On the definitions of Sobolev and BV spaces into singular spaces and the trace problem

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Abstract

The purpose of this paper is to relate two notions of Sobolev and BV spaces into metric spaces, due to N. Korevaar and R. Schoen on the one hand, and J. Jost on the other hand. We prove that these two notions coincide and define the same p-energies. We review also other definitions, due to L. Ambrosio (for BV maps into metric spaces), Y.G. Reshetnyak and finally to the notion of Newtonian-Sobolev spaces. These last approaches define the same Sobolev (or BV) spaces, but with a different energy, which does not extend the standard Dirichlet energy. We also prove a characterization of Sobolev spaces in the spirit of J. Bourgain, H. Brezis and P. Mironescu in terms of "limit" of the space $W^{s,p}$ as $s \to 1$, $0 < s < 1$, and finally following the approach proposed by H.M. Nguyen. We also establish the $W^{s-\frac{1}{p},p}$ regularity of traces of maps in $W^{s,p}$ ($0 < s \leq 1 < sp$).

Keywords: Sobolev spaces; BV maps; metric spaces; traces.

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1 Introduction

The aim of this paper is to relate various definitions of Sobolev and BV spaces into metric spaces. The first one is due to N. Korevaar and R. Schoen (see [12]), which follows the pioneering work of M. Gromov and R. Schoen [9]. The second one is the approach of J. Jost ([11]). We will next focus on other approaches. For the domain, we restrict ourselves to an open bounded and smooth subset Ω of $\mathbb{R}^N$, $N \in \mathbb{N}^*$. Many results extend straightforwardly when Ω is a smooth riemannian manifold. Extensions are also possible when Ω is a measured metric space (see [11], [20], [10]). However, we do not aim such a generality. For the target space, we will work with a complete metric space (X, d).

First, we recall the definition of $L^p(\Omega, X)$, for $1 \leq p \leq \infty$. A measurable map with separable essential range $u : \Omega \to X$ is said to be in $L^p(\Omega, X)$ if there exists $z \in X$ such that $d(u(\cdot), z) \in L^p(\Omega, \mathbb{R})$. If Ω is bounded (and more generally if $|\Omega| < \infty$), then the point z is not relevant: if $u \in L^p(\Omega, X)$, then for any $z \in X$, $d(u(.), z) \in L^p(\Omega, \mathbb{R})$. The space $L^p(\Omega, X)$ is complete for the distance

$$d_{L^p}(u, v) \equiv \Big(\int_\Omega d(u(x), v(x))^p \ dx\Big)^{\frac{1}{p}}$$

if $1 \leq p < \infty$ and for $p = \infty$, is complete for the distance

$$d_{L^\infty}(u,v) \equiv \text{supess}_{x\in\Omega} d(u(x), v(x)).$$

We recall the definitions of Sobolev spaces $W^{1,p}(\Omega, X)$, $1 < p < \infty$ and $BV(\Omega, X)$ spaces proposed by these authors. These spaces naturally appeared in the theory of harmonic functions with values into spaces coming from complex group actions, or into infinite dimensional spaces. This is the reason why [12] developped in the context of metric spaces targets the well-known theory of Sobolev maps: compact embeddings, Poincaré inequality, traces, regularity results for minimizing maps for non-positively curved metric spaces (even though X is only a metric space, the notion of "non-positively curved metric space" does make sense, and we refer to [12] for instance for the definition)... Since we allow the target space X to be singular, one can not reasonably define Sobolev spaces of higher order. Our interest for these spaces was motivated by the study of topological defects in ordered media, such as liquid crystal, where an energy of the type Dirichlet integral plus a potential term appears naturally for maps with values into cones (see [6]). It is then important that this Sobolev theory extends naturally the usual Dirichlet integral.

1.1 Definition of $W^{1,p}(\Omega, X)$ and $BV(\Omega, X)$.

Let $1 \leq p < \infty$. We recall in this subsection the definition of Sobolev spaces of N. Korevaar and R. Schoen given in [12], naturally based on limits of finite differences in L^p. For $\varepsilon > 0$, we set

$$\Omega^\varepsilon \equiv \{x \in \Omega,\ d(x, \partial\Omega) > \varepsilon\}.$$

For $u \in L^p(\Omega, X)$ and $\varepsilon > 0$, we introduce[1]

$$e_\varepsilon(u)(x) \equiv \begin{cases} \dfrac{1}{|\mathbb{S}^{N-1}|} \displaystyle\int_{S_\varepsilon(x)} \dfrac{d(u(x), u(y))^p}{\varepsilon^{N+p-1}}\ d\mathcal{H}^{N-1}(y) & \text{if } x \in \Omega^\varepsilon, \\ \\ 0 & \text{otherwise.} \end{cases}$$

Here, $S_R(x)$ is the sphere in Ω of center x and radius $R > 0$. For $u \in L^p(\Omega, X)$, $e_\varepsilon(u)$ has a meanning as an $L^1(\Omega, \mathbb{R})$ function. We then consider the linear functional E^u_ε on $\mathcal{C}_c(\Omega)$ defined for $f \in \mathcal{C}_c(\Omega)$ by

$$E^u_\varepsilon(f) \equiv \int_\Omega f(x) e_\varepsilon(u)(x)\ dx.$$

Next, we set

$$E^p(u) \equiv \sup_{f\in\mathcal{C}_c(\Omega),\ 0\leq f\leq 1} \Big(\limsup_{\varepsilon\to 0} E^u_\varepsilon(f) \Big) \in \mathbb{R}_+ \cup \{\infty\}.$$

Definition 1 **([12])** *Let $1 \leq p < \infty$. A map $u \in L^p(\Omega, X)$ is said to be in $W^{1,p}(\Omega, X)$ for $1 < p < \infty$ or in $BV(\Omega, X)$ for $p = 1$ if and only if $E^p(u) < \infty$.*

We summarize now some of the main properties of the spaces $W^{1,p}(\Omega, X)$ and $BV(\Omega, X)$, which come from the theory developped in [12] (section 1 there).

[1]Notice that we have chosen to divide the original density measure $e_\varepsilon(u)$ of [12] by the factor $|\mathbb{S}^{N-1}|$ so that all approximate derivatives are based on *averages*.

Proposition 1 **([12])** *Let $1 \le p < \infty$. Then, for every $u \in L^p(\Omega, X)$ such that $E^p(u) < \infty$, there exists a non-negative Radon measure $|\nabla u|_p$ in Ω such that*

$$e_\varepsilon(u) \rightharpoonup |\nabla u|_p$$

weakly as measures as $\varepsilon \to 0$ (hence $E^p(u) = |\nabla u|_p(\Omega)$). Moreover, if $1 < p < \infty$,

$$|\nabla u|_p \in L^1(\Omega, \mathbb{R}_+).$$

Remark 1 We lay the emphasis on the fact that the weak convergence of $e_\varepsilon(u)$ to $|\nabla u|_p$ as $\varepsilon \to 0$ holds for the *real parameter* $\varepsilon \to 0$ and not for a *sequence* $\varepsilon_n \to 0$. This is due to a "monotonicity" property (see Lemma 5 below).

The last statement in Proposition 1 is valid only for $p > 1$, which motivates the definition of the space $W^{1,1}(\Omega, X)$.

Definition 2 **([12])** *We define $W^{1,1}(\Omega, X) \equiv \{u \in BV(\Omega, X),\ |\nabla u|_1 \in L^1(\Omega, \mathbb{R}_+)\}$.*

Let us define for $1 \le p < \infty$, the constants $0 < K_{p,N} \le 1$ by

$$K_{p,N} \equiv \frac{1}{|\mathbb{S}^{N-1}|} \int_{\mathbb{S}^{N-1}} |\omega \cdot \vec{e}|^p \, d\mathcal{H}^{N-1}(\omega), \tag{1}$$

$\vec{e}$ denoting any unit vector in $\mathbb{R}^N$. Definitions 1 and 2 extend the classical Sobolev spaces $W^{1,p}(\Omega, \mathbb{R})$ and $BV(\Omega, \mathbb{R})$.

Theorem 1 **([12])** *Assume $X = \mathbb{R}$ is endowed with the standard distance. Then, for $1 \le p < \infty$,*

$$W^{1,p}(\Omega, X) = W^{1,p}(\Omega, \mathbb{R}) \qquad \text{and} \qquad BV(\Omega, X) = BV(\Omega, \mathbb{R}).$$

Moreover, if $u \in W^{1,p}(\Omega, \mathbb{R})$ for $1 \le p < \infty$ or $u \in BV(\Omega, \mathbb{R})$, then

$$E^p(u) = K_{p,N} \int_\Omega |\nabla u|^p \qquad \text{or} \qquad E^1(u) = K_{1,N} |\nabla u|(\Omega),$$

Remark 2 If $p = 2$ and $X = \mathbb{R}^n$ (euclidean), $n \in \mathbb{N}^*$ arbitrary, then, by Theorem 1, we have

$$E^2(u) = K_{2,N} \int_\Omega |\nabla u|^2,$$

hence E^2 coincides, up to a constant factor $K_{2,N}$ with the usual Dirichlet energy. This remains true if X is a smooth complete riemannian manifold. However, for $X = \mathbb{R}^n$ but $p \neq 2$, $|\nabla u|_p$ or $|\nabla u|_1$ (in the sense of measures) may not be equal (even up to a constant factor) to the standard quantities $|\nabla u|^p$ or $|\nabla u|$ (in the sense of measures).

The second result is the lower-semicontinuity of E^p for the $L^p(\Omega, X)$ topology.

Theorem 2 **([12])** *Let $1 \le p < \infty$. Then, the p-energy is lower semicontinuous for the strong $L^p(\Omega, X)$ topology. In other words, if $u_n \to u$ in $L^p(\Omega, X)$ as $n \to +\infty$, then*

$$E^p(u) \le \liminf_{n \to +\infty} E^p(u_n) \in \mathbb{R}_+ \cup \{+\infty\}.$$

1.2 Alternative definition (J. Jost)

We recall now the alternate definition proposed by J. Jost ([11]). We mention that this work considers a metric measured space as a domain instead of an open subset of $\mathbb{R}^N$ or a riemannian manifold as in [12]. We will however not consider this general setting. Let, for $x \in \mathbb{R}^N$ and $\varepsilon > 0$,

$$\mu_x^\varepsilon(y) \equiv dy \llcorner B_\varepsilon(x), \tag{2}$$

and consider the functionals for $1 \leq p < \infty$

$$J_\varepsilon^p(u) \equiv \int_\Omega \frac{\int_\Omega d(u(x), u(y))^p \; d\mu_x^\varepsilon(y)}{\int_\Omega |x-y|^p \; d\mu_x^\varepsilon(y)} \; dx.$$

Note that the initial point of view of J. Jost was to deal with harmonic maps, and thus he only considered the case $p = 2$. The definition relies on Γ-convergence (see [7]), and is as follows. First, we define $J^p : L^p(\Omega, X) \to \mathbb{R}_+ \cup \{\infty\}$ to be the Γ-limit of J_ε^p as $\varepsilon \to 0$, or for a subsequence $\varepsilon_n \to 0$ as $n \to +\infty$, for the $L^p(\Omega, X)$ topology. We recall that this means that for any sequence $u_\varepsilon \to u$ as $\varepsilon \to 0$ (or $u_n \to u$ as $n \to +\infty$) in $L^p(\Omega, X)$,

$$\liminf_{\varepsilon \to 0} J_\varepsilon^p(u_\varepsilon) \geq J^p(u) \qquad \Big(\text{or } \liminf_{n \to +\infty} J_{\varepsilon_n}^p(u_n) \geq J^p(u)\Big),$$

and there exists a sequence $u_\varepsilon \to u$ as $\varepsilon \to 0$ (or $u_n \to u$ as $n \to +\infty$) in $L^p(\Omega, X)$ such that

$$\lim_{\varepsilon \to 0} J_\varepsilon^p(u_\varepsilon) = J^p(u) \qquad \Big(\text{or } \lim_{n \to +\infty} J_{\varepsilon_n}^p(u_n) = J^p(u)\Big).$$

The existence of this Γ-limit for *some* subsequence $\varepsilon_n \to 0$ as $n \to +\infty$ is guaranteed if $L^p(\Omega, X)$ satisfies the second axiom of countability (see [7]).

Definition 3 ([11]) *Let be given $1 \leq p < \infty$ and $u \in L^p(\Omega, X)$. Then, u is said to be in $\mathcal{W}^{1,p}(\Omega, X)$ for $1 < p < \infty$ or in $\mathcal{BV}(\Omega, X)$ for $p = 1$ if and only if*

$$J^p(u) < \infty.$$

We denote for the moment $\mathcal{W}^{1,p}(\Omega, X)$ and $\mathcal{BV}(\Omega, X)$ since we do not know yet that these spaces are actually $W^{1,p}(\Omega, X)$ and $BV(\Omega, X)$. Notice that the approach of J. Jost does not allow to define directly $W^{1,1}(\Omega, X)$.

Let us point out that the notion of Γ-convergence is in general stated for a *countable* sequence $\varepsilon_n \to 0$, and not for a *real* parameter $\varepsilon \to 0$. One main problem with Definition 3 is that we do not know at this stage if the Γ-limit has to be taken for the full family $\varepsilon \to 0$, or for a subsequence $\varepsilon_n \to 0$. In particular, it is not shown in [11] that the functional J does not depend on the choice of the subsequence $\varepsilon_n \to 0$. This will be however a consequence of our result, and in fact that the Γ-convergence holds for the full family $\varepsilon \to 0$.

