The KdV/KP-I limit of the Nonlinear Schrödinger equation

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Abstract

We justify rigorously the convergence of the amplitude of solutions of Nonlinear-Schrödinger type Equations with non zero limit at infinity to an asymptotic regime governed by the Korteweg-de Vries equation in dimension 1 and the Kadomtsev-Petviashvili I equation in dimensions 2 and more. We get two types of results. In the one-dimensional case, we prove directly by energy bounds that there is no vortex formation for the global solution of the NLS equation in the energy space and deduce from this the convergence towards the unique solution in the energy space of the KdV equation. In arbitrary dimensions, we use an hydrodynamic reformulation of NLS and recast the problem as a singular limit for an hyperbolic system. We thus prove that smooth H^s solutions exist on a time interval independent of the small parameter. We then pass to the limit by a compactness argument and obtain the KdV/KP-I equation.

1 Introduction

We consider the n-dimensional nonlinear Schrödinger equation

$$i\frac{\partial\Psi}{\partial\tau} + \frac{1}{2}\Delta_z\Psi = \Psi f(|\Psi|^2) \qquad \Psi = \Psi(\tau, z) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{C}. \tag{NLS}$$

This equation is used as a model in nonlinear Optics (see for instance [19]) and in superfluidity and Bose-Einstein condensation (see, *e.g.* [23], [10], [13]).

We assume that, for some $\rho_0 > 0$, $f(\rho_0^2) = 0$, so that $\Psi \equiv \rho_0$ is a particular solution of (NLS). We are interested in solutions Ψ of (NLS) such that $|\Psi| \simeq \rho_0$. In the sequel, we take $\rho_0 = 1$, the general case follows changing Ψ for $\tilde{\Psi} \equiv \rho_0^{-1}\Psi$ and f for $\tilde{f}(R) \equiv f(\rho_0^2 R)$. Then, from now on, we consider smooth nonlinearities $f \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ such that

$$f(1) = 0, \qquad f'(1) > 0 \tag{1}$$

and will be interested in situations where $|\Psi| \simeq 1$. Note that this means thanks to (1) that we shall study the equation in a defocusing regime. A typical example of nonlinearity is simply $f(R) = R-1$ for which (NLS) is termed the Gross-Pitaevskii equation. Equation (NLS) is an Hamiltonian flow associated to the Ginzburg-Landau type energy (when it makes sense)

$$\mathcal{E}(\Psi) \equiv \frac{1}{2}\int_{\mathbb{R}^n} |\nabla_z\Psi|^2 + F\big(|\Psi|^2\big)\ dz,$$

where $F(R) \equiv 2\int_1^R f(r)\ dr$.

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1.1 KdV and KP-I asymptotic regimes for NLS

In a suitable scaling corresponding to $|\Psi| \simeq 1$, the dynamics for the amplitude of Ψ converges, in dimension $n = 1$, to the Korteweg-de Vries equation

$$2\partial_t v + k\, v\partial_x v - \frac{1}{4c^2}\,\partial_x^3 v = 0, \tag{KdV}$$

and in dimensions $n \geq 2$ to the Kadomtsev-Petviashvili - I equation

$$\partial_x\Big(2\partial_t v + kv\partial_x v - \frac{1}{4c^2}\,\partial_x^3 v\Big) + \Delta_\perp v = 0 \tag{KP-I}$$

where $v = v(t, X) \in \mathbb{R}$, $X = (x, x_\perp) \in \mathbb{R} \times \mathbb{R}^{n-1}$. The coefficients c and k are related to the nonlinearity f by

$$c \equiv \sqrt{f'(1)} > 0 \qquad \text{and} \qquad k \equiv 6 + \frac{2}{c^2} f''(1). \tag{2}$$

Note that the KP-I equation reduces to the KdV equation if v does not depend on $x_\perp$.

The formal derivation of this regime is as follows. First, we consider a small parameter ε, and rescale time and space according to

$$t = c\varepsilon^3\tau, \quad X_1 = x = \varepsilon(z_1 - c\tau), \quad X_j = \varepsilon^2 z_j, \quad j \in \{2, ..., n\}, \quad \Psi(\tau, z) = \psi^\varepsilon(t, X). \tag{3}$$

In this long wave asymptotics, the nonlinear Schrödinger equation for ψ^ε reads now

$$ic\varepsilon^3 \frac{\partial\psi^\varepsilon}{\partial t} - ic\varepsilon\partial_x\psi^\varepsilon + \frac{\varepsilon^2}{2}\partial_x^2\psi^\varepsilon + \frac{\varepsilon^4}{2}\Delta_\perp\psi^\varepsilon = \psi^\varepsilon f(|\psi^\varepsilon|^2), \qquad X = (x, x_\perp) \in \mathbb{R} \times \mathbb{R}^{n-1}. \tag{4}$$

We shall use the following ansatz for ψ^ε

$$\psi^\varepsilon(t, X) = \big(1 + \varepsilon^2 A^\varepsilon(t, X)\big) \exp\big(i\varepsilon\varphi^\varepsilon(t, X)\big) \tag{5}$$

where the amplitude $A^\varepsilon \in \mathbb{R}$ is assumed to be of order 1 and the real phase $\varphi^\varepsilon \in \mathbb{R}$ is also assumed to be of order 1. This ansatz is natural in the stability analysis of the particular solution $\psi^\varepsilon = 1$ to slowly modulated perturbations (see [18], [19]). We focus on perturbation that travels to the right and are slowly modulated in the transverse direction thanks to (3). Important solutions of NLS that arise in this framework are the travelling waves. The use of the ansatz (5) to study their qualitative properties is classical in the physics litterature.

The ansatz (3), (5) is adapted so that nonlinear and dispersive effects are all of order one on the chosen time scale. Note that the occurence of the KdV or KP-I equation as enveloppe equations in such regimes is expected. We refer for example to [2] and references therein for the derivation of these equations from the water-waves system.

By plugging (5) in (4) and by separating real and imaginary parts, we can rewrite (4) as the system

$$\begin{cases} \varepsilon^2 c\partial_t A^\varepsilon - c\partial_x A^\varepsilon + \varepsilon^2\partial_x A^\varepsilon\partial_x\varphi^\varepsilon + \dfrac{1}{2}\big(1 + \varepsilon^2 A^\varepsilon\big)\partial_x^2\varphi^\varepsilon + \varepsilon^4\nabla_\perp A^\varepsilon \cdot \nabla_\perp\varphi^\varepsilon \\ \qquad + \dfrac{\varepsilon^2}{2}\big(1 + \varepsilon^2 A^\varepsilon\big)\Delta_\perp\varphi^\varepsilon = 0 \\ \varepsilon^2 c\partial_t\varphi^\varepsilon - c\partial_x\varphi^\varepsilon - \varepsilon^2\dfrac{\partial_x^2 A^\varepsilon}{2\big(1 + \varepsilon^2 A^\varepsilon\big)} - \varepsilon^4\dfrac{\Delta_\perp A^\varepsilon}{2\big(1 + \varepsilon^2 A^\varepsilon\big)} + \dfrac{\varepsilon^2}{2}\big(\partial_x\varphi^\varepsilon\big)^2 + \dfrac{\varepsilon^4}{2}|\nabla_\perp\varphi^\varepsilon|^2 \\ \qquad + \dfrac{1}{\varepsilon^2}\, f\big((1 + \varepsilon^2 A^\varepsilon)^2\big) = 0. \end{cases} \tag{6}$$

Now, assuming that $A^\varepsilon \to A$ and $\varphi^\varepsilon \to \varphi$ as $\varepsilon \to 0$, we formally obtain from the two equations of the above system that

$$-c\partial_x A + \frac{1}{2}\,\partial_x^2\varphi = 0, \qquad\qquad -c\partial_x\varphi + 2f'(1)A = 0. \tag{7}$$

Note that we have used that $f(1) = 0$ and thus that $f\big((1+\varepsilon^2 A^\varepsilon)^2\big) \simeq 2\varepsilon^2 f'(1)A$ at leading order. In (7) and from the definition (2) of c, the first equation is just $-\dfrac{1}{2c}$ times the derivative of the second equation with respect to x, hence, we have found for the limit the constraint

$$2cA = \partial_x\varphi. \tag{8}$$

To get the limit equation satisfied by A, we can add the first equation in (6) and $\dfrac{1}{2c}$ times the derivative of the second equation with respect to x in order to cancel the most singular term. This yields the equation

$$\begin{aligned} &c\partial_t\Big(A^\varepsilon + \frac{1}{2c}\,\partial_x\varphi^\varepsilon\Big) - \frac{1}{4c}\partial_x\Big(\frac{\partial_x^2 A^\varepsilon}{1+\varepsilon^2 A^\varepsilon}\Big) + \frac{1}{2}\big(1+\varepsilon^2 A^\varepsilon\big)\Delta_\perp\varphi^\varepsilon + \frac{c}{\varepsilon^4}\,\partial_x\big(Q(\varepsilon^2 A^\varepsilon)\big) \\ &\quad +\Big\{\partial_x A^\varepsilon\partial_x\varphi^\varepsilon + \frac{1}{2}\,A^\varepsilon\partial_x^2\varphi^\varepsilon + \frac{1}{4c}\,\partial_x\big((\partial_x\varphi^\varepsilon)^2\big) + \frac{1}{2c}\big[f'(1) + 2f''(1)\big]\partial_x\big((A^\varepsilon)^2\big)\Big\} \\ &= \frac{\varepsilon^2}{4c}\partial_x\Big(\frac{\Delta_\perp A^\varepsilon}{1+\varepsilon^2 A^\varepsilon}\Big) - \frac{\varepsilon^2}{4c}\partial_x\big(|\nabla_\perp\varphi^\varepsilon|^2\big) - \varepsilon^2\nabla_\perp A^\varepsilon\cdot\nabla_\perp\varphi^\varepsilon, \end{aligned} \tag{9}$$

where

$$c^2 Q(r) \equiv f\big((1+r)^2\big) - 2f'(1)r - \big(f'(1) + 2f''(1)\big)r^2 = \mathcal{O}(r^3) \quad r\to 0.$$

Still on a formal level, if $A^\varepsilon \to A$ and $\varphi^\varepsilon \to \varphi$ as $\varepsilon \to 0$, this yields

$$2\partial_t A + \Big[6 + \frac{2}{c^2}f''(1)\Big]A\partial_x A - \frac{1}{4c^2}\partial_x^3 A + \frac{1}{2c}\,\Delta_\perp\varphi = 0$$

by using the relation (8). Consequently, we have obtained the sytem

$$\begin{cases} \partial_x\varphi = 2cA \\ 2\partial_t A + \Big[6 + \dfrac{2}{c^2}f''(1)\Big]A\partial_x A - \dfrac{1}{4c^2}\partial_x^3 A + \dfrac{1}{2c}\,\Delta_\perp\varphi = 0 \end{cases} \tag{10}$$

which is a reformulation of the KP-I equation. Note that in dimension 1, *i.e.* when $n = 1$, this amounts to assume that all the functions involved in the derivation do not depend on $x_\perp$, then the equation for A in (10) just reduces to the KdV equation since $\Delta_\perp\varphi = 0$.

Finally, let us notice that because of the scaling (3), for the solution Ψ of the original (NLS) equation with time-scale 1, the convergence to KdV or KP-I dynamics takes place for times of order ε^{-3}.

In dimension $n = 1$, the formal derivation of the KdV equation from the (NLS) equation in this asymptotic regime is well-known in the physics literature (see, for example, [18]), and is useful in the stability analysis of dark solitons or travelling waves of small energy. In the case of the

Gross-Pitaevskii equation, for instance (that is for $f(R) = R-1$), the travelling waves are solutions to (NLS) of the form $\Psi(\tau, z) = U(z - \sigma\tau)$, so that U solves

$$-i\sigma\partial_z U + \frac{1}{2}\partial_{zz} U = U(|U|^2 - 1), \qquad z \in \mathbb{R} \tag{11}$$

with the condition $|U|(z) \to 1$ as $z \to \pm\infty$. For this nonlinearity, explicit integration (see, *e.g.* [26]) gives for $0 < \sigma < 1$ the nontrivial solution

$$U_\sigma(z) = \sigma - i\sqrt{1-\sigma^2}\,\mathrm{th}\Big(z\sqrt{1-\sigma^2}\Big).$$

In this scaling, the speed of sound is 1, hence the travelling waves are subsonic. In the transonic limit $\sigma \simeq 1$, thus we set $\sigma^2 = 1 - \varepsilon^2$, $\varepsilon > 0$ small, and we obtain

$$U_\sigma(z) = -i\varepsilon\,\mathrm{th}(\varepsilon z) + \sqrt{1-\varepsilon^2} = \sqrt{1 - \frac{\varepsilon^2}{\mathrm{ch}^2(\varepsilon z)}}\,\exp\big(i\varepsilon\varphi^\varepsilon(\varepsilon z)\big),$$

with $\varphi^\varepsilon(\varepsilon z) = -\mathrm{th}(\varepsilon z) + \mathcal{O}(\varepsilon^3)$, and we see that this corresponds to the ansatz (5) as $\varepsilon \to 0$. Furthermore, here, $A^\varepsilon = -1/\mathrm{ch}^2$ does not depend on ε and is the soliton of the KdV equation ($c = 1$, $k = 6$). Note that (11) is also often adimensionalized in the form

$$-i\sigma\partial_z U + \partial_{zz} U = U(|U|^2 - 1).$$

In this case, the critical speed one which is the sound speed, is changed for $\sqrt{2}$.

In higher dimensions $n = 2$, 3, the convergence of the travelling waves to the Gross-Pitaevskii equation (*i.e.* (NLS) with $f(R) = R - 1$) with speed $\simeq 1$ to a soliton of the KP-I equation is formally derived in the paper [15], while in [3], this KP-I asymptotic regime for (NLS) in dimension $n = 3$ is used to investigate the linear instability of the solitary waves of speed $\simeq 1$. On the mathematical level, in dimension $n = 2$, the convergence of the travelling waves of speed $\simeq 1$ for the Gross-Pitaevskii equation to a ground state of the KP-I equation is proved in [5].

Here, we shall study the rigorous derivation of KdV/KP-I from (NLS) for arbitrary time dependent solutions. All our results are in particular valid for the Gross-Pitaevskii equation $f(R) = R-1$.

When $n = 1$, there are global in time solutions of (NLS) in the energy space and we shall prove that the smallness of the energy prevents Ψ from having zeros. This will allow us to justify the ansatz (5). By using the conservation of energy and momentum, we shall get directly that if the initial datum is well-prepared in the sense that $\|\partial_x\varphi_0^\varepsilon - 2cA_0^\varepsilon\|_{L^2(\mathbb{R})}/\varepsilon$ tends to zero (*cf.* (8)), then we can pass to the limit directly on arbitrarily large interval of time from the global solution of (NLS) in the energy space towards the solution of KdV in the energy space without assuming additional regularity of the initial data (Theorem 3). When the dimension is larger than one, it does not seem possible to prove that if Ψ has a small energy then Ψ does not vanish. As a first step, to deal with the higher dimensional case, we thus need to prove directly that there indeed exists a smooth solution of (4) under the form (5) with $(A^\varepsilon, \varphi^\varepsilon)$ bounded in H^s for s sufficiently large on an interval of time independent of ε. This existence result (Theorem 4) is established by recasting (4) as an hydrodynamical type system close to (6) but with a particular singular PDE limit structure as in [20], [12], [24]. Next, we shall justify the KdV/KP-I limit by using weak compactness arguments. For general initial data (*i.e.* "ill-prepared" data in the terminology of singular PDE limits), that is without assuming that $2cA^\varepsilon - \partial_x\varphi^\varepsilon$ tends to zero at the initial time (in order to be compatible with the constraint (8)), we are able to pass to the limit (Theorem 5) in a weak sense: the amplitude A^ε converges to the solution of the KdV/KP-I equation weakly in $L^2([0,T]\times\mathbb{R}^n)$. If the data are better prepared according to the constraint (8), we can justify the KdV/KP-I asymptotic limit with stronger convergences, namely pointwise in time and global strong in space (Theorem 6).

1.2 KdV asymptotic regime for (NLS) in the energy space

We first focus on the description of our result in the one dimensional case $n = 1$, and work only in the energy space for (NLS) and the H^1 energy space for KdV. The Cauchy problem for (NLS) is not standard because of the condition at infinity $|\Psi| \to 1$ (see [9], [27], [8]) which is expected in order to give a meaning to the energy $\mathcal{E}(\Psi)$. We have the following:

Theorem 1 ([27]) *There exists $\mathcal{E}_0 > 0$ such that for every $\Psi_0 \in H^1_{loc}(\mathbb{R})$ verifying $\mathcal{E}(\Psi_0) \leq \mathcal{E}_0$, and $|\Psi_0|(z) \to 1$ as $|z| \to +\infty$, there exists a unique solution Ψ to (NLS) such that $\Psi - \Psi_0 \in \mathcal{C}\big(\mathbb{R}_+, H^1(\mathbb{R})\big)$. Moreover, $\mathcal{E}\big(\Psi(t)\big) = \mathcal{E}(\Psi_0)$ for $t \geq 0$.*

This Theorem is not exactly formulated under this form in [27] (Theorem III.3.1). Nevertheless, as we shall see in Lemma 1, if $\mathcal{E}(\Psi) \leq \mathcal{E}_0$ is sufficiently small and $|\Psi| \to 1$ at infinity, then we can write $\Psi = \rho e^{i\phi}$ with

$$\|\partial_x\rho\|_{L^2(\mathbb{R})} + \|\rho - 1\|_{L^\infty(\mathbb{R})} + \|\partial_x\phi\|_{L^2(\mathbb{R})}$$

sufficiently small and hence we can indeed use [27] (Theorem III.3.1).

It is also known that the Cauchy problem for the KdV equation[1] [16] is well-posed in the energy space:

Theorem 2 ([16]) *We consider the Cauchy problem for the KdV equation*

$$2\partial_t v + k\, v\partial_x v - \frac{1}{4c^2}\,\partial_x^3 v = 0, \qquad v_{|t=0} = v_0.$$

If $v_0 \in H^1(\mathbb{R})$, then there exists a unique solution of the KdV equation satisfying $v \in \mathcal{C}_b\big(\mathbb{R}_+, H^1(\mathbb{R})\big)$ and $\partial_x v \in L^4_{loc}\big(\mathbb{R}_+, L^\infty(\mathbb{R})\big)$.

Note that it is possible to prove the well-posedness of KdV in spaces of much lower regularity than H^1 (see [17] for example) but we shall not use these results here.

