0+\mathbf{R}_{00})+\sqrt{\rho}\psi(\rho)\mathbf{R}&
\end{aligned}
$$

with $\hat{\mathbf{Y}}^s\overset{\text{def}}{=}\psi(\rho)\hat{\mathbf{Z}}^s=\sqrt{\rho}\psi(\rho)\Lambda^s\hat{\mathbf{z}}$, and $\mathcal{P}\overset{\text{def}}{=}I-\mathcal{Q}$. From this expression and previous estimates, we see that the loss of spatial derivatives in the time derivative of $\|\hat{\mathbf{Y}}^s\|_{L^2}$ is only due to the terms in the third row. Of course, these terms vanish when $\hat{\mathbf{u}}$, and thus also $\hat{\mathbf{z}}$, is potential (hence $\mathcal{P}\Lambda^s\hat{\mathbf{z}}=\Lambda^s\mathcal{P}\hat{\mathbf{z}}=0$), provided that we choose $\psi^2(\rho)=a^s(\rho)$. Provided that the term $\varepsilon^{-1}b(\rho)\sqrt{\rho}\Lambda^s\hat{\mathbf{w}}$ is properly handled, this gives a shorter proof, compared to that of Proposition 4, of uniform bounds in the potential case. In the general case, the idea is to estimate separately $\|\mathcal{Q}\hat{\mathbf{Y}}^s\|_{L^2}$ and $\|\mathcal{P}\hat{\mathbf{X}}^s\|_{L^2}$ where $\hat{\mathbf{X}}^s\overset{\text{def}}{=}\varphi(\rho)\hat{\mathbf{Z}}^s=\sqrt{\rho}\varphi(\rho)\Lambda^s\hat{\mathbf{z}}$ for some other weight φ. These estimates will be based, as in [4], on a preliminary observation using only integration by parts and the properties $\nabla\cdot\mathcal{P}\equiv0$, $\mathrm{curl}\mathcal{Q}\equiv0$, which gives that

$$
\begin{aligned}
\frac{d}{dT}\tfrac12\|\mathcal{Q}\hat{\mathbf{Y}}^s\|_{L^2}^2=\int_{\mathbb{R}^d}\langle\mathcal{Q}\hat{\mathbf{Y}}^s,(\partial_t+\eta\hat{\mathbf{u}}\cdot\nabla)\hat{\mathbf{Y}}^s\rangle\,dX+\eta\int_{\mathbb{R}^d}(\nabla\cdot\hat{\mathbf{u}})\langle\mathcal{Q}\hat{\mathbf{Y}}^s,\tfrac12\mathcal{Q}\hat{\mathbf{Y}}^s+\mathcal{P}\hat{\mathbf{Y}}^s\rangle\,dX&\\
-\eta\int_{\mathbb{R}^d}\langle\mathcal{P}\hat{\mathbf{Y}}^s,(\nabla\hat{\mathbf{u}})\mathcal{Q}\hat{\mathbf{Y}}^s\rangle\,dX\,,&
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dT}\tfrac12\|\mathcal{P}\hat{\mathbf{X}}^s\|_{L^2}^2=\int_{\mathbb{R}^d}\langle\mathcal{P}\hat{\mathbf{X}}^s,(\partial_t+\eta\hat{\mathbf{u}}\cdot\nabla)\hat{\mathbf{X}}^s\rangle\,dX+\eta\int_{\mathbb{R}^d}(\nabla\cdot\hat{\mathbf{u}})\tfrac12\langle\mathcal{P}\hat{\mathbf{X}}^s,\mathcal{P}\hat{\mathbf{X}}^s\rangle\,dX&\\
+\eta\int_{\mathbb{R}^d}\langle\mathcal{P}\hat{\mathbf{X}}^s,(\nabla\hat{\mathbf{u}})\mathcal{Q}\hat{\mathbf{X}}^s\rangle\,dX\,.&
\end{aligned}
$$

Using the equations satisfied by $\hat{\mathbf{Y}}^s$ and $\hat{\mathbf{X}}^s$ - the latter being identical to the former if we substitute $\hat{\mathbf{X}}^s$ for $\hat{\mathbf{Y}}^s$ and φ for ψ -, and summing the equations here above, we are left with harmless remainder terms, bounded in by $C(r,s,d)\eta\|(\nabla\hat{\mathbf{z}},\nabla\hat{\rho})\|_{L^\infty}\|(\hat{\rho},\hat{\mathbf{z}})\|_{H^s}^2$, plus a number of terms that must be handled carefully. Among these delicate terms is

$$
\mathcal{I}\overset{\text{def}}{=}-\frac{1}{\varepsilon}\int_{\mathbb{R}^d}\langle\mathcal{Q}\hat{\mathbf{Y}}^s,b(\rho)\sqrt{\rho}\psi(\rho)\Lambda^s\hat{\mathbf{w}}\rangle+\langle\mathcal{P}\hat{\mathbf{X}}^s,b(\rho)\sqrt{\rho}\varphi(\rho)\Lambda^s\hat{\mathbf{w}}\rangle\,dX\,.
$$

Noting that both $\varepsilon^{-1}b(\rho)\sqrt{\rho}\psi(\rho)\Lambda^s\hat{\mathbf{w}}$ and $\varepsilon^{-1}b(\rho)\sqrt{\rho}\varphi(\rho)\Lambda^s\hat{\mathbf{w}}$ are 'almost' potential, that is, equal to a gradient up to a remainder term bounded in L^2 by $C(r,s,d)\eta\|(\nabla\hat{\mathbf{z}},\nabla\hat{\rho})\|_{L^\infty}\|(\hat{\rho},\hat{\mathbf{z}})\|_{H^s}$

(like $\mathbf{R}$), we see that $\mathcal{I}$ reduces to

$$\mathcal{I} = -\frac{1}{\varepsilon}\int_{\mathbb{R}^d} \langle \hat{\mathbf{Y}}^s, b(\rho)\sqrt{\rho}\psi(\rho)\Lambda^s\hat{\mathbf{w}}\rangle \; dX + \mathcal{R} = -\frac{1}{\varepsilon}\int_{\mathbb{R}^d} b(\rho)\rho\psi^2(\rho)\Lambda^s\mathbf{u}\cdot\Lambda^s\hat{\mathbf{w}} \; dX + \mathcal{R}$$

with $|\mathcal{R}| \leqslant C(r,s,d)\eta\|(\nabla\hat{\mathbf{z}},\nabla\hat{\rho})\|_{L^\infty}\|(\hat{\rho},\hat{\mathbf{z}})\|_{H^s}^2$. Similarly as what is done in the potential case (in the previous section), this remaining $\mathcal{O}(\varepsilon^{-1})$ in $\mathcal{I}$ can be cancelled out by adding to $\frac{d}{dT}\frac{1}{2}(\|\mathcal{Q}\hat{\mathbf{Y}}^s\|_{L^2}^2 + \|\mathcal{P}\hat{\mathbf{X}}^s\|_{L^2}^2)$ the time derivative

$$\begin{aligned}\frac{d}{dT}\int_{\mathbb{R}^d} \tfrac{1}{2}\psi^2(\rho)g'(\rho)(\Lambda^s\hat{\rho})^2 \; dX &= -\eta\int_{\mathbb{R}^d}(\hat{\mathbf{u}}\cdot\nabla(\psi^2(\rho)g'(\rho)) + \rho\partial_\rho(\psi^2(\rho)g'(\rho))\nabla\cdot\hat{\mathbf{u}})(\Lambda^s\hat{\rho})^2 \; dX \\ &+ \eta\int_{\mathbb{R}^d}\nabla\cdot(\psi^2(\rho)g'(\rho))\hat{\mathbf{u}})(\Lambda^s\hat{\rho})^2 \; dX - \int_{\mathbb{R}^d}\rho\psi^2(\rho)g'(\rho)(\Lambda^s\hat{\rho})\nabla\cdot\Lambda^s\hat{\mathbf{u}} \; dX = \\ &\mathcal{R}_1 + \int_{\mathbb{R}^d}\rho\psi^2(\rho)g'(\rho)\nabla(\Lambda^s\hat{\rho})\cdot\Lambda^s\hat{\mathbf{u}} \; dX\end{aligned}$$

with $|\mathcal{R}_1| \leqslant C(r,s,d)\eta\|(\nabla\hat{\mathbf{z}},\nabla\hat{\rho})\|_{L^\infty}\|(\hat{\rho},\hat{\mathbf{z}})\|_{H^s}^2$. Now, observing that

$$\nabla(\Lambda^s\hat{\rho}) = \frac{1}{\varepsilon}\Lambda^s\Big(\frac{\rho\hat{\mathbf{w}}}{a(\rho)}\Big) = \frac{\rho}{\varepsilon a(\rho)}\Lambda^s\hat{\mathbf{w}} + \mathbf{R}_1$$

with $\|\mathbf{R}_1\|_{L^2} \leqslant C(r,s,d)\eta\|(\nabla\hat{\mathbf{z}},\nabla\hat{\rho})\|_{L^\infty}\|(\hat{\rho},\hat{\mathbf{z}})\|_{H^s}$ (like $\mathbf{R}$ again), and recalling that $b(\rho) = \frac{\rho g'(\rho)}{a(\rho)}$, we arrive at

$$\left|\mathcal{I} + \frac{d}{dT}\int_{\mathbb{R}^d}\tfrac{1}{2}\psi^2(\rho)g'(\rho)(\Lambda^s\hat{\rho})^2 \; dX\right| \leqslant C(r,s,d)\eta\|(\nabla\hat{\mathbf{z}},\nabla\hat{\rho})\|_{L^\infty}\|(\hat{\rho},\hat{\mathbf{z}})\|_{H^s}^2\,.$$

Appropriate choices of ψ and φ will enable us to get rid of the other tricky terms, exactly as in [4]. Of course, there is no reason to change ψ, and we set $\psi^2(\rho) = a^s(\rho)$ as in the potential case. As regards φ, it turns out that a good choice is

$$\varphi^2(\rho) = \frac{A(\rho)}{\rho}\,, \text{ with } A'(\rho) = a^s(\rho) - \rho\partial_\rho(a^s(\rho))\,.$$

For convenience, we keep abstract notations for ψ and φ in what follows, and use only that $\psi^2(\rho) = a^s(\rho)$ to simplify the equation satisfied by $\hat{\mathbf{Y}}^s$. From the computations above and the fact that the second order terms do not contribute - indeed, because $\nabla\cdot\mathcal{P}\equiv 0$,

$$\int_{\mathbb{R}^d}\langle\mathcal{Q}\hat{\mathbf{Y}}^s, i\nabla(a(\rho)\nabla\cdot\hat{\mathbf{Y}}^s)\rangle = \int_{\mathbb{R}^d}\langle\hat{\mathbf{Y}}^s, i\nabla(a(\rho)\nabla\cdot\hat{\mathbf{Y}}^s)\rangle = 0\,,$$

$$\int_{\mathbb{R}^d}\langle\mathcal{P}\hat{\mathbf{X}}^s, i\nabla(a(\rho)\nabla\cdot\hat{\mathbf{X}}^s)\rangle = 0\,,$$

we find that

$$\frac{d}{dT}\frac{1}{2}\left(\|\mathcal{Q}\hat{\mathbf{Y}}^s\|_{L^2}^2+\|\mathcal{P}\hat{\mathbf{X}}^s\|_{L^2}^2+\int_{\mathbb{R}^d}\psi^2(\rho)g'(\rho)(\Lambda^s\hat{\rho})^2\,dX\right)=\mathcal{R}_2 \tag{13}$$
$$+\int_{\mathbb{R}^d}\langle\mathcal{Q}\hat{\mathbf{Y}}^s,-i\varepsilon a(\rho)(\nabla_0\hat{\mathbf{Y}}^s)\nabla\log\left(\sqrt{\rho}\psi(\rho)\right)+i\varepsilon s\sqrt{\rho}\psi(\rho)(\nabla\mathcal{P}\Lambda^s\hat{\mathbf{z}})\nabla a(\rho)\rangle\,dX$$
$$+\int_{\mathbb{R}^d}\left\langle\mathcal{P}\hat{\mathbf{X}}^s,-i\varepsilon a(\rho)(\nabla_0\hat{\mathbf{X}}^s)\nabla\log\left(\frac{\sqrt{\rho}a^s(\rho)}{\varphi(\rho)}\right)+i\varepsilon s\sqrt{\rho}\varphi(\rho)(\nabla\mathcal{P}\Lambda^s\hat{\mathbf{z}})\nabla a(\rho)\right\rangle\,dX$$
$$-\int_{\mathbb{R}^d}\left\langle\mathcal{P}\hat{\mathbf{X}}^s,i\varepsilon a(\rho)(\nabla\cdot\hat{\mathbf{X}}^s)\nabla\log\left(\frac{a^s(\rho)}{\varphi^2(\rho)}\right)\right\rangle\,dX$$

with $|\mathcal{R}_2|\leqslant C(r,s,d)\eta\|(\nabla\hat{\mathbf{z}},\nabla\hat{\rho})\|_{L^\infty}\|(\hat{\rho},\hat{\mathbf{z}})\|_{H^s}^2$. Let us concentrate for a while on the second line in (13) here above. By the same computations as in [4, pp.1516-1517], which heavily use that $\psi^2(\rho)=a^s(\rho)$ and rely on successive integrations by parts together with commutator estimates, it is found to be equal to

$$\mathcal{R}_3+\frac{\varepsilon}{2}\int_{\mathbb{R}^d}\rho a^{s+1}(\rho)\left\langle\mathcal{Q}\Lambda^s\hat{\mathbf{z}},i(\nabla\mathcal{P}\Lambda^s\hat{\mathbf{z}})\nabla\log\left(\frac{a^s(\rho)}{\rho}\right)\right\rangle\,dX$$

with $|\mathcal{R}_3|\leqslant C(r,s,d)\eta\,\varepsilon\,\|D^2\hat{\rho}\|_{L^\infty}\|\hat{\mathbf{z}}\|_{H^s}^2$. Furthermore, by a similar approach - as in [4, p.1518]-, the last two lines in (13) can be written as

$$\mathcal{R}_4-\frac{\varepsilon}{2}\int_{\mathbb{R}^d}\rho a(\rho)\varphi^2(\rho)\langle\mathcal{Q}\Lambda^s\hat{\mathbf{z}},i(\nabla\mathcal{P}\Lambda^s\hat{\mathbf{z}})\nabla\log(\rho\varphi^2(\rho))\rangle\,dX$$

with $|\mathcal{R}_4|\leqslant C(r,s,d)\eta\,\varepsilon\,\|D^2\hat{\rho}\|_{L^\infty}\|\hat{\mathbf{z}}\|_{H^s}^2$. Therefore, the appropriate choice of φ is dictated by the fact that we want to get rid of the terms involving $s+1$ derivatives of $\hat{\mathbf{z}}$. If we set φ so that

$$a^s(\rho)\nabla\log\left(\frac{a^s(\rho)}{\rho}\right)-\varphi^2(\rho)\nabla\log(\rho\varphi^2(\rho))=0\,,$$

which is merely equivalent to

$$\nabla(\rho\varphi^2(\rho))=\,(a^s(\rho)-\rho\partial_\rho(a^s(\rho)))\nabla\rho$$

we deduce from (13) that

$$\frac{d}{dT}\frac{1}{2}\left(\|\mathcal{Q}\hat{\mathbf{Y}}^s\|_{L^2}^2+\|\mathcal{P}\hat{\mathbf{X}}^s\|_{L^2}^2+\int_{\mathbb{R}^d}\psi^2(\rho)g'(\rho)(\Lambda^s\hat{\rho})^2\,dX\right)=\mathcal{R}_2+\mathcal{R}_3+\mathcal{R}_4\,.$$

The estimates of $\mathcal{R}_{2,3,4}$ mentioned above and the comparison result below complete the proof of Theorem 2 by a standard, Gronwall-type argument.

Proposition 5 *Let s be a positive real number. Let $r\in(0,\varrho/2]$ be such that $g'(\rho)>0$ and $K(\rho)>0$ if $|\rho-\varrho|\leqslant r$. We denote by ψ and φ the positive functions defined for $|\rho-\varrho|\leqslant r$ by*

$$a^2(\rho)=\rho K(\rho)\,,\qquad\psi^2(\rho)=a^s(\rho)\,,\qquad\rho\varphi^2(\rho)=2\int_{\varrho-r}^{\rho}a^s+2\max_{|\theta-\varrho|\leqslant r}(\theta a^s(\theta))-\rho a^s(\rho)\,,$$

by Λ^s the operator $(1-\Delta)^{s/2}$, by $\mathcal{Q}$ the L^2-orthogonal projector onto potential vector fields, and by $\mathcal{P} = I - \mathcal{Q}$ the L^2-orthogonal projector onto solenoidal vector fields. Then for all $(\hat\rho, \hat{\mathbf{u}}) \in H^{s+1}(\mathbb{R}^d) \times (H^s(\mathbb{R}^d))^d$ such that $\|\hat\rho\|_{W^{1,\infty}} \leqslant r$, for all $\eta \in (0,1]$, for all $\varepsilon > 0$,

$$\begin{aligned} c(\|\hat{\mathbf{u}}\|_{H^s}^2 + \|\hat\rho\|_{H^s}^2 + \varepsilon^2\|\hat\rho\|_{H^{s+1}}^2) \leqslant \\ \|\mathcal{Q}(\sqrt{\rho}\psi(\rho)\Lambda^s\hat{\mathbf{z}})\|_{L^2}^2 + \|\mathcal{P}(\sqrt{\rho}\varphi(\rho)\Lambda^s\hat{\mathbf{z}})\|_{L^2}^2 + \|\sqrt{g'(\rho)}\psi(\rho)\Lambda^s\hat\rho\|_{L^2}^2 \leqslant \\ C(\|\hat{\mathbf{u}}\|_{H^s}^2 + \|\hat\rho\|_{H^s}^2 + \varepsilon^2\|\hat\rho\|_{H^{s+1}}^2)\,, \end{aligned}$$

where $\rho = \varrho + \eta\hat\rho$, $\hat{\mathbf{z}} = \hat{\mathbf{u}} + i\varepsilon\frac{a(\rho)}{\rho}\nabla\hat\rho$, and the constants $c > 0$ and $C > 0$ depend only on r, s, d (and the functions g, K).

3 Free wave regime

s:linear

mparaisonde **Theorem 3** *We choose a real number s with $s > 1 + \dfrac{d}{2}$, and a positive real number M. For $\eta > 0$, $\varepsilon > 0$, any initial data*

$$(\hat\rho^{\mathrm{in}}, \hat{\mathbf{u}}^{\mathrm{in}}) \in B_\varepsilon(M) = \{(\hat\rho, \hat{\mathbf{u}}) \in H^{s+1}(\mathbb{R}^d) \times (H^s(\mathbb{R}^d))^d\,;\ \|(\hat\rho, \hat{\mathbf{u}})\|_{(H^s(\mathbb{R}^d))^{d+1}} + \varepsilon\|\hat\rho\|_{H^{s+1}(\mathbb{R}^d)} \leqslant M\}$$

is associated with the solution $(\hat\rho, \hat{\mathbf{u}}) \in \mathscr{C}([0, T_*/\eta], B_\varepsilon(2M))$ of $(\mathrm{EK}_{\varepsilon,\eta})$ given by Theorem 2. Let $(\mathfrak{r}, \mathfrak{u}) \in \mathscr{C}_b(\mathbb{R}_+, H^s(\mathbb{R}^d) \times (H^s(\mathbb{R}^d))^d)$ solve the acoustic equations

$$\text{(W)} \qquad \begin{cases} \partial_T \mathfrak{r} + \varrho \nabla_X \cdot \mathfrak{u} = 0 \\ \\ \partial_T \mathfrak{u} + g'(\varrho)\nabla_X \mathfrak{r} = 0\,, \end{cases}$$

with initial data $(\mathfrak{r}, \mathfrak{u})_{|T=0} = (\hat\rho^{\mathrm{in}}, \hat{\mathbf{u}}^{\mathrm{in}})$. Then at each time $T \in [0, T_*/\eta]$ we have

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$$\|(\hat\rho, \hat{\mathbf{u}}) - (\mathfrak{r}, \mathfrak{u})\|_{H^{s-2}(\mathbb{R}^d) \times (H^{s-2}(\mathbb{R}^d))^d} \leqslant C(\eta + \varepsilon)T\,, \tag{14}$$

and

JamesBonde
$$\|(\hat\rho, \hat{\mathbf{u}}) - (\mathfrak{r}, \mathfrak{u})\|_{H^{s-3}(\mathbb{R}^d) \times (H^{s-3}(\mathbb{R}^d))^d} \leqslant C(\eta + \varepsilon^2)T\,, \tag{15}$$

where C depends only on M, s and d.

Remark 2 It is clear from (W) that the divergence free part of the vector field $\mathfrak{u}$ remains constant in time.

Notice that the difference between the estimates in (14) and (15) is the regularity index. When $\varepsilon \lesssim \eta$, both estimates (14) and (15) provide an $\mathcal{O}(\eta T)$ error. By contrast, when $\varepsilon^2 \lesssim \eta \ll \varepsilon$, they yield respectively an $\mathcal{O}(\varepsilon T)$ and an $\mathcal{O}(\eta T)$ error, so that the second one is smaller. Therefore, $\eta \approx \varepsilon$ appears to be a threshold at which we lose one derivative. Note finally that estimates (14) (resp. (15)) provide an L^∞ error bound only for $s > 2 + \dfrac{d}{2}$ (resp. $s > 3 + \dfrac{d}{2}$).

In the special case corresponding to the Gross–Pitaevskii equation ($K(\rho) = 1/(4\rho)$, $g(\varrho) = \varrho - 1$), the first rigorous justification of the free wave regime was given by Colin and Soyeur [16] in terms of weak convergence. Strong convergence was proved much more recently by Béthuel, Danchin and Smets [6].