Theorem 3 *Let $1 \leq p < \infty$. As $\varepsilon \to 0$ (resp. for any sequence $\varepsilon_n \to 0$ as $n \to +\infty$), the functional E^p is the pointwise and the Γ-limit, in the $L^p(\Omega, X)$ topology, of the functionals J_ε^p (resp. $J_{\varepsilon_n}^p$). The functional J^p is now well-defined and*

$$J^p = E^p.$$

In particular, for $1 < p < \infty$,

$$W^{1,p}(\Omega, X) = \mathcal{W}^{1,p}(\Omega, X) \qquad \text{and} \qquad BV(\Omega, X) = \mathcal{BV}(\Omega, X).$$

This theorem clarifies the notion of $W^{1,p}(\Omega, X)$ (for $1 < p < \infty$) and $BV(\Omega, X)$ (for $p = 1$) of J. Jost: J^p does not depend on the choice of some subsequence.

Although natural, we have not been able to find in the literature a result concerning the fact that these two definitions coincide. If the definition of J. Jost allows to derive existence results more general than in [12] for harmonic maps into homotopy classes in non-positively curved metric spaces, [12] gives regularity results (namely lipschitzian). Fortunately, these two notions coincide for the Sobolev space $H^1(\Omega, X) \equiv W^{1,2}(\Omega, X)$, with the same energy thus, one can apply, for harmonic maps, the existence results of [11] and the regularity results in [12]. Notice however that [13] establishes existence results similar to those of [11] in a slightly different context.

Our second result concerns the pointwise and Γ-limit of the functional $E_\varepsilon : L^p(\Omega, \mathbb{R}) \to \mathbb{R}^+$, defined as

$$E_\varepsilon(u) \equiv \int_\Omega e_e(u)(x)\ dx = E^u_\varepsilon(1) = \sup_{f \in \mathcal{C}_c(\Omega),\ 0 \le f \le 1} E^u_\varepsilon(f),$$

as $\varepsilon \to 0$ in the $L^p(\Omega, X)$ topology.

Proposition 2 *Let $1 \le p < \infty$ and ε_n be a positive sequence, $\varepsilon_n \to 0$ as $n \to +\infty$. Then, the functional E^p is the pointwise and the Γ-limit for the $L^p(\Omega, X)$-topology of the functionals E_ε (resp. E_{ε_n}) as $\varepsilon \to 0$ (resp. $n \to +\infty$).*

We turn now to other approaches to the definition of Sobolev spaces into metric spaces, based on a characterization through post-composition with lipschitz maps.

1.3 Characterizations of Sobolev and BV maps by post-composition

In this Section, we are interested in characterizations of BV and Sobolev maps with values in X by composition with 1-lipschitzian maps $\varphi : X \to \mathbb{R}$.

1.3.1 BV maps into metric spaces

In [1], L. Ambrosio has proposed another approach to define the space $BV(\Omega, X)$ when X is a *locally compact and separable* metric space. First, let us define for $u \in L^1(\Omega, X)$ the measure (of possibly infinite mass) $|Du|$ on Ω to be the least measure such that for every borelian set $B \subset \Omega$ and every 1-lipschitzian map $\varphi : X \to \mathbb{R}$,

$$|\varphi \circ u|_{BV(\Omega,\mathbb{R})}(B) \le |Du|(B).$$

Here, $|\varphi \circ u|_{BV(\Omega,\mathbb{R})}$ is the usual measure in the BV sense. From [1], such a measure exists.

Definition 4 ([1]) *We define* $\mathrm{BV}(\Omega, X) \equiv \{u \in L^1(\Omega, X),\ |Du|(\Omega) < \infty\}$.

Here, we do not know at this stage that $\mathrm{BV}(\Omega, X) = BV(\Omega, X)$ in the sense of Definition 1, which justifies that we denote $\mathrm{BV}(\Omega, X)$ this space to avoid confusions.

Proposition 3 *Assume X is locally compact and separable. Then,*

$$\mathrm{BV}(\Omega, X) = BV(\Omega, X).$$

Moreover, for $u \in BV(\Omega, X)$, we have in the sense of measures

$$K_{1,N}|Du| \le |\nabla u|_1 \le |Du|.$$

Remark 3 In the case $N = 1$ (since then $K_{1,N} = 1$) or $X = \mathbb{R}$, then $E^1(u) = |\nabla u|(\Omega)$ and the two BV energies are equal. However, in general, the two constants $K_{1,N}$ and 1 are optimal in the sense that for X the euclidean space $\mathbb{R}^N$ and with v_1, $v_2 : \Omega \to \mathbb{R}^N$ defined by $v_1(x) \equiv (x_1, 0, ..., 0)$ and $v_2(x) \equiv x$, then

$$K_{1,N}|Dv_1| = K_{1,N}\ dx = |\nabla v_1|_1 > 0 \qquad \text{and} \qquad |\nabla v_2|_1 = dx = |Dv_2| > 0.$$

Therefore, the two BV energies are only equivalent and not equal up to a constant factor even in the usual euclidean (vectorial) case: the minimizers of $|Du|(\Omega)$ or $E^1(u)$ (in some subset of $BV(\Omega, X)$) may then be different. To overcome this difficulty, when X has dimension n, we should choose maps φ with values into $\mathbb{R}^n$ instead of $\mathbb{R}$.

L. Ambrosio has developped in this framework the well-known results concerning BV maps, in particular the definition of the regular part of the measure $|Du|$, the definition of the jump set of of a BV map as a rectifiable set of locally finite $\mathcal{H}^{N-1}$ measure and the notion of approximate limit along the orthogonal direction to this jump set. By Proposition 3, one can use equivalently any of these two definitions, when X is locally compact and separable.

1.3.2 Reshetnyak's characterization of Sobolev spaces

In [19], Y.G. Reshetnyak proposed a similar approach to define the Sobolev maps for a metric space target. Let $1 \le p < \infty$ and $u \in L^p(\Omega, X)$. Set

$$R(u) \equiv \inf\left\{ ||w||^p_{L^p(\Omega,\mathbb{R})},\ \forall z \in X,\ |\nabla(d(u(\cdot), z))| \le w \text{ a.e. in } \Omega \right\} \in \mathbb{R}_+ \cup \{\infty\}.$$

Definition 5 ([19]) *Let $1 \le p < \infty$ be given. The Reshetnyak-Sobolev space is defined to be the set $\mathcal{R}^{1,p}(\Omega, X) \equiv \{u \in L^p(\Omega, X),\ R(u) < \infty\}$.*

Paralleling the proof of Proposition 3, we have the following result (see also [10] for a similar result when X is a Banach space).

Proposition 4 *Let $1 \le p < \infty$. Then, we have*

$$\mathcal{R}^{1,p}(\Omega, X) = W^{1,p}(\Omega, X).$$

Morever, for $u \in W^{1,p}(\Omega, X)$, we have

$$K_{p,N}R(u) \le E^p(u) \le R(u).$$

Remark 4 When X is locally compact and separable, we could also have defined the Reshetnyak-Sobolev space as the set of maps $u \in \mathrm{BV}(\Omega, X)$ in the sense of L. Ambrosio such that the measure $|Du|$ belongs to $L^p(\Omega, \mathbb{R})$, with p-energy $||\ |Du|\ ||^p_{L^p(\Omega)}$. This gives, in this case, the existence of a minimizer w for $R(u)$ for any $1 \le p < \infty$ (this is not obvious for $p = 1$).

Remark 5 As in Remark 3, if $N = 1$ or $X = \mathbb{R}$, then $E^p = R$. Moreover, the constants $K_{p,N}$ and 1 are optimal (consider the maps v_1, $v_2 : \Omega \to \mathbb{R}^N$ defined in Remark 3). Therefore, if we are interested in the minimization of the Dirichlet energy into a riemannian manifold with singularities, then R does not extend the standard Dirichlet integral (up to a constant factor).

1.4 Link with the Newtonian-Sobolev and Cheeger-Sobolev spaces

The paper [10] generalizes a definition of Sobolev spaces introduced in [20] as Newtonian-Sobolev spaces, for an arbitrary metric space as target space.

Definition 6 ([10]) *Let $1 \le p < \infty$ be given and $u \in L^p(\Omega, X)$. Then, u is said to be in the Newtonian-Sobolev space $\mathcal{N}^{1,p}(\Omega, X)$ if there exist ρ, $w \in L^p(\Omega, \mathbb{R})$ such that for any 1-lipschitzian curve $\gamma : [0, \ell] \to \Omega$,*

$$d(u \circ \gamma(\ell), u \circ \gamma(0)) \le \int_0^\ell \rho \circ \gamma \qquad \text{or} \qquad \int_0^\ell w \circ \gamma = \infty.$$

The Newtonian p-energy of u is then the infimum of $||\rho||^p_{L^p(\Omega)}$ for all the possible ρ's:

$$N(u) \equiv \inf_\rho \int_\Omega \rho^p.$$

The maps ρ are called *upper gradients* for u. Theorem 3.17 in [10] establishes that the Newtonian-Sobolev energy is the same as the Reshetnyak-Sobolev energy. In particular, for $p = 2$, this energy $R(u)$ is not equal to the standard Dirichlet energy $\int_\Omega |\nabla u|^2 \, dx$ (up to a constant factor).

Proposition 5 ([10]) *Let $1 \le p < \infty$. Then, in $L^p(\Omega, X)$, we have $R = N$. In particular,*

$$\mathcal{N}^{1,p}(\Omega, X) = \mathcal{R}^{1,p}(\Omega, X) = W^{1,p}(\Omega, X).$$

We emphasize that neither [19] nor [10] (extending [20]) allow to define the space $BV(\Omega, X)$. Moreover, neither compactness result (as Theorem 1.13 in [12], when X is locally compact) nor lower-semicontinuity (see Theorem 2) result for the energy are given. In fact, $R = N$ is *not* lower semi-continuous for $p = 1$ and for the $L^1(\Omega, X)$ topology.

Finally, [16] extends the notion of Cheeger type Sobolev spaces (see [5]) to a metric space target. Let $1 \le p < \infty$ be given and $u \in L^p(\Omega, X)$. We set

$$H(u) \equiv \inf \left\{ \liminf_{j \to +\infty} \int_\Omega g_j^p \right\} \in \mathbb{R}_+ \cup \{\infty\},$$

where the infimum is computed over all sequences $(u_j) \in L^p(\Omega, X)$ and $(g_j) \in L^p(\Omega, \mathbb{R})$ such that $u_j \to u$ in $L^p(\Omega, X)$ as $j \to +\infty$ and for any $j \in \mathbb{N}$ and any 1-lipschitzian curve $\gamma : [0, \ell] \to \Omega$,

$$d(u_j \circ \gamma(\ell), u_j \circ \gamma(0)) \le \int_0^\ell g_j \circ \gamma. \tag{3}$$

Definition 7 ([16]) *Let $1 \le p < \infty$. The Cheeger type Sobolev space is defined to be the set $H^{1,p}(\Omega, X) \equiv \{u \in L^p(\Omega, X), H(u) < \infty\}$.*

Remark 6 We would like to emphasize the role played by the target space for all these definitions of Sobolev and BV maps. Let $(\bar{X}, \bar{d})$ be a complete metric space and let X be a closed subset of $\bar{X}$ endowed with the induced distance, so that X is also a complete metric space. Then, if we view u as a map $\bar{u}$ in $L^p(\Omega, \bar{X})$ instead of $L^p(\Omega, X)$ the quantities $E^p(u)$, $J^p(u)$, $|Du|(\Omega)$, $R(u)$, and $N(u)$ do not change[2]. However, it is not clear whether $H(u) = H(\bar{u})$ or not since the computation of $H(\bar{u})$ uses sequences (u_j) $\bar{X}$-valued instead of X-valued. In view of Proposition 6 below, we have at least $H(u) \le C_0 H(\bar{u})$, for some constant $C_0 \ge 1$, when X is a length space.

[2] This follows from the definitions. For $|Du|$, we use the well-known fact that any 1-lipschitzian map $\varphi : X \to \mathbb{R}$ can be extended in a 1-lipschitzian map $\bar{\varphi} : \bar{X} \to \mathbb{R}$, for instance by the formula $\bar{\varphi}(x) \equiv \sup_{\xi \in X} (\varphi(\xi) - d(x, \xi))$.

Actually, Proposition 5 is a direct consequence of Remark 6 and Theorem 3.17 in [10], using an embedding of X into $\ell^\infty(X)$. In the statement of the following Proposition, we will need the definition of a *length space*.

Definition 8 *Let (X,d) be a complete metric space. Then, (X,d) is said to be a* length space *if, for every points x, $y \in X$, there exists a 1-lipschitzian map $\gamma : [0, d(x,y)] \to X$ such that $\gamma(0) = x$ and $\gamma(d(x,y)) = y$.*

The result in the next Proposition is already known for $1 < p < \infty$ and X a Banach space in [20] (Theorem 2.3.2.). However, for sake of completeness and in view of Remark 6, we have included a proof here when X is an arbitrary metric space. Furthermore, the following Proposition takes into account the case $p = 1$ when X is a length space.

Proposition 6 *Assume $1 < p < \infty$. Then, in $L^p(\Omega, X)$, we have $H = N = R$. In particular,*

$$H^{1,p}(\Omega, X) = \mathcal{N}^{1,p}(\Omega, X) = \mathcal{R}^{1,p}(\Omega, X) = W^{1,p}(\Omega, X).$$

Assume $p = 1$ and, moreover, that X is a length space. Then $H^{1,1}(\Omega, X) = BV(\Omega, X)$, and there exists a constant $C = C(\Omega, N) \geq 1$ such that for every $u \in BV(\Omega, X)$,

$$K_{1,N} E^1(u) \leq H(u) \leq C(\Omega, N) E^1(u).$$

Remark 7 We emphasize that for $p = 1$, the result may be false if X is not a length space. Indeed (see [16]), if $X = \{0, 1\} \subset \mathbb{R}$, then $u : (-1, 1) \to X$ defined by $u(x) = 1$ if $x > 0$ and 0 if $x \leq 0$ belongs to $BV((-1,1), \mathbb{R}) = H^{1,1}((-1,1), \mathbb{R})$ (by Proposition 6) but not to $H^{1,1}((-1,1), X)$, since X is not path-connected (see also Remark 6).