Our first result relates the solution of (NLS) obtained in Theorem 1 in the scaling (3) and the solution of KdV obtained in Theorem 2:

Theorem 3 ($n = 1$) *Assume that $(A_0^\varepsilon)_{0<\varepsilon<1} \in H^1$ and $(\varphi_0^\varepsilon)_{0<\varepsilon<1} \in \dot{H}^1$ enjoy the uniform estimate*

$$M \equiv \sup_{0<\varepsilon<1} \Big\{ \big\|A_0^\varepsilon\big\|_{H^1(\mathbb{R})} + \frac{1}{\varepsilon}\big\|\partial_x\varphi_0^\varepsilon - 2cA_0^\varepsilon\big\|_{L^2(\mathbb{R})} \Big\} < +\infty \tag{12}$$

and that

$$A_0^\varepsilon \to A_0 \quad in \quad L^2(\mathbb{R}) \quad as \quad \varepsilon \to 0.$$

[1] Here, it might happen that $k = 0$, in which case the KdV equation reduces to the so-called (linear) Airy equation $2\partial_t v - \frac{1}{4c^2}\partial_x^3 v = 0$ and the Cauchy problem is then trivial to solve.

Consider the initial datum

$$\psi_0^\varepsilon = \big(1+\varepsilon^2 A_0^\varepsilon\big)\exp\big(i\varepsilon\varphi_0^\varepsilon\big) \tag{13}$$

for (4)*, and let* $\psi^\varepsilon \in \psi_0^\varepsilon + \mathcal{C}\big(\mathbb{R}_+, H^1(\mathbb{R})\big)$ *be the associated solution to* (4)*(given by Theorem 1).*

Then, there exists $\varepsilon_0 > 0$*, depending only on* M*, such that, for* $0 < \varepsilon \leq \varepsilon_0$*, there exist two real-valued functions* φ^ε*,* $A^\varepsilon \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$ *such that* $(A^\varepsilon, \varphi^\varepsilon)_{|t=0} = (A_0^\varepsilon, \varphi_0^\varepsilon)$*, and*

$$\psi^\varepsilon = \big(1+\varepsilon^2 A^\varepsilon\big)\exp\big(i\varepsilon\varphi^\varepsilon\big) \tag{14}$$

with $1+\varepsilon^2 A^\varepsilon \geq \frac{1}{2}$ *. Furthermore, as* $\varepsilon \to 0$*, we have the convergences*

$$A^\varepsilon \to A \quad \text{in} \quad \mathcal{C}\big([0,T], H^s(\mathbb{R})\big), \quad \partial_x\varphi^\varepsilon \to 2cA, \quad \text{in} \quad \mathcal{C}\big([0,T], L^2(\mathbb{R})\big)$$

for every $s < 1$ *and every* $T > 0$*, where* A *is the solution of KdV with initial value* A_0*.*

Note that the convergence holds for arbitrarily large interval of times $[0,T]$. Moreover, let us emphasize that the initial data are well-prepared (see (8)) in the sense that

$$\big\|\partial_x\varphi_0^\varepsilon - 2cA_0^\varepsilon\big\|_{L^2(\mathbb{R})} = \mathcal{O}(\varepsilon). \tag{15}$$

Under a stronger assumption on the preparedness of the initial data, namely

$$\big\|\partial_x\varphi_0^\varepsilon - 2cA_0^\varepsilon\big\|_{L^2(\mathbb{R})} = o(\varepsilon), \tag{16}$$

one can reach the convergence in H^1 for the amplitude (see Theorem 7 in Subsect. 2.5). This assumption will not be needed when we work with more regular data as in Theorem 4 below. Finally, note that the usual assumption of well-prepared data for a singular system (see [20] for example) like (6) in order to get that $\partial_t A^\varepsilon = \mathcal{O}(1)$ would be that

$$\big\|\partial_x\varphi_0^\varepsilon - 2cA_0^\varepsilon\big\|_{L^2(\mathbb{R})} = \mathcal{O}(\varepsilon^2).$$

Consequently, we note that our assumptions (15) and even (16) are weaker.

Related results have been obtained simultaneously in [6] for the Gross-Pitaevskii equation ($f(R) = R - 1$) by using different methods, namely the complete integrability of the equation through the conservation of higher order energies.

The strategy of the proof is as follows. By using the conservation of the energy and of the momentum

$$\mathcal{P} = \frac{1}{2}\int_{\mathbb{R}} \big(i\Psi, \partial_z\Psi\big)\, dz$$

(actually one of its variants since $\mathcal{P}$ is not well-defined for functions which tend to 1 at infinity), we shall prove that one can write

$$\psi^\varepsilon = \big(1+\varepsilon^2 A^\varepsilon\big)\exp\big(i\varepsilon\varphi^\varepsilon\big),$$

with $1+\varepsilon^2 A^\varepsilon \geq \frac{1}{2}$ and the uniform bounds

$$\sup_{0<\varepsilon<\varepsilon_0,\ t\in\mathbb{R}_+} \Big\{\big\|A^\varepsilon\big\|_{H^1(\mathbb{R})} + \frac{1}{\varepsilon}\big\|\partial_x\varphi^\varepsilon - 2cA^\varepsilon\big\|_{L^2(\mathbb{R})}\Big\} < +\infty.$$

The H^1 bound on A^ε will provide compactness locally in space. Then we shall get compactness in time by using the properties of the singular part of the equation (6) namely properties of the transport equation with high speeds

$$\begin{cases} \partial_t A^\varepsilon - \dfrac{1}{\varepsilon^2}\partial_x\big(A^\varepsilon - u^\varepsilon\big) = S_A^\varepsilon \\ \\ \partial_t u^\varepsilon - \dfrac{1}{\varepsilon^2}\partial_x\big(u^\varepsilon - A^\varepsilon\big) = S_u^\varepsilon. \end{cases} \tag{17}$$

This will allow to extract a subsequence which converges strongly in $L^2_{loc}(\mathbb{R}_+ \times \mathbb{R})$ towards the solution of the KdV equation. Finally we shall prove that we actually have a better convergence which is in particular global in space as stated in the theorem. This uses the conservation of the energy $\mathcal{E}$ for Ψ and the conservation of the L^2 norm for the solution of KdV.

1.3 KdV and KP-I asymptotic regimes for smooth initial data

In arbitrary dimension, we will work with H^s norms and local in time smooth solutions in $H^s(\mathbb{R}^n)$, with s sufficiently large. Our first result is:

Theorem 4 *Let $n \geq 1$ and let s be such that $s > 1 + \dfrac{n}{2}$. Assume that*

$$M_s \equiv \sup_{0<\varepsilon<1} \big\|\big(A_0^\varepsilon, \partial_x\varphi_0^\varepsilon, \varepsilon\nabla_\perp\varphi_0^\varepsilon\big)\big\|_{H^{s+1}(\mathbb{R}^n)} < +\infty \tag{18}$$

and consider the initial datum for (4)

$$\psi_0^\varepsilon = \big(1 + \varepsilon^2 A_0^\varepsilon\big)\exp\big(i\varepsilon\varphi_0^\varepsilon\big).$$

Then, there exist $T > 0$ and $\varepsilon_0 \in (0,1)$, depending on M_s, such that for every $\varepsilon \in (0, \varepsilon_0)$, there exists a unique solution ψ^ε to (4) *with $\psi^\varepsilon_{|t=0} = \psi_0^\varepsilon$ such that $\psi^\varepsilon - \psi_0^\varepsilon \in \mathcal{C}\big([0,T], H^{s+1}(\mathbb{R}^n)\big)$. Furthermore, there exist two real-valued functions $A^\varepsilon \in \mathcal{C}\big([0,T], H^{s+1}(\mathbb{R}^n)\big)$ and $\varphi^\varepsilon \in \mathcal{C}\big([0,T], \dot{H}^{s+1}(\mathbb{R}^n)\big) \cap \mathcal{C}\big([0,T]\times\mathbb{R}^n\big)$ such that $(A^\varepsilon, \varphi^\varepsilon)_{|t=0} = (A_0^\varepsilon, \varphi_0^\varepsilon)$ and, for $0 \leq t \leq T$,*

$$\psi^\varepsilon = \big(1 + \varepsilon^2 A^\varepsilon\big)\exp\big(i\varepsilon\varphi^\varepsilon\big), \qquad\qquad 1 + \varepsilon^2 A^\varepsilon \geq 1/2 \tag{19}$$

and

$$\sup_{0<\varepsilon<\varepsilon_0,\ t\in[0,T]} \Big\{\big\|A^\varepsilon\big\|_{H^{s+1}(\mathbb{R}^n)} + \big\|\big(\partial_x\varphi^\varepsilon, \varepsilon\nabla_\perp\varphi^\varepsilon\big)\big\|_{H^s(\mathbb{R}^n)}\Big\} < +\infty. \tag{20}$$

The important result in Theorem 4 is the qualitative information that there exists a uniform time T for which the representation (19) and the uniform bounds (20) hold.

To prove Theorem 4 we shall rewrite (4) as a hydrodynamical equation. As in [11], we shall use a modified Madelung transform where we allow the amplitude to be complex. This allows to get an hydrodynamic system with a much simpler structure than (6). It is a first order hyperbolic system with a singular perturbation made of a skew-symmetric zero order term and a skew-symmetric second order term. The uniform time existence for the obtained system will then follow from uniform H^s estimates as in the works [20], [11], [24].

In the recent work [4], the linear wave regime for the Gross-Pitaevskii equation is investigated. This regime occurs for larger data on a shorter time. In this regime the equivalent of Theorem 4 is obtained in [4]. The proof in [4] is different from ours since the uniform bounds are obtained through the study of a different hydrodynamical system (namely the one obtained by the standard Madelung transform).

The next step will be the study of the convergence towards solutions of the KP-I equation of the solutions constructed in Theorem 4. Note that for an initial datum in H^s with $s > 1+n/2$, the Cauchy problem for the KP-I equation is well-posed: there exists a unique local in time H^s weak solution. Note that it is actually known to be well-posed in spaces of much lower regularity [14], [22]. Moreover, in dimension $n = 2$, the solutions are global in time whereas in dimension $n = 3$, the solution of KP-I may blow-up (in H^1) in finite time (see [21]).

Our first convergence result is:

Theorem 5 *Under the assumptions of Theorem 4 and the additional assumption*

$$(A_0^\varepsilon, \partial_x \varphi_0^\varepsilon) \to (A_0, \partial_x \varphi_0) \qquad \text{in} \quad L^2(\mathbb{R}^n) \quad \text{as } \varepsilon \to 0, \tag{21}$$

let A be the solution of the KdV/KP-I equation

$$\partial_x \Big(2\partial_t A + kA\partial_x A - \frac{1}{4c^2}\partial_x^3 A \Big) + \Delta_\perp A = 0$$

with initial value $A_{|t=0} = \frac{1}{2}\big(A_0 + \frac{1}{2c}\partial_x \varphi_0\big) \in H^{s+1}(\mathbb{R}^n)$. Then, we have the weak convergences, as $\varepsilon \to 0$,

$$A^\varepsilon \rightharpoonup A \qquad \partial_x \varphi^\varepsilon \rightharpoonup 2cA \qquad \text{weakly in} \quad L^2\big([0,T] \times \mathbb{R}^n\big)$$

and, with the additional assumption

$$\varepsilon \nabla_\perp \varphi_0^\varepsilon \to 0 \qquad \text{in} \quad L^2(\mathbb{R}^n),$$

the strong convergence, for $\sigma < s$,

$$\frac{1}{2}\Big(A^\varepsilon + \frac{1}{2c}\partial_x \varphi^\varepsilon\Big) \to A \qquad \text{in} \qquad L^2\big([0,T], H^\sigma(\mathbb{R}^n)\big).$$

Note that the result of Theorem 5 holds for smooth but ill-prepared initial data in the sense that they do not satisfy the constraint (8). We shall actually get in the proof of Theorem 5 a stronger type of convergence. Namely, we get that $\partial_x A^\varepsilon$ and $(\partial_x^2 \varphi^\varepsilon)/2c$ converge strongly to $\partial_x A$ in $L^2_{loc}\big(0, T, H^\sigma_{loc}(\mathbb{R}^n)\big)$ for every $\sigma < s$ if $n \geq 2$ and that A^ε and $(\partial_x \varphi^\varepsilon)/2c$ converge strongly to A in $L^2_{loc}\big(0, T, H^{\sigma+1}_{loc}(\mathbb{R}^n)\big)$ if $n = 1$.

Finally, for slightly well prepared data, we are able to recover global strong convergence in space:

Theorem 6 *Under the assumptions* (18) *and* (21)*, and assuming moreover that*

$$\begin{aligned} &(n = 1) \qquad \big\|\partial_x \varphi_0^\varepsilon - 2cA_0^\varepsilon\big\|_{L^2(\mathbb{R})} \to 0 \quad \text{as} \quad \varepsilon \to 0, \\ &(n \geq 2) \qquad \big\|\partial_x \varphi_0^\varepsilon - 2cA_0^\varepsilon\big\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(\varepsilon), \quad \|\nabla_\perp \varphi_0^\varepsilon\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(1), \end{aligned} \tag{22}$$

then we have the convergences, as $\varepsilon \to 0$,

$$A^\varepsilon \to A \quad \text{strongly in } \mathcal{C}\big([0,T], H^\sigma(\mathbb{R}^n)\big), \qquad \partial_x \varphi^\varepsilon \to 2cA \quad \text{strongly in } \mathcal{C}\big([0,T], H^{\sigma-1}(\mathbb{R}^n)\big)$$

for every $\sigma < s+1$. *Furthermore, if* $n \geq 2$, *there exists* $K > 0$ *such that, for* $0 \leq t \leq T$, $0 < \varepsilon < \varepsilon_0$,

$$\int_{\mathbb{R}^n} |\nabla_\perp \varphi^\varepsilon|^2 \, dX \leq K. \tag{23}$$

We emphasize that in dimensions $n \geq 2$, the hypothesis in the last theorem is stronger than in dimension $n = 1$ in order to ensure the bound for $\int_{\mathbb{R}^n} |\nabla_\perp \varphi^\varepsilon|^2 \, dX$. Moreover, in dimension $n = 1$, (22) is weaker than the hypothesis in Theorem 3.

The paper is organized as follows. Section 2 is devoted to the proof of Theorem 3, and section 3.1 to the proof of Theorem 4. The proofs of Theorems 5, 6 are finally given in sections 3.2, 3.3.

2 Proof of Theorem 3

We shall split the proof in many steps. In the first step we prove that the modulus of a solution of (NLS) remains away from zero if its energy is sufficiently small so that it can be written as (14) and we prove that one can define a variant of the momentum which is well-defined. Then we shall use the energy and the momentum to get uniform $H^1 \times \dot{H}^1$ estimates for $(A^\varepsilon, \varphi^\varepsilon)$. The third step will be the study of the system (17) in order to get compactness in time. Finally, the last part will be devoted to the passage to the limit in the equation.

2.1 Preliminaries

For the regime of interest to us, the energy is small. In this case, we shall prove that the modulus $|\Psi|$ remains close to 1. A first useful remark is that since $F'(1) = 2f(1) = 0$ and $F''(1) = 2f'(1) = 2c^2 > 0$, we have for some $\delta \in (0, 1/2)$ sufficiently small,

$$F(R) \geq \frac{c^2}{2}(R-1)^2, \qquad |R-1| \leq \delta \tag{24}$$

and also

$$F(R) \leq C(R-1)^2, \qquad |R-1| \leq \delta \tag{25}$$

for some $C > 0$. Our first lemma shows that if the energy $\mathcal{E}(\Psi)$ is sufficiently small, then $|\Psi|^2$ remains uniformly in the interval $[1-\delta, 1+\delta]$ where $F(R) \simeq c^2(R-1)^2$.

Lemma 1 *For every* $\delta \in (0, 1/2)$ *as above, there exists* $\mathcal{E}_0 > 0$, *depending only on the nonlinearity* f, *such that if* $\Psi \in H^1_{loc}(\mathbb{R})$ *verifies* $\mathcal{E}(\Psi) < \mathcal{E}_0$ *and* $|\Psi|(z) \to 1$ *for* $z \to +\infty$, *then*

$$\big\| |\Psi|^2 - 1 \big\|_{L^\infty(\mathbb{R})} \leq \delta.$$

Note that for an initial value under the form (13), we have, since M is finite, that

$$\mathcal{E}(\psi_0^\varepsilon) \leq C\varepsilon \Big(\int_{\mathbb{R}} \varepsilon^4 (\partial_x A_0^\varepsilon)^2 + \varepsilon^2 (1 + \varepsilon^2 A_0^\varepsilon)^2 (\partial_x \varphi_0^\varepsilon)^2 + \varepsilon^2 (A_0^\varepsilon)^2 \, dx \Big) \leq C\varepsilon^3$$

where C depends only on M. Consequently, since the energy is conserved, we can indeed use Lemma 1 for ε sufficiently small to write the solution ψ^ε of NLS given by Theorem 1 under the form $\psi^\varepsilon = \rho e^{i\phi}$ with ρ, $\phi \in H^1_{loc}$ and $|\rho^2 - 1| \leq 1/2$. The functions ρ and ϕ are indeed in $H^1_{loc}(\mathbb{R})$ as can be easily seen using the continuity of $\psi^\varepsilon : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{C}^*$ and local complex logarithms. Note that ρ and ϕ depend on ε but we omit this dependence in our notation.