Proof of Theorem 3. By ($\mathrm{EK}_{\varepsilon,\eta}$) and (W), we see that $(\hat\rho - \mathfrak{r}, \hat{\mathbf{u}} - \mathfrak{u})$ solves

$$\left\{\begin{array}{ll} \partial_T(\hat\rho - \mathfrak{r}) + \varrho \nabla_X \cdot (\hat{\mathbf{u}} - \mathfrak{u}) & = -\eta \nabla_X \cdot (\hat\rho \hat{\mathbf{u}}) \\ \partial_T(\hat{\mathbf{u}} - \mathfrak{r}) + g'(\varrho)\nabla_X(\hat\rho - \mathfrak{r}) & = -\eta \hat{\mathbf{u}} \cdot \nabla_X \hat{\mathbf{u}} - [g'(\varrho + \eta\hat\rho) - g'(\varrho)]\nabla_X \hat\rho \\ & \quad + \varepsilon^2 \nabla_X \Big(K(\varrho + \eta\hat\rho)\Delta_X \hat\rho + \dfrac{\eta}{2} K'(\varrho + \eta\hat\rho)|\nabla_X \hat\rho|^2 \Big) \end{array}\right.$$

with null initial data. As in [6], the proof of Theorem 3 amounts to estimating the source terms in this system. By Theorem 2, we have

$$\|\eta \nabla_X \cdot (\hat\rho \hat{\mathbf{u}})\|_{H^{s-1}} \leqslant C(s,d,M)\eta\,,$$

$$\begin{aligned} &\Big\| -\eta \hat{\mathbf{u}} \cdot \nabla_X \hat{\mathbf{u}} - [g'(\varrho + \eta\hat\rho) - g'(\varrho)]\nabla_X \hat\rho + \varepsilon^2 \nabla_X \Big(K(\varrho + \eta\hat\rho)\Delta_X \hat\rho + \frac{\eta}{2} K'(\varrho + \eta\hat\rho)|\nabla_X \hat\rho|^2 \Big) \Big\|_{H^{s-2}} \\ &\qquad \leqslant C(s,d,M)(\eta + \varepsilon + \varepsilon^2\eta) \leqslant C(s,d,M)(\eta + \varepsilon)\,, \end{aligned}$$

where we have used the bound on $\varepsilon\hat\rho$ in H^{s+1} for the term $\varepsilon^2 \nabla_X \Delta_X \hat\rho$, and

$$\begin{aligned} &\Big\| -\eta \hat{\mathbf{u}} \cdot \nabla_X \hat{\mathbf{u}} - [g'(\varrho + \eta\hat\rho) - g'(\varrho)]\nabla_X \hat\rho + \varepsilon^2 \nabla_X \Big(K(\varrho + \eta\hat\rho)\Delta_X \hat\rho + \frac{\eta}{2} K'(\varrho + \eta\hat\rho)|\nabla_X \hat\rho|^2 \Big) \Big\|_{H^{s-3}} \\ &\qquad \leqslant C(s,d,M)(\eta + \varepsilon^2 + \varepsilon^2\eta) \leqslant C(s,d,M)(\eta + \varepsilon^2)\,, \end{aligned}$$

using this time that $\varepsilon^2 \nabla_X \Delta_X \hat\rho$ in H^{s-3} is bounded by $\varepsilon^2 \|\hat\rho\|_{H^s}$. The conclusion follows from Duhamel's formula and the fact that the wave group is unitary on H^s. $\square$

In one space dimension ($d = 1$), solutions to the acoustic equations in (W) are exactly combinations of left-going and right-going waves. More precisely, there exist w^+ and w^- such that

$$\frac{1}{2}\Big(\mathfrak{r} + \frac{\varrho}{\mathfrak{c}}\mathfrak{u}\Big)(T,X) = w^+(X - \mathfrak{c}T) \qquad \text{and} \qquad \frac{1}{2}\Big(\mathfrak{r} - \frac{\varrho}{\mathfrak{c}}\mathfrak{u}\Big)(T,X) = w^-(X + \mathfrak{c}T)\,.$$

In what follows, we aim at characterizing the counterpart of these linear waves at the weakly nonlinear, and possibly weakly dispersive level.

4 One-way propagating waves on the line

In this section, the space dimension is $d = 1$, and the fluid velocities are denoted by u instead of the bold letter $\mathbf{u}$. We are going to show that the evolution of the two weakly nonlinear / weakly dispersive counter propagating waves is governed by Burgers equations if the parameters η and ε are of the same order, by weakly dispersive Korteweg–de Vries (KdV) equations if $\varepsilon^2 \ll \eta$, and by regular KdV equations if η and ε^2 are of the same order. What remains of the reference density ϱ in these equations lies in the two quantities pointed out in the introduction and defined by

$$\Gamma \overset{\mathrm{def}}{=} \frac{3\mathfrak{c}}{2\varrho} + \frac{\varrho g''(\varrho)}{2\mathfrak{c}}\,, \qquad \kappa \overset{\mathrm{def}}{=} \frac{\varrho}{2\mathfrak{c}} K(\varrho)\,,$$

which already appeared in a special form in earlier results on NLS by the first author [15, 14, 12, 11].

4.1 Statement of errors bounds in various asymptotic regimes

A first, simpler result holds when the left-going wave w^- is negligible, so that $\mathfrak{c}\hat\rho \approx \varrho\hat u$ (at least for small enough T) by (14) and (15). More precisely, we are going to show that, if the initial norm of the difference $\hat\rho - \varrho\hat u/\mathfrak{c}$ is small enough, then both $\hat\rho$ and $\varrho\hat{\mathbf{u}}/\mathfrak{c}$ are either close to solutions $\mathfrak{z}$ to the (inviscid) Burgers equation

$$\partial_\theta \mathfrak{z} + \Gamma \mathfrak{z} \partial_X \mathfrak{z} = 0$$

if $\varepsilon \lesssim \eta$ or $\varepsilon^2 \ll \eta$, or close to solutions ζ to the KdV equation

$$\partial_\theta \zeta + \Gamma \zeta \partial_X \zeta = \frac{\varepsilon^2}{\eta} \kappa \partial_X^3 \zeta$$

if $\varepsilon^2 = \mathcal{O}(\eta)$. Note that this equation is clearly weakly dispersive if $\varepsilon^2 \ll \eta$, and reduces to

$$\partial_\theta \zeta + \Gamma \zeta \partial_X \zeta = \kappa \partial_X^3 \zeta$$

when $\eta = \varepsilon^2$. Precise error bounds are given in the following.

Theorem 4 *We assume $d = 1$, and take an integer $s \geqslant 4$, and a real number $M > 0$. For $\eta \in (0,1]$, $\varepsilon \in (0,1]$, any initial data*

$$(\hat\rho^{\mathrm{in}}, \hat u^{\mathrm{in}}) \in B_\varepsilon(M) = \{(\hat\rho, \hat u) \in H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R})\,;\ \|(\hat\rho, \hat u)\|_{(H^s(\mathbb{R}))^2} + \varepsilon \|\hat\rho\|_{H^{s+1}(\mathbb{R})} \leqslant M\}$$

is associated with the solution $(\hat\rho, \hat u) \in \mathscr{C}([0, T_*/\eta], B_\varepsilon(2M))$ of $(\mathrm{EK}_{\varepsilon,\eta})$ such that $(\rho, u)(0) = (\rho^{\mathrm{in}}, u^{\mathrm{in}})$, as given by Theorem 2 in the case $d = 1$. We also introduce $\mathfrak{z} \in \mathscr{C}([0, \theta_*), H^s(\mathbb{R}))$ the maximal solution of the inviscid Burgers equation

(16) $$\partial_\theta \mathfrak{z} + \Gamma \mathfrak{z} \partial_X \mathfrak{z} = 0$$

such that $\mathfrak{z}(0) = \hat\rho^{\rm in}$*, where the maximal time of existence* θ_* *depends continuously on* M*, and* $\zeta \in \mathscr{C}([0,+\infty), H^s(\mathbb{R}))$ *the global solution of the KdV equation*

`plainKdV`
$$\partial_\theta \zeta + \Gamma \zeta \partial_X \zeta = \frac{\varepsilon^2}{\eta} \kappa \partial_X^3 \zeta \tag{17}$$

such that $\zeta(0) = \hat\rho^{\rm in}$*. Then there exists a constant* C*, depending only on* s *and* M*, so that for* $0 \leqslant T \leqslant \min(T_*, \theta_*)/\eta$*, the following hold:*

`it:exii` *(i). For all integers* σ *such that* $0 \leqslant \sigma \leqslant s-4$*,*

$$\|\hat\rho - \mathfrak{z}(\eta T, \cdot - \mathfrak{c}T)\|_{H^\sigma(\mathbb{R})} + \Big\| \frac{\varrho}{\mathfrak{c}} \hat u - \mathfrak{z}(\eta T, \cdot - \mathfrak{c}T) \Big\|_{H^\sigma(\mathbb{R})} \leqslant C\Big(\eta + \varepsilon + \frac{\varepsilon^2}{\eta} + \Big\| \hat\rho^{\rm in} - \frac{\varrho}{\mathfrak{c}} \hat u^{\rm in} \Big\|_{H^\sigma(\mathbb{R})} \Big),$$

$$\|\hat\rho - \zeta(\eta T, \cdot - \mathfrak{c}T)\|_{H^\sigma(\mathbb{R})} + \Big\| \frac{\varrho}{\mathfrak{c}} \hat u - \zeta(\eta T, \cdot - \mathfrak{c}T) \Big\|_{H^\sigma(\mathbb{R})} \leqslant C\Big(\eta + \varepsilon + \Big\| \hat\rho^{\rm in} - \frac{\varrho}{\mathfrak{c}} \hat u^{\rm in} \Big\|_{H^\sigma(\mathbb{R})} \Big).$$

`it:exi` *(ii). If in addition* $s \geqslant 5$*, for all integers* σ *such that* $0 \leqslant \sigma \leqslant s-5$*,*

$$\|\hat\rho - \mathfrak{z}(\eta T, \cdot - \mathfrak{c}T)\|_{H^\sigma(\mathbb{R})} + \Big\| \frac{\varrho}{\mathfrak{c}} \hat u - \mathfrak{z}(\eta T, \cdot - \mathfrak{c}T) \Big\|_{H^\sigma(\mathbb{R})} \leqslant C\Big(\eta + \frac{\varepsilon^2}{\eta} + \Big\| \hat\rho^{\rm in} - \frac{\varrho}{\mathfrak{c}} \hat u^{\rm in} \Big\|_{H^\sigma(\mathbb{R})} \Big),$$

$$\|\hat\rho - \zeta(\eta T, \cdot - \mathfrak{c}T)\|_{H^\sigma(\mathbb{R})} + \Big\| \frac{\varrho}{\mathfrak{c}} \hat u - \zeta(\eta T, \cdot - \mathfrak{c}T) \Big\|_{H^\sigma(\mathbb{R})} \leqslant C\Big(\eta + \varepsilon^2 + \Big\| \hat\rho^{\rm in} - \frac{\varrho}{\mathfrak{c}} \hat u^{\rm in} \Big\|_{H^\sigma(\mathbb{R})} \Big).$$

Observe that both $\mathfrak{z}$ and ζ are shifted to the right at speed $\mathfrak{c}$ in the estimates above. This theorem provides various types of errors, depending on the relation between η and ε. Roughly speaking and neglecting the term $\|\hat\rho^{\rm in} - \varrho \hat u^{\rm in}/\mathfrak{c}\|_{H^\sigma(\mathbb{R})}$, which be small enough provided that the initial data are *well-prepared*, Theorem 4 ensures that $\hat\rho \approx \varrho\hat u/\mathfrak{c}$ is close, up to a rescaling in time and space shifting, to the solution to

- the Burgers equation (16) if $\varepsilon \lesssim \eta \ll 1$, with an $\mathcal{O}(\eta)$ error;
- still the Burgers equation (16) if $\varepsilon^2 \ll \eta \ll \varepsilon \ll 1$, with an $\mathcal{O}(\varepsilon^2/\eta)$ error;
- but also the KdV equation (17) if $\varepsilon^2 \ll \eta \ll \varepsilon \ll 1$ (which makes (17) weakly dispersive), with a smaller error $\mathcal{O}(\eta)$ (because $\eta \ll \varepsilon^2/\eta$) if we use (ii);
- and the KdV equation (17) if $\varepsilon^2 \approx \eta \ll 1$, with an $\mathcal{O}(\eta)$ error.

When $\eta \ll \varepsilon^2$, the comparison estimates with the solution $\mathfrak{z}$ of the Burgers equation given in the first inequalities of (i) and (ii) are meaningless since $\varepsilon^2/\eta \gg 1$ in the right-hand side. Note also that both statements *(i)* and *(ii)* hold true if $s \geqslant 5$. For instance in the case $\varepsilon \lesssim \eta \ll 1$, both *(i)* and *(ii)* yield $\mathcal{O}(\eta)$ errors, but the advantage of *(i)* is that it controls one more derivative. However, this advantage is lost in the case $\varepsilon^2 \ll \eta \ll \varepsilon \ll 1$, for which *(i)* provides $\mathcal{O}(\varepsilon)$ errors instead of the 'natural' $\mathcal{O}(\eta)$. In this sense, the case $\varepsilon \approx \eta$ corresponds to a threshold across which the natural, $\mathcal{O}(\eta)$ estimates lose the control of one derivative.

The coefficient ε^2/η in the dispersive term in (17) may be removed using the scaling invariances of KdV. Indeed, if ζ solves (17), then the function

$$\zeta_\sharp(\theta, X) \stackrel{\text{def}}{=} \frac{\eta}{\varepsilon^2}\zeta\left(\frac{\eta}{\varepsilon^2}\theta, X\right)$$

solves

$$\partial_\theta \zeta_\sharp + \Gamma\zeta_\sharp\partial_X\zeta_\sharp = \kappa\partial_X^3\zeta_\sharp$$

with associated initial datum $\zeta_\sharp(\theta = 0, X) = (\eta/\varepsilon^2)\zeta(0, X)$, which may be large or small depending on η/ε^2. In particular, $\|\zeta_\sharp(\theta = 0)\|_{H^s} = (\eta/\varepsilon^2)\|\zeta_\sharp(\theta = 0)\|_{H^s}$ for any s.

Our result recover the results of [8] and [11] as particular cases. In Corollary 1 of [11], there is a misprint: the two H^s norms in the statement of the result should be H^{s-5} norms instead.

4.2 Proof of error bounds for right-going waves

Since we concentrate on right-going waves, we can work in a moving frame, and introduce

$$\tilde\rho(T, Y) \stackrel{\text{def}}{=} \hat\rho(T, Y + \mathfrak{c}T) \qquad\qquad \tilde u(T, Y) \stackrel{\text{def}}{=} \hat u(T, Y + \mathfrak{c}T)\,.$$

This changes $(\mathrm{EK}_{\varepsilon,\eta})$ into

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$$\begin{cases} \partial_T\tilde\rho - \mathfrak{c}\partial_Y\tilde\rho + \partial_Y((\varrho + \eta\tilde\rho)\tilde{\mathbf{u}}) = 0 \\[2ex] \partial_T\tilde u - \mathfrak{c}\partial_Y\tilde u + \eta\tilde u\partial_Y\tilde u + g'(\varrho + \eta\tilde\rho)\partial_Y\tilde\rho \\ \qquad = \varepsilon^2\partial_Y\Big(K(\varrho + \eta\tilde\rho)\partial_Y^2\tilde\rho + \dfrac{\eta}{2}K'(\varrho + \eta\tilde\rho)(\partial_Y\tilde\rho)^2\Big)\,. \end{cases} \tag{18}$$

In a first step, we estimate the L^2 norm of $\tilde\rho - \varrho\mathfrak{c}^{-1}\tilde u$.

EKravane **Lemma 1** *We assume that $s > 3.5$. Then, in the framework of Theorem 4, there exists a constant C, depending only on s and M, so that*

$$\sup_{0\leqslant T\leqslant T_*/\eta}\left\|\tilde\rho - \frac{\varrho}{\mathfrak{c}}\tilde u\right\|_{L^2} \leqslant C\left\|\tilde\rho^{\text{in}} - \frac{\varrho}{\mathfrak{c}}\tilde u^{\text{in}}\right\|_{L^2} + C(\eta + \varepsilon^2)\,.$$

Proof. We argue as in the proof of Proposition 14 in [8] (see also Proposition 2 in [11]). Throughout the following computations, C will denote a positive constant depending only on s and Λ that may change from line to line. From (18), we see that the difference $v \stackrel{\text{def}}{=} \tilde\rho - \varrho\mathfrak{c}^{-1}\tilde u$ satisfies

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$$\partial_T v - 2\mathfrak{c}\partial_Y v - \eta\frac{\mathfrak{c}}{\varrho}v\partial_Y v = \partial_Y G\,, \tag{19}$$

where

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$$G \stackrel{\text{def}}{=} \frac{\varrho}{\eta\mathfrak{c}}(g(\varrho + \eta\tilde\rho) - g(\varrho) - \eta g'(\varrho)\tilde\rho) - \varepsilon^2\frac{\varrho}{\mathfrak{c}}\Big(K(\varrho + \eta\tilde\rho)\partial_Y^2\tilde\rho + \frac{\eta}{2}K'(\varrho + \eta\tilde\rho)(\partial_Y\tilde\rho)^2\Big) - \eta\frac{\mathfrak{c}}{\varrho}\frac{\tilde\rho^2}{2}\,.$$

The terms $v\partial_Y v$ and $\mathfrak{c}\partial_Y v$ in (19) do not contribute to the time evolution of $\|v\|_{L^2}$, and more precisely, (19) implies that

$$\frac{d}{dT}\int_{\mathbb{R}} v^2 \; dY = 2\int_{\mathbb{R}} v\partial_Y G \; dY = -\frac{2}{\mathfrak{c}}\int_{\mathbb{R}} (\mathfrak{c}\partial_Y v) \; G \; dY \, .$$

Using again (19), we can substitute $\partial_T v - \eta\frac{\mathfrak{c}}{\varrho}v\partial_Y v - \partial_Y G$ for $2\mathfrak{c}\partial_Y v$ in the previous identity, which gives after integration in time,

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$$\int_{\mathbb{R}} v^2 \; dY = \Big\|\hat{\rho}^{\rm in} - \frac{\varrho}{\mathfrak{c}}\hat{\mathbf{u}}^{\rm in}\Big\|^2_{L^2} + \frac{1}{\mathfrak{c}}\int_0^T\int_{\mathbb{R}}\Big(-\partial_\tau v + \eta\frac{\mathfrak{c}}{\varrho}v\partial_Y v + \partial_Y G\Big) \; G \; dY \; d\tau \, . \tag{21}$$

Of course the integral $\int_{\mathbb{R}} G\partial_Y G \; dY$ vanishes at all times. Moreover, by another integration by parts and the following L^∞ estimate for $\partial_Y G$,

$$\|\partial_Y G\|_{L^\infty} \lesssim \eta\|\tilde{\rho}\|_{L^\infty}\|\tilde{\rho}\|_{W^{1,\infty}} + \varepsilon^2\|\tilde{\rho}\|_{W^{3,\infty}}$$

in which all norms of $\tilde{\rho}$ are controlled since it is in H^s and we have assumed that $s > 3.5$, we obtain

$$\int_{\mathbb{R}} \eta(v\partial_Y v) \; G \; dY = -\frac{\eta}{2}\int_{\mathbb{R}} v^2\partial_Y G \; dY \leqslant C\eta(\eta+\varepsilon^2)\int_{\mathbb{R}} v^2 \; dY \, .$$

Therefore, integrating by parts in time the remaining integral in (21), we deduce

$$\begin{aligned}\int_{\mathbb{R}} v^2 \; dY \leqslant \Big\|\hat{\rho}^{\rm in} - \frac{\varrho}{\mathfrak{c}}\hat{u}^{\rm in}\Big\|^2_{L^2} &+ \frac{1}{\mathfrak{c}}\int_{\mathbb{R}}\Big[v(0)G(0) - v(T)G(T)\Big] \; dY \\ &+ \frac{1}{\mathfrak{c}}\int_0^T\int_{\mathbb{R}} v \; \partial_\tau G \; dY \; d\tau + C\eta(\eta+\varepsilon^2)\int_0^T\int_{\mathbb{R}} v^2 \; dY \; d\tau \, .\end{aligned}$$

We now use that $\|G\|_{L^2} \leqslant C(\eta+\varepsilon^2)$, and substitute the actual expression in (20) of G in the integral of $v\partial_\tau G$ to infer the inequality

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$$\begin{aligned}\int_{\mathbb{R}} v^2 \; dY \leqslant \Big\|\hat{\rho}^{\rm in} - \frac{\varrho}{\mathfrak{c}}\hat{u}^{\rm in}\Big\|^2_{L^2} &+ C(\eta+\varepsilon^2)\Big\|\hat{\rho}^{\rm in} - \frac{\varrho}{\mathfrak{c}}\hat{u}^{\rm in}\Big\|_{L^2} + C(\eta+\varepsilon^2)\|v\|_{L^2} \\ &+ C\eta(\eta+\varepsilon^2)\int_0^T\int_{\mathbb{R}} v^2 \; dY \; d\tau + \frac{\varrho}{\mathfrak{c}^2}\int_0^T\int_{\mathbb{R}}(g'(\varrho+\eta\tilde{\rho}) - g'(\varrho))(\partial_\tau\tilde{\rho}) \; v \; dY \; d\tau \\ &- \varepsilon^2\frac{\varrho}{\mathfrak{c}^2}\int_0^T\int_{\mathbb{R}} v \; K(\varrho+\eta\tilde{\rho})\partial_Y^2\partial_\tau\tilde{\rho} \; dY \; d\tau \\ &- \eta\varepsilon^2\frac{\varrho}{\mathfrak{c}^2}\int_0^T\int_{\mathbb{R}} v \; K'(\varrho+\eta\tilde{\rho})(\partial_\tau\tilde{\rho}) \; (\partial_Y^2\tilde{\rho}) \; dY \; d\tau \\ &- \eta\varepsilon^2\frac{\varrho}{\mathfrak{c}^2}\int_0^T\int_{\mathbb{R}} v \; K'(\varrho+\eta\tilde{\rho})(\partial_Y\tilde{\rho})(\partial_Y\partial_\tau\tilde{\rho}) \; dY \; d\tau \\ &- \eta^2\varepsilon^2\frac{\varrho}{2\mathfrak{c}^2}\int_0^T\int_{\mathbb{R}} v \; K''(\varrho+\eta\tilde{\rho})(\partial_\tau\tilde{\rho})(\partial_Y\tilde{\rho})^2 \; dY \; d\tau - \frac{\eta}{\varrho}\int_0^T\int_{\mathbb{R}} v \; \tilde{\rho}\partial_\tau\tilde{\rho} \; dY \; d\tau \, .\end{aligned} \tag{22}$$

Now, notice that from the first equation in (18) we have $\partial_\tau\tilde\rho - \mathfrak{c}\partial_Y v = -\eta\partial_Y(\tilde\rho\tilde u)$, so that

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$$\|\partial_\tau\tilde\rho - \mathfrak{c}\partial_Y v\|_{L^2} \leqslant C\eta\,. \tag{23}$$

Therefore, we find that

$$\begin{aligned}
\int_0^T\int_{\mathbb{R}} (g'(\varrho+\eta\tilde\rho)-g'(\varrho))(\partial_\tau\tilde\rho)\ v\ dY\ d\tau &\leqslant C\eta^2\int_0^T \|v\|_{L^2}\ d\tau \\
&\quad + \int_0^T\int_{\mathbb{R}} (g'(\varrho+\eta\tilde\rho)-g'(\varrho))\ v\partial_Y v\ dY\ d\tau \\
&\leqslant C\eta^3 T + C\eta\int_0^T \|v\|_{L^2}^2\ d\tau
\end{aligned}$$

after using Young's inequality in the first integral and integrating by parts the second one. We can deal with the last integral in (22) in the same manner. For the remaining integrals, we may use the rough estimate $\|\partial_\tau\tilde\rho\|_{H^1}\leqslant C$, and deduce

$$\begin{aligned}
&\eta\varepsilon^2\int_0^T\int_{\mathbb{R}} v\ K'(\varrho+\eta\tilde\rho)(\partial_\tau\tilde\rho)\ (\partial_Y^2\tilde\rho)\ dY\ d\tau + \eta\varepsilon^2\int_0^T\int_{\mathbb{R}} v\ K'(\varrho+\eta\tilde\rho)(\partial_Y\tilde\rho)(\partial_Y\partial_\tau\tilde\rho)\ dY\ d\tau \\
&+\eta^2\varepsilon^2\int_0^T\int_{\mathbb{R}} v\ K''(\varrho+\eta\tilde\rho)(\partial_\tau\tilde\rho)(\partial_Y\tilde\rho)^2\ dY\ d\tau \\
&\leqslant C(\eta\varepsilon^2+\eta^2\varepsilon^2)\int_0^T\|v\|_{L^2}\ d\tau \leqslant \eta\int_0^T\|v\|_{L^2}^2\ d\tau + C\eta\varepsilon^4 T\,,
\end{aligned}$$

by Young's inequality again. Inserting these estimates into (22), we obtain

$$\begin{aligned}
\int_{\mathbb{R}} v^2\ dY \leqslant{}& \left\|\hat\rho^{\text{in}} - \frac{\varrho}{\mathfrak{c}}\hat u^{\text{in}}\right\|_{L^2}^2 + C(\eta+\varepsilon^2)\left\|\hat\rho^{\text{in}} - \frac{\varrho}{\mathfrak{c}}\hat u^{\text{in}}\right\|_{L^2} + C(\eta+\varepsilon^2)\|v\|_{L^2} \\
&+ C\eta(\eta^2+\varepsilon^4)T + C\eta(1+\eta+\varepsilon^2)\int_0^T\int_{\mathbb{R}} v^2\ dY\ d\tau \\
\leqslant{}& \left\|\hat\rho^{\text{in}} - \frac{\varrho}{\mathfrak{c}}\hat u^{\text{in}}\right\|_{L^2}^2 + C(\eta+\varepsilon^2)^2 + \frac{1}{2}\int_{\mathbb{R}} v^2\ dY + C\eta(\eta^2+\varepsilon^4)T + C\eta\int_0^T\int_{\mathbb{R}} v^2\ dY\ d\tau\,.
\end{aligned}$$

Finally, we can absorb the term $\frac{1}{2}\int_{\mathbb{R}} v^2\ dY$ in the left-hand side and use Gronwall's lemma to arrive at

$$\int_{\mathbb{R}} v^2\ dY \leqslant C\Big(\left\|\hat\rho^{\text{in}} - \frac{\varrho}{\mathfrak{c}}\hat u^{\text{in}}\right\|_{L^2}^2 + (\eta+\varepsilon^2)^2\Big)\mathsf{e}^{C\eta T}\,,$$

which completes the proof. □

We can also estimate higher order, Sobolev norms of $\tilde\rho_\varepsilon - \varrho\mathfrak{c}^{-1}\tilde u$. The natural idea would be to differentiate the equation with respect to Y and argue as for Lemma 1. However, this yields a non optimal result in terms of loss of derivatives. Indeed, we used the L^∞ bound on $\partial_Y G$, which would become, for an H^σ estimate on v, an L^∞ bound on $\partial_Y\Lambda^\sigma G$. This naturally requires G to be bounded in $H^{\sigma+1.5+0}$. Since we only have $\|G\|_{H^{s-2}} \leqslant C(\eta+\varepsilon^2)$ and $\|G\|_{H^{s-1}} \leqslant C(\eta+\varepsilon)$, this restrict us to $\sigma < s-3.5$ or $\sigma < s-2.5$. In the following lemma, we argue as in [11]: we apply ∂_T to the equation and perform an $H^{\sigma-1}$ estimate on $\partial_T(\tilde\rho,\tilde u)$ to derive a better result where we may consider $\sigma < s-$ or $\sigma < s-$. In comparison with Proposition 4 in [8], we gain derivatives.