Remark 8 Actually, the 1-energy in the space $H^{1,1}(\Omega, X)$ is "close" to the BV energy in the sense of L. Ambrosio (see Lemma 8 in section 2.7).

1.5 Other characterizations of Sobolev spaces

In [2], J. Bourgain, H. Brezis and P. Mironescu have introduced a new characterization of the usual Sobolev spaces $W^{1,p}(\Omega, \mathbb{R})$ of real-valued maps, viewed as the limit of the spaces $W^{s,p}$, $0 < s < 1$, as $s \to 1$. This does not require a notion of gradient, and is very close to the two previous definitions ([12] and [11]). We generalize in the following Theorem the results of [2] and [8] in the case of a metric space target.

We consider a family of *radial* mollifiers $(\rho_\varepsilon)_{\varepsilon > 0}$ (or a sequence (ρ_n)). We recall that this means that ρ_ε is in $L^1(\mathbb{R}^N)$, radial, $\rho_\varepsilon \geq 0$, $\int_{\mathbb{R}^N} \rho_\varepsilon = 1$ and for all $\delta > 0$,

$$\int_{|x| > \delta} \rho_\varepsilon(x)\, dx \to 0 \qquad \text{as } \varepsilon \to 0.$$

Then, given $1 \leq p < \infty$, we define, for $u \in L^p(\Omega, X)$,

$$F_\varepsilon(u) \equiv \int_\Omega \int_\Omega \frac{d^p(u(x), u(y))}{|x-y|^p} \rho_\varepsilon(x-y)\, dxdy \in \mathbb{R}_+.$$

Theorem 4 *Let $1 \leq p < \infty$ and (ε_n) be any positive sequence, $\varepsilon_n \to 0$ as $n \to +\infty$. Then, for every $u \in L^p(\Omega, X)$, $\lim_{\varepsilon \to 0} F_\varepsilon(u)$ and $\lim_{n \to +\infty} F_{\varepsilon_n}(u)$ exist in $\mathbb{R}_+ \cup \{+\infty\}$ and*

$$\lim_{\varepsilon \to 0} F_\varepsilon(u) = \lim_{\varepsilon \to 0} \int_\Omega \int_\Omega \frac{d^p(u(x), u(y))}{|x-y|^p} \rho_\varepsilon(|x-y|)\, dxdy = \lim_{n \to +\infty} F_{\varepsilon_n}(u) = E^p(u).$$

Moreover, the functionals (F_ε) (and (F_{ε_n})) Γ-converge as $\varepsilon \to 0$ (and as $n \to +\infty$), in the $L^p(\Omega, X)$ topology, to E^p.

Theorem 4 combined with Theorem 1 extend the results of [2] and [8] which correspond to the case of real-valued maps. Indeed, from [2] and [8], given $1 \leq p < \infty$ and $f \in L^p(\Omega, \mathbb{R})$, then

$$\liminf_{n\to+\infty} \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^p} \rho_n(|x-y|)\, dxdy = K_{p,N} \int_\Omega |\nabla f|^p \, dx \in \mathbb{R}_+ \cup \{\infty\}. \tag{4}$$

The quantity (4) is ∞ if $f \notin W^{1,p}(\Omega, \mathbb{R})$ (if $1 < p < \infty$) or $f \notin BV(\Omega, \mathbb{R})$ (if $p = 1$). Moreover, the right-hand of (4) has to be understood as the mass of the measure $|\nabla u|$ in the BV case. This approach defines the same p-energy as [12] (thus extends the classical Dirichlet energy), but does not allow to define $W^{1,1}(\Omega, X)$.

Remark 9 Theorem 4 emphasizes that the result of J. Bourgain, H. Brezis and P. Mironescu (and also J. Dávila) does not require a linear structure, neither usual Sobolev tools, such that density of smooth functions or integration by parts.

Remark 10 It is a natural question to wonder what may occur if one chooses non-radial functions ρ_n. The problem is that in the non-radial case, there can be privileged directions for ρ_n, for instance, in $\mathbb{R}^2$, ρ_n can be the characteristic function of the thin rectangle in the x_1 direction $\left[-\frac{1}{n}, \frac{1}{n}\right] \times \left[\frac{1}{n^2}, \frac{1}{n^2}\right]$ suitably normalized, and then we have

$$\liminf_{n\to+\infty} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|f(x) - f(y)|^p}{|x-y|^p} \rho_n(|x-y|)\, dxdy = \int_{\mathbb{R}^2} \left|\frac{\partial f}{\partial x_1}\right|^p \, dx.$$

However, for ρ_n's which do not privilege any direction, A. C. Ponce (see [17]) proves the existence of a non-negative measure μ on the sphere such that the results of [2] still hold with the right-hand side of (4) replaced by

$$\int_\Omega \int_{\mathbb{S}^{N-1}} \left|\omega \cdot \nabla f\right|^p \, d\mu(\omega) dx.$$

Therefore, it is reasonable to take radial distributions of masses to define $W^{1,p}(\Omega, X)$ and $BV(\Omega, X)$.

Remark 11 Finally, we mention that if $N \geq 2$, then A.C. Ponce in [18] (theorem 1.2) showed that for a family $u_\varepsilon \in L^p(\Omega, \mathbb{R})$ (or for a sequence $\varepsilon_n \to 0$), the inequality

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) < \infty$$

yields precompactness in $L^p(\Omega, \mathbb{R})$ for (u_ε) as $\varepsilon \to 0$ (if Ω is bounded). This result is false if $N = 1$ (see [2], counterexample 1). It is plausible that these result extend to the case where $\mathbb{R}$ is replaced by a complete metric space X, at least locally compact.

We conclude this section with a generalization of the paper [15] of H-M. Nguyen. This article gives a generalization of the previous work [14] of H-M. Nguyen and the paper of J. Bourgain and H-M. Nguyen [3]. For this approach, we consider a sequence of functions $g_n : \mathbb{R}_+ \to \mathbb{R}_+$ such that:

- (a) for every $n \in \mathbb{N}$, g_n is *non-decreasing*;
- (b) $\sup_{n\in\mathbb{N}} \displaystyle\int_0^{+\infty} \frac{g_n(t)}{t^{p+1}}\, dt < \infty$ and for every $n \in \mathbb{N}$, $\displaystyle\int_0^1 \frac{g_n(t)}{t^{p+1}}\, dt = 1$;
- (c) $(g_n)_{n\in\mathbb{N}}$ converges locally uniformly in $(0, +\infty)$ to 0.

Define then for $n \in \mathbb{N}$ and $u \in L^p(\Omega, X)$ (comparing with the quantity used in [15], we have just introduced the factor $\frac{1}{|\mathbb{S}^{N-1}|}$)

$$G_n(u) \equiv \frac{1}{|\mathbb{S}^{N-1}|} \int\int_{\Omega\times\Omega} \frac{g_n(d(u(x), u(y)))}{|x-y|^{N+p}} \, dxdy.$$

Theorem 5 *Let $1 < p < \infty$, and (g_n) be a sequence verifying (a), (b) and (c). Then, for every $u \in L^p(\Omega, X)$, $\lim_{n\to+\infty} G_n(u)$ exists in $\mathbb{R}_+ \cup \{+\infty\}$ and*

$$\lim_{n\to+\infty} G_n(u) = E^p(u).$$

The key point in the proof of Theorem 5 is the generalization of the result by J. Bourgain and H-M. Nguyen in [3] (Lemma 1) given in theorem 4 in [15].

Remark 12 The above theorem remains true when we replace the sequence g_n by a familly g_δ, $0 < \delta < 1$ satisfying the corresponding hypothesis to (a), (b) and (c). It would be interesting to know whether the functionals G_n Γ-converge as $n \to +\infty$, in the $L^p(\Omega, X)$ topology, to E^p. When $X = \mathbb{R}$, see the partial result Theorem 5 in [15].

Remark 13 As shown by an example due to A. Ponce, these results do not hold for $p = 1$. Namely, there exists a map $f \in W^{1,1}(0,1), \mathbb{R})$ such that $G_\delta(f) \to +\infty$ as $\delta \to 0$, with $g_\delta(t) = p\delta^p \chi_{t\geq\delta}$. However, the inequality $E^1(u) \leq \liminf_{n\to+\infty} G_n(u)$ for $u \in L^1(\Omega, X)$ remains true.

1.6 Metric topology for $W^{1,p}(\Omega, X)$ and $BV(\Omega, X)$

It is a natural question to wonder whether the spaces $W^{1,p}(\Omega, X)$, $1 \leq p < \infty$, and $BV(\Omega, X)$ can be endowed with a metric topology. Actually, one may propose (at least) two distances on these spaces. Then, we define:

$$D_p^1(u,v) \equiv d_{L^p(\Omega,X)}(u,v) + \left| E^p(u)^{\frac{1}{p}} - E^p(v)^{\frac{1}{p}} \right| \qquad \text{if } E^p(u) + E^p(v) < \infty,$$

$$D^2_{W^{1,p}}(u,v) \equiv d_{L^p(\Omega,X)}(u,v) + \left|\left| |\nabla u|_p^{\frac{1}{p}} - |\nabla v|_p^{\frac{1}{p}} \right|\right|_{L^p(\Omega,\mathbb{R})} \qquad \text{if } u,\ v \in W^{1,p}(\Omega,\mathbb{R}), \quad 1 \leq p < \infty$$

and

$$D^2_{BV}(u,v) \equiv d_{L^1(\Omega,X)}(u,v) + \left|\left| |\nabla u|_1 - |\nabla v|_1 \right|\right|_{[\mathcal{C}_0(\Omega,\mathbb{R})]^*} \qquad \text{if } u,\ v \in BV(\Omega,\mathbb{R}).$$

Note that these quantities are well-defined on the spaces $W^{1,p}(\Omega, X)$ for $1 \leq p < \infty$, and on the space $BV(\Omega, X)$. Moreover, it is easily checked that they define distances on these spaces.

The first natural question is to ask if D_p^1, $D^2_{W^{1,p}}$ and D^2_{BV} induce, for $X = \mathbb{R}$, the same topology as the classical one on $W^{1,p}(\Omega, \mathbb{R})$ and $BV(\Omega, \mathbb{R})$, induced by the norm

$$||f||_{W^{1,p}} \equiv ||f||_{L^p(\Omega,\mathbb{R})} + ||\nabla f||_{L^p(\Omega,\mathbb{R})} \qquad \text{if } f \in W^{1,p}(\Omega,\mathbb{R}), \quad 1 \leq p < \infty,$$

$$||f||_{BV} \equiv ||f||_{L^1(\Omega,\mathbb{R})} + ||\nabla f||_{[\mathcal{C}_0(\Omega,\mathbb{R})]^*} \qquad \text{if } f \in BV(\Omega,\mathbb{R}).$$

Given u, $v \in W^{1,p}(\Omega, \mathbb{R})$ or $BV(\Omega, \mathbb{R})$, there holds (by Theorem 1 and note that $K_{p,N} \leq 1$)

$$D_p^1(u,v) \leq D^2_{W^{1,p}}(u,v) \leq ||u-v||_{W^{1,p}} \qquad \text{and} \qquad D_1^1(u,v) \leq D^2_{BV}(u,v) \leq ||u-v||_{BV}. \quad (5)$$

Lemma 1 *Assume $X = \mathbb{R}$. If $1 < p < \infty$, then D^1_p and $D^2_{W^{1,p}}$ induce the same topology as the classical one on $W^{1,p}(\Omega, \mathbb{R})$.*

Proof. Assume first $1 < p < \infty$. If $u_n \to u$ in $W^{1,p}(\Omega, \mathbb{R})$ in the topology of the norm, then (5) implies that $D^1_p(u_n, u) \to 0$ and $D^2_{W^{1,p}}(u_n, u) \to 0$. Assume then that $D^1_p(u_n, u) \to 0$ as $n \to +\infty$. Then, $u_n \to u$ in $L^p(\Omega, \mathbb{R})$ and (∇u_n) is bounded in $L^p(\Omega, \mathbb{R}^N)$, hence $(1 < p < \infty)$ $\nabla u_n \rightharpoonup \nabla u$ in $L^p(\Omega, \mathbb{R}^N)$. Furthermore, since $D^1_p(u_n, u) \to 0$ and using Theorem 1,

$$K_{p,N} \int_\Omega |\nabla u_n|^p \, dx = E^p(u_n) \to E^p(u) = K_{p,N} \int_\Omega |\nabla u|^p \, dx,$$

thus by strict convexity of $L^p(\Omega, \mathbb{R}^N)$ (since $1 < p < \infty$), we have $\nabla u_n \to \nabla u$ in $L^p(\Omega, \mathbb{R}^N)$, so that $||u_n - u||_{W^{1,p}} \to 0$. $\square$

We have nevertheless some negative results for these metrics.