Proof of Lemma 1. Since $|\Psi|(z) \to 1$ for $z \to +\infty$, we have

$$\forall z \in \mathbb{R}, \qquad \big(|\Psi|^2(z) - 1\big)^2 = -4 \int_z^{+\infty} |\Psi|\big(|\Psi|^2 - 1\big)\partial_z|\Psi|, \tag{26}$$

and we can define the maximal interval $I = [a, +\infty)$ such that $\big||\Psi|^2 - 1\big| \leq \delta$ in I. Then,

$$\int_I \big|\partial_z|\Psi|\big|^2 + \frac{c^2}{2}\big(|\Psi|^2 - 1\big)^2 \, dz \leq \int_{\mathbb{R}} |\partial_z\Psi|^2 + F\big(|\Psi|^2\big) \, dz = 2\mathcal{E}(\Psi).$$

As a consequence, by (26) and Cauchy-Schwarz,

$$\big\| \, |\Psi|^2 - 1\big\|^2_{L^\infty(I)} \leq 4\sqrt{1+\delta} \| \, |\Psi^2| - 1\|_{L^2(I)} \cdot \big\|\partial_z|\Psi|\big\|_{L^2(I)} \leq K_0\mathcal{E}(\Psi),$$

where K_0 depends only on f. The result follows from an easy continuation argument, taking $\mathcal{E}_0 \equiv \delta^2/K_0$. $\square$

Next, we recall that the Schrödinger flow also formally preserves the *momentum*, that should be defined by

$$\mathcal{P} = \frac{1}{2}\int_{\mathbb{R}} \big(i\Psi, \partial_z\Psi\big) \, dz.$$

However, this quantity does not make sense as a Lebesgue integral for a map Ψ which is just of finite energy with $|\Psi| \to 1$ at infinity. Notice that if $\Psi = \rho \exp(i\phi)$, then

$$\mathcal{P} = \frac{1}{2}\int_{\mathbb{R}} \rho^2\partial_z\phi \, dz.$$

Variants of the momentum $\mathcal{P}$ are also formally conserved by the Schrödinger equation (NLS), namely

$$\frac{1}{2}\int_{\mathbb{R}} \big(i(\Psi - 1), \partial_z\Psi\big) \, dz \qquad \text{if} \quad \Psi \to 1 \quad \text{at infinity}$$

and

$$\frac{1}{2}\int_{\mathbb{R}} \big(\rho^2 - 1\big)\partial_z\phi \, dz.$$

This last integral has the advantage to be a Lebesgue integral if $\Psi \in H^1_{loc}(\mathbb{R})$ satisfies

$$\mathcal{E}(\Psi) < +\infty, \quad |\Psi|(x) \to 1 \;\text{ as } x \to +\infty \quad \text{ and } \quad \big|\,|\Psi|^2 - 1\big| \leq \delta,$$

since then

$$\frac{1}{2}\int_{\mathbb{R}} \big(\rho^2 - 1\big)\partial_z\phi \, dz = \frac{1}{2}\int_{\mathbb{R}} \big(|\Psi|^2 - 1\big)\frac{\mathrm{Im}(\partial_z\Psi)}{|\Psi|} \, dz.$$

As we have seen in the remark after Lemma 1, in our regime, the map ψ^ε satisfy the bound $\| \, |\psi^\varepsilon|^2 - 1 \|_{L^\infty(\mathbb{R})} \leq \delta$ and hence, we have a well-defined momentum, if we take this last definition.

Finally, in view of the scaling (3), it is usefull to introduce a rescaled version of the energy:

$$
\begin{aligned}
E^\varepsilon(\psi^\varepsilon) &= \frac{\mathcal{E}(\Psi)}{\varepsilon} = \frac{1}{2}\int_{\mathbb{R}} |\partial_x \psi^\varepsilon|^2 + \frac{1}{\varepsilon^2} F(|\psi^\varepsilon|^2)\, dx \\
&= \frac{1}{2}\int_{\mathbb{R}} |\partial_x \rho|^2 + \rho^2 |\partial_x \phi|^2 + \frac{1}{\varepsilon^2} F(\rho^2)\, dx \qquad (27)
\end{aligned}
$$

since $\psi^\varepsilon = \rho e^{i\phi}$. In a similar way, we define a rescaled momentum

$$
P^\varepsilon(\psi^\varepsilon) \equiv \frac{\varepsilon}{2}\int_{\mathbb{R}} \big(\rho^2 - 1\big)\partial_x \phi \; dx. \qquad (28)
$$

Note that both quantities are conserved.

2.2 Uniform estimates

We shall prove the following:

Lemma 2 *Under the assumptions of Theorem 3, there exists $\varepsilon_0 > 0$, depending only on M, such that, for $0 < \varepsilon \leq \varepsilon_0$, there exist two real-valued functions φ^ε, $A^\varepsilon \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$ such that $(A^\varepsilon, \varphi^\varepsilon)_{|t=0} = (A_0^\varepsilon, \varphi_0^\varepsilon)$,*

$$
\psi^\varepsilon = \big(1 + \varepsilon^2 A^\varepsilon\big) \exp\big(i\varepsilon\varphi^\varepsilon\big), \qquad\qquad 1 + \varepsilon^2 A^\varepsilon \geq \frac{1}{2}
$$

and

$$
\sup_{0<\varepsilon<\varepsilon_0,\ t\in\mathbb{R}_+} \Big\{ \|A^\varepsilon\|_{H^1(\mathbb{R})} + \frac{1}{\varepsilon}\|\partial_x \varphi^\varepsilon - 2cA^\varepsilon\|_{L^2(\mathbb{R})} \Big\} < +\infty.
$$

Proof of Lemma 2. The proof relies on the use of the conservation of E^ε and P^ε as noticed in [5]. In particular, the quantity $E^\varepsilon - 2cP^\varepsilon$ gives valuable information.

As we have already noticed in the remark after Lemma 1, we can write $\psi^\varepsilon = \rho \exp(i\phi)$ for some continuous real-valued functions $\rho \geq 1/2$ and ϕ in $H^1_{loc}(\mathbb{R})$ satisfying $\rho_{[|t=0} = 1 + \varepsilon^2 A_0^\varepsilon$ and $\phi_{|t=0} = \varepsilon\varphi_0^\varepsilon$. Note that

$$
|\partial_x \psi^\varepsilon|^2 = (\partial_x \rho)^2 + \rho^2 (\partial_x \phi)^2.
$$

Next, we set

$$
F(R) = c^2\big(R-1\big)^2 + F_3(R), \quad \text{with} \quad F_3(1+r) = \mathcal{O}(r^3), \quad r \to 0.
$$

By using (27) and (28), this yields

$$
E^\varepsilon(\psi^\varepsilon) = \frac{1}{2}\int_{\mathbb{R}} (\partial_x\phi)^2 + \frac{c^2}{\varepsilon^2}\big(\rho^2 - 1\big)^2 + (\rho^2 - 1)\cdot(\partial_x \phi)^2 + (\partial_x\rho)^2 + \frac{1}{\varepsilon^2}\, F_3(\rho^2)\; dx \qquad (29)
$$

and

$$
E^\varepsilon(\psi^\varepsilon) - 2cP^\varepsilon(\psi^\varepsilon) = \frac{1}{2}\int_{\mathbb{R}} \big(\rho^2 - 1\big)(\partial_x\phi)^2 + (\partial_x \rho)^2 + \Big(\partial_x \phi - \frac{c}{\varepsilon}(\rho^2 - 1)\Big)^2 + \frac{1}{\varepsilon^2}\, F_3\big(\rho^2\big)\; dx, \quad (30)
$$

where we have used the identity

$$(\partial_x\phi)^2 + \frac{c^2}{\varepsilon^2}(\rho^2-1)^2 - \frac{2c}{\varepsilon}(\rho^2-1)\partial_x\phi = \Big(\partial_x\phi - \frac{c}{\varepsilon}(\rho^2-1)\Big)^2.$$

The proof of Lemma 2 is divided in 3 Steps. In the proof, K stands for a constant depending only on f and M.

Step 1: We first prove the following expansions for $E^\varepsilon(\psi_0^\varepsilon)$ and $E^\varepsilon(\psi_0^\varepsilon) - 2cP^\varepsilon(\psi_0^\varepsilon)$ as $\varepsilon \to 0$:

$$E^\varepsilon(\psi_0^\varepsilon) = \frac{\varepsilon^2}{2}\int_{\mathbb{R}} 4c^2(A_0^\varepsilon)^2 + (\partial_x\varphi_0^\varepsilon)^2\, dx + \mathcal{O}(\varepsilon^4) = 4c^2\varepsilon^2\int_{\mathbb{R}} A_0^2\, dx + o(\varepsilon^2) + \mathcal{O}(\varepsilon^4)$$

and

$$E^\varepsilon(\psi_0^\varepsilon) - 2cP^\varepsilon(\psi_0^\varepsilon) \le K\varepsilon^4.$$

This follows from (29) and (30) with $\rho = 1+\varepsilon^2 A_0^\varepsilon$ and $\phi = \varepsilon\varphi_0^\varepsilon$. Indeed, from the uniform bound in H^1 for A_0^ε, we immediately infer by Sobolev embedding $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$ that $\|A_0^\varepsilon\|_{L^\infty} \le K$ and $|\psi_0^\varepsilon| = |1+\varepsilon^2 A_0^\varepsilon| \in [1/2, 2]$ for $0 < \varepsilon < \varepsilon_0$ sufficiently small, depending on M. Moreover, $\rho^2 - 1 = 2\varepsilon^2 A_0^\varepsilon + \mathcal{O}_{L^\infty(\mathbb{R})}(\varepsilon^4)$. Since $|F_3(R)| \le K|R-1|^3$ for $0 \le R \le 2$, we have $|F_3(\rho^2)| \le K\varepsilon^6(A_0^\varepsilon)^2$, and the expansion for the energy follows. Concerning the expansion for $E_\varepsilon(\psi_0^\varepsilon) - 2cP_\varepsilon(\psi_0^\varepsilon)$, it suffices to use the assumption $\|\partial_x\varphi_0^\varepsilon - 2cA_0^\varepsilon\|_{L^2(\mathbb{R})}^2 \le M^2\varepsilon^2$.

Step 2: We shall prove that for every $t \in \mathbb{R}_+$,

$$\big\|\rho^2 - 1\big\|_{L^\infty(\mathbb{R})} \le K\varepsilon^2.$$

This will be a consequence of the conservation of energy and momentum. Let $t \in \mathbb{R}_+$. We first infer from (30) a better estimate for $\displaystyle\int_{\mathbb{R}} (\partial_x\rho)^2\, dx$. Since $\rho \ge 1/2$, we have, on the one hand,

$$\Big|\int_{\mathbb{R}} (\rho^2-1)(\partial_x\phi)^2\, dx\Big| \le 4\big\|\rho^2-1\big\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}} \rho^2(\partial_x\phi)^2\, dx \le K\varepsilon^2\big\|\rho^2-1\big\|_{L^\infty(\mathbb{R})}, \tag{31}$$

and on the other hand, in view of $|\rho^2 - 1| \le \delta$, $F_3(1+r) = \mathcal{O}(r^3)$ as $r \to 0$ there holds

$$\Big|\int_{\mathbb{R}} \frac{1}{\varepsilon^2} F_3\big(\rho^2\big)\, dx\Big| \le K\big\|\rho^2-1\big\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}} \frac{1}{\varepsilon^2}\big(\rho^2-1\big)^2\, dx \le K\varepsilon^2\big\|\rho^2-1\big\|_{L^\infty(\mathbb{R})}. \tag{32}$$

Since E^ε and P^ε do not depend on time, inserting (31) and (32) into (30) yields

$$\begin{aligned} K\varepsilon^4 \ge E^\varepsilon(\psi^\varepsilon) - 2cP^\varepsilon(\psi^\varepsilon) &\ge \frac{1}{2}\int_{\mathbb{R}} (\partial_x\rho)^2\, dx - \Big|\int_{\mathbb{R}} (\rho^2-1)(\partial_x\phi)^2\, dx\Big| - \Big|\int_{\mathbb{R}} \frac{1}{\varepsilon^2} F_3\big(\rho^2\big)\, dx\Big| \\ &\ge \frac{1}{2}\int_{\mathbb{R}} (\partial_x\rho)^2\, dx - K\varepsilon^2\big\|\rho^2-1\big\|_{L^\infty(\mathbb{R})}, \end{aligned}$$

so that

$$\int_{\mathbb{R}} (\partial_x \rho)^2 \ dx \leq K\varepsilon^4 + K\varepsilon^2 \big\| \rho^2 - 1 \big\|_{L^\infty(\mathbb{R})}. \tag{33}$$

We now write, since $\rho = |\psi^\varepsilon| \to 1$ as $|x| \to +\infty$,

$$\big(\rho^2 - 1\big)^2(x) = -4 \int_x^{+\infty} \rho\big(\rho^2 - 1\big)\partial_x \rho \leq C\varepsilon \sqrt{E_\varepsilon(\psi^\varepsilon)} \Big(\int_{\mathbb{R}} (\partial_x \rho)^2 \ dx \Big)^{1/2}$$

by Cauchy-Schwarz inequality. From the above estimate (33) and letting

$$\eta_\varepsilon \equiv \frac{1}{\varepsilon^2} \big\| \rho^2 - 1 \big\|_{L^\infty(\mathbb{R})},$$

we obtain

$$\varepsilon^4 \eta_\varepsilon^2 \leq K\varepsilon^2 \sqrt{\varepsilon^4 + \varepsilon^4 \eta_\varepsilon},$$

that is

$$\eta_\varepsilon^2 \leq K\sqrt{1 + \eta_\varepsilon}.$$

This estimate provides immediately the result

$$\eta_\varepsilon = \frac{1}{\varepsilon^2} \big\| \rho^2 - 1 \big\|_{L^\infty(\mathbb{R})} \leq K.$$

We then set

$$A^\varepsilon \equiv \frac{1}{\varepsilon^2}\,(\rho - 1) \qquad \text{and} \qquad \varphi^\varepsilon \equiv \frac{\phi}{\varepsilon}.$$

Step 3: We finally prove that

$$\|A^\varepsilon\|_{H^1(\mathbb{R})} \leq K, \qquad \|\partial_x \varphi^\varepsilon\|_{L^2(\mathbb{R})} \leq K \qquad \text{and} \qquad \|2cA^\varepsilon - \partial_x \varphi^\varepsilon\|_{L^2(\mathbb{R})} \leq K\varepsilon. \tag{34}$$

Indeed, from Step 2, (31) and (32) imply

$$\Big| \int_{\mathbb{R}} (\rho^2 - 1)(\partial_x \phi)^2 \ dx \Big| \leq K\varepsilon^4 \qquad \text{and} \qquad \Big| \int_{\mathbb{R}} \frac{1}{\varepsilon^2}\, F_3\big(\rho^2\big) \ dx \Big| \leq K\varepsilon^4.$$

Inserting this into (30) gives

$$\Big\| \frac{1}{\varepsilon^2}(\rho^2 - 1) \Big\|_{H^1(\mathbb{R})} \leq K, \qquad \int_{\mathbb{R}} (\partial_x \phi)^2 \ dx \leq K\varepsilon^2 \qquad \text{and} \qquad \int_{\mathbb{R}} \Big(\partial_x \phi - \frac{c}{\varepsilon}(\rho^2 - 1) \Big)^2 \ dx \leq K\varepsilon^4$$

and the conclusion follows. This finishes the proof of the Lemma. □

2.3 Properties of the wave operator

In the previous subsection, we have obtained uniform bounds which will provide (local) compactness in space. We shall try now to obtain some compactness in time.

Lemma 3 *Consider $(A^\varepsilon(t,x), u^\varepsilon(t,x))$ a solution of the system*

$$\begin{cases} \partial_t A^\varepsilon - \dfrac{1}{\varepsilon^2}\partial_x\big(A^\varepsilon - u^\varepsilon\big) = S_A^\varepsilon \\[2ex] \partial_t u^\varepsilon - \dfrac{1}{\varepsilon^2}\partial_x\big(u^\varepsilon - A^\varepsilon\big) = S_u^\varepsilon, \end{cases} \tag{35}$$

with initial data

$$A^\varepsilon_{|t=0} = A_0^\varepsilon, \quad u^\varepsilon_{|t=0} = u_0^\varepsilon$$

and assume that, for some $\sigma \in \mathbb{N}$,

i) $(A_0^\varepsilon)_{0<\varepsilon<1}$ and $(u_0^\varepsilon)_{0<\varepsilon<1}$ are uniformly bounded in $L^2(\mathbb{R})$;

ii) $(S_A^\varepsilon)_{0<\varepsilon<1}$ and $(S_u^\varepsilon)_{0<\varepsilon<1}$ are uniformly bounded in $L^\infty\big(\mathbb{R}_+, H^{-\sigma}(\mathbb{R})\big)$.

Then, for every $T>0$, $R>0$,

$$(A^\varepsilon)_{0<\varepsilon<1} \text{ and } (u^\varepsilon)_{0<\varepsilon<1} \text{ are uniformly bounded in } H^{\frac{1}{2}}\big((0,T), H^{-\sigma-1}(-R,R)\big).$$

Proof of Lemma 3. These bounds come from the fact that the speed $\dfrac{1}{\varepsilon^2}$ of the characteristics of the transport equation is extremely large compared to the size of the space domain $(-R,R)$.

We start the proof of Lemma 3 with the following lemma, where we take into account only the initial data, and not the source terms.

Lemma 4 *Consider $(A^\varepsilon(t,x), u^\varepsilon(t,x))$ a solution of the system*

$$\begin{cases} \partial_t A^\varepsilon - \dfrac{1}{\varepsilon^2}\partial_x\Big(A^\varepsilon - u^\varepsilon\Big) = 0 \\[2ex] \partial_t u^\varepsilon - \dfrac{1}{\varepsilon^2}\partial_x\Big(u^\varepsilon - A^\varepsilon\Big) = 0, \end{cases}$$

with initial data

$$A^\varepsilon_{|t=0} = A_0^\varepsilon, \quad u^\varepsilon_{|t=0} = u_0^\varepsilon.$$

Assume that $(A_0^\varepsilon)_{0<\varepsilon<1}$, $(u_0^\varepsilon)_{0<\varepsilon<1}$ are uniformly bounded in $L^2(\mathbb{R})$. Then for every $T>0$, $R>0$, A^ε and u^ε are uniformly bounded in $H^{\frac{1}{2}}\big((0,T), H^{-1}(-R,R)\big)$.

Proof of Lemma 4. At first, we notice that

$$\partial_t\big(A^\varepsilon + u^\varepsilon\big) = 0$$

and that

$$\partial_t\big(A^\varepsilon - u^\varepsilon\big) - \frac{2}{\varepsilon^2}\partial_x\big(A^\varepsilon - u^\varepsilon\big) = 0. \tag{36}$$

The resolution of these transport equations gives

$$A^\varepsilon(t,x) + u^\varepsilon(t,x) = A_0^\varepsilon(x) + u_0^\varepsilon(x)$$

and

$$A^\varepsilon(t,x) - u^\varepsilon(t,x) = A_0^\varepsilon(x+2\varepsilon^{-2}t) - u_0^\varepsilon(x+2\varepsilon^{-2}t). \tag{37}$$

This immediately yields that

$$A^\varepsilon + u^\varepsilon \text{ is uniformly bounded in } H^1\big((0,T),L^2(\mathbb{R})\big) \tag{38}$$

and hence by continuous injection, it is in particular bounded in $H^{\frac{1}{2}}\big((0,T),H^{-1}(-R,R)\big)$.