Lemma 2 *In the framework of Theorem 4, the following estimates hold true.*
If $s > 3.5$, then, for any $0 \leqslant \sigma < s-3.5$, there exists C, depending only on s, σ and M, so that

$$\sup_{0\leqslant T\leqslant T_*/\eta} \Big\|\tilde\rho - \frac{\varrho}{\mathfrak{c}}\tilde{\mathbf{u}}\Big\|_{H^\sigma} \leqslant C\Big(\Big\|\hat\rho^{\mathrm{in}} - \frac{\varrho}{\mathfrak{c}}\hat{\mathbf{u}}^{\mathrm{in}}\Big\|_{H^\sigma} + \eta + \varepsilon^2\Big).$$

If $s > 2.5$, then, for every $0 \leqslant \sigma < s-2.5$, there exists C, depending only on s, σ and M, so that

$$\sup_{0\leqslant T\leqslant T_*/\eta} \Big\|\tilde\rho - \frac{\varrho}{\mathfrak{c}}\tilde{\mathbf{u}}\Big\|_{H^\sigma} \leqslant C\Big(\Big\|\hat\rho^{\mathrm{in}} - \frac{\varrho}{\mathfrak{c}}\hat{\mathbf{u}}^{\mathrm{in}}\Big\|_{H^\sigma} + \eta + \varepsilon\Big).$$

Note that, for $s > 3.5$, the second estimate here above may be poorer than the first one (if $\eta \ll \varepsilon$) in terms of the error value, but it controls one more derivative.

Proof. Applying ∂_T to the system (18) written with the complex extended formulation, namely

$$\partial_T\tilde{\mathbf{z}} - \mathfrak{c}\partial_Y\tilde{\mathbf{z}} + \eta\tilde u\partial_Y\tilde{\mathbf{z}} + i\eta(\partial_Y\tilde{\mathbf{z}})\tilde{\mathbf{w}} + \frac{1}{\varepsilon}\,b(\rho)\tilde{\mathbf{w}} + i\varepsilon\partial_Y(a(\rho)\partial_Y\tilde{\mathbf{z}}) = 0,$$

we see that $\tilde{\mathbf{z}}_T \stackrel{\mathrm{def}}{=} \partial_T\tilde{\mathbf{z}}$ solves

$$\begin{aligned}\partial_T\tilde{\mathbf{z}}_T - \mathfrak{c}\partial_Y\tilde{\mathbf{z}}_T + \eta\tilde u\partial_Y\tilde{\mathbf{z}}_T + i\eta(\partial_Y\tilde{\mathbf{z}}_T)\tilde{\mathbf{w}} + \frac{1}{\varepsilon}\,b(\rho)\tilde{\mathbf{w}}_T + i\varepsilon\partial_Y(a(\rho)\partial_Y\tilde{\mathbf{z}}_T)\\ = -\eta\tilde u_T\partial_Y\tilde{\mathbf{z}} - i\eta(\partial_Y\tilde{\mathbf{z}})\tilde{\mathbf{w}}_T - \frac{\eta}{\varepsilon}\,b'(\rho)\partial_T\tilde\rho\;\tilde{\mathbf{w}} - i\varepsilon\eta\partial_Y(a'(\rho)\partial_T\tilde\rho\;\partial_Y\tilde{\mathbf{z}}),\end{aligned} \tag{24}$$

and for $T=0$, since $b(\rho)\tilde{\mathbf{w}} = \varepsilon g'(\rho)\partial_Y\tilde\rho$, $\mathfrak{c}^2 = \varrho g'(\varrho)$ and $\sigma+1 \leqslant s$, we have

$$\begin{aligned}\|\tilde{\mathbf{z}}_T(T=0)\|_{H^{\sigma-1}} &\leqslant \|\mathfrak{c}\partial_Y\tilde{\mathbf{z}}_{|T=0} - \varepsilon^{-1}b(\rho_{|T=0})\tilde{\mathbf{w}}_{|T=0} - i\varepsilon\partial_Y(a(\rho_{|T=0})\partial_Y\tilde u_{|T=0})\|_{H^{\sigma-1}} + C(\eta+\varepsilon^2)\\ &\leqslant \Big\|(\mathfrak{c}\partial_Y\tilde u - g'(\rho)\partial_Y\tilde\rho)_{|T=0} + i\varepsilon\left(\mathfrak{c}\partial_Y(b(\rho)^{-1}g'(\rho)\partial_Y\tilde\rho) - \partial_Y(a(\rho)\partial_Y\tilde u\right)_{|T=0}\Big\|_{H^{\sigma-1}}\\ &\quad + C(\eta+\varepsilon^2)\\ &\leqslant C\|\partial_Y(\tilde\rho - \varrho\mathfrak{c}^{-1}\tilde u)_{|T=0}\|_{H^{\sigma-1}} + C\varepsilon\|\partial_Y(\tilde\rho - \varrho\mathfrak{c}^{-1}\tilde u)_{|T=0}\|_{H^\sigma} + C(\eta+\varepsilon^2).\end{aligned} \tag{25}$$

The source term in the right-hand side of (24) is bounded in $H^{\sigma-1}$ by $C\eta\|\tilde{\mathbf{z}}_T\|_{H^{\sigma-1}}$. Therefore, following the lines of the proof of Theorem 2, we infer, for $0 \leqslant T \leqslant T_*/\eta$,

$$\|\tilde{\mathbf{z}}_T\|_{H^{\sigma-1}} \leqslant C\|\tilde{\mathbf{z}}_T(T=0)\|_{H^{\sigma-1}}.$$

As a consequence, considering the real part of the equation for $\tilde{\mathbf{z}}$,

$$\begin{aligned}\|\partial_Y(\tilde\rho - \varrho\mathfrak{c}^{-1}\tilde u)\|_{H^{\sigma-1}} &\leqslant C\|\mathrm{Re}(\tilde{\mathbf{z}}_T + \eta\tilde u\partial_X\tilde{\mathbf{z}} + i\eta(\partial_X\tilde{\mathbf{z}})\tilde{\mathbf{w}} + i\varepsilon\partial_X(a(\rho)\partial_X\tilde{\mathbf{z}}))\|_{H^{\sigma-1}}\\ &\leqslant C\|\partial_Y(\tilde\rho^{\mathrm{in}} - \varrho\mathfrak{c}^{-1}\tilde u^{\mathrm{in}})\|_{H^{\sigma-1}} + C\varepsilon\|\partial_Y(\tilde\rho^{\mathrm{in}} - \varrho\mathfrak{c}^{-1}\tilde u^{\mathrm{in}})\|_{H^\sigma} + C(\eta+\varepsilon^2)\end{aligned}$$

fff
$(\tilde{\rho}_T, \tilde{u}_T) \stackrel{\text{def}}{=} (\partial_T \tilde{\rho}, \partial_T \tilde{u})$ solves

$$
\left\{
\begin{aligned}
&\partial_T \tilde{\rho}_T - \mathfrak{c}\partial_Y \tilde{\rho}_T + \partial_Y((\varrho + \eta\tilde{\rho})\tilde{\mathbf{u}}_T) = -\eta\partial_Y(\tilde{\rho}_T \tilde{\mathbf{u}}) \\
&\partial_T \hat{\mathbf{z}} + \eta\hat{u}\partial_X \hat{\mathbf{z}} + i\eta(\partial_X \hat{\mathbf{z}})\hat{\mathbf{w}} + \frac{1}{\varepsilon} b(\rho)\hat{\mathbf{w}} + i\varepsilon\partial_X(a(\rho)\partial_X \hat{\mathbf{z}}) = 0\,.
\end{aligned}
\right.
\tag{26}
$$

Proof of Theorem 4 completed. Recall that for (ii), we have $s > 4.5$ and $0 \leqslant \sigma < s - 5.5$. In particular, we may apply Lemma 2 with $\sigma + 3 < s - 2.5$ to deduce $\|v\|_{H^{\sigma+3}} \leqslant C(\eta + \varepsilon^2 + \|\hat{\rho}^{\text{in}} - \varrho\mathfrak{c}^{-1}\hat{\mathbf{u}}^{\text{in}}\|_{H^{\sigma+3}})$. Combining the two equations in (18), we infer that

$$
w \stackrel{\text{def}}{=} \frac{1}{2}\Big(\tilde{\rho} + \frac{\varrho}{\mathfrak{c}}\tilde{\mathbf{u}}\Big)
$$

satisfies

$$
\begin{aligned}
2\partial_T w + 2\eta\Gamma w\partial_Y w - 2\varepsilon^2\kappa\partial_Y^3 w = {}& \frac{\varrho}{\mathfrak{c}}\Big[g'(\varrho + \eta\tilde{\rho}) - g'(\varrho) - \eta g''(\varrho)\tilde{\rho}\Big]\partial_Y \tilde{\rho} \\
&+ \eta\Big(\frac{\mathfrak{c}}{\varrho} - \frac{\varrho}{\mathfrak{c}}g''(\varrho)\Big)v\partial_Y w + \eta\Big(\frac{\mathfrak{c}}{\varrho} - \frac{\varrho}{\mathfrak{c}}g''(\varrho)\Big)\tilde{\rho}\partial_Y v + 2\varepsilon^2\kappa\partial_Y^3 v \\
&+ \varepsilon^2\frac{\varrho}{2\mathfrak{c}}\partial_Y\Big([K(\varrho + \eta\tilde{\rho}) - K(\varrho)]\partial_Y^2\tilde{\rho} + \frac{\eta}{2}K'(\varrho + \eta\tilde{\rho})(\partial_Y\tilde{\rho})^2\Big)\,.
\end{aligned}
\tag{27}
$$

From the uniform bounds of Theorem 2 and the estimates on v provided in Lemmas 2 and 1, we infer the consistency estimate, for $0 \leqslant \sigma \leqslant s - 4$,

$$
\big\|\partial_T w + \eta\Gamma w\partial_Y w - \varepsilon^2\kappa\partial_Y^3 w\big\|_{H^\sigma} \leqslant C(\eta + \varepsilon^2)\Big(\eta + \varepsilon^2 + \Big\|\hat{\rho}^{\text{in}} - \frac{\varrho}{\mathfrak{c}}\hat{\mathbf{u}}^{\text{in}}\Big\|_{H^{\sigma+3}}\Big)\,.
$$

The conclusion

$$
\|w - \zeta(\eta T, \cdot - \mathfrak{c}T)\|_{H^\sigma(\mathbb{R})} \leqslant C\Big(\eta + \varepsilon + \Big\|\hat{\rho}^{\text{in}} - \frac{\varrho}{\mathfrak{c}}\hat{u}^{\text{in}}\Big\|_{H^\sigma(\mathbb{R})}\Big)\,.
$$

then follows from very standard estimates on the KdV equation. If $s \geqslant 4$, we use the second estimate in Lemma 2. $\square$

4.3 The KdV regime for travelling waves

Under fairly general assumptions on the energy density $\mathscr{F}$, (gEK) admits rich families of planar travelling wave solutions. Indeed, for $(\rho, u) = (R, U)(x - \sigma t)$ to solve the one D version of (gEK) the profile (R, U) must solve the ODEs

$$
\left\{
\begin{aligned}
&(R(U - \sigma))' = 0 \\
&\Big(\frac{1}{2}(U - \sigma)^2 + \delta\mathscr{F}[R]\Big)' = 0\,,
\end{aligned}
\right.
$$

which is equivalent to the existence of three constants (j,λ,μ) such that

prof
$$\left\{\begin{array}{l} R(U-\sigma)=j \\ \\ R'\,\dfrac{\partial\mathscr{F}}{\partial\rho_x}(R,R')-\mathscr{F}(R,R')+\dfrac{j^2}{2R}+\lambda R=\mu\,, \end{array}\right. \tag{28}$$

(See Ben [3] for more details.) If $\varrho>0$ and (j,λ) are such that $(\varrho,0)$ is a strict local minimum of the mapping $\mathscr{H}:(R,\dot R)\mapsto \dot R\,\dfrac{\partial\mathscr{F}}{\partial\rho_x}(R,\dot R)-\mathscr{F}(R,\dot R)+\dfrac{j^2}{2R}+\lambda R$ then the level sets

$$\{(R,\dot R)\,;\ \dot R\,\frac{\partial\mathscr{F}}{\partial\rho_x}(R,\dot R)-\mathscr{F}(R,\dot R)+\frac{j^2}{2R}+\lambda R=\mu\}$$

consist of closed curves for μ greater than, and close to $-\mathscr{F}(\varrho,0)+\frac{j^2}{2\varrho}+\lambda\varrho$. These correspond to periodic travelling wave solutions to (gEK) oscillating around ϱ. Note that, since the Hessian matrix of $\mathscr{H}$ at $(\varrho,0)$ is given by

$$\mathrm{Hess}\mathscr{H}(\varrho,0)=\left(\begin{array}{cc} -\dfrac{\partial^2\mathscr{F}}{\partial\rho^2}(\varrho,0)+\dfrac{j^2}{\rho^3} & 0 \\ 0 & \dfrac{\partial^2\mathscr{F}}{\partial\rho_x^2}(\varrho,0) \end{array}\right)=\left(\begin{array}{cc} \dfrac{j^2}{\rho^3}-\dfrac{\mathfrak{c}^2}{\varrho^2} & 0 \\ 0 & \underline{K} \end{array}\right),$$

the strict local minimization condition for $\mathscr{H}$ at $(\varrho,0)$ is ensured by the inequalities $\underline{K}>0$, $j^2>\varrho^2\mathfrak{c}^2$, provided that $(\varrho,0)$ is a critical point of $\mathscr{H}$, which requires that

$$\lambda=\frac{\partial\mathscr{F}}{\partial\rho}(\varrho,0)+\frac{j^2}{2\varrho^2}\,.$$

This means that (gEK) admits periodic travelling waves solutions with large enough momentum in the frame attached to them. Solitary waves with endstate ϱ arise when $(\varrho,0)$ is a saddle-point of $\mathscr{H}$. They are of small amplitude if this saddle-point is close to local minimum of $\mathscr{H}$. This happens only if $(\varrho,0)$ is close to a critical point of $\mathscr{H}$ where the Hessian of $\mathscr{H}$ is singular. In other words, small amplitude solitary waves occur when $\varrho^2\mathfrak{c}^2-j^2$ is positive and close to zero. Note that for small amplitude waves around $(\varrho,0)$, we have $j=R(U-\sigma)\approx-\varrho\sigma$, so that $\varrho^2\mathfrak{c}^2-j^2$ being close to zero is equivalent to σ^2 being close to $\mathfrak{c}^2$.

Let us consider a travelling wave $(\rho,u)(x,t)=(R,U)(x-\sigma t)$ solution to (gEK), of small amplitude around some reference state $(\varrho,0)$. Assume moreover that its speed σ is close to $\mathfrak{c}$, say $\sigma=\mathfrak{c}+\varepsilon^2\tilde\sigma$ with $\varepsilon>0$ small. Then of course we can write

$$x-\sigma t=(\varepsilon(x-\mathfrak{c}t)-\tilde\sigma\varepsilon^3 t)/\varepsilon,$$

so that if we use the KdV rescaling in (KdVansatz 1),

$$(\rho,u)(x,t)=(\varrho,0)+\varepsilon^2(\tilde\rho,\tilde u)(\varepsilon(x-\mathfrak{c}t),\varepsilon^3 t)\,,$$

we have

$$(\tilde\rho,\tilde u)(Y,\theta)=\frac{1}{\varepsilon^2}\left((R,U)\Big(\frac{Y-\tilde\sigma\theta}{\varepsilon}\Big)-(\varrho,0)\right)\;=\;(\hat R,\hat U)(Y-\tilde\sigma\theta)$$

if we set

$$(R,U)(x)=(\varrho,0)+\varepsilon^2(\hat R,\hat U)(\varepsilon x)\,.$$

As far as (EK) is concerned, we know by Theorem 4 that $(\tilde\rho,\tilde u)$ is such that $w\stackrel{\text{def}}{=}\frac12(\tilde\rho+\frac{\varrho}{\mathfrak c}\tilde u)$ approximately solves the KdV equation

$$\partial_\theta w+\Gamma w\partial_Y w=\kappa\partial_Y^3 w\,.$$

Therefore, $W\stackrel{\text{def}}{=}\frac12(\hat R+\frac{\varrho}{\mathfrak c}\hat U)$ is close to the profile of a travelling wave solution to this KdV equation with speed $\tilde\sigma$. This can also be seen on the profile equations themselves, which is interesting for (gEK) since we do not have a result like Theorem 4 for this more general system.

periodic **Theorem 5** *Let $\varrho>0$ be such that $\frac{\partial^2\mathscr F}{\partial\rho^2}(\varrho,0)>0$ and $\frac{\partial^2\mathscr F}{\partial\rho_x^2}(\varrho,0)>0$. We denote as before*

$$\mathfrak c=\sqrt{\varrho\frac{\partial^2\mathscr F}{\partial\rho^2}(\varrho,0)}\,,\ \Gamma=\frac{3\mathfrak c}{2\varrho}+\frac{\varrho}{2\mathfrak c}\,\frac{\partial^3\mathscr F}{\partial\rho^3}(\varrho,0)\,,\ \kappa=\frac{\varrho}{2\mathfrak c}\,\frac{\partial^2\mathscr F}{\partial\rho_x^2}(\varrho,0)\,,$$

and assume that $\Gamma>0$ (which is the case in 'standard' fluids). Let $w=W(Y-\tilde\sigma\theta)$ be a travelling wave of speed $\tilde\sigma>0$ solution to the Korteweg-de Vries equation

$$\partial_\theta w+\Gamma w\partial_Y w=\kappa\partial_Y^3 w\,,$$

and more precisely such that

$$\tfrac12\kappa W'^2-\tfrac16\Gamma W^3+\tfrac12\tilde\sigma W^2=m\in(0,m_0)\,,\ m_0\stackrel{\text{def}}{=}\frac{2\tilde\sigma^3}{3\Gamma^2}\,.$$

Then there exists $\varepsilon_0>0$ such that for all $\varepsilon\in(0,\varepsilon_0]$, there is a one-parameter family of periodic traveling waves $(\rho,u)(x,t)=(R,U)(x-\sigma t)$ solution to (gEK) with $\sigma=\mathfrak c+\varepsilon^2\tilde\sigma$, verifying (28) with

$$j=-\varrho(\mathfrak c+\tfrac12\varepsilon^2\tilde\sigma)\,,\ \lambda=\frac{\partial\mathscr F}{\partial\rho}(\varrho,0)+\frac{j^2}{2\varrho^2}\,,\ \mu=\varrho\frac{\partial\mathscr F}{\partial\rho}(\varrho,0)-\mathscr F(\varrho,0)+\frac{j^2}{\varrho}+\frac{2\mathfrak c}{\varrho}\varepsilon^6 m\,,$$

and

$$(R,U)(x)=(\varrho,0)+\varepsilon^2(\hat R,\hat U)(\varepsilon x)$$

with

$$\inf_s\|\hat R(\cdot+s)-W\|_{W^{1,\infty}}=\mathcal O(\varepsilon^2)\,.$$

From a straightforward phase portrait analysis of the KdV travelling wave ODEs, using that $\tilde\sigma$ is positive we see that the wave profile W is indeed periodic for $m \in (0, m_0)$, and homoclinic to $\rho_0 \stackrel{\text{def}}{=} 2\tilde\sigma/\Gamma$ in the limiting case $m = m_0$. As explained above, there is no hope to get a solitary wave solution to (gEK) that is homoclinic to ϱ if $j^2 > \varrho^2\mathfrak{c}^2$. This explains why the KdV regime for solitary waves requires $\tilde\sigma < 0$, so that $j = -\varrho(\mathfrak{c} + \frac{1}{2}\varepsilon^2\tilde\sigma)$ implies $j^2 < \varrho^2\mathfrak{c}^2$ for ε small enough. The KdV regime for solitary waves is a little simpler than for periodic waves, and can be justified as follows.

Theorem 6 *With the same notations as in Theorem 5, we consider a travelling wave of speed $\tilde\sigma < 0$, $w = W(Y - \tilde\sigma\theta)$ such that*

$$\tfrac{1}{2}\kappa W'^2 - \tfrac{1}{6}\Gamma W^3 + \tfrac{1}{2}\tilde\sigma W^2 = 0\,.$$

Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$, there is a one-parameter family of solitary traveling waves $(\rho, u)(x,t) = (R, U)(x - \sigma t)$ solution to (gEK) with $\sigma = \mathfrak{c} + \varepsilon^2\tilde\sigma$, verifying (28) with

$$j = -\varrho(\mathfrak{c} + \tfrac{1}{2}\varepsilon^2\tilde\sigma)\,,\ \lambda = \frac{\partial\mathscr{F}}{\partial\rho}(\varrho, 0) + \frac{j^2}{2\varrho^2}\,,\ \mu = \varrho\frac{\partial\mathscr{F}}{\partial\rho}(\varrho, 0) - \mathscr{F}(\varrho, 0) + \frac{j^2}{\varrho}\,,$$

and

$$(R, U)(x) = (\varrho, 0) + \varepsilon^2(\hat R, \hat U)(\varepsilon x)$$

with

$$\inf_s \|\hat R(Y + s) - W(Y)\|_{W^{1,\infty}} = \mathcal{O}(\varepsilon^2)\,.$$

This result was already known for the KdV regime associated with NLS, see [12].