Lemma 2 *(i) The distance D^1_1 does* **not** *induce the classical topology on $W^{1,1}(\Omega, \mathbb{R})$ and $BV(\Omega, \mathbb{R})$.*
(ii) For $1 \leq p < \infty$, the space $W^{1,p}((0,1), \mathbb{R})$ (resp. $BV((0,1), \mathbb{R})$), endowed with the metrics D^1_p or $D^2_{W^{1,p}}$ (resp. D^1_1 or D^2_{BV}) is **not** *complete.*

Proof. We consider the sequence $v_n : (0,1) \to \mathbb{R}$ defined in the following way. Let, for $0 \leq k < n$, $I^+_k \equiv [\frac{k}{n}, \frac{k}{n} + \frac{1}{2n}[$, and $I^-_k \equiv [\frac{k}{n} + \frac{1}{2n}, \frac{k}{n} + \frac{1}{n}[$. Then, let, for $0 \leq k < n$, $v_n(x) \equiv x - \frac{k}{n}$ if $x \in I^+_k$ and $v_n(x) \equiv \frac{k+1}{n} - x$ if $x \in I^-_k$. The map v_n is Lipschitz continuous, and we have (χ is the characteristic function)

$$||v_n||_{L^\infty((0,1))} = \frac{1}{2n}, \qquad v'_n = \sum_{k=0}^{n-1} \chi_{I^+_k} - \chi_{I^-_k}.$$

As a consequence, $v_n \to 0$ in $L^\infty((0,1))$ as $n \to +\infty$, but $|v'_n| = 1$ a.e. in $(0,1)$. Therefore, (v_n) is not convergent in $BV((0,1), \mathbb{R})$, but (v_n) is a Cauchy sequence for D^1_p, $D^2_{W^{1,p}}$ and D^2_{BV}. This proves (ii). Consider now $u_n(x) \equiv x + v_n(x)$ in $(0,1)$. We have then $u_n \to u(x) \equiv x$ in $L^1((0,1), \mathbb{R})$ but (u_n) does not converge in $BV((0,1), \mathbb{R})$. However, we have $|u'_n(x)| = 1 + v'_n(x)$ (since $v'_n = \pm 1$ a.e.) and $|u'(x)| = 1$, hence $E^1(u_n) = \int_0^1 1 + v'_n(x) \, dx = 1$ and $E^1(u) = 1$, so that, as $n \to +\infty$,

$$D^1_1(u_n, u) = ||u_n - u||_{L^1} = ||v_n||_{L^1} \to 0.$$

Hence $D^1_1(u_n, u) \to 0$ but $||u_n - u||_{BV} = ||u_n - u||_{W^{1,1}} = ||v'_n||_{L^1} = 1 \not\to 0$. This proves (i). $\square$

1.7 The trace problem

In [12] (Section 1.12), the trace of a map in $W^{1,p}(\Omega, X)$ is defined, namely, for $1 \leq p < \infty$ and a smooth $\Omega \subset \mathbb{R}^N$, there exists a map

$$\mathrm{tr} : W^{1,p}(\Omega, X) \to L^p(\partial\Omega, X)$$

such that $\mathrm{tr}(u) = u_{|\partial\Omega}$ for $u \in W^{1,p}(\Omega, X) \cap \mathcal{C}(\bar{\Omega}, X)$. Moreover, for $1 < p < \infty$, the map *tr* is "continuous" in the sense that if $u_n \in W^{1,p}(\Omega, X)$ is such that $(E^p(u_n))_{n \in \mathbb{N}}$ is bounded and

$$u_n \to u \quad \text{ in } L^p(\Omega, X),$$

then $\mathrm{tr}(u_n) \to \mathrm{tr}(u)$ in $L^p(\partial\Omega, X)$. In view of (5), this implies that the trace map is continuous from $W^{1,p}(\Omega, X)$ endowed with the distances D_1 or D_2 into $L^p(\Omega, X)$, $1 < p < \infty$. We complete this notion of trace by proving the $W^{1-\frac{1}{p},p}$ regularity of the trace. First, given a compact submanifold $\mathcal{M}$ of $\mathbb{R}^N$ of dimension M (such as $\bar{\Omega}$ or $\partial\Omega$), we define, for $0 < s < 1$ and $1 \le p < \infty$,

$$W^{s,p}(\mathcal{M}, X) \equiv \left\{ u \in L^p(\mathcal{M}, X),\ |u|^p_{W^{s,p}(\mathcal{M},X)} \equiv \int_{\mathcal{M}\times\mathcal{M}} \frac{d(u(x), u(y))^p}{|x-y|^{sp+M}}\, dxdy < \infty \right\}.$$

Proposition 7 *Let $1 \le p < \infty$ and $0 < s \le 1$ such that $sp > 1$ or $s = 1$. Then the trace map* $\mathrm{tr} : W^{s,p}(\Omega, X) \to L^p(\partial\Omega, X)$ *is well-defined and has values into $W^{s-\frac{1}{p},p}(\partial\Omega, X)$. Moreover, there exists C, depending on Ω, s and p such that for every $\xi \in X$,*

$$|\mathrm{tr}(u)|_{W^{s-\frac{1}{p},p}(\partial\Omega,X)} \le C(|d(u,\xi)|_{L^p(\Omega,\mathbb{R})} + |u|_{W^{s,p}(\Omega,X)}).$$

Similarly to $W^{1,p}(\Omega, X)$, we may endow $W^{s,p}(\mathcal{M}, X)$ $(0 < s < 1)$ with the distances defined by

$$D^1_{s,p}(u, v) \equiv d_{L^p(\mathcal{M},X)}(u, v) + \Big| |u|_{W^{s,p}} - |v|_{W^{s,p}} \Big|,$$

$$D^2_{s,p}(u, v) \equiv d_{L^p(\mathcal{M},X)}(u, v) + \Big|\Big| d(u(x), u(y)) - d(v(x), v(y)) \Big|\Big|_{L^p\left(\frac{dxdy}{|x-y|^{sp+M}}\right)}.$$

The following Lemma summarizes the properties of these distances on $W^{s,p}$.

Lemma 3 *Assume* $0 < s < 1$.
(i) If $X = \mathbb{R}$ and $1 < p < \infty$, $D^1_{s,p}$ and $D^2_{s,p}$ induce the usual topology on $W^{s,p}(\Omega, \mathbb{R})$.
(ii) For $1 \le p < \infty$, the space $W^{s,p}((-1, 1), \mathbb{R})$ endowed with the metric $D^1_{s,p}$ is **not** *complete.*
(iii) If $1 \le p < \infty$, $W^{s,p}(\Omega, X)$ endowed with the metric $D^2_{s,p}$ is complete.

Proof. Assertion (i) follows as in Lemma 1 since, for $1 < p < \infty$, $W^{s,p}(\Omega, \mathbb{R})$ is uniformly convex. To prove (ii), let $w : \mathbb{R} \to \mathbb{R}$ be defined by: $w(x) \equiv 1 - |x|$ if $|x| \le 1$, $w(x) \equiv 0$ otherwise. Consider then $w_n : (-1, 1) \to \mathbb{R}$ defined by: $w_n(x) \equiv n^{-s} w(nx)$. Then, $||w_n||_{L^1} = n^{-s} \to 0$ as $n \to +\infty$. Moreover, simple scaling yields $|w_n|_{W^{s,p}} = |w|_{W^{s,p}} = Cte > 0$. Hence, (w_n) is Cauchy for $D^1_{s,p}$ but does not converge in $W^{s,p}((0, 1), \mathbb{R})$. Assertion (iv) is straightforward: if (u_n) is Cauchy for $D^2_{s,p}$, then (u_n) and $\left(\frac{d(u_n(x),u_n(y))}{|x-y|^{s+\frac{M}{p}}}\right)$ are Cauchy sequences in $L^p(\Omega, X)$ and $L^p(\Omega \times \Omega, \mathbb{R})$, hence converge. Thus, $u_n \to u$ in $L^p(\Omega, X)$ and $\frac{d(u_n(x),u_n(y))}{|x-y|^{s+\frac{M}{p}}} \to g$ in $L^p(\Omega \times \Omega, \mathbb{R})$. Clearly, $g = \frac{d(u(x),u(y))}{|x-y|^{s+\frac{M}{p}}}$, so $D^2_{s,p}(u_n, u) \to 0$. $\square$

We have not investigated whether the trace map is continuous or not, from $W^{s,p}(\Omega, X)$ into $W^{s-\frac{1}{p},p}(\partial\Omega, X)$ for the distances we have defined.

The plan of the paper is the following. Section 2 contains the detailed proofs of Proposition 2 (subsection 2.2) Proposition 3 (subsection 2.4), Proposition 4 (subsection 2.5), Proposition 6 (subsection 2.7). For the main results, Theorem 3 is proved in subsection 2.3, then Theorem 4 in subsection 2.8 and finally Theorem 5 in subsection 2.9. We conclude with the proof of Proposition 7 about traces, given in Section 3.

2 Proofs of the main results

2.1 An extension Lemma

Here is a standard extension lemma for smooth domains (a lipschitz boundary is enough).

Lemma 4 *Let $1 \leq p < \infty$ and $\Omega \subset \mathbb{R}^N$ be smooth. Then, there exists $\delta_0 > 0$, depending only on the smoothness of $\partial\Omega$, and C, depending on p and the smoothness of $\partial\Omega$ such that, for any $u \in W^{1,p}(\Omega, X)$ or $BV(\Omega, X)$, there exists $U \in W^{1,p}(\Omega_{\delta_0}, X)$ or $BV(\Omega_{\delta_0}, X)$ such that*

$$U = u \quad \text{in } \Omega \quad \text{and} \quad E^p(U) \leq CE^p(u).$$

Proof. The extension is done by standard reflection across the boundary. Hence, by standard localization and using local charts, we may assume for simplicity that Ω is locally the half-plane $\mathbb{R}^N_+ \equiv \mathbb{R}^{N-1} \times \mathbb{R}^*_+$. Then, we define $U : \mathbb{R}^N \to X$ as $U(x) \equiv u(x)$ if $x_N > 0$ and if $x_N < 0$, $U(x) \equiv u(x_1, ..., x_{N-1}, -x_N)$. It then follows directly that $e_\varepsilon(U)(x) \leq 2e_\varepsilon(u)(x)$, which establishes the Lemma. □

2.2 Proof of Proposition 2

A useful fact we will need in the sequel is to allow other approximate derivative than $e_\varepsilon(u)$, for instance ball averages instead of sphere averages. Therefore, we consider as in [12]

$$_\nu e_\varepsilon(u)(x) \equiv \int_0^{+\infty} e_{\lambda\varepsilon}(u)(x)\ d\nu(\lambda),$$

where ν is a probability measure on $(0, +\infty)$, and we define $_\nu E_\varepsilon : L^p(\Omega, X) \to \mathbb{R}^+ \cup \{+\infty\}$ by

$$_\nu E_\varepsilon(u) = \int_\Omega {}_\nu e_\varepsilon(u)(x)\ dx.$$

Note that in [12], it is required that ν satisfies

$$\text{Supp } \nu \subset (0, 2) \quad \text{and} \quad \int_0^2 \lambda^{-p}\ d\nu(\lambda) < \infty. \tag{6}$$

When $\nu = \delta_1$, $_\nu E_\varepsilon(u)$ is simply the norm of the linear functional $E_\varepsilon(u) : \mathcal{C}_c(\Omega, \mathbb{R}) \to \mathbb{R}$. Therefore, Proposition 2 is a particular case of Theorem 6 below when $\nu = \delta_1$.

Theorem 6 *Let $1 \leq p < \infty$ and ν be a probability measure on $\mathbb{R}^*_+$. Then, the functional E^p is the pointwise and the Γ-limit for the $L^p(\Omega, X)$-topology of the functionals $_\nu E_\varepsilon$ as $\varepsilon \to 0$ (or for any sequence $\varepsilon_n \to 0$ as $n \to +\infty$). Moreover, the following inequality holds, for $u \in L^p(\Omega, X)$,*

$$_\nu E^p(u) \leq C(\Omega, p)E^p(u).$$

Proof. Let us first assume that $u \in W^{1,p}(\Omega, X)$ or $BV(\Omega, X)$. Using Lemma 4, we first extend u, defined in Ω, in a map U defined in the δ_0 neighborhood Ω_{δ_0} of Ω. The energy of the extension is finite and

$$E^p(U, \Omega_{\delta_0}) \leq C(\Omega, p)E^p(u).$$

Combining Theorem 1.8.1 and Lemma 1.9.1 in [12] about directional energies, we infer, for $|h| < \delta_0$

$$\int_\Omega d^p(U(x+h), U(x))\ dx \leq C(\Omega, p, N)|h|^p E^p(u). \tag{7}$$

Therefore, using Fubini's Theorem,

$$\begin{aligned} _\nu E_\varepsilon(u) &\leq \frac{1}{|\mathbb{S}^{N-1}|} \int_\Omega \int_0^{+\infty} \int_{S_{\lambda\varepsilon}(x)} d^p(U(x), U(y)) \frac{d\mathcal{H}^{N-1}(y)}{(\lambda\varepsilon)^{N+p-1}}\ d\nu(\lambda)\ dx \\ &= \frac{1}{|\mathbb{S}^{N-1}|} \int_0^{+\infty} \int_{S_{\lambda\varepsilon}(0)} \int_\Omega \Big(\frac{d(U(x+h), U(x))}{|h|}\Big)^p\ dx\ \frac{d\mathcal{H}^{N-1}(h)}{|h|^{N-1}}\ d\nu(\lambda). \end{aligned} \tag{8}$$

By (7), we then deduce

$$
{}_\nu E_\varepsilon(u) \le C(\Omega,p)\frac{E^p(u)}{|\mathbb{S}^{N-1}|}\int_0^{+\infty}\int_{S_{\lambda\varepsilon}(0)}\frac{d\mathcal{H}^{N-1}(h)}{|h|^{N-1}}\,d\nu(\lambda) = C(\Omega,p)E^p(u),
$$

which concludes the proof of the last assertion.