Next, we shall study $A^\varepsilon - u^\varepsilon$. From the explicit expression (37), we first get that

$$\int_0^T \int_{-R}^R \big|A^\varepsilon - u^\varepsilon\big|^2(t,x)\,dxdt = \int_0^T \int_{-R}^R \big|A_0^\varepsilon - u_0^\varepsilon\big|^2(x+2\varepsilon^{-2}t)\,dxdt.$$

Consequently, by using Fubini Theorem and then changing the variable t into $\tau = x + 2\varepsilon^{-2}t$, we get

$$\int_0^T \int_{-R}^R \big|A^\varepsilon - u^\varepsilon\big|^2(t,x)\,dxdt \le \frac{\varepsilon^2}{2}\int_{-R}^R \|A_0^\varepsilon - u_0^\varepsilon\|^2_{L^2(\mathbb{R})}\,dx \le CR\varepsilon^2. \tag{39}$$

In the proof, C denotes a constant depending on R and the uniform bounds for $(A_0^\varepsilon)_{0<\varepsilon<1}$ and $(u_0^\varepsilon)_{0<\varepsilon<1}$ in L^2. We have thus in particular proven the uniform bound

$$\big\|A^\varepsilon - u^\varepsilon\big\|_{L^2((0,T),H^{-1}(-R,R))} \le C\varepsilon. \tag{40}$$

To estimate the time derivative, it suffices to remark that (36) yields

$$\big\|\partial_t\big(A^\varepsilon - u^\varepsilon\big)(t,\cdot)\big\|_{H^{-1}(-R,R)} = \frac{2}{\varepsilon^2}\big\|\partial_x\big(A^\varepsilon - u^\varepsilon\big)(t,\cdot)\big\|_{H^{-1}(-R,R)} \le \frac{2}{\varepsilon^2}\big\|\big(A^\varepsilon - u^\varepsilon\big)(t,\cdot)\big\|_{L^2(-R,R)}.$$

Hence, taking the L^2 norm in time and using (39) gives

$$\big\|\partial_t(A^\varepsilon - u^\varepsilon)\big\|_{L^2((0,T),H^{-1}(-R,R))} \le \frac{C}{\varepsilon}. \tag{41}$$

Interpolating in time between (40) and (41), we deduce

$$\big\|A^\varepsilon - u^\varepsilon\big\|_{H^{\frac{1}{2}}((0,T),H^{-1}(-R,R))} \le C. \tag{42}$$

The combination of (38) and (42) ends the proof. $\square$

We shall now give the proof of Lemma 3. Since the system (35) is linear, we can write its solution as the sum of the solution of the homogeneous system and the solution of the nonhomogeneous system with zero initial data. Thanks to Lemma 4, we already know that the first term is uniformly bounded in $H^{\frac{1}{2}}\big((0,T),H^{-1}_{loc}\big)$ and hence in $H^{\frac{1}{2}}\big((0,T),H^{-\sigma-1}_{loc}\big)$. Consequently, we can focus on the second term. This means that we consider the solution of (35) with zero initial value.

We notice that

$$\partial_t(A^\varepsilon + u^\varepsilon) = S_A^\varepsilon + S_u^\varepsilon,$$

and we recall that the initial values are zero. Hence,

$$\big(A^\varepsilon + u^\varepsilon\big)(t) = \int_0^t \big(S_A^\varepsilon + S_u^\varepsilon\big)(s)\ ds,$$

thus we immediately get that

$$A^\varepsilon + u^\varepsilon \text{ is uniformly bounded in } H^1\big((0,T), H^{-\sigma}(\mathbb{R})\big). \tag{43}$$

Similarly, since $A^\varepsilon - u^\varepsilon$ solves

$$\partial_t\big(A^\varepsilon - u^\varepsilon\big) - \frac{2}{\varepsilon^2}\partial_x\big(A^\varepsilon - u^\varepsilon\big) = S_A^\varepsilon - S_u^\varepsilon \tag{44}$$

with zero initial value, we infer

$$(A^\varepsilon - u^\varepsilon)(t,x) = \int_0^t \big(S_A^\varepsilon - S_u^\varepsilon\big)\big(s, x + 2\varepsilon^{-2}(t-s)\big)ds. \tag{45}$$

By assumption $ii)$, $S_A^\varepsilon - S_u^\varepsilon$ is uniformly bounded in $L^\infty\big(\mathbb{R}_+, H^{-\sigma}(\mathbb{R})\big)$, hence, using a standard characterization of $H^{-\sigma}(\mathbb{R})$, $\sigma \in \mathbb{N}$, there exists $g^\varepsilon = (g_0^\varepsilon, g_1^\varepsilon, ..., g_\sigma^\varepsilon) \in L^\infty\big(\mathbb{R}_+, L^2(\mathbb{R}, \mathbb{R}^{\sigma+1})\big)$ such that

$$S_A^\varepsilon - S_u^\varepsilon = \sum_{j=0}^{\sigma} \partial_x^j g_j^\varepsilon \qquad \text{with} \qquad \sup_{0<\varepsilon<1, 0\le j\le\sigma} \|g_j^\varepsilon\|_{L^\infty(\mathbb{R}_+, L^2(I,\mathbb{R}))} = C < \infty.$$

Furthermore, for almost every t and any interval I,

$$\big\|\big(S_A^\varepsilon - S_u^\varepsilon\big)(t,\cdot)\big\|_{H^{-\sigma}(I)} \le \|g^\varepsilon\|_{L^2(I,\mathbb{R}^{\sigma+1})} \le C.$$

Here, C stands for a constant depending on R, T and the uniform bounds for $(A_0^\varepsilon, u_0^\varepsilon)_{0<\varepsilon<1}$ and $(S_A^\varepsilon, S_u^\varepsilon)_{0<\varepsilon<1}$ in $H^{-\sigma}(\mathbb{R})$. As a consequence, we get from (45) that

$$\begin{aligned}
\big\|\big(A^\varepsilon - u^\varepsilon\big)(t,\cdot)\big\|^2_{H^{-\sigma}(-R,R)} &\le \Big(\int_0^t \big\|g^\varepsilon\big(s, \cdot + 2\varepsilon^{-2}(t-s)\big)\big\|_{L^2([-R,R],\mathbb{R}^{\sigma+1})}\, ds\Big)^2 \\
&\le t\int_0^t \big\|g^\varepsilon\big(s, \cdot + 2\varepsilon^{-2}(t-s)\big)\big\|^2_{L^2([-R,R],\mathbb{R}^{\sigma+1})}\, ds
\end{aligned}$$

and hence that

$$\int_0^T \big\|\big(A^\varepsilon - u^\varepsilon\big)(t,\cdot)\big\|^2_{H^{-\sigma}(-R,R)}\, dt \le T\int_0^T \int_0^t \int_{-R}^R \big|g^\varepsilon\big|^2\big(s, x + 2\varepsilon^{-2}(t-s)\big)\, dxdsdt,$$

which we can rewrite, by using Fubini Theorem, as:

$$\int_0^T \big\|\big(A^\varepsilon - u^\varepsilon\big)(t,\cdot)\big\|^2_{H^{-\sigma}(-R,R)}\, dt \le T\int_{-R}^R \int_0^T \int_s^T \big|g^\varepsilon\big|^2\big(s, x + 2\varepsilon^{-2}(t-s)\big)\, dtdsdx.$$

By changing t into $\tau = x + 2\varepsilon^{-2}(t-s)$, this yields

$$\int_0^T \big\|\big(A^\varepsilon - u^\varepsilon\big)(t,\cdot)\big\|^2_{H^{-\sigma}(-R,R)}\, dt \le \frac{1}{2}T\varepsilon^2 \int_{-R}^R \int_0^T \|g^\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})}\, dsdx \le C\varepsilon^2.$$

We have thus proven that

$$\left\|A^\varepsilon - u^\varepsilon\right\|_{L^2((0,T),H^{-\sigma}(-R,R))} \leq C\varepsilon, \tag{46}$$

which implies in particular that

$$\left\|A^\varepsilon - u^\varepsilon\right\|_{L^2((0,T),H^{-\sigma-1}(-R,R))} \leq C\varepsilon. \tag{47}$$

To estimate $\partial_t(A^\varepsilon - u^\varepsilon)$, we infer from (44)

$$\begin{aligned}
\left\|\partial_t\big(A^\varepsilon - u^\varepsilon\big)\right\|_{H^{-\sigma-1}(-R,R)} &\leq \frac{2}{\varepsilon^2}\left\|\partial_x\big(A^\varepsilon - u^\varepsilon\big)\right\|_{H^{-\sigma-1}(-R,R)} + \left\|S_A^\varepsilon - S_u^\varepsilon\right\|_{H^{-\sigma-1}(-R,R)} \\
&\leq \frac{2}{\varepsilon^2}\left\|A^\varepsilon - u^\varepsilon\right\|_{H^{-\sigma}(-R,R)} + C,
\end{aligned}$$

which yields, for $0 < \varepsilon < 1$ and in view of (46),

$$\left\|\partial_t\big(A^\varepsilon - u^\varepsilon\big)\right\|_{L^2((0,T),H^{-\sigma-1}(-R,R))} \leq \frac{C}{\varepsilon}. \tag{48}$$

Interpolation in time between (47) and (48) yields

$$\left\|A^\varepsilon - u^\varepsilon\right\|_{H^{\frac{1}{2}}((0,T),H^{-\sigma-1}(-R,R))} \leq C. \tag{49}$$

To end the proof, it suffices to combine (43) and (49). $\square$

2.4 End of the proof of Theorem 3

Since $\rho^\varepsilon = 1 + \varepsilon^2 A^\varepsilon \geq 1/2$ in $\mathbb{R}_+ \times \mathbb{R}$ for $0 < \varepsilon < \varepsilon_0$, we may then rewrite (4) under the form (6). In dimension 1, this reads

$$\left\{\begin{aligned}
&\varepsilon^2 c\partial_t A^\varepsilon - c\partial_x A^\varepsilon + \varepsilon^2\partial_x A^\varepsilon\partial_x\varphi^\varepsilon + \frac{1}{2}\big(1+\varepsilon^2 A^\varepsilon\big)\partial_x^2\varphi^\varepsilon = 0 \\
&\varepsilon^2 c\partial_t\varphi^\varepsilon - c\partial_x\varphi^\varepsilon - \varepsilon^2\frac{\partial_x^2 A^\varepsilon}{2\big(1+\varepsilon^2 A^\varepsilon\big)} + \frac{\varepsilon^2}{2}\big(\partial_x\varphi^\varepsilon\big)^2 + \frac{1}{\varepsilon^2}\, f\big((1+\varepsilon^2 A^\varepsilon)^2\big) = 0,
\end{aligned}\right. \tag{50}$$

and we wish to pass to the limit as $\varepsilon \to 0$. Let us define

$$u^\varepsilon \equiv \frac{1}{2c}\partial_x\varphi^\varepsilon.$$

We shall first prove that the functions $(A^\varepsilon)_{0<\varepsilon<\varepsilon_0}$ and $(u^\varepsilon)_{0<\varepsilon<\varepsilon_0}$ are strongly precompact in $L^2_{loc}(\mathbb{R}_+ \times \mathbb{R})$. Indeed, we may rewrite (50) as

$$\left\{\begin{aligned}
&\partial_t A^\varepsilon - \frac{1}{\varepsilon^2}\partial_x\big(A^\varepsilon - u^\varepsilon\big) = S_A^\varepsilon \\
&\partial_t u^\varepsilon - \frac{1}{\varepsilon^2}\partial_x\big(u^\varepsilon - A^\varepsilon\big) = S_u^\varepsilon,
\end{aligned}\right.$$

where

$$\left\{\begin{aligned}
S_A^\varepsilon &\equiv -2u^\varepsilon\partial_x A^\varepsilon - A^\varepsilon\partial_x u^\varepsilon \\
S_u^\varepsilon &\equiv -\partial_x\big((u^\varepsilon)^2\big) + \partial_x\Big(\frac{\partial_x^2 A^\varepsilon}{4c^2(1+\varepsilon^2 A^\varepsilon)}\Big) - \frac{1}{\varepsilon^4}\partial_x\big(\tilde{f}(\varepsilon^2 A^\varepsilon)\big)
\end{aligned}\right.$$

and

$$\tilde{f}(r) \equiv \frac{1}{c^2} f\big((1+r)^2\big) - 2r = \mathcal{O}(r^2) \qquad \text{as} \quad r \to 0. \tag{51}$$

In order to use Lemma 3, we shall prove that for some constant K depending only on M, we have

$$\big\|S_A^\varepsilon\big\|_{L^\infty(\mathbb{R}_+, H^{-2}(\mathbb{R}))} + \big\|S_u^\varepsilon\big\|_{L^\infty(\mathbb{R}_+, H^{-2}(\mathbb{R}))} \leq K. \tag{52}$$

We first note that, if $t \in \mathbb{R}_+$ and $\zeta \in \mathcal{C}_c^\infty(\mathbb{R})$,

$$\begin{aligned}\langle S_A^\varepsilon(t), \zeta \rangle =& - \langle u^\varepsilon(t) \partial_x A^\varepsilon(t), \zeta \rangle + \langle u^\varepsilon(t) A^\varepsilon(t), \partial_x \zeta \rangle \\ \leq& \big\|u^\varepsilon(t)\big\|_{L^2(\mathbb{R})} \big\|\partial_x A^\varepsilon(t)\big\|_{L^2(\mathbb{R})} \big\|\zeta\big\|_{L^\infty(\mathbb{R})} + \big\|u^\varepsilon(t)\big\|_{L^2(\mathbb{R})} \big\|A^\varepsilon(t)\big\|_{L^\infty(\mathbb{R})} \big\|\partial_x \zeta\big\|_{L^2(\mathbb{R})}.\end{aligned}$$

Hence, by using the embedding $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$ and Lemma 2, we get:

$$\big\|S_A^\varepsilon(t)\big\|_{H^{-1}(\mathbb{R})} \lesssim \big\|u^\varepsilon(t)\big\|_{L^2(\mathbb{R})} \big\|A^\varepsilon(t)\big\|_{H^1(\mathbb{R})} \leq K.$$

In a similar way, we have, for $t \in \mathbb{R}_+$ and $\zeta \in \mathcal{C}_c^\infty(\mathbb{R})$,

$$\begin{aligned}\langle S_u^\varepsilon(t), \zeta \rangle =& \int_\mathbb{R} \Big[(u^\varepsilon)^2 + \frac{1}{\varepsilon^4} \tilde{f}(\varepsilon^2 A^\varepsilon) - \frac{\varepsilon^2 (\partial_x A^\varepsilon)^2}{4c^2 (1 + \varepsilon^2 A^\varepsilon)^2} \Big] \partial_x \zeta + \frac{\partial_x A^\varepsilon}{4c^2 (1 + \varepsilon^2 A^\varepsilon)} \partial_x^2 \zeta. \\ \leq& K \Big(\Big[\big\|u^\varepsilon(t)\big\|_{L^2(\mathbb{R})}^2 + \big\|A^\varepsilon(t)\big\|_{L^2(\mathbb{R})}^2 + \varepsilon^2 \big\|\partial_x A^\varepsilon(t)\big\|_{L^2(\mathbb{R})}^2 \Big] \big\|\partial_x \zeta\big\|_{L^\infty(\mathbb{R})} \\ & + \big\|\partial_x A^\varepsilon(t)\big\|_{L^2(\mathbb{R})} \big\|\partial_x^2 \zeta\big\|_{L^2(\mathbb{R})} \Big),\end{aligned}$$

where we have used that $\tilde{f}(r) = \mathcal{O}(r^2)$ as $r \to 0$, and $\varepsilon^2 \|A^\varepsilon\|_{L^\infty(\mathbb{R})} \leq 1/2$. Using again the embedding $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$ and Lemma 2, this yields, for $0 < \varepsilon < \varepsilon_0$,

$$\big\|S_u^\varepsilon(t)\big\|_{H^{-2}(\mathbb{R})} \lesssim \big\|u^\varepsilon\big\|_{L^2(\mathbb{R})}^2 + \big\|A^\varepsilon\big\|_{H^1(\mathbb{R})}^2 + \big\|A^\varepsilon\big\|_{H^1(\mathbb{R})} \leq K.$$

Consequently, thanks to (52) and the fact that by our assumptions, A_0^ε and u_0^ε are uniformly bounded in L^2, we may apply Lemma 3 with $\sigma = 2$ and deduce that $(A^\varepsilon)_{0<\varepsilon<\varepsilon_0}$ and $(u^\varepsilon)_{0<\varepsilon<\varepsilon_0}$ are uniformly bounded in $H_{loc}^{\frac{1}{2}}\big(\mathbb{R}_+, H_{loc}^{-3}(\mathbb{R})\big)$. In particular, since $(A^\varepsilon)_{0<\varepsilon<\varepsilon_0}$ is uniformly bounded in $L^\infty\big(\mathbb{R}_+, H^1(\mathbb{R})\big)$ and in $H_{loc}^{\frac{1}{2}}\big(\mathbb{R}_+, H_{loc}^{-3}(\mathbb{R})\big)$, we can use Corollary 7 of [25] to get that $(A^\varepsilon)_{0<\varepsilon<\varepsilon_0}$ is strongly compact in $L_{loc}^2\big(\mathbb{R}_+, L_{loc}^2(\mathbb{R})\big) = L_{loc}^2(\mathbb{R}_+ \times \mathbb{R})$. Since, by Lemma 2, $A^\varepsilon - u^\varepsilon$ tends to zero strongly in $L^\infty\big(\mathbb{R}_+, L^2(\mathbb{R})\big)$, we also get that $(u^\varepsilon)_{0<\varepsilon<\varepsilon_0}$ is strongly compact in $L_{loc}^2\big(\mathbb{R}_+, L_{loc}^2(\mathbb{R})\big)$.

Let now $A \in L_{loc}^2\big(\mathbb{R}_+, L_{loc}^2(\mathbb{R})\big)$ and $0 < \varepsilon_j \to 0$ as $j \to +\infty$ such that

$$A^{\varepsilon_j} \text{ converges to } A \text{ strongly in } L_{loc}^2\big(\mathbb{R}_+, L_{loc}^2(\mathbb{R})\big) \text{ and weakly in } L_{loc}^2\big(\mathbb{R}_+, H_{loc}^1(\mathbb{R})\big); \tag{53}$$

$$u^{\varepsilon_j} \text{ converges to } A \text{ in } L_{loc}^2\big(\mathbb{R}_+, L_{loc}^2(\mathbb{R})\big). \tag{54}$$

Note that the weak convergence of A^ε just comes from the uniform H^1 bound which comes from Lemma 2.