5 The approximation by counter propagating waves on the line

5.1 Statement of error bounds for counter propagating waves

We can extend Theorem 4 by taking into account both left and right-going waves. In order to secure the interaction between these two waves, we shall assume an additional bound on the initial data, in terms of the $\mathcal{M}$-norm defined as in [24, 8] by

$$\|h\|_{\mathcal{M}} \stackrel{\text{def}}{=} \sup_{a,b\in\mathbb{R}} \left| \int_a^b h \right|\,.$$

Theorem 7 *We assume $d = 1$, and take an integer $s \geqslant 4$, and a real number $M > 0$. For $\eta \in (0, 1]$, $\varepsilon \in (0, 1]$, any initial data*

$$(\hat\rho^{\text{in}}, \hat u^{\text{in}}) \in B_\varepsilon(M) = \{(\hat\rho, \hat u) \in H^{s+1}(\mathbb{R}) \times H^s(\mathbb{R})\,;\ \|(\hat\rho, \hat u)\|_{(H^s(\mathbb{R}))^2} + \varepsilon\|\hat\rho\|_{H^{s+1}(\mathbb{R})} \leqslant M\}$$

is associated with the solution $(\hat\rho,\hat u)\in\mathscr{C}([0,T_*/\eta],B_\varepsilon(2M))$ *of* $(\mathrm{EK}_{\varepsilon,\eta})$ *such that* $(\hat\rho,\hat u)(0)=(\hat\rho^{\rm in},\hat u^{\rm in})$*, as given by Theorem 2 in the case* $d=1$*. We also introduce* $\mathfrak{z}^\pm\in\mathscr{C}([0,\theta_*^\pm],H^s(\mathbb{R}))$ *solutions of the uncoupled, inviscid Burgers equations*

$$\partial_\theta\mathfrak{z}^\pm\pm\Gamma\mathfrak{z}^\pm\partial_X\mathfrak{z}^\pm=0 \tag{29}$$

such that $\mathfrak{z}^\pm(0)=(\hat\rho^{\rm in}\pm\varrho\hat u^{\rm in}/\mathfrak{c})/2$*, where the maximal times of existence* $\theta_*^\pm$ *depend continuously on* M*, and* $\zeta^\pm\in\mathscr{C}([0,+\infty),H^s(\mathbb{R}))$ *the global solutions of the uncoupled KdV equations*

$$\partial_\theta\zeta^\pm\pm\Gamma\zeta^\pm\partial_X\zeta^\pm=\pm\frac{\varepsilon^2}{\eta}\kappa\partial_X^3\zeta^\pm \tag{30}$$

such that $\zeta^\pm(0)=(\hat\rho^{\rm in}\pm\varrho\hat u^{\rm in}/\mathfrak{c})/2$*. If, in addition,*

$$\|(\hat\rho^{\rm in},\hat{\mathbf{u}}^{\rm in})\|_{\mathcal{M}}\leqslant M\,,$$

then there exists a constant C*, depending only on* s *and* M*, so that for* $0\leqslant T\leqslant\min(T_*,\theta_*^+,\theta_*^-,)/\eta$*, the following hold:*

(i). For all integers σ *such that* $0\leqslant\sigma\leqslant s-4$*,*

$$\Big\|\tfrac12\big(\hat\rho+\frac{\varrho}{\mathfrak{c}}\hat{\mathbf{u}}\big)(T)-\mathfrak{z}^+(\eta T,\cdot-\mathfrak{c}T)\Big\|_{H^\sigma}+\Big\|\tfrac12\big(\hat\rho-\frac{\varrho}{\mathfrak{c}}\hat{\mathbf{u}}\big)(T)-\mathfrak{z}^-(\eta T,\cdot+\mathfrak{c}T)\Big\|_{H^\sigma}\leqslant C\left(\eta+\varepsilon+\frac{\varepsilon^2}{\eta}\right),$$

$$\Big\|\tfrac12\big(\hat\rho+\frac{\varrho}{\mathfrak{c}}\hat{\mathbf{u}}\big)(T)-\zeta^+(\eta T,\cdot-\mathfrak{c}T)\Big\|_{H^\sigma}+\Big\|\tfrac12\big(\hat\rho-\frac{\varrho}{\mathfrak{c}}\hat{\mathbf{u}}\big)(T)-\zeta^-(\eta T,\cdot+\mathfrak{c}T)\Big\|_{H^\sigma}\leqslant C\left(\eta+\varepsilon+\frac{\varepsilon^4}{\eta}\right).$$

(ii). If in addition $s\geqslant5$*, for all integers* σ *such that* $0\leqslant\sigma\leqslant s-5$*,*

$$\Big\|\tfrac12\big(\hat\rho+\frac{\varrho}{\mathfrak{c}}\hat{\mathbf{u}}\big)(T)-\mathfrak{z}^+(\eta T,\cdot-\mathfrak{c}T)\Big\|_{H^\sigma}+\Big\|\tfrac12\big(\hat\rho-\frac{\varrho}{\mathfrak{c}}\hat{\mathbf{u}}\big)(T)-\mathfrak{z}^-(\eta T,\cdot+\mathfrak{c}T)\Big\|_{H^\sigma}\leqslant C\left(\eta+\frac{\varepsilon^2}{\eta}\right),$$

$$\Big\|\tfrac12\big(\hat\rho+\frac{\varrho}{\mathfrak{c}}\hat{\mathbf{u}}\big)(T)-\zeta^+(\eta T,\cdot-\mathfrak{c}T)\Big\|_{H^\sigma}+\Big\|\tfrac12\big(\hat\rho-\frac{\varrho}{\mathfrak{c}}\hat{\mathbf{u}}\big)(T)-\zeta^-(\eta T,\cdot+\mathfrak{c}T)\Big\|_{H^\sigma}\leqslant C\left(\eta+\varepsilon^2+\frac{\varepsilon^4}{\eta}\right).$$

The proof relies in particular on a careful estimate of the interaction terms, as in [8]. However, we believe that our computation is simpler. The idea is the following. Assume that ζ^+ and ζ^- are smooth functions of (T,X) approximately satisfying transport equations with opposite propagation speeds,

$$\partial_T\zeta^+-\mathfrak{c}\partial_X\zeta^+=o(1)\qquad\text{and}\qquad\partial_T\zeta^-+\mathfrak{c}\partial_X\zeta^-=o(1)\,,$$

with sufficiently fast decay at $|X|\to\infty$, as it will turn out to be the case for

$$z^+(T,X)\stackrel{\rm def}{=}\tfrac12\big(\hat\rho+\frac{\varrho}{\mathfrak{c}}\hat{\mathbf{u}}\big)(T,X)-\zeta^+(\eta T,X-\mathfrak{c}T)\quad\text{and}\quad z^-(T,X)\stackrel{\rm def}{=}\tfrac12\big(\hat\rho-\frac{\varrho}{\mathfrak{c}}\hat{\mathbf{u}}\big)(T,X)-\zeta^-(\eta T,X+\mathfrak{c}T)\,.$$

The point is to show that quantities like the interaction term $\int_0^T\int_{\mathbb{R}}(\partial_X\zeta^+)\zeta^- dX d\tau$ are small. This is done by successively integrating by parts in space, substituting $-\mathfrak{c}^{-1}\partial_T\zeta^- + o(1)$ for $\partial_X\zeta^-$, integrating by parts in time, and finally substituting $\mathfrak{c}\partial_X\zeta^+ + o(1)$ for $\partial_T\zeta^+$, which formally gives

$$
\begin{aligned}
\int_0^T\int_{\mathbb{R}}(\partial_X\zeta^+)\zeta^-\ dXd\tau &= -\int_0^T\int_{\mathbb{R}}\zeta^+(\partial_X\zeta^-)\ dXd\tau = \mathfrak{c}^{-1}\int_0^T\int_{\mathbb{R}}\zeta^+(\partial_\tau\zeta^-)\ dXd\tau + o(T) \\
&= \mathfrak{c}^{-1}\int_0^T\int_{\mathbb{R}}(\partial_\tau\zeta^+)\zeta^-\ dXd\tau - \mathfrak{c}^{-1}\int_{\mathbb{R}}\zeta^+(T)\zeta^-(T)\ dX + \mathfrak{c}^{-1}\int_{\mathbb{R}}\zeta^+(0)\zeta^-(0)\ dX + o(T) \\
&= -\int_0^T\int_{\mathbb{R}}(\partial_X\zeta^+)\zeta^-\ dXd\tau - \mathfrak{c}^{-1}\int_{\mathbb{R}}\zeta^+(T)\zeta^-(T)\ dX + \mathfrak{c}^{-1}\int_{\mathbb{R}}\zeta^+(0)\zeta^-(0)\ dX + o(T)\,.
\end{aligned}
$$

As a consequence,

$$
2\int_0^T\int_{\mathbb{R}}(\partial_X\zeta^+)\zeta^-\ dXd\tau = \mathfrak{c}^{-1}\int_{\mathbb{R}}\zeta^+(0)\zeta^-(0)\ dX - \mathfrak{c}^{-1}\int_{\mathbb{R}}\zeta^+(T)\zeta^-(T)\ dX + o(T)\,,
$$

which means that the integral in time $\int_0^T\int_{\mathbb{R}}(\partial_X\zeta^+)\zeta^-\ dXd\tau$ is controlled by $\int_{\mathbb{R}}\zeta^+(T)\zeta^-(T)\ dX$ and $\int_{\mathbb{R}}\zeta^+(0)\zeta^-(0)\ dX$, even though T may be large. This is the basic idea used to bound the interaction terms.

5.2 Detailed proof of error bounds for counter propagating waves

In order to handle the two counter propagating waves we can no longer change frame. We just define

$$
w \stackrel{\text{def}}{=} \frac{1}{2}\Big(\hat\rho + \frac{\varrho}{\mathfrak{c}}\hat{\mathbf{u}}\Big)\,, \qquad v \stackrel{\text{def}}{=} \frac{1}{2}\Big(\hat\rho - \frac{\varrho}{\mathfrak{c}}\hat{\mathbf{u}}\Big)\,.
$$

From ($\mathrm{EK}_{\varepsilon,\eta}$), we deduce that w satisfies

$$
\begin{aligned}
2\partial_T w + 2\mathfrak{c}\partial_X w + \eta\partial_X(\hat\rho\hat{\mathbf{u}}) + \frac{\varrho}{\mathfrak{c}}\eta\hat{\mathbf{u}}\partial_X\hat{\mathbf{u}} + \frac{\varrho}{\mathfrak{c}}\Big[g'(\varrho+\eta\hat\rho) - g'(\varrho)\Big]\partial_X\hat\rho \\
= \varepsilon^2\frac{\varrho}{\mathfrak{c}}\partial_X\Big(K(\varrho+\eta\hat\rho)\partial_X^2\hat\rho + \frac{\eta}{2}K'(\varrho+\eta\hat\rho)(\partial_X\hat\rho)^2\Big)\,,
\end{aligned}
$$

that is

$$
\begin{aligned}
2\partial_T w + 2\mathfrak{c}\partial_X w + \eta\partial_X(\hat\rho\hat{\mathbf{u}}) + \frac{\varrho}{\mathfrak{c}}\eta\hat{\mathbf{u}}\partial_X\hat{\mathbf{u}} + \eta\frac{\varrho}{\mathfrak{c}}g''(\varrho)\hat\rho\partial_X\hat\rho \\
= \varepsilon^2\frac{\varrho}{\mathfrak{c}}K(\varrho)\partial_X^3\hat\rho - \frac{\varrho}{\mathfrak{c}}\Big[g'(\varrho+\eta\hat\rho) - g'(\varrho) - \eta g''(\varrho)\hat\rho\Big]\partial_X\hat\rho \\
+ \varepsilon^2\frac{\varrho}{\mathfrak{c}}\partial_X\Big([K(\varrho+\eta\hat\rho) - K(\varrho)]\partial_X^2\hat\rho + \frac{\eta}{2}K'(\varrho+\eta\hat\rho)(\partial_X\hat\rho)^2\Big)\,.
\end{aligned}
$$

Using that $\hat\rho = w + v$ and $\hat{\mathbf{u}} = \mathfrak{c}\varrho^{-1}(w - v)$, we obtain

$$\begin{aligned}
\partial_X(\hat\rho\hat{\mathbf{u}}) + \frac{\varrho}{\mathfrak{c}}\hat{\mathbf{u}}\partial_X\hat{\mathbf{u}} + \frac{\varrho}{\mathfrak{c}}g''(\varrho)\hat\rho\partial_X\hat\rho &= \frac{\mathfrak{c}}{\varrho}\partial_X((w+v)(w-v)) + \frac{\mathfrak{c}}{\varrho}(w-v)\partial_X(w-v)\\
&\quad + \frac{\varrho}{\mathfrak{c}}g''(\varrho)(w+v)\partial_X(w+v)\\
&= \left(3\frac{\mathfrak{c}}{\varrho} + \frac{\varrho}{\mathfrak{c}}g''(\varrho)\right)w\partial_X w + \left(-\frac{\mathfrak{c}}{\varrho} + \frac{\varrho}{\mathfrak{c}}g''(\varrho)\right)v\partial_X v\\
&\quad + \left(-\frac{\mathfrak{c}}{\varrho} + \frac{\varrho}{\mathfrak{c}}g''(\varrho)\right)\partial_X(vw)\,.
\end{aligned}$$

Therefore, using the definitions of κ and Γ, w solves

EKw
$$\begin{aligned}
2\partial_T w + 2\mathfrak{c}\partial_X w + 2\eta\Gamma w\partial_X w - 2\varepsilon^2\kappa\partial_X^3 w = &-\frac{\varrho}{\mathfrak{c}}\Big[g'(\varrho+\eta\hat\rho) - g'(\varrho) - \eta g''(\varrho)\hat\rho\Big]\partial_X\hat\rho\\
&+\eta\Big(\frac{\mathfrak{c}}{\varrho} - \frac{\varrho}{\mathfrak{c}}g''(\varrho)\Big)\partial_X(vw) + \eta\Big(\frac{\mathfrak{c}}{\varrho} - \frac{\varrho}{\mathfrak{c}}g''(\varrho)\Big)v\partial_X v + 2\varepsilon^2\kappa\partial_X^3 v\\
&+\varepsilon^2\frac{\varrho}{\mathfrak{c}}\partial_X\Big([K(\varrho+\eta\hat\rho) - K(\varrho)]\partial_X^2\hat\rho + \frac{\eta}{2}K'(\varrho+\eta\hat\rho)(\partial_X\hat\rho)^2\Big)
\end{aligned}\tag{31}$$

and, similarly, v satisfies

eqpourv
$$\begin{aligned}
2\partial_T v - 2\mathfrak{c}\partial_X v - 2\eta\Gamma v\partial_X v + 2\varepsilon^2\kappa\partial_X^3 v = &\frac{\varrho}{\mathfrak{c}}\Big[g'(\varrho+\eta\hat\rho) - g'(\varrho) - \eta g''(\varrho)\hat\rho\Big]\partial_X\hat\rho\\
&-\eta\Big(\frac{\mathfrak{c}}{\varrho} - \frac{\varrho}{\mathfrak{c}}g''(\varrho)\Big)\partial_X(vw) - \eta\Big(\frac{\mathfrak{c}}{\varrho} - \frac{\varrho}{\mathfrak{c}}g''(\varrho)\Big)w\partial_X w - 2\varepsilon^2\kappa\partial_X^3 w\\
&-\varepsilon^2\frac{\varrho}{\mathfrak{c}}\partial_X\Big([K(\varrho+\eta\hat\rho) - K(\varrho)]\partial_X^2\hat\rho + \frac{\eta}{2}K'(\varrho+\eta\hat\rho)(\partial_X\hat\rho)^2\Big)\,.
\end{aligned}\tag{32}$$

As a consequence, the function $W \stackrel{\text{def}}{=} w - z^+$ with $z^+(T,X) \stackrel{\text{def}}{=} \zeta^+(\eta T, X - \mathfrak{c}T)$, satisfies $W(T=0) = 0$ and solves

EKrambar
$$\partial_T W + \mathfrak{c}\partial_X W + \eta\Gamma W\partial_X W + \eta\Gamma\partial_X(z^+W) - \varepsilon^2\kappa\partial_X^3 W = \partial_X Q + 2\varepsilon^2\kappa\partial_X^3 v\,,\tag{33}$$

where

$$\begin{aligned}
Q \stackrel{\text{def}}{=} &-\frac{\varrho}{2\eta\mathfrak{c}}\Big[g(\varrho+\eta\hat\rho) - g(\varrho) - \eta g'(\varrho)\hat\rho - \frac{\eta^2}{2}g''(\varrho)\hat\rho^2\Big]\\
&+\varepsilon^2\frac{\varrho}{2\mathfrak{c}}\Big([K(\varrho+\eta\hat\rho) - K(\varrho)]\partial_X^2\hat\rho + \frac{\eta}{2}K'(\varrho+\eta\hat\rho)(\partial_X\hat\rho)^2\Big)\\
&+\eta\Big(\frac{\mathfrak{c}}{\varrho} - \frac{\varrho}{\mathfrak{c}}g''(\varrho)\Big)\partial_X(vw) + \eta\Big(\frac{\mathfrak{c}}{\varrho} - \frac{\varrho}{\mathfrak{c}}g''(\varrho)\Big)v\partial_X v\,.
\end{aligned}$$

Note that the term Q enjoys the estimates (since $s > 5/2$)

EKartouche
$$\|Q\|_{H^{s-2}} \leqslant C\eta^2\|\hat\rho\|_{H^{s-2}}^3 + C\varepsilon^2\eta\|\hat\rho\|_{H^s}^2 \leqslant C\eta(\varepsilon^2+\eta) \qquad \text{and} \qquad \|Q\|_{H^{s-1}} \leqslant C\eta(\varepsilon+\eta)\,.\tag{34}$$

We shall work for $s \geqslant 5$, the modifications for $s \geqslant 4$ being straightforward. We now assume $0 \leqslant \sigma \leqslant s-5$ and perform an H^σ estimate on (33). Using once again (??) for the term $W\partial_X W$ and (34), and then integrating in time, we deduce:

$$
\begin{aligned}
\int_{\mathbb{R}} (\partial_X^\sigma W)^2(T)\, dX \leqslant C\eta \int_0^T \|\partial_X W(\tau)\|_{L^\infty} \|W(\tau)\|_{H^\sigma}^2 \, d\tau - \eta\Gamma \int_0^T \int_{\mathbb{R}} \partial_X^\sigma \partial_X (z^+(\tau) W(\tau)) \partial_X^\sigma W(\tau)\, dX d\tau \\
+ C(\varepsilon^2+\eta)^2 \eta T + \eta\Big(\frac{\mathfrak{c}}{\varrho} - \frac{\varrho}{\mathfrak{c}} g''(\varrho)\Big) \int_0^T \int_{\mathbb{R}} \partial_X^\sigma [\partial_X(vw) + v\partial_X v](\tau) \partial_X^\sigma W(\tau)\, dX d\tau \\
+ \varepsilon^2\kappa \int_0^T \int_{\mathbb{R}} \partial_X^\sigma \partial_X^3 v(\tau) \partial_X^\sigma W(\tau)\, dX d\tau \,.
\end{aligned} \tag{35}
$$

In view of the uniform bounds given by Theorem 2, we may bound $\|\partial_X W(\tau)\|_{L^\infty}$ for $\tau \leqslant T_*/\eta$ by C. The last term in the first line of (35) is estimated by using (A.3), which provides

$$
\begin{aligned}
-\eta\Gamma \int_0^T \int_{\mathbb{R}} \partial_X^\sigma \partial_X (z^+(\tau) W(\tau)) \partial_X^\sigma W(\tau)\, dX d\tau &\leqslant C\eta \int_0^T \|z^+(\tau)\|_{H^s} \|W(\tau)\|_{H^\sigma}^2 \, d\tau \\
&\quad - \eta\Gamma \int_0^T \int_{\mathbb{R}} z^+(\tau) \partial_X^\sigma \partial_X W(\tau) \partial_X^\sigma W(\tau)\, dX d\tau \\
&\leqslant C\eta \int_0^T \|W(\tau)\|_{H^\sigma}^2 \, d\tau \,,
\end{aligned} \tag{36}
$$

after integration by parts in space for the last integral. For the last integral in (35), we argue as in the proof of Lemma 1: we integrate by parts once in space, then report $\partial_X W$ from (33) and finally integrate by parts in time or space. This yields

$$
\begin{aligned}
\varepsilon^2\kappa \int_0^T \int_{\mathbb{R}} \partial_X^\sigma \partial_X^3 v(\tau) \partial_X^\sigma W(\tau)\, dX d\tau = \frac{\varepsilon^2\kappa}{\mathfrak{c}} \int_{\mathbb{R}} \partial_X^{\sigma+2} v(T) \partial_X^\sigma W(T)\, dX \\
- \frac{\varepsilon^2\kappa}{\mathfrak{c}} \int_0^T \int_{\mathbb{R}} \partial_X^{\sigma+2} \partial_\tau v(\tau) \partial_X^\sigma W(\tau)\, dX d\tau \\
- \frac{\varepsilon^2\kappa\eta\Gamma}{2\mathfrak{c}} \int_0^T \int_{\mathbb{R}} \partial_X^{\sigma+3} v(\tau) \partial_X^\sigma [W^2(\tau) + 2W(\tau) z^+(\tau)]\, dX d\tau \\
+ \frac{\varepsilon^4\kappa^2}{\mathfrak{c}} \int_0^T \int_{\mathbb{R}} \partial_X^{\sigma+5} v(\tau) \partial_X^\sigma W(\tau)\, dX d\tau \\
+ \frac{\varepsilon^2\kappa}{\mathfrak{c}} \int_0^T \int_{\mathbb{R}} \partial_X^{\sigma+2} Q(\tau) \partial_X^\sigma W(\tau)\, dX d\tau \,.
\end{aligned}
$$

For the last integral in the first line, we use that $\partial_\tau v = \mathfrak{c}\partial_X v + \mathcal{O}_{H^{\sigma+2}}(\eta+\varepsilon^2)$, hence

$$
\begin{aligned}
-\frac{\varepsilon^2\kappa}{\mathfrak{c}} \int_0^T \int_{\mathbb{R}} \partial_X^{\sigma+2} \partial_\tau v(\tau) \partial_X^\sigma W(\tau)\, dX d\tau \leqslant & - \varepsilon^2\kappa \int_0^T \int_{\mathbb{R}} \partial_X^{\sigma+3} v(\tau) \partial_X^\sigma W(\tau)\, dX d\tau \\
& + \eta \int_0^T \|W(\tau)\|_{H^\sigma}^2 \, d\tau + C\varepsilon^4 + C\frac{\varepsilon^8}{\eta^2} \,.
\end{aligned}
$$

Reporting the term $\varepsilon^2\kappa\int_0^T\int_{\mathbb{R}}\partial_X^{\sigma+3}v(\tau)\partial_X^\sigma W(\tau)\ dXd\tau$ in the left-hand side and estimating the integrals as before, it follows that

$$\varepsilon^2\kappa\int_0^T\int_{\mathbb{R}}\partial_X^\sigma\partial_X^3 v(\tau)\partial_X^\sigma W(\tau)\ dXd\tau\leqslant\frac{1}{4}\int_{\mathbb{R}}(\partial_X^\sigma W)^2(T)\ dX+C\eta\int_0^T\|W(\tau)\|_{H^\sigma}^2\ d\tau+C\varepsilon^4+C\frac{\varepsilon^8}{\eta^2}\,. \tag{37}$$

Concerning the integral in the second line of (35), the term involving $v\partial_X v=\partial_X(v^2/2)$ is treated similarly. This gives

$$\begin{aligned}\eta\int_0^T\int_{\mathbb{R}}\partial_X^\sigma\partial_X(v^2)(\tau)\partial_X^\sigma W(\tau)\ dXd\tau=&\frac{\eta}{\mathfrak{c}}\int_{\mathbb{R}}\partial_X^\sigma(v^2)(T)\partial_X^\sigma W(T)\ dX\\&-\frac{\eta}{\mathfrak{c}}\int_0^T\int_{\mathbb{R}}\partial_X^\sigma\partial_\tau(v^2)(\tau)\partial_X^\sigma W(\tau)\ dXd\tau\\&-\frac{\eta^2\Gamma}{2\mathfrak{c}}\int_0^T\int_{\mathbb{R}}\partial_X^\sigma(v^2)(\tau)\partial_X^\sigma[W^2(\tau)+2W(\tau)z^+(\tau)]\ dXd\tau\\&+\frac{\varepsilon^2\eta\kappa}{\mathfrak{c}}\int_0^T\int_{\mathbb{R}}\partial_X^{\sigma+3}(v^2)(\tau)\partial_X^\sigma W(\tau)\ dXd\tau\\&+\frac{\eta}{\mathfrak{c}}\int_0^T\int_{\mathbb{R}}\partial_X^\sigma Q(\tau)\partial_X^\sigma W(\tau)\ dXd\tau\,.\end{aligned}$$