Proof of the liminf inequality. Let u, $u_\varepsilon \in L^p(\Omega, X)$ be such that $u_\varepsilon \to u$ in $L^p(\Omega, X)$ (the case of a sequence is analogous). We wish to prove that

$$
E^p(u) \le \liminf_{\varepsilon\to 0} {}_\nu E_\varepsilon(u_\varepsilon), \tag{9}
$$

and we may assume without loss of generality that the right-hand side is finite. There exists a decreasing sequence $\varepsilon_j \to 0$ such that

$$
\liminf_{\varepsilon\to 0} {}_\nu E_\varepsilon(u_\varepsilon) = \lim_{j\to+\infty} {}_\nu E_{\varepsilon_j}(u_{\varepsilon_j}) < \infty.
$$

We argue as in Lemma 1.4.2 in [12], letting, for $\varepsilon > 0$, n_j be the integer part of $\varepsilon/\varepsilon_j$. Therefore, $n_j\varepsilon_j \to \varepsilon$ as $j \to +\infty$. Since $u_{\varepsilon_j} \to u$ in $L^p(\Omega, X)$, then ${}_\nu e_{n_j\varepsilon_j}(u_{\varepsilon_j}) \to {}_\nu e_\varepsilon(u)$ pointwise almost everywhere, thus by Fatou's lemma,

$$
{}_\nu E_\varepsilon(u) = \int_\Omega {}_\nu e_\varepsilon(u) \le \liminf_{j\to+\infty}\int_\Omega {}_\nu e_{n_j\varepsilon_j}(u_{\varepsilon_j}) = \liminf_{j\to+\infty} {}_\nu E_{n_j\varepsilon_j}(u_{\varepsilon_j}). \tag{10}
$$

We apply now the following Lemma, stating a result analogous to the fact that the approximate length of a lipschitz curve with respect to a partition is increased if the partition is refined. This Lemma comes from [12], Lemma 1.3.1, with the only change that we take $f \equiv 1$ (the proof clearly works for this f).

Lemma 5 *Assume ν is a probability measure on $(0, +\infty)$. Let $1 \le p < \infty$, and $u \in L^p(\Omega, X)$. Then, given $n \in \mathbb{N}^*$ and $\lambda_i > 0$, $1 \le i \le n$ such that $\sum_{i=1}^n \lambda_i = 1$, we have for every $\varepsilon > 0$*

$$
{}_\nu E_\varepsilon(u)^{\frac{1}{p}} \le \sum_{i=1}^n \lambda_i\Big({}_\nu E_{\lambda_i\varepsilon}(u)\Big)^{\frac{1}{p}}.
$$

Applying Lemma 5 with $u = u_{\varepsilon_j}$, $n = n_j$, $\varepsilon = n_j\varepsilon_j$ and $\lambda_i = \frac{1}{n_j} > 0$ (thus $\sum \lambda_i = 1$) we obtain

$$
{}_\nu E_{n_j\varepsilon_j}(u_{\varepsilon_j})^{\frac{1}{p}} \le {}_\nu E_{\varepsilon_j}(u_{\varepsilon_j})^{\frac{1}{p}} \le {}_\nu E_{\varepsilon_j}(u_{\varepsilon_j})^{\frac{1}{p}}.
$$

Passing to the limit as $j \to +\infty$ and using (10), we infer

$$
{}_\nu E_\varepsilon(u) \le \liminf_{j\to+\infty} {}_\nu E_{n_j\varepsilon_j}(u_{\varepsilon_j}) \le \liminf_{j\to+\infty} {}_\nu E_{\varepsilon_j}(u_{\varepsilon_j}) = \liminf_{\epsilon\to 0} {}_\nu E_\epsilon(u_\epsilon).
$$

As a consequence,

$$
\limsup_{\varepsilon\to 0} {}_\nu E_\varepsilon(u) \le \liminf_{\epsilon\to 0} {}_\nu E_\epsilon(u_\epsilon) < \infty,
$$

and (9) follows.

Proof of the pointwise limit. Let $u \in L^p(\Omega, X)$. We wish to prove

$$
\lim_{\varepsilon\to 0} {}_\nu E_\varepsilon(u) = E^p(u) \in \mathbb{R}^+ \cup \{+\infty\}.
$$

We know from (9) (with $u_\varepsilon = u$) that the equality is true if $E^p(u) = +\infty$, hence we may assume $E^p(u) < +\infty$, and, still with (9) (and $u_\varepsilon = u$), it suffices to show

$$\limsup_{\varepsilon \to 0} {}_\nu E_\varepsilon(u) \le E^p(u) < +\infty. \tag{11}$$

First, by monotone convergence, $\nu((1/R, R)) \to 1$ if $R \to +\infty$, hence for $R > 1$ large enough (so that $\nu((1/R, R)) \ge 1/2$ for instance), we can define the measure

$$\nu_R \equiv \frac{\nu \llcorner (1/R, R)}{\nu((1/R, R))}.$$

This probability measure satisfies

$$\text{Supp } \nu_R \subset (0, R) \qquad \text{and} \qquad \int_{\mathbb{R}_+^*} \lambda^{-p} \, d\nu_R(\lambda) \le R^{-p} \int_{\mathbb{R}_+^*} d\nu_R = R^{-p} < \infty.$$

This is not exactly (6), but this is sufficient however to apply Theorem 1.7 in [12], which asserts that $\lim_{\varepsilon \to 0} \Big({}_{\nu_R} E_\varepsilon(u) \Big)$ exists and

$$\lim_{\varepsilon \to 0} \Big({}_{\nu_R} E_\varepsilon(u) \Big) = E^p(u). \tag{12}$$

Therefore, using (7) and arguing as for (8), we deduce that, for $R > 1$ large enough,

$$\begin{aligned}
&\Big| {}_\nu E_\varepsilon(u) - \nu((1/R, R)) \Big({}_{\nu_R} E_\varepsilon(u) \Big) \Big| \\
&\qquad \le \frac{1}{|\mathbb{S}^{N-1}|} \int_{\mathbb{R}_+^* \setminus (1/R, R)} \int_{S_{\lambda\varepsilon}(0)} \int_\Omega \Big(\frac{d(U(x+h), U(x))}{|h|} \Big)^p \, dx \, \frac{d\mathcal{H}^{N-1}(h)}{|h|^{N-1}} \, d\nu(\lambda) \\
&\qquad \le C(\Omega, p) \nu(\mathbb{R}_+^* \setminus (1/R, R)) E^p(u).
\end{aligned}$$

Now, letting $\varepsilon \to 0$ and using (12) gives

$$\limsup_{\varepsilon \to 0} {}_\nu E_\varepsilon(u) \le \nu((1/R, R)) E^p(u) + C(\Omega, p, N) \nu(\mathbb{R}_+^* \setminus (1/R, R)) E^p(u).$$

We may now let R go to $+\infty$ to deduce, since $\nu(\mathbb{R}_+^* \setminus (1/R, R)) \to 0$ and $\nu((1/R, R)) \to 1$, that

$$\limsup_{\varepsilon \to 0} {}_\nu E_\varepsilon(u) \le E^p(u).$$

This finishes the proof of (11), and hence of the pointwise limit. Combining the liminf inequality together with the pointwise limit, we deduce Theorem 6. □

2.3 Proof of Theorem 3

Proof of the liminf inequality. First, we have, for $x \in \Omega^\varepsilon$ and μ_x^ε defined by (2),

$$\int_\Omega |x - y|^p \, d\mu_x^\varepsilon(y) = |\mathbb{S}^{N-1}| \int_0^\varepsilon r^{p+N-1} \, dr = \frac{|\mathbb{S}^{N-1}|}{N+p} \varepsilon^{N+p}.$$

Thus, denoting $d\nu = (N+p)\lambda^{N+p-1} \, d\lambda \llcorner (0,1)$ the measure giving ball averages, we infer that for every $u \in L^p(\Omega, X)$

$$J_\varepsilon^p(u) \ge \int_{\Omega^\varepsilon} {}_\nu e_\varepsilon(u) \, dx = {}_\nu E_\varepsilon(u).$$

Therefore, for any family $u_\varepsilon \to u$ in $L^p(\Omega, X)$ as $\varepsilon \to 0$ (or for a sequence $u_n \to u$ in $L^p(\Omega, X)$ with $\varepsilon_n \to 0$ as $n \to +\infty$), we deduce from Theorem 6 that

$$\liminf_{\varepsilon\to 0} J^p_\varepsilon(u_\varepsilon) \geq \liminf_{\varepsilon\to 0} {}_\nu E_\varepsilon(u_\varepsilon) \geq E^p(u),$$

or the corresponding statement for the sequence $\varepsilon_n \to 0$. In particular, if $J^p(u) < \infty$, that is if $u \in \mathcal{W}^{1,p}(\Omega, X)$ (or $u \in \mathcal{BV}^{1,p}(\Omega, X)$), then $u \in W^{1,p}(\Omega, X)$ (or $u \in BV^{1,p}(\Omega, X)$).

Proof of the pointwise limit. We now prove that there will be equality in the above inequality for the particular family $u_\varepsilon = u$ for all $\varepsilon > 0$ (or $u_n = u$ for all n). First, we extend by Lemma 4 u in U of finite p-energy in the δ_0-neighborhood of Ω. Furthermore, we have for any $\varepsilon > 0$ and $x \in \Omega \setminus \Omega^\varepsilon$, since Ω is smooth,

$$\int_\Omega |x-y|^p \; d\mu^\varepsilon_x(y) \geq \frac{1}{C(\Omega)} \int_{B_\varepsilon(x)} |x-y|^p \; d\mu^\varepsilon_x(y) = \frac{1}{C(\Omega)} \frac{|\mathbb{S}^{N-1}|}{N+p} \varepsilon^{N+p},$$

where $C(\Omega)$ depends only on the smoothness of $\partial\Omega$. As a consequence, fixing $0 < \eta < \delta_0$, we have for every $0 < \varepsilon < \eta/2$

$$0 \leq J^p_\varepsilon(u) - {}_\nu E_\varepsilon(u) = \int_{\Omega\setminus\Omega^\varepsilon} \frac{\int_\Omega d(U(x), U(y))^p \; d\mu^\varepsilon_x(y)}{\int_\Omega |x-y|^p \; d\mu^\varepsilon_x(y)} \; dx \leq C(\Omega) \int_{\Omega_\eta\setminus\Omega^\eta} {}_\nu e_\varepsilon(U)(x) \; dx,$$

from which we deduce, using Theorem 6,

$$\begin{aligned} E^p(u) = \liminf_{\varepsilon\to 0} {}_\nu E_\varepsilon(u) \leq \liminf_{\varepsilon\to 0} J^p_\varepsilon(u) &\leq \limsup_{\varepsilon\to 0} J^p_\varepsilon(u) \\ &\leq \limsup_{\varepsilon\to 0} \Big({}_\nu E_\varepsilon(u) + C(\Omega) \int_{\Omega_\eta\setminus\Omega^\eta} {}_\nu e_\varepsilon(U)(x) \; dx \Big) \\ &= E^p(u) + C(\Omega) \int_{\Omega_\eta\setminus\Omega^\eta} |\nabla U|_p \; dx. \end{aligned}$$

Letting $\eta \to 0$, the last term tends to 0. Indeed, $|\nabla U|_p \in L^1$ if $1 < p < \infty$, and if $p = 1$, $|\nabla U|_1$ is a measure such that $|\nabla U|_1(\partial\Omega) = 0$ (since U is the extension by reflection across the boundary). Thus, we infer

$$\lim_{\varepsilon\to 0} J^p_\varepsilon(u) = E^p(u),$$

which is the desired equality: J^p_ε Γ-converge at u to $E^p(u)$, which is finite if $u \in W^{1,p}(\Omega, X)$ or $u \in BV(\Omega, X)$ and infinite otherwise. $\square$

2.4 Proof of Proposition 3

In this subection, we prove the equivalence of the two definitions of BV maps into a locally compact metric space X, namely Definitions 1 and 4.

First, if $u \in BV(\Omega, X)$, and $\varphi : X \to \mathbb{R}$ is a 1-lipschitzian map, then for every $\varepsilon > 0$ and $x \in \Omega^\varepsilon$,

$$\begin{aligned} e_\varepsilon(\varphi \circ u)(x) &= \frac{1}{|\mathbb{S}^{N-1}|\varepsilon^N} \int_{S_\varepsilon(x)} |\varphi \circ u(x) - \varphi \circ u(y)| \; d\mathcal{H}^{N-1}(y) \\ &\leq \frac{1}{|\mathbb{S}^{N-1}|\varepsilon^N} \int_{S_\varepsilon(x)} d(u(x), u(y)) \; d\mathcal{H}^{N-1}(y) = e_\varepsilon(u)(x). \end{aligned}$$

Passing to the limit as $\varepsilon \to 0$ and using Proposition 1 and Theorem 1, we obtain

$$K_{1,N}|D(\varphi \circ u)|_{BV} \leq |\nabla u|_1.$$

Since this is true for any 1-lipschitzian map $\varphi : X \to \mathbb{R}$, we deduce that $u \in \mathrm{BV}(\Omega, X)$ and $K_{1,N}|Du|(\Omega) \leq |\nabla u|_1$.

Let now $u \in \mathrm{BV}(\Omega, X)$ be given. We claim that for every $\varepsilon > 0$ and $|h| \leq \varepsilon$, we have

$$\int_{\Omega^\varepsilon} d(u(x), u(x+h))\ dx \leq |h| \cdot |Du|(\Omega). \tag{13}$$

Proof of (13). If $N = 1$ and $\Omega = (a, b)$, then (13) follows from the inequality for $x \in (a+\varepsilon, b-\varepsilon)$ and $0 < h < \varepsilon$

$$d(u_+(x), u_+(x+h)) \leq |Du|(]x, x+h])$$

valid for the right-hand side continuous representative u_+ of u. Integrating this inequality yields

$$\int_{a+\varepsilon}^{b-\varepsilon} d(u(x), u(x+h))\ dx = \int_{a+\varepsilon}^{b-\varepsilon} d(u_+(x), u_+(x+h))\ dx \leq h|Du|([a,b]),$$

which is the desired inequality for $N = 1$. The general case follows by slicing in the direction h and using (2.5) in Proposition 2.1 in [1] (see the analogous proof of (ii) in Lemma 3.2 in [1]).