The next step in the proof is to obtain that A is a weak solution to the KdV equation.

For that purpose, let us write from (50) the equation satisfied by $A^{\varepsilon_j} + u^{\varepsilon_j}$ in the weak form:

$$\int_{\mathbb{R}_+\times\mathbb{R}} \big(A^{\varepsilon_j} + u^{\varepsilon_j}\big)\partial_t\zeta\, dtdx + \int_{\mathbb{R}_+\times\mathbb{R}} \Big((u^{\varepsilon_j})^2 + \frac{1}{\varepsilon_j^4}\tilde{f}(\varepsilon_j^2 A^{\varepsilon_j}) - \frac{\varepsilon_j^2(\partial_x A^{\varepsilon_j})^2}{4c^2(1+\varepsilon_j^2 A^{\varepsilon_j})^2}\Big)\partial_x\zeta\, dtdx$$
$$+ \int_{\mathbb{R}_+\times\mathbb{R}} \frac{\partial_x A^{\varepsilon_j}}{4c^2(1+\varepsilon_j^2 A^{\varepsilon_j})}\partial_x^2\zeta\, dtdx + \int_{\mathbb{R}_+\times\mathbb{R}} \big(-u^{\varepsilon_j}\partial_x A^{\varepsilon_j}\zeta + A^{\varepsilon_j} u^{\varepsilon_j}\partial_x\zeta\big)\, dtdx$$
$$= \int_{\mathbb{R}} \big(A_0^{\varepsilon_j} + u_0^{\varepsilon_j}\big)\zeta(0,x)\, dx$$

for every $\zeta \in \mathcal{C}_c^\infty(\mathbb{R}\times\mathbb{R})$. One can pass to the limit easily in most of the terms by the strong convergence. Moreover, we can use that

$$\int_{\mathbb{R}_+\times\mathbb{R}} u^{\varepsilon_j}\partial_x A^{\varepsilon_j}\zeta \to \int_{\mathbb{R}_+\times\mathbb{R}} A\,\partial_x A\,\zeta$$

since $u^{\varepsilon_j} \to A$ strongly and $\partial_x A^\varepsilon \to \partial_x A$ weakly in $L^2(\mathbb{R}_+\times\mathbb{R})$. Since A^ε is uniformly bounded in $L^\infty\big(\mathbb{R}_+, H^1(\mathbb{R})\big)$, we have that

$$\Big|\int_{\mathbb{R}_+\times\mathbb{R}} \frac{\varepsilon_j^2(\partial_x A^{\varepsilon_j})^2}{4c^2(1+\varepsilon_j^2 A^{\varepsilon_j})^2}\partial_x\zeta\, dtdx\Big| \leq K\varepsilon_j^2 \to 0.$$

Moreover, since

$$\int_{\mathbb{R}_+\times\mathbb{R}} \frac{\partial_x A^{\varepsilon_j}}{4c^2(1+\varepsilon_j^2 A^{\varepsilon_j})}\partial_x^2\zeta\, dtdx = \int_{\mathbb{R}_+\times\mathbb{R}} \frac{\partial_x A^{\varepsilon_j}}{4c^2}\partial_x^2\zeta\, dtdx - \varepsilon_j^2\int_{\mathbb{R}_+\times\mathbb{R}} \frac{A^{\varepsilon_j}\partial_x A^{\varepsilon_j}}{4c^2(1+\varepsilon_j^2 A^{\varepsilon_j})}\partial_x^2\zeta\, dtdx,$$

we get that the first term converges to

$$\int_{\mathbb{R}_+\times\mathbb{R}} \frac{\partial_x A}{4c^2}\partial_x^2\zeta\, dtdx$$

by weak convergence and that the second term converges to zero because of the uniform bounds. Therefore,

$$\int_{\mathbb{R}_+\times\mathbb{R}} \frac{\partial_x A^{\varepsilon_j}}{4c^2(1+\varepsilon_j^2 A^{\varepsilon_j})}\partial_x^2\zeta\, dtdx \to \int_{\mathbb{R}_+\times\mathbb{R}} \frac{\partial_x A}{4c^2}\partial_x^2\zeta\, dtdx.$$

Finally, we write

$$\tilde{f}(r) = \big[1 + \frac{2}{c^2}f''(1)\big]r^2 + \mathcal{O}(r^3) \qquad \text{as} \quad r \to 0$$

to infer

$$\int_{\mathbb{R}_+\times\mathbb{R}} \frac{1}{\varepsilon_j^4}\tilde{f}(\varepsilon_j^2 A^{\varepsilon_j})\partial_x\zeta\, dtdx \to \big[c^2 + 2f''(1)\big]\int_{\mathbb{R}_+\times\mathbb{R}} A^2\partial_x\zeta\, dtdx.$$

Consequently, we finally obtain that A satisfies

$$\int_{\mathbb{R}_+\times\mathbb{R}} \Big(2A\partial_t\zeta + \frac{k}{2}A^2\partial_x\zeta + \frac{1}{4c^2}\partial_x A\partial_x^2\zeta\Big)\, dtdx = \int_{\mathbb{R}} 2A_0(x)\zeta(0,x)\, dx,$$

which is the weak form of the KdV equation.

Next, by passing to the limit in the bound of Lemma 2, we get that $A \in L^\infty(\mathbb{R}_+, H^1(\mathbb{R}))$. Moreover, since it is a solution of the KdV equation, we deduce that

$$\partial_t A = \frac{1}{8c^2}\partial_x^3 A - \frac{k}{2} A\partial_x A \in L^\infty(\mathbb{R}_+, H^{-2}(\mathbb{R})).$$

Hence $A \in \mathrm{Lip}(\mathbb{R}_+, H^{-2}(\mathbb{R}))$, and by interpolation in space, we get that $A \in \mathcal{C}_b^0(\mathbb{R}_+, H^s(\mathbb{R}))$ for any $0 \le s < 1$.

We shall now prove that $A = v$ the unique solution of the KdV equation given by Theorem 2. This fact can be deduced from a general uniqueness theorem for the KdV equation [28]. Nevertheless, here, by using that the solution v given by [16] verifies the additional property $\partial_x v \in L^4_{loc}(\mathbb{R}_+, L^\infty(\mathbb{R}))$, one can get that $A = v$ by a very simple weak strong uniqueness argument. Indeed, let us set $\theta \equiv A - v$ and observe that $\theta \in L^\infty(\mathbb{R}_+, H^1(\mathbb{R})) \cap \mathcal{C}_b^0(\mathbb{R}_+, H^s(\mathbb{R}))$ for $0 < s < 1$ solves

$$2\partial_t\theta - \frac{1}{4c^2}\partial_x^3\theta = -kA\partial_x\theta - k\theta\partial_x v = -k\theta\partial_x\theta - k\theta\partial_x v - kv\partial_x\theta, \qquad \theta_{|t=0} = 0.$$

Consequently, the standard L^2 energy estimate for this equation gives

$$\frac{d}{dt}\int_{\mathbb{R}} \theta^2 \, dx \le 2|k| \|\partial_x v\|_{L^\infty} \|\theta\|_{L^2}^2.$$

By the standard Gronwall inequality, this yields immediately that $\theta = 0$, since $\theta_{|t=0} = 0$, and $\partial_x v \in L^4_{loc}(\mathbb{R}_+, L^\infty(\mathbb{R})) \subset L^1_{loc}(\mathbb{R}_+, L^\infty(\mathbb{R}))$.

As a consequence of the uniqueness of the limit, the full sequence A^ε converges to v as $\varepsilon \to 0$ strongly in $L^2_{loc}(\mathbb{R}_+, L^2_{loc}(\mathbb{R}))$ and weakly in $L^2_{loc}(\mathbb{R}_+, H^1_{loc}(\mathbb{R}))$, where v is *the* H^1-solution of the KdV equation of Theorem 2.

It remains to improve the convergence of A^ε i.e. to prove that we actually have the local in time global in space strong convergence, as $\varepsilon \to 0$,

$$A^\varepsilon \to v \qquad \text{in} \quad \mathcal{C}([0,T], L^2(\mathbb{R}))$$

for every $T > 0$.

From Lemma 2 and the proof of Lemmas 4 and 3, we infer that

$$A^\varepsilon + u^\varepsilon \quad \text{is uniformly bounded in } L^\infty(\mathbb{R}_+, L^2(\mathbb{R})) \cap \mathrm{Lip}(\mathbb{R}_+, H^{-2}(\mathbb{R})).$$

In particular,

$$A^\varepsilon + u^\varepsilon \quad \text{is uniformly bounded in } \mathcal{C}^{0,1/2}(\mathbb{R}_+, H^{-1}(\mathbb{R})) \cap L^\infty(\mathbb{R}_+, L^2(\mathbb{R})).$$

Since we already have that

$$A^\varepsilon + u^\varepsilon \to 2A = 2v \qquad \text{in} \quad L^2_{loc}(\mathbb{R}_+, L^2_{loc}(\mathbb{R})),$$

it follows by a new use of the Aubin-Lions lemma that

$$A^\varepsilon + u^\varepsilon \to 2v \qquad \text{in} \quad \mathcal{C}^0_{loc}(\mathbb{R}_+, H^{-1}_{loc}(\mathbb{R})). \tag{55}$$

Consequently, we can write for every $T > 0$, $R > 0$,

$$\sup_{[0,T]} \|A^\varepsilon - v\|_{H^{-1}(-R,R)} \leq \frac{1}{2} \sup_{[0,T]} \Big(\|A^\varepsilon + u^\varepsilon - 2v\|_{H^{-1}(-R,R)} + \|A^\varepsilon - u^\varepsilon\|_{H^{-1}(-R,R)} \Big)$$

and since by Lemma 2, we have that $A^\varepsilon - u^\varepsilon \to 0$ in $L^\infty\big(\mathbb{R}_+, L^2(\mathbb{R})\big)$, this yields thanks to (55) that

$$A^\varepsilon \to v \qquad \text{in} \quad \mathcal{C}^0_{loc}\big(\mathbb{R}_+, H^{-1}_{loc}(\mathbb{R})\big).$$

Let us now fix $T > 0$. We then prove that, as $\varepsilon \to 0$,

$$\sup_{[0,T]} \big|\langle A^\varepsilon - v, v\rangle_{L^2(\mathbb{R})}\big| \to 0.$$

Indeed, let $\eta > 0$ be given. Since $v \in \mathcal{C}^0_b\big(\mathbb{R}_+, L^2(\mathbb{R})\big)$, there exists $R > 0$ such that

$$\sup_{[0,T]} \int_{|x| \geq R} v^2 \; dx \leq \eta^2.$$

Next, with $\zeta \in \mathcal{C}^\infty_c(-2R, 2R)$ such that $0 \leq \zeta \leq 1$ and $\zeta = 1$ on $[-R, R]$, we split

$$\sup_{[0,T]} \big|\langle A^\varepsilon - v, v\rangle_{L^2(\mathbb{R})}\big| \leq \sup_{[0,T]} \big|\langle A^\varepsilon - v, \zeta v\rangle_{L^2(\mathbb{R})}\big| + \sup_{[0,T]} \big|\langle A^\varepsilon - v, (1-\zeta) v\rangle_{L^2(\mathbb{R})}\big|.$$

The first term tends to 0 as $\varepsilon \to 0$ since $\zeta v \in \mathcal{C}^0_b\big(\mathbb{R}_+, H^1(\mathbb{R})\big)$ is compactly supported and $A^\varepsilon \to v$ in $\mathcal{C}^0_{loc}\big(\mathbb{R}_+, H^{-1}_{loc}(\mathbb{R})\big)$. Since $1 - \zeta$ is supported in $\{|x| \geq R\}$, the second term is $\leq \eta \sup_{[0,T]} \big\|A^\varepsilon - v\big\|_{L^2(\mathbb{R})} \leq K\eta$, and the limit follows.

Therefore,

$$\begin{aligned} \sup_{[0,T]} \big\|A^\varepsilon - v\big\|^2_{L^2(\mathbb{R})} &= \sup_{[0,T]} \Big\{ \big\|A^\varepsilon\big\|^2_{L^2} - \big\|v\big\|^2_{L^2(\mathbb{R})} - 2\langle A^\varepsilon - v, v\rangle_{L^2(\mathbb{R})} \Big\} \\ &= \sup_{[0,T]} \Big\{ \big\|A^\varepsilon\big\|^2_{L^2(\mathbb{R})} - \big\|v\big\|^2_{L^2(\mathbb{R})} \Big\} + o(1). \end{aligned} \tag{56}$$

We now use that $E^\varepsilon(\psi^\varepsilon)$ and $\mathcal{I}_0(A) = \int_{\mathbb{R}} A^2 \; dx$ are independent of t, thus

$$\big\|A(t)\big\|^2_{L^2(\mathbb{R})} = \big\|A_0\big\|^2_{L^2(\mathbb{R})} \tag{57}$$

and, using Lemma 2 and the same expansion as in Step 1 of the proof of Lemma 2, we infer

$$E^\varepsilon\big(\psi^\varepsilon(t)\big) = \frac{\varepsilon^2}{2} \int_{\mathbb{R}} 4c^2 \big(A^\varepsilon(t)\big)^2 + \big(\partial_x \varphi^\varepsilon(t)\big)^2 \; dx + \mathcal{O}(\varepsilon^4) = 4c^2 \varepsilon^2 \int_{\mathbb{R}} \big(A^\varepsilon(t)\big)^2 \; dx + \mathcal{O}(\varepsilon^3).$$

Note that the $\mathcal{O}(\varepsilon^3)$ is uniform with respect to $t \in \mathbb{R}_+$. Since $E^\varepsilon\big(\psi^\varepsilon(t)\big) = E^\varepsilon\big(\psi^\varepsilon_0\big)$ and the same expansion holds at $t = 0$ (this is Step 1 in the proof of Lemma 2), we deduce

$$\int_{\mathbb{R}} \big(A^\varepsilon(t)\big)^2 \; dx = \int_{\mathbb{R}} \big(A^\varepsilon_0\big)^2 \; dx + \mathcal{O}(\varepsilon), \tag{58}$$

where $\mathcal{O}(\varepsilon)$ is uniform with respect to $t \in \mathbb{R}_+$. Consequently, thanks to (56), (57), (58), we obtain that

$$\sup_{[0,T]} \big\|A^\varepsilon - v\big\|^2_{L^2(\mathbb{R})} = \big\|A^\varepsilon_0\big\|^2_{L^2(\mathbb{R})} - \big\|A_0\big\|^2_{L^2(\mathbb{R})} + o(1),$$

and since $A_0^\varepsilon \to A_0$ in $L^2(\mathbb{R})$ by assumption, the result in L^2 follows.

The proof of Theorem 3 is now complete, since the convergence of A^ε in $L^\infty_{loc}\big(\mathbb{R}_+, H^s(\mathbb{R})\big)$, $0 < s < 1$ follows by interpolation in space using the convergence in $L^\infty_{loc}\big(\mathbb{R}_+, L^2(\mathbb{R})\big)$ and the uniform bounds in $L^\infty\big(\mathbb{R}_+, H^1(\mathbb{R})\big)$.

2.5 Convergence in H^1

In this subsection, we shall put a more restrictive assumption on the initial data, namely

$$\big\|\partial_x \varphi_0^\varepsilon - 2cA_0^\varepsilon\big\|_{L^2(\mathbb{R})} = o(\varepsilon)$$

instead of $\mathcal{O}(\varepsilon)$ in order to get the strong convergence in H^1 of the amplitude A^ε.

Theorem 7 *Under the assumptions of Theorem 3, if, at the initial time, we have the additional assumptions*

$$A_0^\varepsilon \to A_0 \qquad \text{in} \quad H^1(\mathbb{R})$$

and

$$\big\|\partial_x \varphi_0^\varepsilon - 2cA_0^\varepsilon\big\|_{L^2(\mathbb{R})} = o(\varepsilon), \tag{59}$$

then

$$A^\varepsilon \to A \qquad \text{in} \quad \mathcal{C}^0_{loc}\big(\mathbb{R}_+, H^1(\mathbb{R})\big).$$

Proof. The idea follows the one in the end of the proof of Theorem 3, but relies on the conservation of the energy for (KdV)

$$\mathcal{I}_1\big(A(t)\big) \equiv \int_{\mathbb{R}} \frac{1}{4c^2}(\partial_x A)^2 + \frac{k}{3}A^3 \, dx$$

on the one hand, and of $E_\varepsilon\big(\psi^\varepsilon(t)\big) - 2cP^\varepsilon\big(\psi^\varepsilon(t)\big)$ for (4) on the other hand. First, we expand to third order

$$F(R) = c^2(R-1)^2 + \frac{1}{3}f''(1)(R-1)^3 + F_4(R), \quad \text{with} \quad F_4(1+r) = \mathcal{O}(r^4), \quad r \to 0,$$

so that (30) becomes now

$$E^\varepsilon(\psi^\varepsilon) - 2cP^\varepsilon(\psi^\varepsilon) = \frac{1}{2}\int_{\mathbb{R}} (\rho^2-1)(\partial_x\phi)^2 + (\partial_x\rho)^2 + \Big(\partial_x\phi - \frac{c}{\varepsilon}(\rho^2-1)\Big)^2 + \frac{f''(1)}{3\varepsilon^2}\,(\rho^2-1)^3 + \frac{1}{\varepsilon^2}\,F_4(\rho^2)\,dx.$$

By using the hypothesis (59) , we infer

$$\int_{\mathbb{R}} \Big(\partial_x\varphi_0^\varepsilon - 2cA_0^\varepsilon - c\varepsilon^2(A_0^\varepsilon)^2\Big)^2 \, dx = \int_{\mathbb{R}} \Big(\partial_x\varphi_0^\varepsilon - 2cA_0^\varepsilon\Big)^2 \, dx + \mathcal{O}(\varepsilon^3) = o(\varepsilon^2).$$

Therefore, at time $t = 0$, we infer, as in Step 1 of the proof of Lemma 2, that

$$\begin{aligned}
E^\varepsilon(\psi_0^\varepsilon) - 2cP^\varepsilon(\psi_0^\varepsilon) \; &= \frac{\varepsilon^4}{2}\int_{\mathbb{R}} 2A_0^\varepsilon\big(\partial_x\varphi_0^\varepsilon\big)^2 + (\partial_x A_0^\varepsilon)^2 + \frac{8f''(1)}{3}(A_0^\varepsilon)^3 \, dx \\
&\quad + \frac{\varepsilon^2}{2}\int_{\mathbb{R}} \Big(\partial_x\varphi_0^\varepsilon - 2cA_0^\varepsilon - c\varepsilon^2(A_0^\varepsilon)^2\Big)^2 \, dx + \mathcal{O}(\varepsilon^6) \\
&= \frac{\varepsilon^4}{2}\int_{\mathbb{R}} (\partial_x A_0^\varepsilon)^2 + 8\Big[c^2 + \frac{f''(1)}{3}\Big](A_0^\varepsilon)^3 \, dx + o(\varepsilon^4) \\
&= 2c^2\varepsilon^4\mathcal{I}_1\big(A_0^\varepsilon\big) + o(\varepsilon^4) = 2c^2\varepsilon^4\mathcal{I}_1\big(A_0\big) + o(\varepsilon^4),
\end{aligned}$$

since $A_0^\varepsilon \to A_0$ in $H^1(\mathbb{R}) \subset L^3(\mathbb{R})$. Similarly, given $t \in \mathbb{R}_+$ and using Lemma 2, we have

$$E^\varepsilon\big(\psi^\varepsilon(t)\big) - 2cP^\varepsilon\big(\psi^\varepsilon(t)\big) = 2c^2\varepsilon^4\mathcal{I}_1\big(A^\varepsilon(t)\big) + \frac{\varepsilon^2}{2}\int_{\mathbb{R}}\Big(\partial_x\varphi^\varepsilon(t) - 2cA^\varepsilon(t)\Big)^2\,dx + \mathcal{O}(\varepsilon^5),$$

where $\mathcal{O}(\varepsilon^5)$ is uniform with respect to time. Since $\mathcal{I}_1\big(A(t)\big)$ and $E^\varepsilon(\psi^\varepsilon) - 2cP^\varepsilon(\psi^\varepsilon)$ are independent of time, this implies,

$$\mathcal{I}_1\big(A(t)\big) = \mathcal{I}_1\big(A^\varepsilon(t)\big) + \frac{1}{4c^2\varepsilon^2}\int_{\mathbb{R}}\Big(\partial_x\varphi^\varepsilon(t) - 2cA^\varepsilon(t)\Big)^2\,dx + o(1) \tag{60}$$

uniformly in time.