Using once again that $\partial_\tau v=\mathfrak{c}\partial_X v+\mathcal{O}_{H^{\sigma+2}}(\eta+\varepsilon^2)$, hence $\partial_\tau(v^2)=\mathfrak{c}\partial_X(v^2)+\mathcal{O}_{H^{\sigma+2}}(\eta+\varepsilon^2)$, we infer from estimates similar to those already used that

$$\begin{aligned}2\eta\Big(\frac{\mathfrak{c}}{\varrho}-\frac{\varrho}{\mathfrak{c}}g''(\varrho)\Big)\int_0^T\int_{\mathbb{R}}\partial_X^\sigma\partial_X(v^2)(\tau)\partial_X^\sigma W(\tau)\ dXd\tau\leqslant&\frac{1}{4}\int_{\mathbb{R}}(\partial_X^\sigma W)^2(T)\ dX+C\eta\int_0^T\|W(\tau)\|_{H^\sigma}^2d\tau\\&+C\eta(\eta+\varepsilon^2)^2+C(\eta+\varepsilon^2)^2\eta T\,.\end{aligned}\tag{38}$$

Inserting (36), (37) and (38) into (35) provides, for $T\leqslant T_*/\eta$,

$$\begin{aligned}\int_{\mathbb{R}}(\partial_X^\sigma W)^2(T)\ dX\leqslant&\, C\eta\int_0^T\|W(\tau)\|_{H^\sigma}^2\ d\tau+\frac{1}{4}\int_{\mathbb{R}}(\partial_X^\sigma W)^2(T)\ dX+C(\varepsilon^2+\eta)^2+C\frac{\varepsilon^8}{\eta^2}\\&+\eta\Big(\frac{\mathfrak{c}}{\varrho}-\frac{\varrho}{\mathfrak{c}}g''(\varrho)\Big)\int_0^T\int_{\mathbb{R}}\partial_X^\sigma\partial_X(vw)(\tau)\partial_X^\sigma W(\tau)\ dXd\tau\,.\end{aligned}\tag{39}$$

For the remaining integral in (39), we can no longer argue exactly as we did, since the interaction term vw does not solve a transport equation. Nonetheless, since $\partial_T v=\mathfrak{c}\partial_X v+\mathcal{O}(\eta+\varepsilon^2)$ and $\partial_T w=-\mathfrak{c}\partial_X w+\mathcal{O}(\eta+\varepsilon^2)$, we may observe that $\partial_T(vw)=\mathfrak{c}\partial_X(vw)-2\mathfrak{c}v\partial_X w+$

$\mathcal{O}(\eta+\varepsilon^2)$ (the error $\mathcal{O}(\eta+\varepsilon^2)$ being uniform for $T \leqslant T_*/\eta$ in H^{s-3}). As a consequence,

$$
\begin{aligned}
\eta\Big(\frac{\mathfrak{c}}{\varrho}-\frac{\varrho}{\mathfrak{c}}g''(\varrho)\Big)&\int_0^T\int_{\mathbb{R}}\partial_X^\sigma\partial_X(vw)(\tau)\partial_X^\sigma W(\tau)\;dXd\tau\\
&\leqslant -\,\eta\Big(\frac{\mathfrak{c}}{\varrho}-\frac{\varrho}{\mathfrak{c}}g''(\varrho)\Big)\int_0^T\int_{\mathbb{R}}\partial_X^\sigma\partial_X(vw)(\tau)\partial_X^\sigma W(\tau)\;dXd\tau\\
&\quad+2\eta\Big(\frac{\mathfrak{c}}{\varrho}-\frac{\varrho}{\mathfrak{c}}g''(\varrho)\Big)\int_0^T\int_{\mathbb{R}}\partial_X^\sigma(v\partial_X w)(\tau)\partial_X^\sigma W(\tau)\;dXd\tau\\
&\quad+C\eta\int_0^T\|W(\tau)\|_{H^\sigma}^2\;d\tau+\frac{1}{4s}\int_{\mathbb{R}}(\partial_X^\sigma W)^2(T)\;dX+C(\varepsilon^2+\eta)^2
\end{aligned}
$$

and reporting the first term in the left-hand side and combining the result with (39) gives

EKftiere

$$
\begin{aligned}
\int_{\mathbb{R}}(\partial_X^\sigma W)^2(T)\;dX\leqslant C\eta\int_0^T\|W(\tau)\|_{H^\sigma}^2\;d\tau+\left(\frac14+\frac{1}{4s}\right)\int_{\mathbb{R}}(\partial_X^\sigma W)^2(T)\;dX+C(\varepsilon^2+\eta)^2+C\frac{\varepsilon^8}{\eta^2}\\
+\,\eta\Big(\frac{\mathfrak{c}}{\varrho}-\frac{\varrho}{\mathfrak{c}}g''(\varrho)\Big)\int_0^T\int_{\mathbb{R}}\partial_X^\sigma(v\partial_X w)(\tau)\partial_X^\sigma W(\tau)\;dXd\tau\,.
\end{aligned}
\tag{40}
$$

Repeating this argument σ times, we arrive at

EKsstoidla

$$
\begin{aligned}
\int_{\mathbb{R}}(\partial_X^\sigma W)^2(T)\;dX\leqslant C\eta\int_0^T\|W(\tau)\|_{H^\sigma}^2\;d\tau+\left(\frac14+\frac{\sigma}{4s}\right)\int_{\mathbb{R}}(\partial_X^\sigma W)^2(T)\;dX+C(\varepsilon^2+\eta)^2+C\frac{\varepsilon^8}{\eta^2}\\
+\,\eta\Big(\frac{\mathfrak{c}}{\varrho}-\frac{\varrho}{\mathfrak{c}}g''(\varrho)\Big)\int_0^T\int_{\mathbb{R}}v(\tau)\partial_X^\sigma\partial_X w(\tau)\partial_X^\sigma W(\tau)\;dXd\tau\,.
\end{aligned}
\tag{41}
$$

At this stage, we can not apply the same argument to the remaining integral in (41). The idea is then to write $v(\tau)$ as a X-derivative. The final argument in the proof relies as in [8] (proof of Lemma 1 there) on the following estimate for the $\mathcal{M}$-norm.

EKchalot **Lemma 3** *There exists C, depending only on M, such that, for any $0\leqslant T\leqslant T_*/\eta$,*

M

$$
\|(\hat\rho,\hat{\mathbf{u}})(T)\|_{\mathcal{M}}\leqslant C\|(\hat\rho^{\rm in},\hat{\mathbf{u}}^{\rm in})\|_{\mathcal{M}}+C(\eta+\varepsilon^2)T\,.
\tag{42}
$$

Proof. Recall that $(\mathrm{EK}_{\varepsilon,\eta})$ becomes, in dimension $d=1$,

EKzimir

$$
\left\{
\begin{aligned}
&\partial_T\hat\rho+\partial_X((\varrho+\eta\hat\rho)\hat{\mathbf{u}})=0\\
&\partial_T\hat{\mathbf{u}}+\eta\hat{\mathbf{u}}\partial_X\hat{\mathbf{u}}+g'(\varrho+\eta\hat\rho)\partial_X\hat\rho=\varepsilon^2\partial_X\Big(K(\varrho+\eta\hat\rho)\partial_X^2\hat\rho+\frac{\eta}{2}K'(\varrho+\eta\hat\rho)[\partial_X\hat\rho]^2\Big)\,,
\end{aligned}
\right.
\tag{43}
$$

in which we recognize the acoustic equations

$$
\left\{
\begin{aligned}
&\partial_T\hat\rho+\varrho\partial_X\hat{\mathbf{u}}=0\\
&\partial_T\hat{\mathbf{u}}+g'(\varrho)\partial_X\hat\rho=0\,,
\end{aligned}
\right.
$$

which is diagonal in the variables $(\hat\rho+\varrho\hat{\mathbf{u}}/\mathfrak{c},\hat\rho-\varrho\hat{\mathbf{u}}/\mathfrak{c})$ (recall that $\mathfrak{c}^2=\varrho g'(\varrho)$). We thus compute

$$\begin{aligned}[\partial_T+\mathfrak{c}\partial_X]\Big(\hat\rho+\frac{\varrho}{\mathfrak{c}}\hat{\mathbf{u}}\Big) = &-\eta\partial_X(\hat\rho\hat{\mathbf{u}})-\eta^{-1}\partial_X\left(g(\varrho+\eta\hat\rho)-\eta g'(\varrho)\hat\rho\right)\\ &+\varepsilon^2\partial_X\Big(K(\varrho+\eta\hat\rho)\partial_X^2\hat\rho+\frac{\eta}{2}K'(\varrho+\eta\hat\rho)[\partial_X\hat\rho]^2\Big)=\partial_X\Upsilon\,.\end{aligned}$$

We fix $a<b$. Using the characteristic method, we obtain

$$\Big(\hat\rho+\frac{\varrho}{\mathfrak{c}}\hat{\mathbf{u}}\Big)(T,X)=\Big(\hat\rho^{\rm in}+\frac{\varrho}{\mathfrak{c}}\hat{\mathbf{u}}^{\rm in}\Big)(X-\mathfrak{c}T)+\int_0^T\partial_X\Upsilon(\bar T,X-\mathfrak{c}(T-\bar T))\;d\bar T,$$

hence, integrating in space,

$$\begin{aligned}\left|\int_a^b\Big(\hat\rho+\frac{\varrho}{\mathfrak{c}}\hat{\mathbf{u}}\Big)(T,X)\;dX\right|\leqslant&\left|\int_a^b\Big(\hat\rho^{\rm in}+\frac{\varrho}{\mathfrak{c}}\hat{\mathbf{u}}^{\rm in}\Big)(X-\mathfrak{c}T)\;dX\right|\\ &+\left|\int_a^b\int_0^T\partial_X\Upsilon(\bar T,X-\mathfrak{c}(T-\bar T))\;d\bar TdX\right|\\ \leqslant&\left|\int_{a-\mathfrak{c}T}^{b-\mathfrak{c}T}\Big(\hat\rho^{\rm in}+\frac{\varrho}{\mathfrak{c}}\hat{\mathbf{u}}^{\rm in}\Big)(X)\;dX\right|\\ &+\left|\int_0^T\Upsilon(\bar T,b-\mathfrak{c}(T-\bar T))-\Upsilon(\bar T,a-\mathfrak{c}(T-\bar T))\;d\bar T\right|\\ \leqslant&\, C\|(\hat\rho^{\rm in},\hat{\mathbf{u}}^{\rm in})\|_{\mathcal{M}}+C(\eta+\varepsilon^2)T,\end{aligned}$$

which implies

$$\left\|\hat\rho(T)+\frac{\varrho}{\mathfrak{c}}\hat{\mathbf{u}}(T)\right\|_{\mathcal{M}}\leqslant C\|(\hat\rho^{\rm in},\hat{\mathbf{u}}^{\rm in})\|_{\mathcal{M}}+C(\eta+\varepsilon^2)T.$$

Since a similar estimate holds for $\hat\rho(T)-\varrho\mathfrak{c}^{-1}\hat{\mathbf{u}}(T)$, the proof is finished. □

We fix arbitrarily some $a\in\mathbb{R}$ and we consider

$$\mathcal{V}_a(T,X)\stackrel{\rm def}{=}\int_a^X v(T,\bar X)\;d\bar X\,,$$

which solves, in view of (32),

$$\begin{aligned}2\partial_T\mathcal{V}_a-2\mathfrak{c}\partial_X\mathcal{V}_a=&\,\eta\Gamma[v^2-v^2(T,a)]-2\varepsilon^2\kappa[\partial_X^2v-\partial_X^2v(T,a)]\\ &-\Big[g'(\varrho+\eta\hat\rho)-g'(\varrho)-\eta g''(\varrho)\hat\rho\Big]\partial_X\hat\rho+2\varepsilon^2\kappa[\partial_X^2w-\partial_X^2w(T,a)]\\ &-\eta\Big(\frac{\mathfrak{c}}{\varrho}-\frac{\varrho}{\mathfrak{c}}g''(\varrho)\Big)[vw-vw(T,a)]+\frac{\eta}{2}\Big(\frac{\mathfrak{c}}{\varrho}-\frac{\varrho}{\mathfrak{c}}g''(\varrho)\Big)[w^2-w^2(T,a)]\\ &+\varepsilon^2\frac{\varrho}{\mathfrak{c}}\partial_X\Big([K(\varrho+\eta\hat\rho)-K(\varrho)]\partial_X^2\hat\rho+\frac{\eta}{2}K'(\varrho+\eta\hat\rho)(\partial_X\hat\rho)^2\Big)\,,\end{aligned}$$

hence

$$\|\partial_T \mathcal{V}_a - \mathfrak{c}\partial_X \mathcal{V}_a(T)\|_{L^\infty} = \|\partial_T \mathcal{V}_a - \mathfrak{c} v(T)\|_{L^\infty} \leqslant C(\eta + \varepsilon^2)\,. \tag{44}$$

Therefore,

$$\begin{aligned}
\eta\Big(\frac{\mathfrak{c}}{\varrho} - \frac{\varrho}{\mathfrak{c}} g''(\varrho)\Big) & \int_0^T \int_{\mathbb{R}} v(\tau)\partial_X^{\sigma+1} w(\tau)\partial_X^\sigma W(\tau)\ dXd\tau \\
&= \eta\Big(\frac{\mathfrak{c}}{\varrho} - \frac{\varrho}{\mathfrak{c}} g''(\varrho)\Big) \int_0^T \int_{\mathbb{R}} \partial_X \mathcal{V}_a(\tau)\partial_X^{\sigma+1} w(\tau)\partial_X^\sigma W(\tau)\ dXd\tau \\
&= -\eta\Big(\frac{\mathfrak{c}}{\varrho} - \frac{\varrho}{\mathfrak{c}} g''(\varrho)\Big) \int_0^T \int_{\mathbb{R}} \mathcal{V}_a(\tau)\partial_X [\partial_X^{\sigma+1} w(\tau)\partial_X^\sigma W(\tau)]\ dXd\tau \\
&\leqslant \frac{\eta}{\mathfrak{c}}\Big(\frac{\mathfrak{c}}{\varrho} - \frac{\varrho}{\mathfrak{c}} g''(\varrho)\Big) \int_0^T \int_{\mathbb{R}} \mathcal{V}_a(\tau)\partial_\tau [\partial_X^{\sigma+1} w(\tau)\partial_X^\sigma W(\tau)]\ dXd\tau \\
&\quad + C\eta \int_0^T \|W(\tau)\|_{H^\sigma}^2\ d\tau + \frac14 \int_{\mathbb{R}} (\partial_X^\sigma W)^2(T)\ dX + C(\varepsilon^2+\eta)^2 \\
&\leqslant -\frac{\eta}{\mathfrak{c}}\Big(\frac{\mathfrak{c}}{\varrho} - \frac{\varrho}{\mathfrak{c}} g''(\varrho)\Big) \int_0^T \int_{\mathbb{R}} \partial_\tau \mathcal{V}_a(\tau) [\partial_X^{\sigma+1} w(\tau)\partial_X^\sigma W(\tau)]\ dXd\tau \\
&\quad - \frac{\eta}{\mathfrak{c}}\Big(\frac{\mathfrak{c}}{\varrho} - \frac{\varrho}{\mathfrak{c}} g''(\varrho)\Big) \int_{\mathbb{R}} \mathcal{V}_a(T) [\partial_X^{\sigma+1} w(T)\partial_X^\sigma W(T)]\ dX \\
&\quad + C\eta \int_0^T \|W(\tau)\|_{H^\sigma}^2\ d\tau + \frac14 \int_{\mathbb{R}} (\partial_X^\sigma W)^2(T)\ dX + C(\varepsilon^2+\eta)^2 \\
&\leqslant -\eta\Big(\frac{\mathfrak{c}}{\varrho} - \frac{\varrho}{\mathfrak{c}} g''(\varrho)\Big) \int_0^T \int_{\mathbb{R}} v(\tau) [\partial_X^{\sigma+1} w(\tau)\partial_X^\sigma W(\tau)]\ dXd\tau \\
&\quad - \frac{\eta}{\mathfrak{c}}\Big(\frac{\mathfrak{c}}{\varrho} - \frac{\varrho}{\mathfrak{c}} g''(\varrho)\Big) \int_{\mathbb{R}} \mathcal{V}_a(T) [\partial_X^{\sigma+1} w(T)\partial_X^\sigma W(T)]\ dX \\
&\quad + C\eta \int_0^T \|W(\tau)\|_{H^\sigma}^2\ d\tau + \frac14 \int_{\mathbb{R}} (\partial_X^\sigma W)^2(T)\ dX + C(\varepsilon^2+\eta)^2\,,
\end{aligned}$$

where we have used (44) for the last inequality. Since $\|\mathcal{V}_a(T)\|_{L^\infty} \leqslant \|v(T)\|_{\mathcal{M}} \leqslant C$ by Lemma 3, we infer that the integral in the second line is $\leqslant C\eta\|W(T)\|_{H^\sigma} \leqslant \frac14 \|W(T)\|_{H^\sigma}^2 + C\eta^2$. Consequently, we may report the first term in the left-hand side and deduce

$$\begin{aligned}
\eta\Big(\frac{\mathfrak{c}}{\varrho} - \frac{\varrho}{\mathfrak{c}} g''(\varrho)\Big) & \int_0^T \int_{\mathbb{R}} v(\tau)\partial_X^{\sigma+1} w(\tau)\partial_X^\sigma W(\tau)\ dXd\tau \\
&\leqslant C\eta \int_0^T \|W(\tau)\|_{H^\sigma}^2\ d\tau + \frac14 \int_{\mathbb{R}} (\partial_X^\sigma W)^2(T)\ dX + C(\varepsilon^2+\eta)^2\,.
\end{aligned} \tag{45}$$

Inserting (45) into (41) yields, since $\sigma \leqslant s$,

$$\int_{\mathbb{R}} (\partial_X^\sigma W)^2(T)\ dX \leqslant C\eta \int_0^T \|W(\tau)\|_{H^\sigma}^2\ d\tau + \frac34 \int_{\mathbb{R}} (\partial_X^\sigma W)^2(T)\ dX + C(\varepsilon^2+\eta)^2 + C\frac{\varepsilon^8}{\eta^2}\,.$$

Incorporating the second term of the right-hand side in the left-hand side, we arrive at

$$\int_{\mathbb{R}} (\partial_X^\sigma W)^2(T)\; dX \leqslant C\eta \int_0^T \|W(\tau)\|_{H^\sigma}^2 \; d\tau + C(\varepsilon^2+\eta)^2 + C\frac{\varepsilon^8}{\eta^2}\,,$$

and conclude by the Gronwall lemma.

Since the error estimate between v and $\zeta^-(\eta T, \cdot + \mathfrak{c}T)$ is analogous, the proof of Theorem 7 is completed. $\square$

6 The (KP-I) asymptotic regime

In this section, we concentrate on the case $\eta = \varepsilon^2$. In one space dimension, we have obtained as asymptotic equations the KdV equation for well-prepared initial data, and two decoupled KdV equations for more general initial data. In higher dimensions, if one considers a weakly transverse perturbation, we expect Kadomtsev-Petviashvili (KP-I) type equations

$$\text{(KP-I)} \qquad \partial_\theta \zeta + \Gamma \zeta \partial_{z_1} \zeta = \kappa \partial_{z_1}^3 \zeta - \frac{\mathfrak{c}}{2} \Delta_{z_\perp} \partial_{z_1}^{-1} \zeta\,.$$

Throughout this section, we shall assume that the vector field u is curl-free, which is a natural hypothesis for the KP-I asymptotic regime.

6.1 Main results

We replace the long wave ansatz (2) by the weakly transverse long wave ansatz

$$(46) \quad \rho(t,x) = \varrho + \eta\hat\rho(T,z) \qquad \mathbf{u}(t,x) = \eta(\hat{\mathbf{u}}_1, \delta\hat{\mathbf{u}}_\perp)(T,z)\,, \qquad T=\varepsilon t\,, \quad z = (\varepsilon x_1, \varepsilon\delta x_\perp)\,,$$

where δ is another small parameter (we have changed X to z to keep in mind that the scaling is now weakly transverse). Usually, we take $\eta = \varepsilon^2 = \delta^2$ to derive (KP-I) but we may consider weakly dispersive KP-I equations similar to weakly dispersive KdV equation we have already obtained. Then, the Euler–Korteweg system (EK) becomes

$$(47) \quad \begin{cases} \partial_T \hat\rho + \partial_{z_1}((\varrho+\eta\hat\rho)\hat{\mathbf{u}}_1) + \delta^2 \nabla_{z_\perp}\cdot((\varrho+\eta\hat\rho)\hat{\mathbf{u}}_\perp) = 0 \\[2ex] \partial_T \hat{\mathbf{u}}_1 + \eta\hat{\mathbf{u}}_1 \partial_{z_1}\hat{\mathbf{u}}_1 + \eta\delta^2 \hat{\mathbf{u}}_\perp \cdot \nabla_{z_\perp} \hat{\mathbf{u}}_1 + g'(\varrho+\eta\hat\rho)\partial_{z_1}\hat\rho \\ \qquad = \varepsilon^2 \partial_{z_1}\Big(K(\varrho+\eta\hat\rho)[\partial_{z_1}^2 + \delta^2\Delta_{z_\perp}]\hat\rho + \dfrac{\eta}{2} K'(\varrho+\eta\hat\rho)[(\partial_{z_1}\hat\rho)^2 + \delta^2 |\nabla_{z_\perp}\hat\rho|^2]\Big) \\[2ex] \partial_T \hat{\mathbf{u}}_\perp + \eta\hat{\mathbf{u}}_1 \partial_{z_1}\hat{\mathbf{u}}_\perp + \eta\delta^2 \hat{\mathbf{u}}_\perp \cdot \nabla_{z_\perp} \hat{\mathbf{u}}_\perp + g'(\varrho+\eta\hat\rho)\nabla_{z_\perp}\hat\rho \\ \qquad = \varepsilon^2 \nabla_{z_\perp}\Big(K(\varrho+\eta\hat\rho)[\partial_{z_1}^2 + \delta^2\Delta_{z_\perp}]\hat\rho + \dfrac{\eta}{2} K'(\varrho+\eta\hat\rho)[(\partial_{z_1}\hat\rho)^2 + \delta^2 |\nabla_{z_\perp}\hat\rho|^2]\Big)\,. \end{cases}$$

We first state a result providing uniform bounds on the time scale $T \approx \eta^{-1}$ (that is $t \approx \varepsilon^{-1}\eta^{-1}$) and need to define, for $s \geqslant 0$ and $M > 0$, the set

$$\tilde B_{s,\varepsilon,\delta}(M) \stackrel{\text{def}}{=} \{(\hat\rho, \hat{\mathbf{u}}) \in H^{s+1}(\mathbb{R}^d) \times (H^s(\mathbb{R}^d))^d\,;\; \|(\hat\rho, \hat{\mathbf{u}}, \varepsilon\partial_1\hat\rho, \varepsilon\delta\nabla_\perp\hat\rho)\|_{(H^s(\mathbb{R}^d))^{d+1}} \leqslant M\}\,.$$