We conclude now the proof of Proposition 3. For $\varepsilon > 0$ and $\nu = (N+1)\lambda^N\ d\lambda \llcorner (0,1)$ giving ball averages, we have, using (13),

$$\begin{aligned}
\int_{\Omega^\varepsilon} {}_\nu e_\varepsilon(u)(x)\ dx &= \frac{N+1}{|\mathbb{S}^{N-1}|\varepsilon^N} \int_{B_\varepsilon(0)} \int_{\Omega^\varepsilon} d(u(x), u(x+h))\ dx\ dh \\
&\leq \frac{N+1}{|\mathbb{S}^{N-1}|\varepsilon^N} \int_{B_\varepsilon(0)} |h| \cdot |Du|(\Omega)\ dh = \frac{N+1}{\varepsilon^N}|Du|(\Omega) \int_0^\varepsilon r^N\ dr = |Du|(\Omega)
\end{aligned}$$

From Definition 1, we infer $u \in BV(\Omega, X)$ and

$$E^1(u) \leq |Du|(\Omega).$$

We have therefore $K_{1,N}|Du|(\Omega) \leq E^1(u) \leq |Du|(\Omega)$ for any $u \in L^1(\Omega, X)$, which implies that $BV(\Omega, X) = \mathrm{BV}(\Omega, X)$. Given now $u \in BV(\Omega, X)$ and an open ball $B \subset \Omega$, we may apply this to $u \in BV(B, X)$ to infer $K_{1,N}|Du|(B) \leq |\nabla u|_1(B) \leq |Du|(B)$, hence $K_{1,N}|Du| \leq |\nabla u|_1 \leq |Du|$ as borelian measures. This concludes the proof of Proposition 3. $\square$

2.5 Proof of Proposition 4

Given $1 \leq p < \infty$ and $u \in W^{1,p}(\Omega, X)$, we have $K_{p,N}R(u) \leq E^p(u)$. Indeed (as in section 2.4), for any $z \in X$, since $d(\cdot, z)$ is 1-lipschitzian,

$$K_{p,N}|\nabla(d(u(\cdot), z))|^p = |\nabla(d(u(\cdot), z))|_p \leq |\nabla u|_p \in L^1(\Omega, \mathbb{R}).$$

Hence, one can take $w \equiv (K_{p,N}^{-1}|\nabla u|_p)^{\frac{1}{p}}$ and infer $K_{p,N}R(u) \leq E^p(u)$ for $u \in W^{1,p}(\Omega, X)$. The inequality $E^p(u) \leq R(u)$ will follow as in section 2.4 from the inequality, for $|h| < \varepsilon$,

$$\int_{\Omega^\varepsilon} d^p(u(x), u(x+h))\ dx \leq |h|^p R(u), \tag{14}$$

which is the analogue of (13). Let us first consider some $w \in L^p(\Omega, \mathbb{R})$ such that for all $z \in X$,

$$|\nabla(d(u(\cdot), z))| \leq w$$

a.e. in Ω. Using Fubini's Theorem, for a.e. $x \in \Omega^\varepsilon$ and $|h| < \varepsilon$, $[0,1] \ni t \mapsto d(u(x+th), u(x)) \in W^{1,p}$ and $|\frac{d}{dt}(d(u(x+th), u(x)))| \leq |h| w(x+th)$, thus by integration

$$d^p(u(x), u(x+h)) \leq \Big(\int_0^1 |h| w(x+th)\ dt\Big)^p \leq |h|^p \int_0^1 w^p(x+th)\ dt. \tag{15}$$

Integrating over Ω^ε and using Fubini's Theorem, we obtain $\int_{\Omega^\varepsilon} d^p(u(x), u(x+h))\ dx \leq |h|^p ||w||_{L^p}^p$, and the conclusion follows by taking the infimum over all such w's.

This proves (as in section 2.4) that $E^p \leq R$ in $L^p(\Omega, X)$. This concludes the proof of the second assertion of Proposition 4, together with the fact that if $p > 1$, $\mathcal{R}^{1,p}(\Omega, X) = W^{1,p}(\Omega, X)$, and (if $p = 1$) $W^{1,1}(\Omega, X) \subset \mathcal{R}^{1,1}(\Omega, X) \subset BV(\Omega, X)$. In order to finish the proof, we have then just to prove that $\mathcal{R}^{1,1}(\Omega, X) \subset W^{1,1}(\Omega, X)$. Consider then $u \in \mathcal{R}^{1,1}(\Omega, X)$, and a function $w \in L^1(\Omega, \mathbb{R})$ associated, so that (15) holds (with $p = 1$). Therefore, by integration and changing variables, one has for $\varepsilon > 0$, $\chi_{\Omega^\varepsilon}$ standing for the characteristic function of Ω^ε,

$$e_\varepsilon(u)(x) \leq \frac{\chi_{\Omega^\varepsilon}}{|\mathbb{S}^{N-1}|} \int_0^1 \int_{\mathbb{S}^{N-1}} w(x + \varepsilon t y)\ dt d\mathcal{H}^{N-1}(y).$$

We notice that the right-hand side is in $L^1(\Omega, \mathbb{R})$ and converges to w in $L^1(\Omega, \mathbb{R})$ by dominated convergence, thus, by Proposition 1, $e_\varepsilon(u)$ converges as measure to $|\nabla u|_1 \leq w \in L^1(\Omega, \mathbb{R})$, thus $|\nabla u|_1 \in L^1(\Omega, \mathbb{R})$ and $u \in W^{1,1}(\Omega, X)$. This ends the proof. $\square$

2.6 An approximation result

Before going to the proof of Proposition 6, we prove the following approximation result. Such a result is analogous to Theorem 3.3 in [1], which is an approximation result in $L^1(\Omega, X)$ of BV maps by maps constant on cubes, with a control on the energy. Theorem 3.3 in [1] is stated for X a locally compact separable metric space, and we will state first this approximation result when X is just a complete metric space.

Lemma 6 *Let $N \geq 1$. For $n \in \mathbb{N}^*$, $\mathcal{P}_n$ is the usual partition of $]0,1[^N$ in n^N cubes of edge $\frac{1}{n}$. There exists a constant $C_0 = C_0(N, X)$ such that, for every $n \in \mathbb{N}^*$ and every $u \in BV(\Omega, X)$, there exists $v_n \in BV(\Omega, X)$ such that v_n is constant on each cube of $\mathcal{P}_n$,*

$$d_{L^1(]0,1[^N, X)}(u, v_n) \leq \frac{C_0}{n} E^1(u) \qquad \textit{and} \qquad E^1(v_n) \leq C_0 E^1(u).$$

Proof. The proof is based on the following Poincaré type inequality (see Theorem 1.4.1 in [13] for a general version in $W^{1,p}(\Omega, X)$, $1 \leq p < \infty$). There exists a constant C_0, depending only on N such that, for every $\ell > 0$ and every $u \in BV(]0, \ell[^N, X)$, there holds

$$\inf_{\xi \in X} \int_{]0,\ell[^N} d(u(x), \xi)\ dx \leq C_0 \ell |\nabla u|_1(]0, \ell[^N). \tag{16}$$

Proof of (16). By simple scaling, it suffices to treat the case $\ell = 1$. For $u \in BV(]0,1[^N, X)$, we first extend u as a map $U :]-2, 3[^N \to X$ by successive reflections across the boundary (as in Lemma 4).

Next, we may write, using (13),

$$\begin{aligned}\int_{]0,1[^N\times]0,1[^N} d(u(x),u(y))\ dxdy &\le \int_{]-1,1[^N}\int_{]0,1[^N} d(U(x),U(x+h))\ dxdh\\ &\le \int_{]-1,1[^N} |h|\cdot|\nabla U|_1(]-2,3[^N)\ dh \le K_N|\nabla u|_1(]0,1[^N).\end{aligned}$$

As a consequence, by the mean value formula, there exists $y\in]0,1[^N$ such that $\xi\equiv u(y)$ verifies (16).

We complete now easily the proof of Lemma 6. Let $u\in BV(]0,1[^N,X)$ and $n\in\mathbb{N}^*$ be given. For any cube $Q\in\mathcal{P}_n$, there exists by (16) some $\xi_n(Q)\in X$ such that

$$\int_Q d(u(x),\xi_n(Q))\ dx \le \frac{2K_N}{n}|\nabla u|_1(Q). \tag{17}$$

Therefore, we may define $v_n:]0,1[^N\to X$, constant on each cube Q of $\mathcal{P}_n$, of value $\xi_n(Q)$. Summing (17) over all Q's in $\mathcal{P}_n$ yields the first estimate for v_n

$$d_{L^1(]0,1[^N,X)}(u,v_n)\le\frac{2K_N}{n}E^1(u).$$

We turn now to the second estimate. We have easily

$$E^1(v_n)=\frac{1}{2}\sum_{Q\sim Q'\in\mathcal{P}_n}\frac{1}{n^{N-1}}\ d(\xi_n(Q),\xi_n(Q')),$$

where $Q\sim Q'$ means $Q\neq Q'$ and $\bar{Q}$, $\bar{Q}'$ have one face in common. Moreover, for $Q\sim Q'\in\mathcal{P}_n$, say $Q=[0,\frac{1}{n}]^N$ and $Q'=[\frac{1}{n},\frac{2}{n}]\times[0,\frac{1}{n}]^{N-1}=Q+\frac{1}{n}\vec{e}_1$, with $\vec{e}_1=(1,0,...,0)\in\mathbb{R}^N$ for simplicity, we have by the triangle inequality

$$\frac{1}{n^N}\ d(\xi_n(Q),\xi_n(Q'))\le\int_Q d(u(x),\xi_n(Q))\ dx+\int_Q d(u(x),u(x+\frac{1}{n}\vec{e}_1))\ dx+\int_Q d(u(x+\frac{1}{n}\vec{e}_1),\xi_n(Q'))\ dx.$$

The last integral is simply $\int_{Q'} d(u(x),\xi_n(Q'))\ dx$. Thus, using (13) and (17),

$$\frac{1}{n^N}\ d(\xi_n(Q),\xi_n(Q'))\le\frac{2K_N}{n}|\nabla u|_1(Q)+\frac{1}{n}|\nabla u|_1\Big([0,\frac{2}{n}]\times[0,\frac{1}{n}]^{N-1}\Big)+\frac{2K_N}{n}|\nabla u|_1(Q').$$

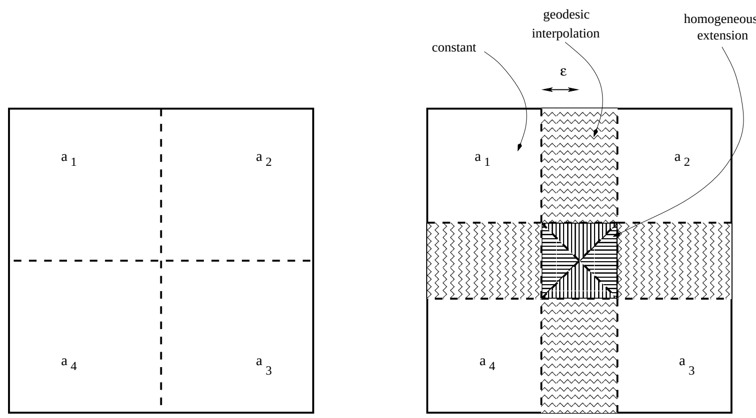
Summing this inequality for $Q\sim Q'\in\mathcal{P}_n$ yields the desired estimate. $\square$

This Lemma 6 will enable us to establish the following approximation result when X is a length space (the proof is not very different from that of Theorem 3.3 in [1]).

Lemma 7 *Let $N\ge 1$ and assume X is a length space. Then, there exists a constant C_0, depending only on N such that for every $u\in BV(]0,1[^N,X)$, there exists a sequence $u_n\in W^{1,1}(]0,1[^N,X)$ such that $|\nabla u_n|_1(]0,1[^N)\le C_0|\nabla u|_1(]0,1[^N)$ for every $n\in\mathbb{N}$ and, as $n\to+\infty$, $u_n\to u$ in $L^1(]0,1[^N,X)$.*

Remark 14 The map $u_n\in W^{1,1}(]0,1[^N,X)$ we construct is actually in a smaller space. If $N=1$, u_n is by construction Lipschitz continuous. For $N\ge 2$, u_n is continuous except at Σ_n, where Σ_n is a finite union of subsets Σ_n^i, $1\le i\le q_n$, where each Σ_n^i is the image by a linear injective map $\mathbb{R}^{N-2}\to\mathbb{R}^N$ of $[0,1]^{N-2}$. If $N=2$, Σ_n^i is by convention a point.

Proof. Let $u \in BV(]0,1[^N, X)$. For $n \in \mathbb{N}^*$, we consider the approximation v_n given by Lemma 6, constant on each cube of the partition $\mathcal{P}_n$. The map v_n is not in $W^{1,1}(]0,1[^N, X)$, thus we now approximate the v_n's. We denote $v = v_n$ for simplicity, and let $0 < \varepsilon < 1/(2n)$. Assume first $N = 1$. For $1 \le j \le n$, $v = a_j$ on $Q_j = (\frac{j-1}{n}, \frac{j}{n})$. Let $2\ell_j \equiv d(a_{j-1}, a_j)$. We fix, for every $1 \le j \le n$, a unit speed geodesic $\gamma_j : [-\ell_j, \ell_j]$ in X from a_{j-1} to a_j. These geodesics exist since X is assumed to be a length space. We define then naturally $u_\varepsilon(x) = a_j$ if $x \in (\frac{j-1}{n} + \varepsilon, \frac{j}{n} - \varepsilon)$ for some $1 < j < n$, $u_\varepsilon(x) = a_1$ if $x \in (0, \frac{1}{n} - \varepsilon)$, $u_\varepsilon(x) = a_n$ if $x \in (\frac{n-1}{n} - \varepsilon, 1)$, and for $1 < j < n$ and $x \in (\frac{j}{n} - \varepsilon, \frac{j}{n} + \varepsilon) \equiv I_j^\varepsilon$, $u_\varepsilon(x) = \gamma_j(\frac{\ell_j}{\varepsilon}(x - \frac{j}{n}))$. We easily check that $u_\varepsilon \to v$ a.e. in $[0,1]$, hence in $L^1(]0,1[^N, X)$ by dominated convergence, and $|\nabla u_\varepsilon|_1 = \sum_{j=1}^n \frac{\ell_j}{\varepsilon} \chi_{I_j^\varepsilon} \to |\nabla v|_1 = \sum_{j=1}^n \ell_j$ in measure as $\varepsilon \to 0$, with $|\nabla u_\varepsilon|_1(]0,1[^N) = |\nabla v|_1(]0,1[^N)$ (here, $\chi_{I_j^\varepsilon}$ stands for the characteristic function of I_j^ε). Since $u_\varepsilon \in W^{1,1}(]0,1[^N, X)$, the proof is complete if $N = 1$.