Now, let us study the term involving the L^3-norm term in $\mathcal{I}_1$. Let $T > 0$ be fixed. From Lemma 2, A^ε is uniformly bounded in $L^\infty(\mathbb{R}_+ \times \mathbb{R})$. Moreover, we have proved in Step 4 that $A^\varepsilon \to A$ in $\mathcal{C}\big([0,T], L^2(\mathbb{R})\big)$. As a consequence, $A^\varepsilon \to A$ in $\mathcal{C}\big([0,T], L^3(\mathbb{R})\big)$. Inserting this in (60) yields, uniformly for $t \in [0,T]$,

$$\int_{\mathbb{R}}\big(\partial_x A(t)\big)^2\,dx = \int_{\mathbb{R}}\big(\partial_x A^\varepsilon(t)\big)^2\,dx + \frac{1}{\varepsilon^2}\int_{\mathbb{R}}\Big(\partial_x\varphi^\varepsilon(t) - 2cA^\varepsilon(t)\Big)^2\,dx + o(1). \tag{61}$$

We now consider

$$\nu^\varepsilon(T) \equiv \sup_{[0,T]}\Big\{\big\|\partial_x A^\varepsilon - \partial_x A\big\|^2_{L^2(\mathbb{R})} + \frac{1}{\varepsilon^2}\big\|\partial_x\varphi^\varepsilon - 2cA^\varepsilon\big\|^2_{L^2(\mathbb{R})}\Big\}.$$

Since $A \in \mathcal{C}\big([0,T], H^1(\mathbb{R})\big)$, arguing as in the end of the proof of Theorem 3, we infer

$$\nu^\varepsilon(T) = \sup_{[0,T]}\Big\{\big\|\partial_x A^\varepsilon\big\|^2_{L^2(\mathbb{R})} - \big\|\partial_x A\big\|^2_{L^2(\mathbb{R})} + \frac{1}{\varepsilon^2}\big\|\partial_x\varphi^\varepsilon - 2cA^\varepsilon\big\|^2_{L^2(\mathbb{R})}\Big\} + o(1).$$

Combining this with (61) gives $\nu^\varepsilon(T) = o(1)$ as desired. This ends the proof of Theorem 7.

3 The general n dimensional case

3.1 Proof of Theorem 4

It is more convenient to use a different hydrodynamic form of (NLS). As in [11], we shall seek for a solution of (4) under the form

$$\psi^\varepsilon = \big(1 + \varepsilon^2 a^\varepsilon(t,X)\big)e^{i\varepsilon\theta^\varepsilon(t,X)}, \quad a^\varepsilon \in \mathbb{C}, \quad \theta^\varepsilon \in \mathbb{R}, \quad \varepsilon^2|a^\varepsilon| \le \frac{1}{2} \tag{62}$$

that is to say that we allow the amplitude to be complex at positive times. The reason for this choice is that we can obtain an hydrodynamic equation for $(a^\varepsilon, \theta^\varepsilon)$ which is much simpler. We shall prove that a^ε and θ^ε are well defined on $[0,T]$ for some $T > 0$ independent of ε and satisfy for $s > 1 + n/2$ the uniform estimate

$$\big\|a^\varepsilon(t)\big\|_{H^{s+1}(\mathbb{R}^n)} + \big\|\partial_x\theta^\varepsilon(t)\big\|_{H^{s+1}(\mathbb{R}^n)} + \varepsilon\big\|\nabla_\perp\theta^\varepsilon(t)\big\|_{H^{s+1}(\mathbb{R}^n)} \le C, \quad \forall t \in [0,T], \forall\varepsilon \in (0,\varepsilon_0] \tag{63}$$

for some $C > 0$ independent of ε.

Note that once this estimate is proven, the representation (19) and the estimate (20) immediately follow. Indeed, for ε sufficently small, we get that $|\psi^\varepsilon|$ remains far from zero on $[0,T]$ and we have the relations

$$A^\varepsilon = \frac{|1+\varepsilon^2 a^\varepsilon| - 1}{\varepsilon^2}, \qquad \partial_j \varphi^\varepsilon = \partial_j \theta^\varepsilon + \frac{\varepsilon}{i}\Big(\frac{\partial_j a^\varepsilon}{1+\varepsilon^2 a^\varepsilon} - \frac{\partial_j A^\varepsilon}{1+\varepsilon^2 A^\varepsilon}\Big) \quad 1 \le j \le n \tag{64}$$

from which we deduce by standard Sobolev-Gagliardo-Nirenberg-Moser estimates that

$$\big\|A^\varepsilon(t)\big\|_{H^{s+1}(\mathbb{R}^n)} + \big\|\partial_x \varphi^\varepsilon(t)\big\|_{H^s(\mathbb{R}^n)} + \varepsilon\big\|\nabla_\perp \varphi^\varepsilon(t)\big\|_{H^s(\mathbb{R}^n)} \le C \quad \forall t \in [0,T], \quad \forall \varepsilon \in (0,\varepsilon_0]$$

for some C independent of ε since $s > 1 + n/2$.

Let us now write down the equation for $(a^\varepsilon, \theta^\varepsilon)$. By plugging the anzatz (62) in (4), we get

$$\begin{aligned}
&ic\varepsilon^3\Big(\varepsilon^2 \partial_t a^\varepsilon + i\varepsilon(1+\varepsilon^2 a^\varepsilon)\partial_t \theta^\varepsilon\Big) - ic\varepsilon\Big(\varepsilon^2 \partial_x a^\varepsilon + i\varepsilon(1+\varepsilon^2 a^\varepsilon)\partial_x \theta^\varepsilon\Big)\\
&+\frac{\varepsilon^2}{2}\Big(\varepsilon^2 \Delta^\varepsilon a^\varepsilon + 2i\varepsilon^3 \nabla^\varepsilon \theta^\varepsilon \cdot \nabla^\varepsilon a^\varepsilon + i\varepsilon(1+\varepsilon^2 a^\varepsilon)\Delta^\varepsilon \theta^\varepsilon - \varepsilon^2(1+\varepsilon^2 a^\varepsilon)|\nabla^\varepsilon \theta^\varepsilon|^2\Big)\\
&-(1+\varepsilon^2 a^\varepsilon) f\big(|1+\varepsilon^2 a^\varepsilon|^2\big) = 0
\end{aligned}$$

where we use the notation

$$\nabla^\varepsilon \equiv (\partial_x, \varepsilon \partial_\perp)^t, \quad \Delta^\varepsilon \equiv \nabla^\varepsilon \cdot \nabla^\varepsilon = \partial_x^2 + \varepsilon^2 \Delta_\perp.$$

Since we allow the amplitude a^ε to be complex, we have some freedom to write down hydrodynamic equations. As noticed in [11], it is convenient to split the above equation into the system

$$\left\{\begin{aligned}
&\partial_t a^\varepsilon - \frac{1}{\varepsilon^2}\partial_x a^\varepsilon + \frac{1}{c}\nabla^\varepsilon \theta^\varepsilon \cdot \nabla^\varepsilon a^\varepsilon + \frac{1}{2c\varepsilon^2}(1+\varepsilon^2 a^\varepsilon)\Delta^\varepsilon \theta^\varepsilon = \frac{i}{2\varepsilon c}\Delta^\varepsilon a^\varepsilon\\
&\partial_t \theta^\varepsilon - \frac{1}{\varepsilon^2}\partial_x \theta^\varepsilon + \frac{1}{2c}|\nabla^\varepsilon \theta^\varepsilon|^2 + \frac{1}{c\varepsilon^4} f\big(|1+\varepsilon^2 a^\varepsilon|^2\big) = 0.
\end{aligned}\right.$$

Consequently, by using the new unknown $v^\varepsilon \equiv \dfrac{1}{2c}\nabla^\varepsilon \theta^\varepsilon$, we get

$$\left\{\begin{aligned}
&\partial_t a^\varepsilon - \frac{1}{\varepsilon^2}\partial_x a^\varepsilon + 2v^\varepsilon \cdot \nabla^\varepsilon a^\varepsilon + \frac{1}{\varepsilon^2}(1+\varepsilon^2 a^\varepsilon)\nabla^\varepsilon \cdot v^\varepsilon = \frac{i}{2\varepsilon c}\Delta^\varepsilon a^\varepsilon\\
&\partial_t v^\varepsilon - \frac{1}{\varepsilon^2}\partial_x v^\varepsilon + 2v^\varepsilon \cdot \nabla^\varepsilon v^\varepsilon + \frac{1}{2c^2\varepsilon^2} f'\big(|1+\varepsilon^2 a^\varepsilon|^2\big)\Big(2\nabla^\varepsilon \mathrm{Re}\, a^\varepsilon + \varepsilon^2 \nabla^\varepsilon |a^\varepsilon|^2\Big) = 0.
\end{aligned}\right. \tag{65}$$

We add to this system the initial condition

$$a^\varepsilon(0,X) = A_0^\varepsilon(X), \quad v^\varepsilon(0,X) = \frac{1}{2c}\nabla^\varepsilon \varphi_0^\varepsilon(X). \tag{66}$$

Consequently, we can set $U^\varepsilon \equiv (\mathrm{Re}\, a^\varepsilon, \mathrm{Im}\, a^\varepsilon, v^\varepsilon)^t \in \mathbb{R}^{2+n}$, $\partial^\varepsilon \equiv (\partial_x, \varepsilon \partial_\perp)$ and write the above system under the abstract form:

$$\partial_t U^\varepsilon + \frac{1}{\varepsilon^2} H(\varepsilon^2 U^\varepsilon, \partial^\varepsilon) U^\varepsilon = \frac{1}{\varepsilon} L(\partial^\varepsilon) U^\varepsilon \tag{67}$$

where $L(\partial^\varepsilon)$ is a constant coefficients second order differential operator

$$L(\partial^\varepsilon) \equiv \frac{1}{2c}\begin{pmatrix} J\Delta^\varepsilon & 0 \\ 0 & 0 \end{pmatrix}, \quad J \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and $H(\varepsilon^2 U^\varepsilon, \partial^\varepsilon)$ is a first order hyperbolic operator

$$H(\varepsilon^2 U^\varepsilon, \partial^\varepsilon) \equiv \sum_{k=1}^{n} H^k(\varepsilon^2 U^\varepsilon)\partial_k^\varepsilon,$$

with symbol

$$H(\varepsilon^2 U^\varepsilon, \xi) = \sum_{k=1}^{n} H^k(\varepsilon^2 U^\varepsilon)\xi_k = \begin{pmatrix} (-\xi_1 + 2\varepsilon^2 v^\varepsilon \cdot \xi) I_2 & (e + \varepsilon^2 a^\varepsilon)\xi^t \\ \big(1 + g(\varepsilon^2 a^\varepsilon)\big)\xi\big(e + \varepsilon^2 a^\varepsilon\big)^t & \big(-\xi_1 + 2\varepsilon^2 v^\varepsilon \cdot \xi\big) I_n \end{pmatrix}$$

where

$$e \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{R}^2$$

and g is defined by the expansion:

$$\frac{1}{c^2} f'(1 + 2a \cdot e + |a|^2) = 1 + g(a), \quad g(a) = \mathcal{O}(|a|),\ |a| \leq 1 \tag{68}$$

since $f'(1) = c^2$.

Note that the structure of (67) is much simpler than the one of the usual hydrodynamic system for $(A^\varepsilon, \nabla^\varepsilon \varphi^\varepsilon)^t$ that is obtained from (6) by the standard Madelung transform. Indeed, (67) is a simple skew-symmetric constant coefficient perturbation of an hyperbolic system.

Note that the difficulties du to the presence of vacuum which arise in the study of (NLS) with solutions which tends to zero at infinity (as in [1], [7]) are not present here. The above system can be easily symmetrized by using

$$S(\varepsilon^2 U^\varepsilon) \equiv \begin{pmatrix} I_2 & 0 \\ 0 & \frac{1}{1+g(\varepsilon^2 a^\varepsilon)} \end{pmatrix}$$

which is positive. Indeed, we have

$$S(\varepsilon^2 U^\varepsilon) L(\partial^\varepsilon) = \frac{1}{2c}\begin{pmatrix} J\Delta^\varepsilon & 0 \\ 0 & 0 \end{pmatrix}$$

which is a skew symmetric operator:

$$\Big(S^\varepsilon(\varepsilon^2 U^\varepsilon) L(\partial^\varepsilon) V, V\Big) = 0, \quad \forall V \in H^2(\mathbb{R}^n) \tag{69}$$

where we use the notation $(\cdot,\cdot)$ for the $L^2(\mathbb{R}^n)$ scalar product. Moreover, we also have that

$$S(\varepsilon^2 U^\varepsilon) H(\varepsilon^2 U^\varepsilon, \xi) = \begin{pmatrix} (-\xi_1 + 2\varepsilon^2 v^\varepsilon \cdot \xi) I_2 & (e + \varepsilon^2 a^\varepsilon)\xi^t \\ \xi\big(e + \varepsilon^2 a^\varepsilon\big)^t & \frac{1}{1 + g(\varepsilon^2 a^\varepsilon)}\big(-\xi_1 + 2\varepsilon^2 v^\varepsilon \cdot \xi\big) I_n \end{pmatrix}$$

is symmetric for every $\xi \in \mathbb{R}$.

The local existence and uniqueness of a smooth solution $U^\varepsilon \in \mathcal{C}([0,T^\varepsilon), H^{s+1}(\mathbb{R}^n))$ for this system is classical. Moreover, let us define

$$T_*^\varepsilon = \sup\Big\{T \in [0,T^\varepsilon),\ \forall t \in [0,T], \quad |\varepsilon^2 a^\varepsilon|_{L^\infty} \leq \delta, \quad \|U^\varepsilon\|_{H^{s+1}(\mathbb{R}^n)} < +\infty\Big\}.$$

We shall prove that T_*^ε is bounded from below by a positive number when ε tends to zero. This will be achieved by proving H^{s+1} estimates uniform in ε.

Note that for $t \leq T_*^\varepsilon$, by (24), the symmetrizer $S(\varepsilon^2 U^\varepsilon)$ is well defined and verifies

$$\big(S(\varepsilon^2 U^\varepsilon)V, V\big) \geq c_0 \|V\|^2_{L^2(\mathbb{R}^n)}, \quad \forall t \in [0,T_*^\varepsilon], \quad \forall V \in L^2(\mathbb{R}^n) \tag{70}$$

for some $c_0 > 0$ independent of ε. Moreover, thanks to an integration by parts, we also have for some $C > 0$ independent of ε that

$$\big|\big(S(\varepsilon^2 U^\varepsilon)H(\varepsilon^2 U^\varepsilon, \partial^\varepsilon)V, V\big)\big| \leq C\varepsilon^2 \big\|\nabla U^\varepsilon\big\|_{L^\infty(\mathbb{R}^n)}\big\|V\big\|^2_{L^2(\mathbb{R}^n)}, \quad \forall t \in [0,T_*^\varepsilon] \tag{71}$$

for every $V \in H^1(\mathbb{R}^n)$.

We can now easily perform for $s > 1+n/2$ an H^{s+1} estimate for (67). Indeed, for every $\alpha \in \mathbb{N}^n$, $|\alpha| \leq s+1$, we have

$$\partial_t \partial^\alpha U^\varepsilon + \frac{1}{\varepsilon^2} H(\varepsilon^2 U^\varepsilon, \partial^\varepsilon)\partial^\alpha U^\varepsilon - \frac{1}{\varepsilon} L(\partial^\varepsilon)\partial^\alpha U^\varepsilon + \frac{1}{\varepsilon^2}\big[\partial^\alpha, H(\varepsilon^2 U^\varepsilon, \partial^\varepsilon)\big]U^\varepsilon = 0. \tag{72}$$

By the standard tame Gagliardo-Nirenberg-Moser estimate, we get that

$$\Big\|\frac{1}{\varepsilon^2}\big[\partial^\alpha, H(\varepsilon^2 U^\varepsilon, \partial^\varepsilon)\big]U^\varepsilon\Big\|_{L^2(\mathbb{R}^n)} \leq C\big\|U^\varepsilon\big\|_{W^{1,\infty}(\mathbb{R}^n)}\big\|U^\varepsilon\big\|_{H^{s+1}(\mathbb{R}^n)}, \quad \forall t \in [0,T_*^\varepsilon]. \tag{73}$$

From now on C is a number independent of ε which may change from line to line.