Theorem 8 *Let s be a real number greater than $1+d/2$ and $\eta \in (0,1]$. If $\varrho > 0$, $g'(\varrho) > 0$, and $(\hat\rho^{\text{in}}, \hat{\mathbf{u}}^{\text{in}}) \in \tilde{B}_{s,\varepsilon;\delta}(M)$, then there exists $T_* > 0$, depending only on M, s and d, such that the maximal solution to (47) such that $(\hat\rho, \hat{\mathbf{u}})(0) = (\hat\rho^{\text{in}}, \hat{\mathbf{u}}^{\text{in}})$ exists at least on $[0, T_*/\eta]$, and $(\hat\rho, \hat{\mathbf{u}})(T) \in \tilde{B}_{s,\varepsilon,\delta}(2M)$ for all $T \in [0, T_*/\eta]$.*

In this asymptotic regime, one might expect an approximation by the two counter propagating waves described by the two uncoupled KP-I equations

$$
\left\{
\begin{aligned}
&\partial_\theta \zeta^+ + \Gamma \zeta^+ \partial_{z_1} \zeta^+ = \frac{\varepsilon^2}{\eta} \kappa \partial_{z_1}^3 \zeta^+ - \frac{\mathfrak{c}}{2} \cdot \frac{\delta^2}{\eta} \Delta_{z_\perp} \partial_{z_1}^{-1} \zeta^+ \\
&\partial_\theta \zeta^- - \Gamma \zeta^- \partial_{z_1} \zeta^- = -\frac{\varepsilon^2}{\eta} \kappa \partial_{z_1}^3 \zeta^- + \frac{\mathfrak{c}}{2} \cdot \frac{\delta^2}{\eta} \Delta_{z_\perp} \partial_{z_1}^{-1} \zeta^-
\end{aligned}
\right.
\tag{48}
$$

instead of the two KdV equations. However, D. Lannes in [18] has shown that, in the case $\eta = \varepsilon^2 = \delta^2$ to fix ideas, the natural $\mathcal{O}(\varepsilon^2)$ error estimate does not hold due to the singularity of the symbol associated with the operator $\Delta_{z_\perp} \partial_{z_1}^{-1}$, unless you impose the zero mass assumption $\int_{\mathbb{R}} A(z_1, z_\perp)\, dz_1 = 0$ for every $z_\perp \in \mathbb{R}^{d-1}$, which is not physical. This is the reason why Lannes and Saut have proposed in [20] weakly transverse Boussinesq type systems for which we can prove the natural error estimate and for which no zero mass assumption is made. This weakly transverse Boussinesq type system is formally equivalent to the system of two uncoupled (KP-I) equations (48), and it can be shown to converge to (48) (without optimal error estimates) under extra regularity and zero mass type hypothesis. In our context, a natural weakly transverse Boussinesq type system is the following:

$$
(\mathcal{B}_{\varepsilon,\delta,\eta}) \qquad
\left\{
\begin{aligned}
&\partial_T \hat\rho + \varrho \partial_{z_1} \hat{\mathbf{u}}_1 + \eta \partial_{z_1}(\hat\rho \hat{\mathbf{u}}_1) + \delta^2 \nabla_{z_\perp} \cdot ((\varrho + \eta \hat\rho) \hat{\mathbf{u}}_\perp) = 0 \\
&\partial_T \hat{\mathbf{u}}_1 + g'(\varrho) \partial_{z_1} \hat\rho + \eta \hat{\mathbf{u}}_1 \partial_{z_1} \hat{\mathbf{u}}_1 + \eta \delta^2 \hat{\mathbf{u}}_\perp \cdot \nabla_{z_\perp} \hat{\mathbf{u}}_1 + \eta \hat\rho g''(\varrho) \partial_{z_1} \hat\rho \\
&\qquad = \varepsilon^2 K(\varrho) \partial_{z_1} [\partial_{z_1}^2 + \delta^2 \Delta_{z_\perp}] \hat\rho \\
&\partial_T \hat{\mathbf{u}}_\perp + g'(\varrho) \nabla_{z_\perp} \hat\rho + \eta \hat{\mathbf{u}}_1 \partial_{z_1} \hat{\mathbf{u}}_\perp + \eta \delta^2 \hat{\mathbf{u}}_\perp \cdot \nabla_{z_\perp} \hat{\mathbf{u}}_\perp + \eta \hat\rho g''(\varrho) \nabla_{z_\perp} \hat\rho \\
&\qquad = \varepsilon^2 K(\varrho) \nabla_{z_\perp} [\partial_{z_1}^2 + \delta^2 \Delta_{z_\perp}] \hat\rho\,.
\end{aligned}
\right.
$$

Let us observe that system $(\mathcal{B}_{\varepsilon,\delta,\eta})$ may be seen as a particular case of system (47) when g is a quadratic polynomial and the capillarity K has constant value $K(\varrho)$. The weakly transverse Boussinesq system $(\mathcal{B}_{\varepsilon,\delta,\eta})$ may also be seen as the weakly transverse analogue to the systems of the (a,b,c,d) class introduced in [9] and [10] when $a = b = d = 0$ and $c < 0$.

Theorem 9 *Let s be a real number such that $s > 1 + d/2$ and $\eta \in (0,1]$.*

(i) If $\varrho > 0$, $g'(\varrho) > 0$, and $(\hat\rho^{\text{in}}, \hat{\mathbf{u}}^{\text{in}}) \in \tilde{B}_{s,\varepsilon,\delta}(M)$, then there exists $T_* > 0$, depending only on M, s and d, such that the system $(\mathcal{B}_{\varepsilon,\delta,\eta})$ with initial datum $(\hat\rho^{\text{in}}, \hat{\mathbf{u}}^{\text{in}})$ has a unique solution $(\hat\rho, \hat{\mathbf{u}}) \in \mathscr{C}([0, T_*/\eta], H^{s+1}(\mathbb{R}^d) \times (H^s(\mathbb{R}^d))^d)$. Moreover, for any $T \in [0, T_*/\eta]$, we have $(\hat\rho, \hat{\mathbf{u}})(T) \in \tilde{B}_{s,\varepsilon,\delta}(2M)$.

(ii) Assume that $\zeta^{\pm,\mathrm{in}}$ belong to $\partial_{z_1} H^{s+7}(\mathbb{R}^d)$ and that $\Delta_{z_\perp}\zeta^{\pm,\mathrm{in}} \in \partial_{z_1}^2 H^{s+3}(\mathbb{R}^d)$. Then, there exists $\theta_* > 0$, depending only on s, d and the initial data $\zeta^{\pm,\mathrm{in}}$ such that the uncoupled system (48) has a unique solution $\zeta^\pm \in \mathscr{C}([0,\theta_*], H^{s+6}(\mathbb{R}^d)) \cap \mathrm{Lip}([0,\theta_*], H^{s+3}(\mathbb{R}^d))$. Moreover, one has $\zeta^\pm \in L^\infty([0,\theta_*], \partial_{z_1} H^{s+6}(\mathbb{R}^d))$. Let us also assume that

$$\frac{1}{2}\Big(\hat\rho^{\mathrm{in}} + \frac{\varrho}{\mathfrak{c}}\hat{\mathbf{u}}_1^{\mathrm{in}}\Big) = \zeta^{+,\mathrm{in}}, \qquad \frac{1}{2}\Big(\hat\rho^{\mathrm{in}} - \frac{\varrho}{\mathfrak{c}}\hat{\mathbf{u}}_1^{\mathrm{in}}\Big) = \zeta^{-,\mathrm{in}} \qquad \frac{\varrho}{\mathfrak{c}}\hat{\mathbf{u}}_\perp^{\mathrm{in}} = \nabla_{z_\perp}\partial_{z_1}^{-1}(\zeta^{+,\mathrm{in}} - \zeta^{-,\mathrm{in}}) \tag{49}$$

and that

$$\delta^2 \leqslant \eta \qquad \text{and} \qquad \varepsilon^2 \leqslant \eta\,. \tag{50}$$

Then, the following comparison estimate with the uncoupled system (48) holds as $\eta \to 0$:

$$\sup_{0\leqslant T\leqslant \min(\theta_*,T_*)/\eta} \left\| \frac{1}{2}\Big(\hat\rho + \frac{\varrho}{\mathfrak{c}}\hat{\mathbf{u}}_1\Big)(T) - \zeta^+(\eta T, \cdot - \mathfrak{c}T)\right\|_{H^{s-1}(\mathbb{R}^d)} \to 0$$

and

$$\sup_{0\leqslant T\leqslant \min(\theta_*,T_*)/\eta} \left\| \frac{1}{2}\Big(\hat\rho - \frac{\varrho}{\mathfrak{c}}\hat{\mathbf{u}}_1\Big)(T) - \zeta^-(\eta T, \cdot + \mathfrak{c}T)\right\|_{H^{s-1}(\mathbb{R}^d)} \to 0\,.$$

Remark 3 The properties of the solution $\zeta^\pm$ to the (KP-I) equation given in (ii) come from [26] and [22]. The compatibility condition (49) on $\hat{\mathbf{u}}_\perp^{\mathrm{in}}$ is natural since the vector field $\hat{\mathbf{u}}$ is curl-free.

Remark 4 Statement (i) is a consequence of Theorem 8 in the particular case where g is quadratic and $K \equiv K(\varrho)$ is constant. An alternative approach would be to use the result given in Theorem 1.1 of [23] with $a = b = d = 0 > c$, *i.e.* case (12). However, this result is stated in the Boussinesq scaling and not the weakly transverse one. It is plausible that their method extends to the weakly transverse case, but we have not checked this fact.

Remark 5 In [11], we have proposed (in the case $\eta = \varepsilon^2 = \delta^2$) another weakly transverse Boussinesq system which is adapted to the case where one wave, say the left-going one, is negligible. This system has the structure of a symmetrizable hyperbolic system plus a constant coefficient skew adjoint term (which is not affected by the symmetrization), which is a simpler structure than $(\mathcal{B}_{\varepsilon,\delta,\eta})$. We would like to point out that one may think that the dispersive terms $\varepsilon^2\delta^2\Delta_{z_\perp}\partial_{z_1}\hat\rho$ and $\varepsilon^2\delta^2\Delta_{z_\perp}\nabla_{z_\perp}\hat\rho$ in the last two equations in $(\mathcal{B}_{\varepsilon,\delta,\eta})$ should be removable in view of their formal order $\mathcal{O}(\eta^2)$ (by (50)). However, our existence and uniqueness result relies on a nonlinear symmetrization type argument which breaks down without these terms. Moreover, our estimates provide a uniform control on $\hat\rho$, $\varepsilon\partial_1\hat\rho$ and $\varepsilon\delta\nabla_{z_\perp}\hat\rho$ in H^s, so that the high order derivatives of $\varepsilon^2\delta^2\Delta_{z_\perp}\partial_{z_1}\hat\rho$ and $\varepsilon^2\delta^2\Delta_{z_\perp}\nabla_{z_\perp}\hat\rho$ are not that small.

Our last result gives a quantitative comparison estimate between system (47) and the weakly transverse system $(\mathcal{B}_{\varepsilon,\delta,\eta})$.

pistoletdor **Theorem 10** *Let $s > 3 + d/2$ (s integer), η, ε, $\delta \in (0, 1]$ and assume that $\varrho > 0$, $g'(\varrho) > 0$, and let $(\hat\rho^{\mathrm{in}}, \hat{\mathbf{u}}^{\mathrm{in}}) \in \tilde B_{s,\varepsilon}(M)$. Then, there exists $T_* > 0$, depending only on M, s and d, such that the two systems (DrNo 47), resp. $(\mathscr{B}_{\varepsilon,\delta,\eta})$, with initial datum $(\hat\rho^{\mathrm{in}}, \hat{\mathbf{u}}^{\mathrm{in}})$ have a unique solution $(\hat\rho, \hat{\mathbf{u}})$, resp. $(\tilde\rho, \tilde{\mathbf{u}})$, in $\mathscr{C}([0, T_*/\eta], H^{s+1}(\mathbb{R}^d) \times (H^s(\mathbb{R}^d))^d)$. Moreover, for any $T \in [0, T_*/\eta]$, $(\hat\rho, \hat{\mathbf{u}})$ and $(\tilde\rho, \tilde{\mathbf{u}})$ belong to $\tilde B_{s,\varepsilon,\delta}(2M)$. Then, there exists a constant C, depending only on s, d and M, such that, for $0 \leqslant T \leqslant T_*/\eta$, we have*

$$\|(\hat\rho, \hat{\mathbf{u}}_1, \delta\hat{\mathbf{u}}_\perp) - (\tilde\rho, \tilde{\mathbf{u}}_1, \delta\tilde{\mathbf{u}}_\perp)\|_{H^{s-3}} \leqslant C(\eta + \varepsilon^2)$$

and

$$\|(\hat\rho, \hat{\mathbf{u}}_1, \delta\hat{\mathbf{u}}_\perp) - (\tilde\rho, \tilde{\mathbf{u}}_1, \delta\tilde{\mathbf{u}}_\perp)\|_{H^{s-2}} \leqslant C(\eta + \varepsilon)\,.$$

6.2 Uniform bounds in the weakly transverse scaling

ss:prelKPI

Proof of Theorem diamants *8.* The complex vector field $\hat{\mathbf{z}}$ is now

EKartilage (51)
$$\hat{\mathbf{z}} = (\hat{\mathbf{z}}_1, \delta\hat{\mathbf{z}}_\perp) = \hat{\mathbf{u}} + i\hat{\mathbf{w}} = (\hat{\mathbf{u}}_1, \delta\hat{\mathbf{u}}_\perp) + i\varepsilon\sqrt{\frac{K(\rho)}{\rho}}(\partial_1\hat\rho, \delta\nabla_\perp\hat\rho),$$

and the assumption that the vector field $\hat{\mathbf{u}}$ is curl free reads now

curly (52)
$$\partial_1\hat{\mathbf{u}}_\perp = \nabla_\perp\hat{\mathbf{u}}_1.$$

The L^2-type functional E_0 reads now

$$E_0^\natural[\hat\rho, \hat{\mathbf{z}}] \stackrel{\mathrm{def}}{=} \frac{1}{2}\int_{\mathbb{R}^d} \rho|\hat{\mathbf{z}}_1|^2 + \delta^2\rho|\hat{\mathbf{z}}_\perp|^2 + g'(\rho)\hat\rho^2\ dy\,, \quad \rho = \varrho + \eta\hat\rho\,,$$

and the H^s-type functional E_s becomes

$$E_s^\natural[\hat\rho, \hat{\mathbf{z}}] \stackrel{\mathrm{def}}{=} \sum_{\sigma=0}^{s} \dot E_\sigma^\natural(\hat\rho, \hat{\mathbf{z}})\,,$$

where we have denoted

$$\dot E_\sigma^\natural[\hat\rho, \hat{\mathbf{z}}] \stackrel{\mathrm{def}}{=} \sum_{\substack{\alpha\in\mathbb{N}_0^d,\\ |\alpha|=\sigma}} \frac{\sigma!}{\alpha!}\int_{\mathbb{R}^d}\frac{1}{2}a(\rho)^\sigma\left(\rho|\partial^\alpha\hat{\mathbf{z}}_1|^2 + \delta^2\rho|\partial^\alpha\hat{\mathbf{z}}_\perp|^2 + g'(\rho)(\partial^\alpha\hat\rho)^2\right)\,dz\,, \quad \rho = \varrho + \eta\hat\rho\,.$$

Under the assumptions of Proposition degresept 3, we now have

$$c(\|\hat{\mathbf{u}}\|_{H^s}^2 + \|\hat\rho\|_{H^s}^2 + \varepsilon^2\|\nabla^\delta\hat\rho\|_{H^s}^2) \leqslant E_s^\natural[\hat\rho, \hat{\mathbf{u}}] \leqslant C(\|\hat{\mathbf{u}}\|_{H^s}^2 + \|\hat\rho\|_{H^s}^2 + \varepsilon^2\|\nabla^\delta\hat\rho\|_{H^s}^2)\,,$$

where $c > 0$ and $C > 0$ depend only on r, s, d (and the functions g, K).

The system $(\mathrm{ES}_{\varepsilon,\eta})$ is now, setting $\nabla^\delta \stackrel{\mathrm{def}}{=} (\partial_1, \delta\nabla_\perp)$,

dedeKP (53)
$$\partial_T\hat{\mathbf{z}} + \eta(\hat{\mathbf{u}}\cdot\nabla^\delta)\hat{\mathbf{z}} + i\eta(\nabla^\delta\hat{\mathbf{z}})\hat{\mathbf{w}} + \frac{1}{\varepsilon}b(\rho)\hat{\mathbf{w}} + i\varepsilon\nabla^\delta(a(\rho)\nabla^\delta\cdot\hat{\mathbf{z}}) = 0\,,$$

and applying ∂^α, where $|\alpha| \leqslant s$, gives

$$\partial_T \partial^\alpha \hat{\mathbf{z}} + \eta(\hat{\mathbf{u}} \cdot \nabla^\delta)\partial^\alpha \hat{\mathbf{z}} + i\eta(\nabla^\delta \hat{\mathbf{z}})\partial^\alpha \hat{\mathbf{w}} + \frac{1}{\varepsilon} b(\rho)\partial^\alpha \hat{\mathbf{w}} + i\varepsilon \partial^\alpha \nabla^\delta (a(\rho)\nabla^\delta \cdot \hat{\mathbf{z}}) = \mathbf{R}\,, \tag{54}$$

where

$$\mathbf{R} = (\mathbf{R}_1, \delta \mathbf{R}_\perp) \overset{\text{def}}{=} \eta[\hat{\mathbf{u}} \cdot \nabla^\delta, \partial^\alpha]\hat{\mathbf{z}} + \frac{1}{\varepsilon}[b(\rho), \partial^\alpha]\hat{\mathbf{w}} + i\eta\Big((\nabla^\delta \partial^\alpha \hat{\mathbf{z}})\hat{\mathbf{w}} - \partial^\alpha((\nabla^\delta \hat{\mathbf{z}})\hat{\mathbf{w}})\Big)\,.$$

In view of (A.3), we have

$$\|(\mathbf{R}_1, \delta \mathbf{R}_\perp)\|_{L^2} \leqslant C(r,s,d)\varepsilon^2 \|(\nabla \hat{\mathbf{z}}, \nabla \hat{\rho})\|_{L^\infty} \sqrt{E_s^\natural[\hat{\rho}, \hat{\mathbf{z}}]},$$

since, recalling that $\delta^2 \leqslant \eta \leqslant 1$ and $\varepsilon^2 \leqslant \eta \leqslant 1$ by (50),

$$\begin{aligned}\eta\|[\hat{\mathbf{u}} \cdot \nabla^\delta, \partial^\alpha]\hat{\mathbf{z}}\|_{L^2} &\leqslant C(s,d)\eta\Big(\|\nabla \hat{\mathbf{u}}\|_{H^{s-1}}\|\nabla^\delta \hat{\mathbf{z}}\|_{L^\infty} + \|\nabla^\delta \hat{\mathbf{z}}\|_{H^{s-1}}\|\nabla \hat{\mathbf{u}}\|_{L^\infty}\Big)\\ &\leqslant C(s,d)\eta\|\nabla \hat{\mathbf{z}}\|_{L^\infty}\sqrt{E_s^\natural[\hat{\rho}, \hat{\mathbf{z}}]}\,,\end{aligned}$$

$$\begin{aligned}\eta\left\|(\nabla^\delta \partial^\alpha \hat{\mathbf{z}})\hat{\mathbf{w}} - \partial^\alpha((\nabla^\delta \hat{\mathbf{z}})\hat{\mathbf{w}})\right\|_{L^2} &\leqslant C(s,d)\eta\Big(\|\nabla \hat{\mathbf{w}}\|_{H^{s-1}}\|\nabla^\delta \hat{\mathbf{z}}\|_{L^\infty} + \|\nabla^\delta \hat{\mathbf{z}}\|_{H^{s-1}}\|\nabla \hat{\mathbf{w}}\|_{L^\infty}\Big)\\ &\leqslant C(s,d)\eta\|\nabla \hat{\mathbf{z}}\|_{L^\infty}\sqrt{E_s^\natural[\hat{\rho}, \hat{\mathbf{z}}]}\end{aligned}$$

and, using that $\|\hat{\mathbf{w}}\|_{H^{s-1}} \leqslant C(r,s,d)\varepsilon\|\hat{\rho}\|_{H^s} \leqslant C(r,s,d)\varepsilon\sqrt{E_s^\natural[\hat{\rho}, \hat{\mathbf{z}}]}$ and that $\|\hat{\mathbf{w}}\|_{L^\infty} \leqslant C(r,s,d)\varepsilon\|\nabla^\delta \hat{\rho}\|_{L^\infty}$,

$$\begin{aligned}\left\|\frac{1}{\varepsilon}[b(\rho), \partial^\alpha]\hat{\mathbf{w}}\right\|_{L^2} &\leqslant C(r,s,d)\frac{\eta}{\varepsilon}\Big(\|\hat{\mathbf{w}}\|_{H^{s-1}}\|\nabla \hat{\rho}\|_{L^\infty} + \|\hat{\mathbf{w}}\|_{L^\infty}\|\nabla \hat{\rho}\|_{H^{s-1}}\Big)\\ &\leqslant C(r,s,d)\eta\|\nabla \hat{\rho}\|_{L^\infty}\sqrt{E_s^\natural[\hat{\rho}, \hat{\mathbf{z}}]}\,.\end{aligned}$$

Then, following the same lines as in the proof of Proposition 4, but working with the variables (51) and the operator $\nabla^\delta = (\partial_1, \delta\nabla_\perp)$, we infer

$$\frac{d}{dT}\dot{E}_\sigma^\natural[\hat{\rho}, \hat{\mathbf{z}}] \leqslant C(r,s,d)\eta\|(\nabla \hat{\rho}, \nabla \hat{\mathbf{z}})\|_{L^\infty}(1 + \varepsilon\eta\|\nabla \hat{\rho}\|_{L^\infty})E_s^\natural[\hat{\rho}, \hat{\mathbf{z}}]\,.$$

Notice that the computation is actually slightly simplified since we assume that the vector field $\hat{\mathbf{u}}$ is curl-free. □

6.3 Proof of Theorem 9 (ii)

As already mentioned, the argument follows the lines of the proof of Theorem 1 in [20]. We briefly recall the ideas.