The map v (left) and its approximation u_ε (right)

We assume now $N = 2$. We follow the approximation scheme given in the figure above, where we assume for simplicity that $n = 2$. By geodesic interpolation, for the upper rectangle for instance, we mean that, as in the 1-dimensional case, $u_\varepsilon(x) = \gamma(\frac{\ell}{\varepsilon}(x_1 - \frac{1}{2}))$, where $2\ell \equiv d(a_1, a_2)$ and $\gamma : [-\ell, \ell] \to X$ is a unit speed geodesic from a_1 to a_2 (in this rectangle, we have $|x_1 - \frac{1}{2}| \le \varepsilon$). Finally, by homogeneous extension, we mean that in the square $C_\varepsilon \equiv a + [-\varepsilon, \varepsilon]^2$, where $a = (\frac{1}{2}, \frac{1}{2})$, we have $u_\varepsilon(x) = u_\varepsilon(a + \varepsilon \frac{x}{|x|_\infty})$. Notice indeed that at this stage, u_ε is already defined on the boundary of the square. The map u_ε is therefore continuous except at a (if we are not in the case $a_1 = a_2 = a_3 = a_4$), and in $W^{1,1}(]0,1[^N, X)$ since u_ε is lipschitz outside C_ε and a direct computation gives

$$\int_{C_\varepsilon} |\nabla u_\varepsilon|_1 \, dx = \varepsilon \int_{\partial C_\varepsilon} |\nabla u_\varepsilon|_1 \, d\mathcal{H}^1(x).$$

Since $|\nabla u_\varepsilon|_1 \le C\varepsilon^{-1}$ on ∂C_ε, where $C = C(a_1, a_2, a_3, a_4)$, we infer $\int_{C_\varepsilon} |\nabla u_\varepsilon|_1 \, dx \le C\varepsilon$. As a consequence, from the computations in the 1-dimensional case, we have

$$|\nabla u_\varepsilon|_1 = \chi_{R_{12}^\varepsilon} \frac{d(a_1, a_2)}{\varepsilon} + \chi_{R_{23}^\varepsilon} \frac{d(a_2, a_3)}{\varepsilon} + \chi_{R_{34}^\varepsilon} \frac{d(a_3, a_4)}{\varepsilon} + \chi_{R_{41}^\varepsilon} \frac{d(a_4, a_1)}{\varepsilon} + \mathcal{O}_{L^1}(\varepsilon),$$

where R_{12}^ε is the rectangle "between" a_1 and a_2, where u_ε is a geodesic interpolation between a_1 and a_2 (and similarly for the other rectangles). Then, as $\varepsilon \to 0$, $u_\varepsilon \to v$ in $L^1(]0,1[^N, X)$ and

$$|\nabla u_\varepsilon|_1 \to |\nabla v|_1$$

as measure. This concludes the proof in the case $N = 2$. Finally, one extends easily this construction to an arbitrary $N \ge 2$. The map u_ε we construct in this case clearly belongs to $W^{1,1}(]0,1[^N, X)$, and this completes the proof of the lemma. $\square$

2.7 Proof of Proposition 6

First, given $1 \leq p < \infty$ and $u \in \mathcal{N}^{1,p}(\Omega, X)$, consider some functions w, $\rho \in L^p(\Omega, \mathbb{R})$ as in Definition 6. With the sequences $u_j = u$ and $g_j = \rho + \frac{w}{j}$, (3) is always satisfied for any 1-lipschitzian curve $\gamma : [0, \ell] \to \Omega$ (even when $\int_0^\ell w \circ \gamma \; ds = \infty$) one has immediately $u \in H^{1,p}(\Omega, X)$ and

$$H(u) \leq N(u).$$

The case $p > 1$. Let us embed X "isometrically" into a Banach space, for instance into $\ell^\infty(X)$ by the Kuratowski embedding

$$\iota : X \to \ell^\infty(X)$$

defined by $\iota(x) \equiv (d(x, z) - d(x, y_0))_{z \in X}$. Here, $y_0 \in X$ is an arbitrary fixed point and "isometric" means that $||\iota(x) - \iota(y)||_{\ell^\infty(X)} = d(x, y)$ for every x, $y \in X$. It is clear that $H(\iota(u)) \leq H(u)$. Hence, by [16], Theorem 3.2, there exists (since $p > 1$) a function $g \in L^p(\Omega, \mathbb{R})$ such that $H(\iota(u)) = \int_\Omega g^p \; dx$ and for any 1-lipschitzian curve $\gamma : [0, \ell] \to \Omega$,

$$||\iota(u) \circ \gamma(\ell) - \iota(u) \circ \gamma(0)||_{\ell^\infty(X)} = d(u \circ \gamma(\ell), u \circ \gamma(0)) \leq \int_0^\ell g \circ \gamma \; ds.$$

Therefore, $u \in \mathcal{N}^{1,p}(\Omega, X)$ and $N(u) \leq \int_\Omega g^p \; dx = H(u)$. Thus, if $1 < p < \infty$, then, in $L^p(\Omega, X)$,

$$H = N.$$

The case $p = 1$. We assume now $p = 1$. Let $u \in H^{1,1}(\Omega, X)$. Then, there exist sequences $u_j \in L^1(\Omega, X)$ and $g_j \in L^1(\Omega, \mathbb{R}_+)$ such that $u_j \to u$ and $\int_\Omega g_j \; dx \to H(u)$ as $j \to +\infty$, and, for any 1-lipschitzian curve $\gamma : [0, \ell] \to \Omega$,

$$d(u_j \circ \gamma(0), u_j \circ \gamma(\ell)) \leq \int_0^\ell g_j \circ \gamma \; ds.$$

In particular, we have $u_j \in \mathcal{N}^{1,1}(\Omega, X)$ (take $w = 0$, $\rho = g_j$) and $N(u_j) \leq \int_\Omega g_j \; dx$. As a consequence of Propositions 4 and 5, we infer $u_j \in W^{1,1}(\Omega, X)$ and

$$K_{1,N} E^1(u_j) \leq R(u_j) = N(u_j) \leq \int_\Omega g_j \; dx.$$

Taking the liminf as $j \to +\infty$ and using the lower semi-continuity of E^1 in $L^1(\Omega, X)$ (Theorem 2), we deduce

$$K_{1,N} E^1(u) \leq K_{1,N} \liminf_{j \to +\infty} E^1(u_j) \leq H(u).$$

Therefore, $H^{1,1}(\Omega, X) \subset BV(\Omega, X)$, and $K_{1,N} E^1 \leq H$ on $H^{1,1}(\Omega, X)$.

Assume now $u \in BV(\Omega, X)$. To prove that $u \in H^{1,1}(\Omega, X)$, we have to construct a sequence in $W^{1,1}(\Omega, X)$ approximating u in $L^1(\Omega, X)$ with controled 1-energy. First, since Ω is smooth, we may extend u by an arbitrary constant outside Ω, so that the energy of the extension is $\leq C(\Omega) E^1(u)$. We may now assume, without loss of generality, that Ω is a cube, and by scaling, that this cube is $]0, 1[^N$. Now, we apply Lemma 7: there exists a sequence $u_n \in W^{1,1}(\Omega, X)$ such that $u_n \to u$ in $L^1(\Omega, X)$ as $n \to +\infty$ and $E^1(u_n) \leq C_0 E^1(u)$ for every $n \in \mathbb{N}^*$. Therefore, $u \in H^{1,1}(\Omega, X)$ (take $g_n = \frac{1}{K_{1,N}} |\nabla u_n|_1$ as in section 2.4) and we obtain the second inequality

$$H(u) \leq \liminf_{n \to +\infty} \int_\Omega g_n \; dx \leq \frac{1}{K_{1,N}} \liminf_{n \to +\infty} E^1(u_n) \leq \frac{C_0}{K_{1,N}} E^1(u). \qquad \square$$

In the case where X is locally compact and separable, the link between $H(U)$ and the BV energy $|Du|(\Omega)$ in the sense of L. Ambrosio can actually be made more precise.

Lemma 8 *Assume that X is locally compact and separable. Then,*

$$|Du|(\Omega) \le H(u) \quad \text{if } u \in BV(\Omega, X), \qquad |Du|(\Omega) = H(u) \quad \text{if } u \in W^{1,1}(\Omega, X).$$

If, furthermore, X is a length space, then there exists $C_0(N, X, \Omega)$ such that for every $u \in BV(\Omega, X)$,

$$H(u) \le C_0 |Du|(\Omega).$$

Proof. The last inequality follows from Propositions 3 and 6. Let us prove the first one. Let $u \in L^1(\Omega, X)$ and $g \in L^1(\Omega, \mathbb{R}_+)$ be such that

$$d(u \circ \gamma(\ell), u \circ \gamma(0)) \le \int_0^\ell g \circ \gamma \tag{18}$$

for any 1-lipschitzian curve $\gamma : [0, \ell] \to \Omega$. Then, if $\varphi : X \to \mathbb{R}$ is 1-lipschitzian, $v \equiv \varphi \circ u : \Omega \to \mathbb{R}$ satisfies

$$|v \circ \gamma(\ell) - v \circ \gamma(0)| \le \int_0^\ell g \circ \gamma.$$

Hence, given $\varepsilon > 0$, $\omega \in \mathbb{S}^{N-1}$ and taking, for $x \in \Omega^\varepsilon$, $\gamma(s) = x + s\omega$, $s \in [0, \varepsilon]$, one obtains

$$|v(x + \varepsilon\omega) - v(x)| \le \varepsilon \int_0^1 g(x + \varepsilon t \omega)\ dt.$$

Assume for the moment that v, $g : \Omega \to \mathbb{R}$ are smooth. Dividing by ε and letting $\varepsilon \to 0$ yields

$$|\omega \cdot \nabla v|(x) \le g(x)$$

for every $\omega \in \mathbb{S}^{N-1}$, which implies

$$|\nabla v| \le g.$$

The general case follows easily by standard regularization. As a consequence, $u \in W^{1,1}(\Omega, X)$ and

$$|Du|(\Omega) \le \int_\Omega g\ dx. \tag{19}$$

If $u \in H^{1,1}(\Omega, X)$, then there exist sequences $u_j \in L^1(\Omega, X)$ and $g_j \in L^1(\Omega, \mathbb{R})$ such that (18) holds for any j and $H(u) = \lim_{j \to +\infty} \int_\Omega g_j\ dx$. Therefore, $|Du_j|(\Omega) \le \int_\Omega g_j\ dx$ by (19) gives at the limit

$$|Du|(\Omega) \le \liminf_{j \to +\infty} |Du_j|(\Omega) \le \liminf_{j \to +\infty} \int_\Omega g_j\ dx = H(u).$$

Here, we have used that $u \mapsto |Du|(\Omega)$ is lower semi-continuous for the $L^1(\Omega, X)$ topology (see [1], section 1). This finishes the proof of the first inequality.

Assume now $u \in W^{1,1}(\Omega, X)$. Let $\gamma : [0, \ell] \to \Omega$ be a 1-lipschitzian curve, and consider the 1-lipschitzian function $\varphi \equiv d(\cdot, u \circ \gamma(0)) : X \to \mathbb{R}$. Then, $v \equiv \varphi \circ u \in W^{1,1}(\Omega, \mathbb{R})$ and $|Dv| \le |Du|$. Moreover, since v is real valued, $|Dv| = |\nabla v| \in L^1(\Omega, \mathbb{R})$ (in the classical sense, see [1], Remark 2.2). This implies

$$d(u \circ \gamma(\ell), u \circ \gamma(0)) = |v \circ \gamma(\ell) - v \circ \gamma(0)| \le \int_0^\ell |\gamma'(s) \cdot (\nabla v) \circ \gamma(t)|\ dt \le \int_0^\ell |Du| \circ \gamma(t)\ dt.$$

Since $|Du| \in L^1(\Omega, \mathbb{R})$, we infer that $u \in H^{1,1}(\Omega, X)$ and $H(u) \le \int_\Omega |Du|\ dx = |Du|(\Omega)$. The second inequality is proved. $\square$

2.8 Proof of Theorem 4

Let (ρ_ε) be a family of mollifiers (or (ρ_n) be a sequence of mollifiers), and recall

$$F_\varepsilon(u) = \int\int_{\Omega\times\Omega}\Big(\frac{d(u(x),u(y))}{|x-y|}\Big)^p \rho_\varepsilon(x-y)\ dxdy.$$

We first prove that for any $u \in L^p(\Omega, X)$,

$$\limsup_{\varepsilon\to 0} F_\varepsilon(u) \le E^p(u). \tag{20}$$

Proof of (20). We may assume $E^p(u) < \infty$, that is $u \in BV(\Omega, X)$ (if $p = 1$) or $u \in W^{1,p}(\Omega, X)$ (if $1 < p < \infty$). First, we have

$$F_\varepsilon(u) = \int_{\mathbb{R}^N}\Big(\int_{\Omega\cap(\Omega-h)}\Big(\frac{d(u(x),u(x+h))}{|h|}\Big)^p \rho_\varepsilon(h)\ dx\Big)dh.$$