By using (69), (71), (73), we get the energy estimate:

$$\frac{d}{dt}\Big(\frac{1}{2}\big(S(\varepsilon^2 U^\varepsilon)\partial^\alpha U^\varepsilon, \partial^\alpha U^\varepsilon\big)\Big) \leq C\Big(\varepsilon^2\|\partial_t U^\varepsilon\|_{L^\infty(\mathbb{R}^n)} + \big\|U^\varepsilon\big\|_{W^{1,\infty}(\mathbb{R}^n)}\Big)\big\|U^\varepsilon\big\|^2_{H^{s+1}(\mathbb{R}^n)}, \quad \forall t \in [0,T_*^\varepsilon].$$

By using (67), we get that

$$\big\|\partial_t U^\varepsilon\big\|_{L^\infty(\mathbb{R}^n)} \leq C\Big(\frac{1}{\varepsilon^2}\big\|U^\varepsilon\big\|_{W^{1,\infty}(\mathbb{R}^n)} + \frac{1}{\varepsilon}\big\|U^\varepsilon\big\|_{W^{2,\infty}(\mathbb{R}^n)}\Big).$$

Consequently, we can integrate in time and use (70) to get

$$\big\|U^\varepsilon(t)\big\|^2_{H^{s+1}(\mathbb{R}^n)} \leq C\Big(\big\|U_0^\varepsilon\big\|^2_{H^{s+1}(\mathbb{R}^n)} + \int_0^t \big\|U^\varepsilon(\tau)\big\|_{W^{2,\infty}(\mathbb{R}^n)}\big\|U^\varepsilon(\tau)\big\|^2_{H^{s+1}(\mathbb{R}^n)}\, d\tau\Big). \tag{74}$$

Finally, by using the Sobolev embedding $H^{s+1} \subset W^{2,\infty}$ for $s > 1 + n/2$, we find in a classical way from (74) that $T_*^\varepsilon > T > 0$ for every $\varepsilon \in (0,\varepsilon_0)$ for some ε_0 sufficiently small. We refer for example to [20], [12], [24] for more details. This ends the proof of Theorem 4.

3.2 Proof of Theorem 5

We shall now study the convergence towards the KdV/KP-I equation. We could pass to the limit directly from (65). Nevertheless, to make a link more clear with the first part of the paper and the formal derivation, we shall pass to the limit directly from the standard hydrodynamic equation (6). As already explained (see (64)), we can deduce from the representation (62) and the bounds (63) that the smooth representation (19) with the uniform bounds (20) hold on $[0,T]$. Consequently, we already have

$$\|A^\varepsilon(t)\|_{H^{s+1}(\mathbb{R}^n)} + \|u^\varepsilon(t)\|_{H^s(\mathbb{R}^n)} \le C, \quad \forall t \in [0,T],\, \forall \varepsilon \in (0,\varepsilon_0) \tag{75}$$

for $s > 1 + n/2$, where $(A^\varepsilon, u^\varepsilon = \frac{1}{2c}\nabla^\varepsilon \varphi^\varepsilon)$ solves the system

$$\begin{cases} \partial_t A^\varepsilon - \dfrac{1}{\varepsilon^2}\partial_x A^\varepsilon + \dfrac{1}{\varepsilon^2}\nabla^\varepsilon \cdot u^\varepsilon + 2u^\varepsilon \cdot \nabla^\varepsilon A^\varepsilon + A^\varepsilon \nabla^\varepsilon \cdot u^\varepsilon = 0 \\ \partial_t u^\varepsilon - \dfrac{1}{\varepsilon^2}\partial_x u^\varepsilon + \dfrac{1}{\varepsilon^2}\nabla^\varepsilon A^\varepsilon + 2u^\varepsilon \cdot \nabla^\varepsilon u^\varepsilon + \dfrac{1}{\varepsilon^2}\nabla^\varepsilon\Big(\tilde{f}(\varepsilon^2 A^\varepsilon)\Big) = \dfrac{1}{4c^2}\nabla^\varepsilon\Big(\dfrac{\Delta^\varepsilon A^\varepsilon}{1+\varepsilon^2 A^\varepsilon}\Big). \end{cases} \tag{76}$$

We recall that $\tilde{f}$ is defined in (51). Note that $\nabla^\varepsilon \times u^\varepsilon = 0$, hence, we obtain in particular that

$$\partial_x u^\varepsilon_\perp = \varepsilon \nabla_\perp u^\varepsilon_1. \tag{77}$$

We can apply ∂_x to the first equation and the first line of the second equation in (76) to get the system:

$$\begin{cases} \partial_t \partial_x A^\varepsilon + \dfrac{1}{\varepsilon^2}\partial_x\big(\partial_x u^\varepsilon_1 - \partial_x A^\varepsilon\big) = \mathrm{S}^\varepsilon_A \\ \partial_t \partial_x u^\varepsilon_1 + \dfrac{1}{\varepsilon^2}\partial_x\big(\partial_x A^\varepsilon - \partial_x u^\varepsilon_1\big) = \mathrm{S}^\varepsilon_u, \end{cases} \tag{78}$$

where

$$\begin{aligned} \mathrm{S}^\varepsilon_A &\equiv -\partial_x\big(2u^\varepsilon \cdot \nabla^\varepsilon A^\varepsilon + A^\varepsilon \nabla^\varepsilon \cdot u^\varepsilon\big) - \frac{1}{\varepsilon}\partial_x \nabla_\perp \cdot u^\varepsilon_\perp \\ \mathrm{S}^\varepsilon_u &\equiv -\partial_x\Big(2u^\varepsilon \cdot \nabla^\varepsilon u^\varepsilon_1 + \frac{1}{\varepsilon^2}\partial_x\Big[\tilde{f}(\varepsilon^2 A^\varepsilon)\Big]\Big) + \frac{1}{4c^2}\partial_x^2\Big(\frac{\Delta^\varepsilon A^\varepsilon}{1+\varepsilon^2 A^\varepsilon}\Big). \end{aligned}$$

By using (77) and the H^{s+1} bound (75) which holds for $s > 1 + n/2 \ge 3/2$, we get the uniform estimate

$$\|(\mathrm{S}^\varepsilon_A, \mathrm{S}^\varepsilon_u)\|_{H^{-2}(\mathbb{R}^n)} \le C, \quad \forall t \in [0,T], \quad \forall \varepsilon \in (0,\varepsilon_0]$$

for some $C > 0$.

Consequently, from the proof of Lemma 3 (it suffices to integrate also with respect to the transverse variable), we get that: $\partial_x A^\varepsilon$ and $\partial_x u^\varepsilon_1$ are uniformly bounded in $H^{\frac{1}{2}}\big((0,T), H^{-3}_{loc}(\mathbb{R}^n)\big)$ and also (see (46)) that

$$\partial_x A^\varepsilon - \partial_x u^\varepsilon_1 = \mathcal{O}(\varepsilon) \quad \text{in } L^2\big((0,T), H^{-2}_{loc}(\mathbb{R}^n)\big). \tag{79}$$

Consequently, we can use again the relative compactness criterion of [25] and (75) to get that $\partial_x A^\varepsilon$ and $\partial_x u^\varepsilon_1$ are strongly compact in $L^2\big((0,T), H^\sigma_{loc}(\mathbb{R}^n)\big)$ and $L^2\big((0,T), H^{\sigma-1}_{loc}(\mathbb{R}^n)\big)$ respectively for every $\sigma < s$. Note that since $s > 1$, one can choose $\sigma > 1$. Consequently, the way to recover the weak form of the KP-I or KdV equation will be very close to what was done in the proof of Theorem 3. We can take a subsequence $\varepsilon_j \to 0$ such that, for $1 < \sigma < s$,

$$\begin{aligned}
\partial_x A^{\varepsilon_j} \to \partial_x A & \qquad \text{strongly in } L^2\big((0,T), H^\sigma_{loc}(\mathbb{R}^n)\big), \\
\partial_x u_1^{\varepsilon_j} \to \partial_x u_1 & \qquad \text{strongly in } L^2\big((0,T), H^{\sigma-1}_{loc}(\mathbb{R}^n)\big), \\
A^{\varepsilon_j} \rightharpoonup A & \qquad \text{weakly in } L^2\big((0,T), H^{s+1}(\mathbb{R}^n)\big), \\
u^{\varepsilon_j} \rightharpoonup u & \qquad \text{weakly in } L^2\big((0,T), H^{s}(\mathbb{R}^n)\big)
\end{aligned}$$

and moreover, from (79), we also have

$$A = u_1 \qquad \text{for almost every } t \in [0,T],\ X \in \mathbb{R}^n. \tag{80}$$

As in the proof of Theorem 3, the above properties are sufficient to pass to the limit in the weak form of the equation satisfied by $\partial_x A^\varepsilon + \partial_x u_1^\varepsilon$. Indeed, by using (77), we get from (76) that

$$\begin{aligned}
&\int_{\mathbb{R}_+\times\mathbb{R}^n} \partial_x\big(A^{\varepsilon_j} + u_1^{\varepsilon_j}\big)\partial_t\zeta + \Big(2u_1^{\varepsilon_j}\big(\partial_x A^{\varepsilon_j} + \partial_x^{\varepsilon_j} u_1^{\varepsilon_j}\big) + A^{\varepsilon_j}\partial_x u_1^{\varepsilon_j} + \partial_x A^{\varepsilon_j}\tilde{f}'\big(\varepsilon_j^2 A^{\varepsilon_j}\big)\Big)\partial_x\zeta \; dtdX \\
&- \int_{\mathbb{R}_+\times\mathbb{R}^n} \Delta_\perp u_1^{\varepsilon_j}\zeta \; dtdX + \frac{1}{4c^2}\int_{\mathbb{R}_+\times\mathbb{R}^n} \partial_x^2 A^{\varepsilon_j}\partial_x^2\zeta \; dtdX = \int_{\mathbb{R}^n} \partial_x\big(A_0^{\varepsilon_j} + (u_0)_1^{\varepsilon_j}\big)\zeta(0,X)\,dX + R^{\varepsilon_j}
\end{aligned}$$

for every $\zeta \in \mathcal{C}_c^\infty(\mathbb{R}\times\mathbb{R}^n)$, where thanks to the uniform bound (75), we have

$$|R^{\varepsilon_j}| \leq C\varepsilon_j.$$

We can easily pass to the limit in the above formulation by using that in the nonlinear terms one converges strongly and one weakly. We thus get, by using again an expansion of $\tilde{f}(\varepsilon^2 A^\varepsilon)$, that

$$\int_{\mathbb{R}_+\times\mathbb{R}^n} \Big(2\partial_x A\partial_t\zeta + kA\partial_x A\partial_x\zeta - \Delta_\perp A\,\zeta + \frac{1}{4c^2}\partial_x^2 A\partial_x^2\zeta\Big)dtdX = \int_{\mathbb{R}^n} \partial_x\big(A_0 + (u_0)_1\big)\zeta(0,X)\,dX,$$

which is the weak form of the KP-I equation (or KdV)

$$\partial_x\Big(2\partial_t A + kA\partial_x A - \frac{1}{4c^2}\partial_x^3 A\Big) + \Delta_\perp A = 0$$

with initial value

$$A_{|t=0} = \frac{1}{2}\big(A_0 + \frac{1}{2c}\partial_x\varphi_0\big).$$

Furthermore, thanks to the uniqueness of H^s solutions, $s > 1 + n/2$ for the KP-I equation, we get that the full sequence A^ε, $\partial_x\varphi^\varepsilon$ converges.

Note that in dimension $n = 1$, we can get compactness in time by writting directly that

$$\left\{\begin{aligned}
&\partial_t A^\varepsilon + \frac{1}{\varepsilon^2}\partial_x\big(u_1^\varepsilon - A^\varepsilon\big) = S_A^\varepsilon \\
&\partial_t u_1^\varepsilon + \frac{1}{\varepsilon^2}\partial_x\big(A^\varepsilon - u_1^\varepsilon\big) = S_u^\varepsilon,
\end{aligned}\right.$$

with

$$\big\|(S_A^\varepsilon, S_u^\varepsilon)\big\|_{H^{-1}(\mathbb{R})} \leq C, \quad \forall t \in [0,T], \quad \forall \varepsilon \in (0,\varepsilon_0]$$

for some $C > 0$ since the apparently singular term $\varepsilon^{-1}\nabla_\perp \cdot u^\varepsilon_\perp$ is absent in dimension 1. Then we can finish as in the proof of Theorem 3 and get in particular that A^ε converges strongly towards A in $L^2\big((0,T), H^{\sigma+1}_{loc}(\mathbb{R})\big)$ for $\sigma < s$ (for $n \geq 2$, we have only proven the strong convergence in $L^2\big((0,T), H^{\sigma}_{loc}(\mathbb{R}^n)\big)$ for $\partial_x A^\varepsilon$).

In the general n-dimensional case, it remains to show that, if $u^\varepsilon_\perp = \varepsilon \nabla_\perp \varphi^\varepsilon \to 0$ in L^2, then

$$\frac{1}{2}\big(A^\varepsilon + u^\varepsilon_1\big) \to A \qquad \text{in} \quad L^2\big([0,T], L^2(\mathbb{R}^n)\big).$$

Indeed, the convergences in $L^2\big([0,T], H^\sigma(\mathbb{R}^n)\big)$ for $0 \leq \sigma < s$ will then follow by interpolation in space by using the bounds (20).

In dimension $n \geq 1$, the scaled energy is defined by

$$E^\varepsilon(\psi^\varepsilon) \equiv \frac{1}{2}\int_{\mathbb{R}^n} |\partial_x \psi^\varepsilon|^2 + \varepsilon^2 |\nabla_\perp \psi^\varepsilon|^2 + \frac{1}{\varepsilon^2} F\big(|\psi^\varepsilon|^2\big)\ dX = \frac{\mathcal{E}(\Psi)}{\varepsilon^{2n-1}},$$

and we recall the expansion to second order

$$F(R) = c^2\big(R-1\big)^2 + F_3(R), \qquad \text{with} \quad F_3(1+r) = \mathcal{O}(r^3), \quad r \to 0.$$

Moreover, we have, on $[0,T]$,

$$\psi^\varepsilon = \rho^\varepsilon \exp\big(i\varepsilon\varphi^\varepsilon\big), \quad \rho^\varepsilon = 1 + \varepsilon^2 A^\varepsilon,$$

and using that for $1 \leq j \leq n$, $|\partial_j \psi^\varepsilon|^2 = \varepsilon^4 (\partial_j A^\varepsilon)^2 + \varepsilon^2 (\rho^\varepsilon)^2 (\partial_j \varphi^\varepsilon)^2$, we infer as in the proof of Lemma 2 the following equality:

$$\begin{aligned} E^\varepsilon(\psi^\varepsilon) &= \frac{\varepsilon^2}{2}\int_{\mathbb{R}^n} (\partial_x\varphi^\varepsilon)^2 + \frac{c^2}{\varepsilon^4}\big((\rho^\varepsilon)^2 - 1\big)^2 + \big((\rho^\varepsilon)^2 - 1\big)\cdot(\partial_x\varphi^\varepsilon)^2 + \varepsilon^2(\partial_x A^\varepsilon)^2\ dX \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^n} \varepsilon^4 (\rho^\varepsilon)^2 |\nabla_\perp \varphi^\varepsilon|^2 + \varepsilon^2 |\nabla_\perp \rho^\varepsilon|^2 + \frac{1}{\varepsilon^2} F_3\big((\rho^\varepsilon)^2\big)\ dX \\ &= \frac{\varepsilon^2}{2}\int_{\mathbb{R}^n} (\partial_x\varphi^\varepsilon)^2 + 4c^2 (A^\varepsilon)^2 + \varepsilon^2 |\nabla_\perp\varphi^\varepsilon|^2\ dX + \mathcal{O}(\varepsilon^4) \end{aligned} \tag{81}$$

uniformly on $[0,T]$. To get the last line, we have used (20), which yields that $\|A^\varepsilon\|_{L^\infty(\mathbb{R}^n)} \leq K$, hence $\|(\rho^\varepsilon)^2 - 1\|_{L^\infty(\mathbb{R}^n)} \leq K\varepsilon^2$,

$$\Big|\int_{\mathbb{R}^n} \big((\rho^\varepsilon)^2 - 1\big)(\partial_x\varphi^\varepsilon)^2\ dX\Big| \leq K\varepsilon^2 \qquad \text{and} \qquad \Big|\int_{\mathbb{R}^n} \frac{1}{\varepsilon^2} F_3\big((\rho^\varepsilon)^2\big)\ dx\Big| \leq K\varepsilon^4.$$

Furthermore, we may define (if $n \geq 2$) the momentum in the x direction by

$$P^\varepsilon\big(\psi^\varepsilon\big) \equiv \frac{\varepsilon}{2}\int_{\mathbb{R}^n} \big((\rho^\varepsilon)^2 - 1\big)\partial_x \varphi^\varepsilon\ dX$$

for maps $\psi^\varepsilon = \rho^\varepsilon e^{i\varepsilon\varphi^\varepsilon}$ with $\rho^\varepsilon = |\psi^\varepsilon| \geq 1/2$. In view of the bounds (20), $|\psi^\varepsilon| \geq 1/2$ on $[0,T]$ (for $0 < \varepsilon \leq \varepsilon_0$), hence ψ^ε has a well-defined momentum, which is independent of $t \in [0,T]$. Moreover, there holds, uniformly on $[0,T]$,

$$\begin{aligned} P^\varepsilon(\psi^\varepsilon) = \frac{\varepsilon}{2}\int_{\mathbb{R}^n} \big((\rho^\varepsilon)^2 - 1\big)\partial_x\varphi^\varepsilon\ dX &= \frac{\varepsilon^2}{2}\int_{\mathbb{R}^n} \big(2A^\varepsilon + \varepsilon^2 (A^\varepsilon)^2\big)\partial_x\varphi^\varepsilon\ dX \\ &= \varepsilon^2 \int_{\mathbb{R}^n} A^\varepsilon \partial_x \varphi^\varepsilon\ dX + \mathcal{O}(\varepsilon^4). \end{aligned} \tag{82}$$