We look for an approximate solution $(\hat\rho^{\rm app}, \hat{\mathbf{u}}^{\rm app})$ to $(\mathcal{B}_{\varepsilon,\delta,\eta})$ under the form

$$(\hat\rho^{\rm app}, \hat{\mathbf{u}}^{\rm app})(T,z) = (\hat\rho^0, \hat{\mathbf{u}}^0)(T, \eta T, z) + \eta (\hat\rho^1, \hat{\mathbf{u}}^1)(T, \eta T, z)\,, \tag{55}$$

where $\hat{\mathbf{u}}^0$ and $\hat{\mathbf{u}}^1$ are curl free. We set $\theta = \eta T$. Recall that $\delta^2 \leqslant \eta$ and $\varepsilon^2 \leqslant \eta$, and we wish to construct an approximate solution so that the consistency error is $o(\eta)$, since we consider $T \lesssim \eta^{-1}$. Notice that we simplify the computations by assuming an expansion in powers of ε^2, but an expansion in powers of ε is also possible (see [20] in this case and also [11] if one considers only one wave propagating to the right). We then compute

$$\begin{aligned}
{\rm Err}_\rho &\overset{\rm def}{=} \partial_T \hat\rho^{\rm app} + \varrho \partial_{z_1} \hat{\mathbf{u}}_1^{\rm app} + \eta \partial_{z_1} (\hat\rho^{\rm app} \hat{\mathbf{u}}_1^{\rm app}) + \delta^2 \nabla_{z_\perp} \cdot ((\varrho + \eta \hat\rho^{\rm app}) \hat{\mathbf{u}}_\perp^{\rm app}) \\
&= \big(\partial_T \hat\rho^0 + \varrho \partial_{z_1} \hat{\mathbf{u}}^0 \big) + \eta \left(\partial_T \hat\rho^1 + \varrho \partial_{z_1} \hat{\mathbf{u}}_1^1 + \partial_\theta \hat\rho^0 + \partial_{z_1} (\hat\rho^0 \hat{\mathbf{u}}_1^0) + \varrho (\delta^2/\eta) \nabla_{z_\perp} \cdot \hat{\mathbf{u}}_\perp^0 \right) + \mathcal{R}\,,
\end{aligned}$$

$$\begin{aligned}
{\rm Err}_1 &\overset{\rm def}{=} \partial_T \hat{\mathbf{u}}_1^{\rm app} + g'(\varrho) \partial_{z_1} \hat\rho^{\rm app} + \varepsilon^2 \hat{\mathbf{u}}_1^{\rm app} \partial_{z_1} \hat{\mathbf{u}}_1^{\rm app} + \varepsilon^4 \hat{\mathbf{u}}_\perp^{\rm app} \cdot \nabla_{z_\perp} \hat{\mathbf{u}}_1^{\rm app} + \varepsilon^2 g''(\varrho) \hat\rho^{\rm app} \partial_{z_1} \hat\rho^{\rm app} \\
&\quad - \varepsilon^2 K(\varrho) \partial_{z_1} [\partial_{z_1}^2 + \varepsilon^2 \Delta_{z_\perp}] \hat\rho^{\rm app} \\
&= \big(\partial_T \hat{\mathbf{u}}_1^0 + g'(\varrho) \partial_{z_1} \hat{\mathbf{u}}_1^0 \big) + \varepsilon^2 \left(\partial_T \hat{\mathbf{u}}_1^1 + g'(\varrho) \partial_{z_1} \hat{\mathbf{u}}_1^1 + \partial_\theta \hat{\mathbf{u}}_1^0 + \hat{\mathbf{u}}_1^0 \partial_{z_1} \hat{\mathbf{u}}_1^0 + g''(\varrho) \partial_{z_1} \hat\rho^0 - K(\varrho) \partial_{z_1}^3 \hat\rho^0 \right) \\
&\quad + \varepsilon^4 \mathcal{S}_{4,1} + \varepsilon^6 \mathcal{S}_{6,1} + \varepsilon^8 \mathcal{S}_{8,1}\,,
\end{aligned}$$

and

$$\begin{aligned}
{\rm Err}_\perp &\overset{\rm def}{=} \partial_T \hat{\mathbf{u}}_\perp^{\rm app} + g'(\varrho) \nabla_{z_\perp} \hat\rho^{\rm app} + \varepsilon^2 \hat{\mathbf{u}}_1^{\rm app} \partial_{z_1} \hat{\mathbf{u}}_\perp^{\rm app} + \varepsilon^4 \hat{\mathbf{u}}_\perp^{\rm app} \cdot \nabla_{z_\perp} \hat{\mathbf{u}}_1^{\rm app} + \varepsilon^2 g''(\varrho) \hat\rho^{\rm app} \nabla_{z_\perp} \hat\rho^{\rm app} \\
&\quad - \varepsilon^2 K(\varrho) \partial_{z_1} [\partial_{z_1}^2 + \varepsilon^2 \Delta_{z_\perp}] \hat\rho^{\rm app} \\
&= \big(\partial_T \hat{\mathbf{u}}_1^0 + g'(\varrho) \partial_{z_1} \hat{\mathbf{u}}_1^0 \big) + \varepsilon^2 \left(\partial_T \hat{\mathbf{u}}_1^1 + g'(\varrho) \partial_{z_1} \hat{\mathbf{u}}_1^1 + \partial_\theta \hat{\mathbf{u}}_1^0 + \hat{\mathbf{u}}_1^0 \partial_{z_1} \hat{\mathbf{u}}_1^0 + g''(\varrho) \partial_{z_1} \hat\rho^0 - K(\varrho) \partial_{z_1}^3 \hat\rho^0 \right) \\
&\quad + \varepsilon^4 \mathcal{S}_{4,\perp} + \varepsilon^6 \mathcal{S}_{6,\perp} + \varepsilon^8 \mathcal{S}_{8,\perp},
\end{aligned}$$

where the $\mathcal{R}_{2j}$ and $\mathcal{S}_{2j}$, $2 \leqslant j \leqslant 4$, will be explicited later on. The point is that we are not able to prove that $\hat{\mathbf{u}}_\perp^1$ remains of order one on the time intervals we consider.

Cancellation of the terms of formal order η^0 yields

$$\partial_T \hat\rho^0 + \varrho \partial_{z_1} \hat{\mathbf{u}}_1^0 = \partial_T \hat{\mathbf{u}}_1^0 + g'(\varrho) \partial_{z_1} \hat\rho^0 = 0\,,$$

with general solution $(\hat\rho^0, \varrho \mathfrak{c}^{-1} \hat{\mathbf{u}}_1^0)(T, \theta, z) = \mathcal{Z}^+(\theta, z_1 - \mathfrak{c}T, z_\perp)(1,1) + \mathcal{Z}^-(\theta, z_1 + \mathfrak{c}T, z_\perp)(1,-1)$ for some functions $\mathcal{Z}^\pm$.

Cancellation of the terms of formal order η provide

$$\left\{ \begin{aligned}
&\partial_T \hat\rho^1 + \varrho \partial_{z_1} \hat{\mathbf{u}}_1^1 + \partial_{z_1} (\hat\rho^0 \hat{\mathbf{u}}_1^0) + \partial_\theta \hat\rho^0 + \varrho (\delta^2/\eta) \nabla_{z_\perp} \cdot \hat{\mathbf{u}}_\perp^0 = 0 \\
&\partial_T \hat{\mathbf{u}}_1^1 + g'(\varrho) \partial_{z_1} \hat\rho^1 + \partial_\theta \hat{\mathbf{u}}_1^0 + \hat{\mathbf{u}}_1^0 \partial_{z_1} \hat{\mathbf{u}}_1^0 + g''(\varrho) \partial_{z_1} \hat\rho^0 - K(\varrho) (\varepsilon^2/\eta) \partial_{z_1}^3 \hat\rho^0 = 0\,.
\end{aligned} \right.$$

Therefore, using the expressions for $\hat{\rho}^0$ and $\varrho\mathfrak{c}^{-1}\hat{\mathbf{u}}_1^0$,

$$\begin{aligned}
&\partial_T \left(\hat{\rho}^1 + \frac{\varrho}{\mathfrak{c}}\hat{\mathbf{u}}_1^1\right) + \mathfrak{c}\partial_{z_1}\left(\hat{\rho}^1 + \frac{\varrho}{\mathfrak{c}}\hat{\mathbf{u}}_1^1\right)\\
&= -\partial_\theta\hat{\rho}^0 - \frac{\varrho}{\mathfrak{c}}\partial_\theta\hat{\mathbf{u}}_1^0 - \partial_{z_1}(\hat{\rho}^0\hat{\mathbf{u}}_1^0) - \varrho(\delta^2/\eta)\nabla_{z_\perp}\cdot\hat{\mathbf{u}}_\perp^0 - \frac{\varrho}{\mathfrak{c}}\hat{\mathbf{u}}_1^0\partial_{z_1}\hat{\mathbf{u}}_1^0\\
&\quad - \frac{\varrho}{\mathfrak{c}}g''(\varrho)\hat{\rho}^0\partial_{z_1}\hat{\rho}^0 + \frac{\varrho}{\mathfrak{c}}(\varepsilon^2/\eta)K(\varrho)\partial_{z_1}^3\hat{\rho}^0\\
&= \left(-2\partial_\theta\mathcal{Z}^+ - 2\Gamma\mathcal{Z}^+\partial_{z_1}\mathcal{Z}^+ - \mathfrak{c}(\delta^2/\eta)\Delta_{z_\perp}\partial_{z_1}^{-1}\mathcal{Z}^+ + 2\kappa(\varepsilon^2/\eta)\partial_{z_1}^3\mathcal{Z}^+\right)(\theta, z_1 - \mathfrak{c}T, z_\perp)\\
&\quad + \left(-2(\Gamma - 2\mathfrak{c}/\varrho)\mathcal{Z}^-\partial_{z_1}\mathcal{Z}^- + \mathfrak{c}(\delta^2/\eta)\Delta_{z_\perp}\partial_{z_1}^{-1}\mathcal{Z}^- + 2\kappa(\varepsilon^2/\eta)\partial_{z_1}^3\mathcal{Z}^-\right)(\theta, z_1 + \mathfrak{c}T, z_\perp)\\
&\quad - \left(\frac{\mathfrak{c}}{\varrho} + 2\kappa(\varepsilon^2/\eta)\right)\partial_{z_1}[\mathcal{Z}^+(\theta, z_1 - \mathfrak{c}T, z_\perp)\mathcal{Z}^-(\theta, z_1 + \mathfrak{c}T, z_\perp)]\,.
\end{aligned}$$

Then, $\hat{\rho}^1 + \varrho\mathfrak{c}^{-1}\hat{\mathbf{u}}_1^1$ solves a transport equation with source terms. Notice that the first source term is a function of $z_1 - \mathfrak{c}T$, thus is a solution to the associated homogeneous transport equation. Therefore, it has to vanish in order to remove secular growth (using the characteristic method). Hence,

$$\partial_\theta\mathcal{Z}^+ + \Gamma\mathcal{Z}^+\partial_{z_1}\mathcal{Z}^+ + \frac{\mathfrak{c}}{2}(\delta^2/\eta)\Delta_{z_\perp}\partial_{z_1}^{-1}\mathcal{Z}^+ - \kappa(\varepsilon^2/\eta)\partial_{z_1}^3\mathcal{Z}^+ = 0\,,$$

which is precisely the right-going KP-I equation: we then choose $\mathcal{Z}^+ = \zeta^+$. In a symmetric way, we shall take $\mathcal{Z}^- = \zeta^-$. Recall that we assume $\Delta_{z_\perp}\zeta^\pm(\theta = 0) \in \partial_{z_1}^2 H^{s+3}(\mathbb{R}^d)$. Therefore, $\partial_\theta\zeta^\pm \in L^\infty([0, \theta_*], \partial_{z_1}H^{s+3})$ and $\Delta_{z_\perp}\zeta^\pm \in L^\infty([0, \theta_*], \partial_{z_1}^2 H^{s+3})$, as it follows from the arguments in [21] (see (3.9) and (3.10) there). Indeed, $\partial_\theta\zeta^\pm$ solves

$$\partial_\theta(\partial_\theta\zeta^\pm) + \Gamma\partial_{z_1}(\zeta^\pm(\partial_\theta\zeta^\pm)) + \frac{\mathfrak{c}}{2}(\delta^2/\eta)\Delta_{z_\perp}\partial_{z_1}^{-1}(\partial_\theta\zeta^\pm) - \kappa(\varepsilon^2/\eta)\partial_{z_1}^3(\partial_\theta\zeta^\pm) = 0$$

and $\partial_\theta\zeta^\pm(\theta = 0) = -\Gamma\partial_{z_1}((\zeta^\pm)^2(0)/2) - (\mathfrak{c}\delta^2/(2\eta))\Delta_{z_\perp}\partial_{z_1}^{-1}(\zeta^\pm(0)) + (\kappa\varepsilon^2/\eta)\partial_{z_1}^3(\zeta^\pm(0)) \in \partial_{z_1}H^{s+3}$ by assumption, hence, $\mathcal{F}$ denoting Fourier transform,

$$\begin{aligned}
\mathcal{F}(\partial_\theta\zeta^\pm)(\theta) &= \exp\left(-i\theta(\kappa(\varepsilon^2/\eta)\xi_1^3 + \mathfrak{c}(\delta^2/\eta)|\xi_\perp|^2/(2\xi_1))\right)\mathcal{F}(\partial_\theta\zeta^\pm(0))\\
&\quad - i\Gamma\xi_1\int_0^\theta \exp\left(-i(\theta - \bar{\theta})(\kappa(\varepsilon^2/\eta)\xi_1^3 + \mathfrak{c}(\delta^2/\eta)|\xi_\perp|^2/(2\xi_1))\right)\mathcal{F}(\zeta^\pm(\partial_\theta\zeta^\pm))(\bar{\theta})\,d\bar{\theta}\,.
\end{aligned}$$

It then follows that $\partial_\theta\zeta^\pm \in L^\infty([0, \theta_*], \partial_{z_1}H^{s+3})$ (and the argument does not depend on the space dimension). Consequently, we may rewrite the source term in the equation for $\hat{\rho}^1 + \varrho\mathfrak{c}^{-1}\hat{\mathbf{u}}_1^1$ as a z_1-derivative:

$$\begin{aligned}
&\partial_T\left(\hat{\rho}^1 + \frac{\varrho}{\mathfrak{c}}\hat{\mathbf{u}}_1^1\right) + \mathfrak{c}\partial_{z_1}\left(\hat{\rho}^1 + \frac{\varrho}{\mathfrak{c}}\hat{\mathbf{u}}_1^1\right)\\
&\quad = \partial_{z_1}\Big\{ -(\Gamma - 2\mathfrak{c}/\varrho)[\zeta^-]^2(\theta, z_1 + \mathfrak{c}T, z_\perp) + 2\kappa(\varepsilon^2/\eta)\partial_{z_1}^2\zeta^-(\theta, z_1 + \mathfrak{c}T, z_\perp)\\
&\qquad\qquad - \left(\frac{\mathfrak{c}}{\varrho} + 2\kappa(\varepsilon^2/\eta)\right)\zeta^+(\theta, z_1 - \mathfrak{c}T, z_\perp)\zeta^-(\theta, z_1 + \mathfrak{c}T, z_\perp)\\
&\qquad\qquad + \mathfrak{c}(\delta^2/\eta)\Delta_{z_\perp}\partial_{z_1}^{-2}\zeta^-(\theta, z_1 + \mathfrak{c}T, z_\perp)\Big\}\,.
\end{aligned}$$

The characteristic method then provides

$$
\begin{aligned}
\left(\hat\rho^1+\frac{\varrho}{\mathfrak{c}}\hat{\mathbf{u}}_1^1\right)(T,\theta,z) = {}& \left(\hat\rho^{1,\mathrm{in}}+\frac{\varrho}{\mathfrak{c}}\hat{\mathbf{u}}_1^{1,\mathrm{in}}\right)(z_1-\mathfrak{c}T,z_\perp)\\
&+\frac{1}{2\mathfrak{c}}(\Gamma-2\mathfrak{c}/\varrho)[\zeta^-]^2(\theta,z_1-\mathfrak{c}T,z_\perp)-\frac{1}{2\mathfrak{c}}(\Gamma-2\mathfrak{c}/\varrho)[\zeta^-]^2(\theta,z_1+\mathfrak{c}T,z_\perp)\\
&+\frac{\kappa}{\mathfrak{c}}(\varepsilon^2/\eta)\partial_{z_1}^2\zeta^-(\theta,z_1+\mathfrak{c}T)-\frac{\kappa}{\mathfrak{c}}(\varepsilon^2/\eta)\partial_{z_1}^2\zeta^-(\theta,z_1-\mathfrak{c}T)\\
&-\left(\frac{1}{2\varrho}+\frac{\kappa}{\mathfrak{c}}(\varepsilon^2/\eta)\right)\partial_{z_1}\left[\zeta^+(\theta,z_1-\mathfrak{c}T,z_\perp)\int_{z_1-\mathfrak{c}T}^{z_1+\mathfrak{c}T}\zeta^-(\theta,y,z_\perp)\,dy\right]\\
&+\frac{\delta^2}{2\eta}\Delta_{z_\perp}\partial_{z_1}^{-2}\zeta^-(\theta,z_1+\mathfrak{c}T,z_\perp)-\frac{\delta^2}{2\eta}\Delta_{z_\perp}\partial_{z_1}^{-2}\zeta^-(\theta,z_1-\mathfrak{c}T,z_\perp)\,.
\end{aligned}
\tag{56}
$$

All the terms in the second, third and fifth lines in (56) belong to $L^\infty([0,\theta_*],H^{s+3})$. For the term in the last line, we do not use (as in [20]) Proposition 3.6 in [19] for an estimate by $o(\sqrt{T})$, but write it under the form

$$
-\left(\frac{1}{2\varrho}+\frac{\kappa}{\mathfrak{c}}(\varepsilon^2/\eta)\right)\partial_{z_1}\Big[\zeta^+(\theta,z_1-\mathfrak{c}T,z_\perp)\left(\partial_{z_1}^{-1}\zeta^-(\theta,z_1+\mathfrak{c}T,z_\perp)-\partial_{z_1}^{-1}\zeta^-(\theta,z_1-\mathfrak{c}T,z_\perp)\right)\Big]\,, \tag{57}
$$

to see that is is bounded in H^{s+3} by a constant uniformly for $0\leqslant\theta,\eta T\leqslant\theta_*$. This allows us to derive the estimate

$$
\sup_{0\leqslant T\leqslant\theta_*/\eta}\left\|\left(\hat\rho^1+\frac{\varrho}{\mathfrak{c}}\hat{\mathbf{u}}_1^1\right)(T,\eta T,\cdot)\right\|_{H^{s+3}}\leqslant C\,.
$$

In a similar way, we show that

$$
\sup_{0\leqslant T\leqslant\theta_*/\eta}\left\|\left(\hat\rho^1-\frac{\varrho}{\mathfrak{c}}\hat{\mathbf{u}}_1^1\right)(T,\eta T,\cdot)\right\|_{H^{s+3}}\leqslant C\,,
$$

from which we deduce

$$
\sup_{0\leqslant T\leqslant\theta_*/\eta}\|(\hat\rho^1,\hat{\mathbf{u}}_1^1)(T,\eta T,\cdot)\|_{H^{s+3}}\leqslant C\,. \tag{58}
$$

As a consequence, the approximate solution (55) enjoys the estimate

$$
\sup_{0\leqslant T\leqslant\theta_*/\eta}\|(\hat\rho^{\mathrm{app}},\hat{\mathbf{u}}^{\mathrm{app}})(T)\|_{H^s}\leqslant C\,. \tag{59}
$$

The error terms $\mathcal{R}$ and $\mathcal{S}$ contain θ-derivatives of $\hat\rho^1$ and $\hat{\mathbf{u}}_1^1$ that we wish now to bound. Let us observe that we have $\partial_\theta\zeta^\pm\in L^\infty([0,\theta_*],\partial_{z_1}H^{s+3})\cap L^\infty([0,\theta_*],H^{s+6})$, but the direct differentiation of (56) with respect to θ would require to have $\partial_\theta\zeta^\pm\in L^\infty([0,\theta_*],\partial_{z_1}^2H^{s+2})$, or at least $\Delta_{z_\perp}\partial_\theta\zeta^\pm\in L^\infty([0,\theta_*],\partial_{z_1}^2H^{s+2})$. This is by no way possible if $d=2$ or 3 since the term $\Delta_{z_\perp}\partial_{z_1}^{-1}[(\zeta^\pm)^2]$ appearing in $\partial_\theta\zeta^\pm$ is meaningless. Indeed, $(\zeta^\pm)^2\in L^1$ has a Fourier transform which is continuous in $\mathbb{R}^d$ and positive at $\xi = 0$ (unless $\zeta^\pm \equiv 0$), but $\dfrac{\xi_1}{|\xi_\perp|^2}$ is not integrable near the origin for $d = 2, 3$. We thus proceed to the estimate for $\partial_\theta(\hat\rho^1, \hat{\mathbf{u}}_1^1)$ by first rewriting the term $\Delta_{z_\perp}\partial_{z_1}^{-2}\zeta^-(\theta, z_1 + \mathfrak{c}T, z_\perp) - \Delta_{z_\perp}\partial_{z_1}^{-2}\zeta^-(\theta, z_1 - \mathfrak{c}T, z_\perp)$ in the right-hand side of (56) under the form $\int_{z_1-\mathfrak{c}T}^{z_1+\mathfrak{c}T} \Delta_{z_\perp}\partial_{z_1}^{-1}\zeta^-(\theta, y, z_\perp)\, dy$. Consequently, differentiation of (56) with respect to θ gives, using (57),

$$
\begin{aligned}
&\partial_\theta\left(\hat\rho^1 + \frac{\varrho}{\mathfrak{c}}\hat{\mathbf{u}}_1^1\right)\\
=&\frac{1}{\mathfrak{c}}\Big\{ - (\Gamma - 2\mathfrak{c}/\varrho)\zeta^+\partial_\theta\zeta^+(\theta, z_1 + \mathfrak{c}T, z_\perp) + (\Gamma - 2\mathfrak{c}/\varrho)\zeta^-\partial_\theta\zeta^-(\theta, z_1 - \mathfrak{c}T, z_\perp)\\
&\qquad + \kappa(\varepsilon^2/\eta)\partial_{z_1}^2\partial_\theta\zeta^-(\theta, z_1 + \mathfrak{c}T, z_\perp) - \kappa(\varepsilon^2/\eta)\partial_{z_1}^2\partial_\theta\zeta^-(\theta, z_1 - \mathfrak{c}T, z_\perp)\Big\}\\
&- \left(\frac{1}{2\varrho} + \frac{\kappa}{\mathfrak{c}}(\varepsilon^2/\eta)\right)\partial_{z_1}\partial_\theta\Big[\zeta^+(\theta, z_1 - \mathfrak{c}T, z_\perp)\left(\partial_{z_1}^{-1}\zeta^-(\theta, z_1 + \mathfrak{c}T, z_\perp) - \partial_{z_1}^{-1}\zeta^-(\theta, z_1 - \mathfrak{c}T, z_\perp)\right)\Big]\\
&+ \mathfrak{c}(\delta^2/\eta)\int_{-\mathfrak{c}T}^{+\mathfrak{c}T} \Delta_{z_\perp}\partial_{z_1}^{-1}\partial_\theta\zeta^-(\theta, y + z_1, z_\perp)\, dy\,,
\end{aligned}
$$

thus the estimates on $\zeta^\pm$ we have at hand and applying Proposition 3.6 in [19] for the last term yield

$$
\left\|\partial_\theta\left(\hat\rho^1 + \frac{\varrho}{\mathfrak{c}}\hat{\mathbf{u}}_1^1\right)(T, \eta T, \cdot)\right\|_{H^s} \leqslant C + o(T) = o(\eta^{-1})\,.
$$

Since a similar estimate holds true for $\partial_\theta(\hat\rho^1 - \varrho\mathfrak{c}^{-1}\hat{\mathbf{u}}_1^1)$, we deduce

EKbillaud
$$
\sup_{0\leqslant T\leqslant \theta_*/\eta} \left\|\partial_\theta(\hat\rho^1, \hat{\mathbf{u}}_1^1)(T, \eta T, \cdot)\right\|_{H^s} = o(\eta^{-1})\,. \tag{60}
$$

On the other hand, the formula (56) provides, since $\hat{\mathbf{u}}^1$ is curl free,

$$
\begin{aligned}
&\left(\nabla_{z_\perp}\hat\rho^1 + \frac{\varrho}{\mathfrak{c}}\partial_{z_1}\hat{\mathbf{u}}_\perp^1\right)(T, \theta, z)\\
&\qquad = \left(\nabla_{z_\perp}\hat\rho^{1,\mathrm{in}} + \frac{\varrho}{\mathfrak{c}}\partial_{z_1}\hat{\mathbf{u}}_\perp^{1,\mathrm{in}}\right)(z_1 - \mathfrak{c}T, z_\perp)\\
&\qquad\quad + \frac{1}{2\mathfrak{c}}\Big\{ - (\Gamma - 2\mathfrak{c}/\varrho)\partial_{z_1}\nabla_{z_\perp}\int_{z_1-\mathfrak{c}T}^{z_1+\mathfrak{c}T} [\zeta^-]^2(\theta, y, z_\perp)\, dy\\
&\qquad\qquad + \mathfrak{c}(\delta^2/\eta)\partial_{z_1}\int_{z_1-\mathfrak{c}T}^{z_1+\mathfrak{c}T} \Delta_{z_\perp}\partial_{z_1}^{-2}\zeta^-(\theta, y, z_\perp)\, dy\\
&\qquad\qquad + 2\kappa(\varepsilon^2/\eta)\partial_{z_1}^2\nabla_{z_\perp}\left[\zeta^-(\theta, z_1 + \mathfrak{c}T) - \zeta^-(\theta, z_1 - \mathfrak{c}T)\right]\Big\}\\
&\qquad\quad - \left(\frac{1}{2\varrho} + \frac{\kappa}{\mathfrak{c}}(\varepsilon^2/\eta)\right)\partial_{z_1}\nabla_{z_\perp}\left[\zeta^+(\theta, z_1 - \mathfrak{c}T, z_\perp)\int_{z_1-\mathfrak{c}T}^{z_1+\mathfrak{c}T} \zeta^-(\theta, y, z_\perp)\, dy\right]
\end{aligned}
$$

and a similar equality holds for $\nabla_{z_\perp}\hat\rho^1 - \varrho\mathfrak{c}^{-1}\partial_{z_1}\hat{\mathbf{u}}_\perp^1$. Thus, taking the difference of the two equations and integrating in z_1, we obtain

EKssoulet
$$
\sup_{0\leqslant T\leqslant \theta_*/\eta} \|\hat{\mathbf{u}}_\perp^1(T, \eta T)\|_{H^s} \leqslant C + o(T) = o(\eta^{-1})\,, \tag{61}
$$

using once again Proposition 3.6 in [19] for the terms involving $\int_{z_1-\mathfrak{c}T}^{z_1+\mathfrak{c}T}$.