As a consequence, in view of (20),

$$E^\varepsilon(\psi^\varepsilon) + 2cP^\varepsilon(\psi^\varepsilon) = 2c^2\varepsilon^2 \int_{\mathbb{R}^n} \big(A^\varepsilon + u_1^\varepsilon\big)^2 + |u_\perp^\varepsilon|^2 \, dX + \mathcal{O}(\varepsilon^4)$$

uniformly on $[0,T]$. At the initial time $t=0$, we have

$$E^\varepsilon(\psi_0^\varepsilon) + 2cP^\varepsilon(\psi_0^\varepsilon) = 2c^2\varepsilon^2 \int_{\mathbb{R}^n} \big(A_0^\varepsilon + (u_0^\varepsilon)_1\big)^2 + |(u_0^\varepsilon)_\perp|^2 \, dX + \mathcal{O}(\varepsilon^4),$$

hence, by conservation of $E^\varepsilon(\psi^\varepsilon) + 2cP^\varepsilon(\psi^\varepsilon)$ for $0 \le t \le T$,

$$\int_{\mathbb{R}^n} \big(A^\varepsilon(t) + u_1^\varepsilon(t)\big)^2 + \big|u_\perp^\varepsilon(t)\big|^2 \, dX = \int_{\mathbb{R}^n} \big(A_0^\varepsilon + (u_0^\varepsilon)_1\big)^2 + \big|(u_0^\varepsilon)_\perp\big|^2 \, dX + \mathcal{O}(\varepsilon^2), \tag{83}$$

uniformly for $t \in [0,T]$. We consider now

$$\nu^\varepsilon \equiv \int_0^T \big\|A^\varepsilon + u_1^\varepsilon - 2A\big\|^2_{L^2(\mathbb{R}^n)} + \big\|u_\perp^\varepsilon\big\|^2_{L^2(\mathbb{R}^n)} \, dt.$$

Expansion gives

$$\nu^\varepsilon = \int_0^T \big\|A^\varepsilon + u_1^\varepsilon\big\|^2_{L^2(\mathbb{R}^n)} + \big\|u_\perp^\varepsilon\big\|^2_{L^2} - 4\big\|A\big\|^2_{L^2(\mathbb{R}^n)} \, dt - 4\int_0^T \big(A^\varepsilon + u_1^\varepsilon - 2A, A\big\rangle_{L^2(\mathbb{R}^n)} \, dt.$$

One can show exactly as in the end of subsect. 2.4 that since $A \in \mathcal{C}\big([0,T], L^2(\mathbb{R}^n)\big)$ and A^ε, u_1^ε converge to A weakly in $L^2\big([0,T], L^2_{loc}(\mathbb{R}^n)\big)$, then

$$\int_0^T \big(A^\varepsilon + u_1^\varepsilon - 2A, A\big)_{L^2(\mathbb{R}^n)} \, dt \to 0 \qquad \text{as} \qquad \varepsilon \to 0.$$

Moreover, since the L^2 norm of the solution A of KdV/KP-I does not depend on time,

$$\big\|2A(t)\big\|_{L^2(\mathbb{R}^n)} = \big\|2A_{|t=0}\big\|_{L^2(\mathbb{R}^n)} = \big\|A_0 + (u_0)_1\big\|_{L^2(\mathbb{R}^n)}.$$

Hence, by using (83), we find after an integration in time that

$$\nu^\varepsilon = T\Big(\big\|A_0^\varepsilon + (u_0^\varepsilon)_1\big\|^2_{L^2(\mathbb{R}^n)} - \big\|A_0 + (u_0)_1\big\|^2_{L^2(\mathbb{R}^n)} + \big\|(u_0^\varepsilon)_\perp\big\|^2_{L^2(\mathbb{R}^n)}\Big) + o(1).$$

Thanks to our assumption (21), we thus get $\nu^\varepsilon \to 0$ as required.

3.3 Proof of Theorem 6

To use the assumption (22) in order to get the convergence in stronger norms, we will follow the lines of the proof of Lemma 2. From (81), we infer

$$\begin{aligned} E^\varepsilon(\psi^\varepsilon) - 2cP^\varepsilon(\psi^\varepsilon) = \frac{\varepsilon^2}{2} \int_{\mathbb{R}^n} & \big((\rho^\varepsilon)^2 - 1\big)(\partial_x \varphi^\varepsilon)^2 + \varepsilon^2(\partial_x A^\varepsilon)^2 \\ & + \Big(\partial_x \varphi^\varepsilon - \frac{c}{\varepsilon^2}\big((\rho^\varepsilon)^2 - 1\big)\Big)^2 + \varepsilon^2(\rho^\varepsilon)^2 |\nabla_\perp \varphi^\varepsilon|^2 \, dX \\ & + \frac{1}{2} \int_{\mathbb{R}^n} \varepsilon^6 |\nabla_\perp A^\varepsilon|^2 + \frac{1}{\varepsilon^2} F_3\big((\rho^\varepsilon)^2\big) \, dX. \end{aligned} \tag{84}$$

Let

$$\delta^\varepsilon \equiv \big\|\partial_x \varphi_0^\varepsilon - 2cA_0^\varepsilon\big\|_{L^2(\mathbb{R}^n)},$$

which tends to zero by assumption. As in the proof of Lemma 2, we have by using (22) in the case $n \geq 2$ the following upper bounds

$$E^\varepsilon(\psi_0^\varepsilon) = \frac{\varepsilon^2}{2}\int_{\mathbb{R}^n} 4c^2(A_0^\varepsilon)^2 + (\partial_x\varphi_0^\varepsilon)^2 \, dX + \mathcal{O}(\varepsilon^4) = 4c^2\varepsilon^2 \int_{\mathbb{R}^n} A_0^2 \, dX + o(\varepsilon^2) \leq K\varepsilon^2 \tag{85}$$

and

$$E^\varepsilon(\psi_0^\varepsilon) - 2cP^\varepsilon(\psi_0^\varepsilon) \leq K\varepsilon^4 + \varepsilon^2(\delta^\varepsilon)^2.$$

Note that here, we have used that

$$\begin{aligned} \Big\|\partial_x\varphi_0^\varepsilon - \frac{c}{\varepsilon^2}\big((\rho_0^\varepsilon)^2 - 1\big)\Big\|_{L^2(\mathbb{R}^n)} &= \big\|\partial_x\varphi_0^\varepsilon - 2cA_0^\varepsilon - c\varepsilon^2(A_0^\varepsilon)^2\big\|_{L^2(\mathbb{R}^n)} \\ &\leq \big\|\partial_x\varphi_0^\varepsilon - 2cA_0^\varepsilon\big\|_{L^2(\mathbb{R}^n)} + c\varepsilon^2\big\|(A_0^\varepsilon)^2\big\|_{L^2(\mathbb{R}^n)} \leq \delta^\varepsilon + K\varepsilon^2. \end{aligned}$$

As a consequence, since $E^\varepsilon(\psi^\varepsilon)$ and $P^\varepsilon(\psi^\varepsilon)$ do not depend on time,

$$\begin{aligned} K\varepsilon^4 + \varepsilon^2(\delta^\varepsilon)^2 &\geq E^\varepsilon\big(\psi^\varepsilon(t)\big) - 2cP^\varepsilon\big(\psi^\varepsilon(t)\big) \\ &\geq \frac{\varepsilon^4}{2}\int_{\mathbb{R}^n}(\partial_x A^\varepsilon)^2 + (\rho^\varepsilon)^2|\nabla_\perp\varphi^\varepsilon|^2 \, dX + \frac{\varepsilon^2}{2}\int_{\mathbb{R}^n}\Big(\partial_x\varphi^\varepsilon - \frac{c}{\varepsilon^2}\big((\rho^\varepsilon)^2-1\big)\Big)^2 \, dX \\ &\quad - \frac{1}{2}\Big|\int_{\mathbb{R}^n}\big((\rho^\varepsilon)^2 - 1\big)(\partial_x\varphi^\varepsilon)^2 \, dX\Big| - \Big|\int_{\mathbb{R}^n}\frac{1}{2\varepsilon^2}F_3\big(\rho^2\big) \, dX\Big| \\ &\geq \frac{\varepsilon^4}{2}\int_{\mathbb{R}^n}(\rho^\varepsilon)^2|\nabla_\perp\varphi^\varepsilon|^2 \, dX + \frac{\varepsilon^2}{2}\int_{\mathbb{R}^n}\Big(\partial_x\varphi^\varepsilon - \frac{c}{\varepsilon^2}\big((\rho^\varepsilon)^2-1\big)\Big)^2 \, dX - K\varepsilon^4. \end{aligned} \tag{86}$$

This gives the estimate

$$\sup_{0\leq t\leq T}\int_{\mathbb{R}^n}\Big(\partial_x\varphi^\varepsilon - \frac{c}{\varepsilon^2}\big((\rho^\varepsilon)^2-1\big)\Big)^2 \, dX \leq K\varepsilon^2 + 2(\delta^\varepsilon)^2 \to 0 \qquad \text{as} \quad \varepsilon \to 0 \tag{87}$$

in all dimensions $n \geq 1$. Furthermore, if $n \geq 2$, since $\delta^\varepsilon = \mathcal{O}(\varepsilon)$, we also get from (86) that

$$\int_{\mathbb{R}^n}(\rho^\varepsilon)^2|\nabla_\perp\varphi^\varepsilon|^2 \, dX \leq K.$$

Thus, we have obtained (23) since $\rho^\varepsilon \geq 1/2$.

From (76), $A^\varepsilon + u_1^\varepsilon$ solves

$$\partial_t\big(A^\varepsilon + u_1^\varepsilon\big) + 2u^\varepsilon\cdot\nabla^\varepsilon\big(A^\varepsilon + u_1^\varepsilon\big) + (k-5)A^\varepsilon\partial_x A^\varepsilon + A^\varepsilon\nabla^\varepsilon\cdot u^\varepsilon + \Delta_\perp\varphi^\varepsilon = \partial_x\Big(\frac{\Delta^\varepsilon A^\varepsilon}{4c^2\rho^\varepsilon}\Big).$$

In view of the the H^s bounds (20) in Theorem 4, and possibly (23) if $n \geq 2$, we then infer

$$\big\|A^\varepsilon + u_1^\varepsilon\big\|_{\mathcal{C}([0,T],H^s(\mathbb{R}^n))} \leq K \qquad \text{and} \qquad \big\|\partial_t\big(A^\varepsilon + u_1^\varepsilon\big)\big\|_{L^\infty([0,T],H^{-1}(\mathbb{R}^n))} \leq K. \tag{88}$$

This implies, by Aubin-Lions's Lemma (see, *e.g.*, [25]), that for any $0 \leq \sigma < s$, $A^\varepsilon + u_1^\varepsilon$ is precompact in $\mathcal{C}\big([0,T], H^\sigma_{loc}(\mathbb{R}^n)\big)$. From (87), we know that

$$\partial_x \varphi^\varepsilon - 2cA^\varepsilon = 2c\big(u_1^\varepsilon - A^\varepsilon\big) \to 0 \qquad \text{in} \quad \mathcal{C}\big([0,T], L^2(\mathbb{R}^n)\big).$$

Combining this with the H^s bounds (20), this yields, by interpolation, for $0 \leq \sigma < s$,

$$\partial_x \varphi^\varepsilon - 2cA^\varepsilon = 2c\big(u_1^\varepsilon - A^\varepsilon\big) \to 0 \qquad \text{in} \quad \mathcal{C}\big([0,T], H^\sigma(\mathbb{R}^n)\big).$$

In particular,

$$A^\varepsilon \to A \qquad \text{and} \qquad \partial_x \varphi^\varepsilon \to 2cA \qquad \text{in} \quad \mathcal{C}\big([0,T], H^\sigma_{loc}(\mathbb{R}^n)\big).$$

We can now prove that, as $\varepsilon \to 0$,

$$A^\varepsilon \to A \qquad \text{in} \quad \mathcal{C}\big([0,T], L^2(\mathbb{R}^n)\big).$$

Indeed, we may follow the lines of the end of the proof of Theorem 3 in Sect. 2.4 since thanks to (20), (23) (if $n \geq 2$) and (87), the expansion

$$E^\varepsilon(\psi^\varepsilon) = \frac{\varepsilon^2}{2} \int_{\mathbb{R}^n} 4c^2 \big(A^\varepsilon(t)\big)^2 + \big(\partial_x \varphi^\varepsilon(t)\big)^2 \, dX + \mathcal{O}(\varepsilon^4) = 4c^2\varepsilon^2 \int_{\mathbb{R}^n} \big(A^\varepsilon(t)\big)^2 \, dX + o(\varepsilon^2)$$

holds uniformly for $0 \leq t \leq T$ and $\mathcal{I}_0\big(A(t)\big) = \|A(t)\|^2_{L^2(\mathbb{R}^n)} = \|A_0\|^2_{L^2(\mathbb{R}^n)}$ do not depend on $t \in [0,T]$. Notice indeed that in this case, the initial datum for KdV/KP-I is

$$A_{|t=0} = \frac{1}{2}\big(A_0 + \frac{1}{2c}\partial_x \varphi_0\big) = A_0,$$

From the H^s bounds (20) and by interpolation in space, we finally get for $0 \leq \sigma < s$ that

$$A^\varepsilon \to A \qquad \text{in} \quad \mathcal{C}\big([0,T], H^{\sigma+1}(\mathbb{R}^n)\big) \qquad \text{and} \qquad \partial_x \varphi^\varepsilon \to 2cA \qquad \text{in} \quad \mathcal{C}\big([0,T], H^\sigma(\mathbb{R}^n)\big).$$

The proof is complete.

References

[1] T. Alazard and R. Carles, *Supercritical geometric optics for Nonlinear Schrödinger equations.* Arch. Ration. Mech. Anal. **194**, no. 1 (2009), 315-347.

[2] B. Alvarez-Samaniego and D. Lannes, *Large time existence for 3D water-waves and asymptotics.* Invent. Math. **171**, no. 3 (2008), 485-541.

[3] N. Berloff and P. Roberts, *Motions in a Bose condensate: X. New results on stability of axisymmetric solitary waves of the Gross-Pitaevskii equation.* J. Phys. A: Math. Gen., **37** (2004), 11333-11351.

[4] F. Béthuel, R. Danchin and D. Smets, *On the linear wave regime of the Gross-Pitaevskii equation.* Preprint.

[5] F. Béthuel, P. Gravejat and J-C. Saut, *On the KP-I transonic limit of two-dimensional Gross-Pitaevskii travelling waves.* Dynamics of PDE **5**, 3 (2008), 241-280.

[6] F. Béthuel, P. Gravejat, J-C. Saut and D. Smets, *On the Korteweg-de Vries long-wave transonic approximation of the Gross-Pitaevskii equation.* To appear in Internat. Math. Res. Notices

[7] D. Chiron and F. Rousset, *Geometric optics and boundary layers for Nonlinear Schrödinger equations*, Comm. Math. Phys. **288**, no. 2 (2009), 503-546.

[8] C. Gallo, *The Cauchy Problem for defocusing Nonlinear Schrödinger equations with non-vanishing initial data at infinity.* Comm. Partial Differential Equations **33**, no. 4-6 (2008), 729-771.

[9] P. Gérard, *The Gross-Pitaevskii equation in the energy space.* Stationary and Time Dependent Gross-Pitaevskii Equations", A. Farina and J.-C. Saut editors, Contemporary Mathematics, American Mathematical Society (2008).

[10] V. Ginzburg and L. Pitaevskii, *On the theory of superfluidity.* Sov. Phys. JETP **34** (1958), 1240.

[11] E. Grenier, *Semiclassical limit of the nonlinear Schrödinger equation in small time.* Proc. Amer. Math. Soc. **126**, no. 2 (1998), 523-530.

[12] E. Grenier, *Pseudo-differential energy estimates of singular perturbations.* Comm. Pure Appl. Math. **50**, no. 9 (1997), 821-865.

[13] E. Gross, *Hydrodynamics of a superfluid condensate,* J. Math. Phys. **4**, (2) (1963), 195-207.

[14] A. Ionescu, C. Kenig and D. Tataru, *Global well-posedness of the KP-I initial-value problem in the energy space.* Invent. Math. **173**, no. 2 (2008), 265-304.

[15] C. Jones and P. Roberts, *Motion in a Bose condensate: IV. Axisymmetric solitary waves.* J. Phys. A: Math. Gen., **15** (1982), 2599-2619.

[16] C. Kenig, G. Ponce and L. Vega, *Well-posedness of the initial value problem for the Korteweg-de Vries equation.* J. Amer. Math. Soc. **4**, no. 2 (1991), 323-347.

[17] C. Kenig, G. Ponce and L. Vega, *A bilinear estimate with applications to the KdV equation* J. Amer. Math. Soc. **9**, no. 2 (1996), 573-603.

[18] Y. Kivshar, D. Anderson and M. Lisak, *Modulational instabilities and dark solitons in a generalized nonlinear Schrödinger-equation.* Phys. Scr. **47**, (1993) 679-681.

[19] Y. S. Kivshar and B. Luther-Davies, *Dark optical solitons: physics and applications.* Physics Reports **298** (1998), 81-197.

[20] S. Klainerman and A. Majda, *Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids.* Comm. Pure Appl. Math. **34**, no. 4 (1981), 481-524.

[21] Y. Liu, *Strong instability of solitary-wave solutions to a Kadomtsev-Petviashvili equation in three dimensions.* J. Differential Equations, **180** no. 1 (2002), 153-170.

[22] L. Molinet, J.-C. Saut and N. Tzvetkov, *Global well-posedness for the KP-I equation.* Math. Ann. **324**, no. 2 (2002), 255-275.

[23] P. Roberts and N. Berloff, *Nonlinear Schrödinger equation as a model of superfluid helium.* In "Quantized Vortex Dynamics and Superfluid Turbulence" edited by C.F. Barenghi, R.J. Donnelly and W.F. Vinen, Lecture Notes in Physics, volume 571, Springer-Verlag, 2001.

[24] S. Schochet, *Asymptotics for symmetric hyperbolic systems with a large parameter.* J. Differential Equations **75**, no. 1 (1988), 1-27.

[25] J. Simon, *Compact sets in the space* $L^p(0,T;B)$. Ann. Mat. Pura Appl. **146**, (4) (1987), 65-96.

[26] T. Tsuzuki, *Nonlinear waves in the Pitaevskii-Gross equation,* J. Low Temp. Phys. **4**, no. 4 (1971) 441-457.

[27] P. Zhidkov, *Korteweg-De Vries and nonlinear Schrödinger equations: qualitative theory,* Volume 1756 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2001.

[28] Y. Zhou, *Uniqueness of weak solution of the KdV equation.* Internat. Math. Res. Notices, no. 6 (1997), 271-283.