Let us now explicit

$$\begin{aligned}\mathcal{R} =& \eta^2\partial_\theta\hat\rho^1 + \eta^2\partial_{z_1}(\hat\rho^1\hat{\mathbf{u}}_1^0 + \hat\rho^0\hat{\mathbf{u}}_1^1) + \delta^2\eta\nabla_{z_\perp}\cdot(\varrho\hat{\mathbf{u}}_\perp^1 + \hat\rho^0\hat{\mathbf{u}}_\perp^0)\\ &+ \eta^2\delta^2\nabla_{z_\perp}\cdot(\hat\rho^0\hat{\mathbf{u}}_\perp^1 + \hat\rho^1\hat{\mathbf{u}}_\perp^0) + \eta^3\delta^2\nabla_{z_\perp}\cdot(\hat\rho^1\hat{\mathbf{u}}_\perp^1)\,.\end{aligned}$$

It follows from (58), (60) and (61) that

$$\sup_{0\leqslant T\leqslant T_*/\eta}\|\mathrm{Err}_\rho\|_{H^s} \leqslant \sup_{0\leqslant T\leqslant \theta_*/\eta}\|\mathcal{R}\|_{H^s} = o(\eta). \tag{62}$$

Similarly, from the explicit relations

$$\begin{aligned}\mathcal{S}_1 =& \eta^2\partial_\theta\hat{\mathbf{u}}_1^1 + \eta^2\partial_{z_1}(\hat{\mathbf{u}}_1^1\hat{\mathbf{u}}_1^0) + \eta\delta^2\hat{\mathbf{u}}_\perp^0\cdot\nabla_{z_\perp}\hat{\mathbf{u}}_1^0 + \eta^2 g''(\varrho)\partial_{z_1}(\hat\rho^1\hat\rho^0)\\ &- \varepsilon^2\eta K(\varrho)\partial_{z_1}^3\hat\rho^1 - \varepsilon^2\delta^2 K(\varrho)\partial_{z_1}\Delta_{z_\perp}\hat\rho^0\\ &+ \eta^3\hat{\mathbf{u}}_1^1\partial_{z_1}\hat{\mathbf{u}}_1^1 + \eta^2\delta^2\hat{\mathbf{u}}_\perp^1\cdot\nabla_{z_\perp}\hat{\mathbf{u}}_1^0 + \eta^2\delta^2\hat{\mathbf{u}}_\perp^0\cdot\nabla_{z_\perp}\hat{\mathbf{u}}_1^1 + \eta^3 g''(\varrho)\hat\rho^1\partial_{z_1}\hat\rho^1 - \eta^2\delta^2K(\varrho)\partial_{z_1}\Delta_{z_\perp}\hat\rho^1\\ &+ \eta^3\delta^2\hat{\mathbf{u}}_\perp^1\cdot\nabla_{z_\perp}\hat{\mathbf{u}}_1^1\,,\end{aligned}$$

we infer

$$\sup_{0\leqslant T\leqslant T_*/\eta}\|\mathrm{Err}_1\|_{H^s} = o(\eta)\,, \tag{63}$$

and from

$$\begin{aligned}\mathcal{S}_\perp =& \eta^2\partial_\theta\hat{\mathbf{u}}_\perp^1 + \eta^2\hat{\mathbf{u}}_1^1\partial_{z_1}\hat{\mathbf{u}}_\perp^0 + \eta^2\hat{\mathbf{u}}_1^0\partial_{z_1}\hat{\mathbf{u}}_\perp^1 + \eta\delta^2\hat{\mathbf{u}}_\perp^0\cdot\nabla_{z_\perp}\hat{\mathbf{u}}_\perp^0 + \eta^2 g''(\varrho)\nabla_{z_\perp}(\hat\rho^1\hat\rho^0)\\ &- \varepsilon^2\eta K(\varrho)\nabla_{z_\perp}\partial_{z_1}^2\hat\rho^1 - \varepsilon^2\delta^2K(\varrho)\nabla_{z_\perp}\Delta_{z_\perp}\hat\rho^0\\ &+ \eta^3\hat{\mathbf{u}}_1^1\partial_{z_1}\hat{\mathbf{u}}_\perp^1 + \eta^2\delta^2\hat{\mathbf{u}}_\perp^1\cdot\nabla_{z_\perp}\hat{\mathbf{u}}_\perp^0 + \eta^2\delta^2\hat{\mathbf{u}}_\perp^0\cdot\nabla_{z_\perp}\hat{\mathbf{u}}_\perp^1 + \eta^3 g''(\varrho)\hat\rho^1\nabla_{z_\perp}\hat\rho^1 - \varepsilon^2\delta^2K(\varrho)\nabla_{z_\perp}\Delta_{z_\perp}\hat\rho^1\\ &+ \eta^3\delta^2\hat{\mathbf{u}}_\perp^1\cdot\nabla_{z_\perp}\hat{\mathbf{u}}_\perp^1\,,\end{aligned}$$

we deduce

$$\sup_{0\leqslant T\leqslant T_*/\varepsilon^2}\|\varepsilon\mathrm{Err}_\perp\|_{H^s} = o(\eta)\,. \tag{64}$$

We complete the proof of Theorem 9 using the comparison estimate given in Proposition 6 below and the consistency errors (62), (63), (64) we have established. □

6.4 A comparison estimate and proof of Theorem 10

Proposition 6 *Assume that $s > 1+d/2$ (s integer). Assume that $(\tilde\rho,\tilde{\mathbf{u}})\in\mathscr{C}([0,\tilde T], H^{s+1}(\mathbb{R}^d)\times[H^s(\mathbb{R}^d)]^d)$ solves*

$$\left\{\begin{aligned}&\partial_T\tilde\rho + \nabla^\delta\cdot((\varrho+\eta\tilde\rho)\tilde{\mathbf{u}}) = \mathrm{Err}_{\tilde\rho}\\ &\partial_T\tilde{\mathbf{u}} + g'(\varrho)\nabla^\delta\tilde\rho + \eta\tilde{\mathbf{u}}\cdot\nabla^\delta\tilde{\mathbf{u}} + \eta g''(\varrho)\tilde\rho\nabla^\delta\tilde\rho = \varepsilon^2K(\varrho)\nabla^\delta[\partial_{z_1}^2+\delta^2\Delta_{z_\perp}]\tilde\rho + \mathrm{Err}_{\tilde{\mathbf{u}}}\,,\end{aligned}\right. \tag{65}$$

where $(\mathrm{Err}_{\tilde\rho}, \mathrm{Err}_{\tilde{\mathbf{u}}}) \in L^\infty([0,\tilde T], H^{s+1}(\mathbb{R}^d)\times[H^s(\mathbb{R}^d)]^d)$*. Let us denote* $(\hat\rho, \hat{\mathbf{u}}) \in \mathscr{C}([0,\hat T], H^{s+1}(\mathbb{R}^d)\times[H^s(\mathbb{R}^d)]^d)$ *a solution of* ($\mathcal{B}_{\varepsilon,\delta,\eta}$) *with* $(\tilde\rho,\tilde{\mathbf{u}})(0)$ *as initial condition. Assume that* $M>0$ *is such that for any* $0\leqslant T\leqslant \min(\tilde T,\hat T)$, $(\hat\rho,\hat{\mathbf{u}})(T)$ *and* $(\tilde\rho,\tilde{\mathbf{u}})(T)\in \tilde B_\varepsilon(2M)$*. Then,*

$$\|(\tilde\rho,\tilde{\mathbf{u}},\varepsilon\nabla^\delta\tilde\rho)-(\hat\rho,\hat{\mathbf{u}},\varepsilon\nabla^\delta\hat\rho)\|_{H^{s-1}}\leqslant \frac{C(s,r,d,M)}{\eta}\|(\mathrm{Err}_{\tilde\rho},\mathrm{Err}_{\tilde{\mathbf{u}}})\|_{L^\infty([0,\min(\tilde T,\hat T)],H^{s-1})}\,.$$

Proof. The difference $(\check\rho,\check{\mathbf{u}})\stackrel{\mathrm{def}}{=}(\hat\rho,\hat{\mathbf{u}})-(\tilde\rho,\tilde{\mathbf{u}})$ satisfies the system

EKasserole
$$\left\{\begin{array}{l}\partial_T\check\rho+\nabla^\delta\cdot((\varrho+\eta\hat\rho)\check{\mathbf{u}})+\eta\nabla^\delta\cdot(\check\rho\hat{\mathbf{u}})=-\mathrm{Err}_{\tilde\rho}\\[2ex] \partial_T\check{\mathbf{u}}+g'(\varrho)\nabla^\delta\check\rho+\eta\hat{\mathbf{u}}\cdot\nabla^\delta\check{\mathbf{u}}+\eta\check{\mathbf{u}}\cdot\nabla^\delta\tilde{\mathbf{u}}+\eta g''(\varrho)\hat\rho\nabla^\delta\check\rho+\eta g''(\varrho)\check\rho\nabla^\delta\tilde\rho\\ \quad=\varepsilon^2K(\varrho)\nabla^\delta[\partial_{z_1}^2+\delta^2\Delta_{z_\perp}]\check\rho-\mathrm{Err}_{\tilde{\mathbf{u}}}\,,\end{array}\right. \tag{66}$$

with null initial condition. Then, the complex vector field

$$\check{\mathbf{z}}\stackrel{\mathrm{def}}{=}\check{\mathbf{u}}+i\check{\mathbf{w}},\qquad \check{\mathbf{w}}\stackrel{\mathrm{def}}{=}\varepsilon\sqrt{\frac{K(\varrho)}{\varrho+\eta\hat\rho}}\nabla^\delta\hat\rho-\varepsilon\sqrt{\frac{K(\varrho)}{\varrho+\eta\tilde\rho}}\nabla^\delta\tilde\rho=\varepsilon\sqrt{\frac{K(\varrho)}{\varrho+\eta\hat\rho}}\nabla^\delta\check\rho+\mathcal{O}_{H^{s-1}}(\varepsilon\eta)$$

is a solution, with zero initial datum, of

$$\begin{aligned}\partial_T\check{\mathbf{z}}\;&+\eta\tilde{\mathbf{u}}\cdot\nabla^\delta\check{\mathbf{z}}+\eta\check{\mathbf{u}}\cdot\nabla^\delta\tilde{\mathbf{z}}+i\eta(\nabla^\delta\check{\mathbf{z}})\hat{\mathbf{w}}+i\eta(\nabla^\delta\tilde{\mathbf{z}})\check{\mathbf{w}}\\ &+\frac{1}{\varepsilon}b_\sharp(\varrho+\eta\hat\rho)\check{\mathbf{w}}+\frac{1}{\varepsilon}\Big(b_\sharp(\varrho+\eta\hat\rho)-b_\sharp(\varrho+\eta\tilde\rho)\Big)\tilde{\mathbf{w}}\\ &+i\varepsilon\nabla^\delta(a_\sharp(\varrho+\eta\hat\rho)\nabla^\delta\cdot\check{\mathbf{z}})+i\varepsilon\nabla^\delta\left((a_\sharp(\varrho+\eta\hat\rho)-a_\sharp(\varrho+\eta\tilde\rho))\,\nabla^\delta\cdot\tilde{\mathbf{z}}\right)\\ &=-\mathrm{Err}_{\tilde{\mathbf{u}}}-i\varepsilon\sqrt{\frac{K(\varrho)}{\varrho+\eta\hat\rho}}\nabla^\delta\mathrm{Err}_{\tilde\rho}\,,\end{aligned}$$

where

$$a_\sharp(\rho)\stackrel{\mathrm{def}}{=}\sqrt{K(\varrho)\rho}\,,\qquad b_\sharp(\rho)\stackrel{\mathrm{def}}{=}\frac{\rho(g'(\varrho)+(\rho-\varrho)g''(\varrho))}{a_\sharp(\rho)}\,.$$

We see that in order to perform an H^σ estimate on $\check{\mathbf{z}}$, we need $\mathrm{Err}_{\tilde\rho}$ bounded in $H^{\sigma+1}$.

The terms $\eta\check{\mathbf{u}}\cdot\nabla^\delta\tilde{\mathbf{z}}$ and $i\eta(\nabla^\delta\tilde{\mathbf{z}})\check{\mathbf{w}}$ are easily estimated in H^{s-1}, using (tameprod A.1), by

$$C(s,d)\eta\|\check{\mathbf{z}}\|_{H^{s-1}}\|\nabla^\delta\tilde{\mathbf{z}}\|_{H^{s-1}}\leqslant C(s,d,M)\eta\|\check{\mathbf{z}}\|_{H^{s-1}}\,.$$

For the term $\varepsilon^{-1}(b_\sharp(\varrho+\eta\hat\rho)-b_\sharp(\varrho+\eta\tilde\rho))\hat{\mathbf{w}}$, we write that its H^{s-1} norm is

$$\leqslant C(s,d,r)\eta\varepsilon^{-1}\|\check\rho\|_{H^{s-1}}\|\tilde{\mathbf{w}}\|_{H^{s-1}}\leqslant C(s,d,r,M)\varepsilon\|\check\rho\|_{H^{s-1}}\|\eta\nabla^\delta\tilde\rho\|_{H^{s-1}}\leqslant C(s,d,r,M)\eta\|\check\rho\|_{H^{s-1}}\,.$$

The H^{s-1} norm of the term $i\varepsilon\nabla^\delta\left((a_\sharp(\varrho+\eta\hat\rho)-a_\sharp(\varrho+\eta\tilde\rho))\,\nabla^\delta\cdot\tilde{\mathbf{z}}\right)$ is

$$\leqslant C(s,d,r)\Big(\varepsilon\eta\|\check\rho\|_{H^{s-1}}+\eta\|\varepsilon\nabla^\delta\check\rho\|_{H^{s-1}}\Big)\|\nabla^\delta\cdot\tilde{\mathbf{z}}\|_{H^s}\leqslant C(s,d,r,M)\eta\|(\check\rho,\varepsilon\nabla^\delta\check\rho)\|_{H^{s-1}}\,.$$

Therefore, $\check{\mathbf{z}}$ is a solution of

$$\partial_T \check{\mathbf{z}} + \eta \hat{\mathbf{u}} \cdot \nabla^\delta \check{\mathbf{z}} + i\eta (\nabla^\delta \check{\mathbf{z}}) \hat{\mathbf{w}} + \frac{1}{\varepsilon} b_\sharp(\varrho + \eta\hat{\rho}) \check{\mathbf{w}} + i\varepsilon \nabla^\delta (a_\sharp(\varrho + \eta\hat{\rho}) \nabla^\delta \cdot \check{\mathbf{z}}) = \mathcal{G} \,,$$

with zero initial datum and where

$$\|\mathcal{G}\|_{H^{s-1}} \leqslant C(s,r,d,M) \Big(\|(\mathrm{Err}_{\tilde{\mathbf{u}}}, \varepsilon \nabla^\delta \mathrm{Err}_{\tilde{\rho}})\|_{H^{s-1}} + \eta \|(\check{\rho}, \varepsilon \nabla^\delta \check{\rho})\|_{H^{s-1}} \Big) \,.$$

Letting

$$E^\sharp_{s-1}(\check{\rho}, \check{\mathbf{z}}) \stackrel{\mathrm{def}}{=} \sum_{\sigma=0}^{s-1} \dot{E}^\sharp_\sigma(\check{\rho}, \check{\mathbf{z}}) \,,$$

where

$$\dot{E}^\sharp_\sigma(\check{\rho}, \check{\mathbf{z}}) \stackrel{\mathrm{def}}{=} \sum_{\substack{\alpha \in \mathbb{N}_0^d, \\ |\alpha| = \sigma}} \frac{\sigma!}{\alpha!} \int_{\mathbb{R}^d} \frac{1}{2} a^\sigma(\varrho + \eta\hat{\rho}) \Big((\varrho + \eta\hat{\rho}) |\partial^\alpha \check{\mathbf{z}}|^2 + (g'(\varrho) + g''(\varrho)\eta\hat{\rho})(\partial^\alpha \check{\rho})^2 \Big) \, dz \,,$$

and arguing as in the proof of Theorem 8, we arrive, for $0 \leqslant T \leqslant \min(\tilde{T}, \hat{T})$, at

$$\frac{d}{dT} E^\sharp_{s-1}(\check{\rho}, \check{\mathbf{z}}) \leqslant \frac{C(s,r,d,M)}{\eta} \|(\mathrm{Err}_{\tilde{\rho}}, \mathrm{Err}_{\tilde{\mathbf{u}}})\|^2_{H^{s-1}} + C(s,r,d,M) \eta E^\sharp_{s-1}(\check{\rho}, \check{\mathbf{z}}) \,,$$

since $\nabla \hat{\mathbf{z}}$ and $\nabla \hat{\rho}$ are uniformly bounded in L^∞. Indeed, there is only one place where we have to pay attention to the extra terms in the first equation in (65), namely when we compute

$$\frac{d}{dT} \int_{\mathbb{R}^d} \frac{1}{2} [g'(\varrho) + \eta \hat{\rho} g''(\varrho)] a^\sigma_\sharp(\varrho_\eta \hat{\rho}) (\partial^\alpha \check{\rho})^2 \, dz \,.$$

These extra terms are controled in the following way:

$$\begin{aligned}
&\int_{\mathbb{R}^d} [g'(\varrho) + \eta \hat{\rho} g''(\varrho)] a^\sigma_\sharp(\varrho + \eta\hat{\rho}) \partial^\alpha \hat{\rho} \left(\partial^\alpha \mathrm{Err}_{\tilde{\rho}} + \eta \partial^\alpha \nabla^\delta \cdot (\check{\rho} \hat{\mathbf{u}}) \right) \, dz \\
&\leqslant C(s,r,M) \sqrt{E^\sharp_{s-1}(\check{\rho}, \check{\mathbf{z}})} \|\mathrm{Err}_{\tilde{\rho}}\|_{H^{s-1}} + C(s,r,M) \eta E^\sharp_{s-1}(\check{\rho}, \check{\mathbf{z}}) \\
&\quad + \eta \int_{\mathbb{R}^d} [g'(\varrho) + \eta \hat{\rho} g''(\varrho)] a^\sigma_\sharp(\varrho + \eta\hat{\rho}) \partial^\alpha \hat{\rho} \; \hat{\mathbf{u}} \cdot \nabla^\delta \partial^\alpha \check{\rho} \, dz \\
&\leqslant \frac{C(s,r,d,M)}{\eta} \|(\mathrm{Err}_{\tilde{\rho}}, \mathrm{Err}_{\tilde{\mathbf{u}}})\|^2_{H^{s-1}} + C(s,r,d,M) \eta E^\sharp_{s-1}(\check{\rho}, \check{\mathbf{z}})
\end{aligned}$$

with another use of (A.3), Young inequality and integration by parts. This implies, by the Gronwall lemma, the result. □

Proof of Theorem pistoletdor **10.** From (DrNo 47) and the uniform bounds $(\hat\rho, \hat{\mathbf{u}}) \in \tilde{B}_\varepsilon(2M)$ for any $0 \leqslant T \leqslant T_*/\eta$, we infer that

$$\begin{cases} \partial_T \hat\rho + \nabla^\delta \cdot ((\varrho + \eta\hat\rho)\hat{\mathbf{u}}) = 0 \\ \partial_T \hat{\mathbf{u}} + \eta \hat{\mathbf{u}} \cdot \nabla^\delta \hat{\mathbf{u}} + g'(\varrho)\nabla^\delta \hat\rho + \eta g''(\varrho)\hat\rho \nabla^\delta \hat\rho - \varepsilon^2 K(\varrho)\nabla^\delta[\partial_{z_1}^2 + \delta^2 \Delta_{z_\perp}]\hat\rho \\ \qquad = -\Big(g'(\varrho + \eta\hat\rho) - g'(\varrho) - \eta g''(\varrho)\hat\rho\Big)\nabla^\delta\hat\rho \\ \qquad\quad + \varepsilon^2 \nabla^\delta \Big((K(\varrho + \eta\hat\rho) - K(\varrho))[\partial_{z_1}^2 + \delta^2\Delta_{z_\perp}]\hat\rho + \dfrac{\varepsilon^2}{2} K'(\varrho + \eta\hat\rho)|\nabla^\delta \hat\rho|^2\Big) \\ \qquad = \mathcal{O}_{L^\infty([0,T_*/\eta],H^{s-3})}(\eta^2 + \eta\varepsilon^2) \qquad \text{and} \qquad \mathcal{O}_{L^\infty([0,T_*/\eta],H^{s-2})}(\eta^2 + \eta\varepsilon). \end{cases}$$

The conclusion then follows from Proposition EKdetrousselle 6. ☐

Acknowledgements: This work is supported by the ANR project *BoND* (Bond-ANR-13-BS01-0009-02). We would like to thank Pr. Q and Dr. MoneyPenny for helpful comments.

Appendix

doucement **Proposition A.1** • *For $s \geqslant 0$, for all $u, v \in H^s(\mathbb{R}^d)$,*

tameprod (A.1)
$$\|uv\|_{H^s} \leqslant C(d,s)\left(\|u\|_{L^\infty}\|v\|_{H^s} + \|v\|_{L^\infty}\|u\|_{H^s}\right).$$

- *For $s \in \mathbb{N}$, if $F \in W^{s,\infty}([-r,r])$ vanishes at zero, for all $v \in H^s(\mathbb{R}^d)$ taking values in $[-r,r]$,*

tamecomp (A.2)
$$\|F(v)\|_{H^s} \leqslant C(d,s,r)\,\|F'\|_{W^{s,\infty}([-r,r])}\left(1 + \|v\|_{L^\infty}\right)^s \|v\|_{H^s}.$$

- *For $s > 0$, for all $u, v \in H^s(\mathbb{R}^d)$, for $\alpha \in \mathbb{N}^d$ such that $|\alpha| \leqslant s$,*

tamecomm (A.3)
$$\|\partial^\alpha(uv) - u\partial^\alpha v\|_{L^2} \leqslant C(d,s)(\|\nabla u\|_{H^{s-1}}\|v\|_{L^\infty} + \|\nabla u\|_{L^\infty}\|v\|_{H^{s-1}}).$$

This is also true when ∂^α is replaced by $\Lambda^s = (1-\Delta)^{s/2}$.

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