Let us fix $\xi \in X$ and $\eta > 0$. Then,

$$\int_{\mathbb{R}^N\setminus B_\eta(0)}\Big(\int_{\Omega\cap(\Omega-h)}\Big(\frac{d(u(x),u(x+h))}{|h|}\Big)^p \rho_\varepsilon(h)\ dx\Big)dh \le \Big(\frac{2d_{L^p}(u,\xi)}{\delta_0}\Big)^p\int_{\mathbb{R}^N\setminus B_\eta(0)}\rho_\varepsilon(h)\ dh$$

and the last integral tends to 0 as $\varepsilon \to 0$. Therefore, for any $\eta > 0$,

$$F_\varepsilon(u) = \int_{B_\eta(0)}\int_{\Omega\cap(\Omega-h)}\Big(\frac{d(u(x),u(x+h))}{|h|}\Big)^p \rho_\varepsilon(h)\ dx\Big)dh + o_{\varepsilon\to 0}(1). \tag{21}$$

Moreover, letting U be the extension by reflection across the boundary of u on Ω_{δ_0} as in Lemma 4, and using Theorem 1.8.1 and Lemma 1.9 in [12], we infer, assuming $|h| \le \eta < \delta_0/2$,

$$\int_{(\Omega\cap(\Omega-h))\setminus\Omega^\eta} d^p(u(x),u(x+h))\ dx \le C(p,\Omega)|h|^p\int_{\Omega^{2\eta}\setminus\Omega_{2\eta}}|\nabla U|_p\ dx.$$

As a consequence,

$$\int_{B_\eta(0)}\int_{(\Omega\cap(\Omega-h))\setminus\Omega^\eta}\Big(\frac{d(u(x),u(x+h))}{|h|}\Big)^p \rho_\varepsilon(h)\ dx\Big)dh \le C(p,\Omega)\int_{\Omega^{2\eta}\setminus\Omega_{2\eta}}|\nabla U|_p\ dx.$$

Inserting this into (21) yields for $\eta < \delta_0/2$

$$\begin{aligned}\limsup_{\varepsilon\to 0} F_\varepsilon(u) \le\ & \limsup_{\varepsilon\to 0}\int_{B_\eta(0)}\int_{\Omega^\eta\cap(\Omega-h)}\Big(\frac{d(u(x),u(x+h))}{|h|}\Big)^p\ dx\Big)\rho_\varepsilon(h)dh \\ & + C(p,\Omega)\int_{\Omega^{2\eta}\setminus\Omega_{2\eta}}|\nabla U|_p\ dx\end{aligned} \tag{22}$$

We denote $\bar\rho_\varepsilon \in L^1(\mathbb{R}_+^*)$ defined by $\rho_\varepsilon(x) = \bar\rho_\varepsilon(|x|)$. Finally, ${}^\omega E^p(u)$ standing for the p-energy in the direction $\omega \in \mathbb{S}^{N-1}$ (see [12], section 1.8), we have, using polar coordinates and [12],

$$\begin{aligned}&\int_{B_\eta(0)}\int_{\Omega^\eta\cap(\Omega-h)}\Big(\frac{d(u(x),u(x+h))}{|h|}\Big)^p\ dx\Big)\rho_\varepsilon(h)dh \\ &\qquad\le \int_0^\eta \rho_\varepsilon(r)r^{N-1}\int_{\mathbb{S}^{N-1}}\int_{\Omega^\eta}\Big(\frac{d(u(x),u(x+r\omega))}{r}\Big)^p\ dxd\mathcal{H}^{N-1}(\omega)dr \\ &\qquad\le |\mathbb{S}^{N-1}|\int_0^\eta \rho_\varepsilon(r)r^{N-1}\Big[\frac{1}{|\mathbb{S}^{N-1}|}\int_{\mathbb{S}^{N-1}}{}^\omega E^p(u)\ d\mathcal{H}^{N-1}(\omega)\Big]\ dr.\end{aligned}$$

From [12], Theorem 1.8.1, the term between brackets is $E^p(u)$, and $|\mathbb{S}^{N-1}|\int_0^\eta \rho_\varepsilon(r) r^{N-1}\, dr = \int_{B_\eta}\rho_\varepsilon \to 1$ as $\varepsilon \to 0$, hence

$$\limsup_{\varepsilon\to 0}\int_{B_\eta(0)}\int_{\Omega^\eta\cap(\Omega-h)}\Big(\frac{d(u(x),u(x+h))}{|h|}\Big)^p\, dx\Big)\rho_\varepsilon(h)dh \le E^p(u).$$

Combining this with (22) gives, for $\eta < \delta_0/2$,

$$\limsup_{\varepsilon\to 0} F_\varepsilon(u) \le E^p(u) + C(p,\Omega)\int_{\Omega^{2\eta}\setminus\Omega_{2\eta}} |\nabla U|_p\, dx.$$

As in the proof of Theorem 3, the last integral tends to 0 as $\eta \to 0$. This finishes the proof of (20).

We now prove that if $u_\varepsilon \to u$ (or $u_n \to u$) in $L^p(\Omega, X)$, then

$$E^p(u) \le \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \qquad \text{or} \qquad E^p(u) \le \liminf_{n\to+\infty} F_n(u_n). \tag{23}$$

This will conclude the proof of Theorem 4, since for the pointwise limit, it suffices to take $u_\varepsilon = u$ (or $u_n = u$) to obtain by (20) and (23) $\lim_{\varepsilon\to 0} F_\varepsilon(u) = E^p(u)$ (or $\lim_{n\to+\infty} F_n(u) = E^p(u)$), and the proof of the Γ-limit follows from (23) and the pointwise limit.

Proof of (23). We will prove (23) only in the case of the sequence $\rho_n \to \delta_0$, since the general case will then follow. We may also assume without loss of generality that the "$\liminf_{n\to+\infty}$" is actually a "$\lim_{n\to+\infty}$". We denote $\bar\rho_n(r) \in L^1_{loc}(\mathbb{R}_+^*)$ defined by $\rho_n(x) = \bar\rho_n(|x|)$. First, for any $u \in L^p(\Omega, X)$,

$$\begin{aligned} F_n(u) &= \int\int_{\Omega\times\Omega}\Big(\frac{d(u(x),u(y))}{|x-y|}\Big)^p\rho_n(x-y)\, dxdy \\ &= \int_\Omega\int_0^\infty\Big[\bar\rho_n(r)\int_{S_r(x)\cap\Omega}\Big(\frac{d(u(x),u(y))}{r}\Big)^p\, d\mathcal{H}^{N-1}(y)\Big]\, drdx \\ &\ge \int_0^\infty |\mathbb{S}^{N-1}| r^{N-1}\bar\rho_n(r)\Big(\int_{\Omega^r} e_r(u)(x)\, dx\Big)\, dr = \int_0^\infty E_r(u)\mu_n(r)\, dr, \end{aligned} \tag{24}$$

where $\mu_n(r) \equiv |\mathbb{S}^{N-1}| r^{N-1}\bar\rho_n(r)$ is a probability measure on $\mathbb{R}_+^*$ such that $\mu_n \rightharpoonup \delta_0$ as $n \to +\infty$, *i.e.* for all $\delta > 0$,

$$\int_\delta^\infty \mu_n(r)\, dr \to 0 \qquad \text{as} \quad n \to +\infty. \tag{25}$$

Suppose for a while that $u_n = u$ for all $n \in \mathbb{N}$. Then, since $E_r(u) \to E^p(u) \in \mathbb{R}_+ \cup \{+\infty\}$ as $r \to 0$, one infers easily that since $\mu_n \rightharpoonup \delta_0$ as $n \to +\infty$, then

$$\int_0^\infty E_r(u)\mu_n(r)\, dr \to E^p(u) \in \mathbb{R}_+ \cup \{+\infty\} \qquad \text{as} \quad n \to +\infty.$$

Hence, $\liminf_{n\to+\infty} F_n(u) \ge E^p(u) \in \mathbb{R}_+ \cup \{+\infty\}$ as desired. To prove the result in the general case, we will prove in fact that there exist a subsequence n_j and a sequence $r_j > 0$ such that

$$\forall j \in \mathbb{N}^*, \qquad r_j \le \frac{1}{j} \qquad \text{and} \qquad E_{r_j}(u_{n_j}) \le F_{n_j}(u_{n_j})(1+\frac{1}{j}). \tag{26}$$

We proceed by induction on j, and for $j = 0$, there is nothing to prove. Assume now $r_{j-1} \in (0, \frac{1}{j-1})$ and n_{j-1} constructed (satisfying (26)), such that $(n_1, ... n_{j-1})$ is increasing. We fix, by (25), n_j large enough so that $n_j > n_{j-1}$ and

$$\int_0^{\frac{1}{j}} \mu_{n_j}(r)\, dr \ge \Big(1+\frac{1}{j}\Big)^{-1}.$$

Then, by (24) with $n = n_j$, one has

$$F_{n_j}(u_{n_j}) \geq \int_0^{\frac{1}{j}} E_r(u_{n_j})\mu_{n_j}(r)\ dr.$$

Thus, by the mean-value formula, there exists $r_j \in (0, \frac{1}{j})$ such that

$$E_{r_j}(u_{n_j}) \leq \frac{\int_0^{\frac{1}{j}} E_r(u_{n_j})\mu_{n_j}(r)\ dr}{\int_0^{\frac{1}{j}} \mu_{n_j}(r)\ dr} \leq F_{n_j}(u_{n_j})(1 + \frac{1}{j}).$$

This concludes the proof of (26) by induction.

From (26), we infer $u_{n_j} \to u$ in $L^p(\Omega, X)$ (it is a subsequence of (u_n)) and $r_j \to 0$, hence we may apply Proposition 2 (with $\nu = \delta_1$) to infer, by (26),

$$E^p(u) \leq \liminf_{j\to+\infty} E_{r_j}(u_{n_j}) \leq \liminf_{j\to+\infty} F_{n_j}(u_{n_j})$$

which concludes the proof. □

2.9 Proof of Theorem 5

The proof is divided in two steps.

Step 1: $u \in W^{1,p}(\Omega, X)$. We assume first $N = 1$, and, by scaling, that $\Omega = (0, 1)$. Let $u \in W^{1,p}((0,1), X)$. Since $p > 1$, we may assume u (Hölder-)continuous up to a modification on a negligible set. The proof of the pointwise limit for G_n relies on the dominated convergence theorem.

Pointwise convergence. We first show that for almost every $x \in (0, 1)$,

$$\Lambda_n(x) \equiv \int_0^1 \frac{g_n(d(u(x), u(y)))}{|x-y|^{p+1}}\ dy \to 2|u'|_p(x) \qquad \text{as } n \to +\infty. \tag{27}$$

Indeed, we know from [12] that for almost every $x \in (0, 1)$,

$$\frac{d(u(x), u(y))}{|x-y|} \to |u'|_1(x) = \Big(|u'|_p(x)\Big)^{\frac{1}{p}} \qquad \text{as } y \to x.$$

We fix such an $x \in (0, 1)$ and assume first $|u'|_1(x) > 0$. Then, for $0 < \varepsilon < |u'|_1(x)$, there exists $\delta > 0$ (depending on ε and x) such that, if $|x - y| \leq \delta$, then

$$0 \leq |u'|_1(x) - \varepsilon \leq \frac{d(u(x), u(y))}{|x-y|} \leq |u'|_1(x) + \varepsilon.$$

Without loss of generality, we may assume $(x - \delta, x + \delta) \subset (0, 1)$. Thus, since g_n is non decreasing,

$$\int_{x-\delta}^{x+\delta} \frac{g_n(|x-y|(|u'|_1(x)-\varepsilon))}{|x-y|^{p+1}}\ dy \leq \int_{x-\delta}^{x+\delta} \frac{g_n(d(u(x), u(y)))}{|x-y|^{p+1}}\ dy \leq \int_{x-\delta}^{x+\delta} \frac{g_n(|x-y|(|u'|_1(x)+\varepsilon))}{|x-y|^{p+1}}\ dy.$$

By simple change of variables, we have

$$\int_{x-\delta}^{x+\delta} \frac{g_n(|x-y|(|u'|_1(x) \pm \varepsilon))}{|x-y|^{p+1}}\ dy = 2(|u'|_1(x) \pm \varepsilon)^p \int_0^{\delta(|u'|_1(x)\pm\varepsilon)} \frac{g_n(t)}{t^{p+1}}\ dt.$$

Using hypothesis (b) and (c) on (g_n), we infer that, on the one hand,

$$\int_0^{\delta(|u'|_1(x)\pm\varepsilon)} \frac{g_n(t)}{t^{p+1}}\ dt \to 1 \qquad \text{as} \ \ n \to +\infty,$$

and on the other hand (u is bounded),

$$\int_{\{|y-x|>\delta\}} \frac{g_n(d(u(x),u(y)))}{|x-y|^{p+1}}\ dy \to 0 \qquad \text{as} \ \ n \to +\infty.$$

Consequently, for every $0 < \varepsilon < |u'|_1(x)$, there holds for n large enough (depending on ε and x)

$$2(|u'|_1(x)-\varepsilon)^p(1-\varepsilon)-\varepsilon \leq \int_0^1 \frac{g_n(d(u(x),u(y)))}{|x-y|^{p+1}}\ dy \leq 2(|u'|_1(x)+\varepsilon)^p(1+\varepsilon)+\varepsilon,$$

and (27) follows. If $|u'|_1(x) = 0$, then it suffices to consider only the right-hand side inequalities to infer (27). We can notice that (27) holds even for $p = 1$ if $u \in W^{1,1}((0,1), X)$.

Domination. We now try to find an integrable function dominating the Λ_n's on $(0,1)$. Here, the hypothesis $p > 1$ is crucial, since we use, as in [15], the theory of maximal functions in L^p. We let $U \in W^{1,p}((-1,2), X)$ be its reflexion across the boundary. We denote, for $x \in (0,1)$, M the maximal function

$$M(x) \equiv \sup_{0<h<1} \frac{1}{2h} \int_{x-h}^{x+h} |U'|(t)\ dt.$$

For $x \in (0,1)$, we have

$$d(u(x),u(y)) \leq \Big| \int_x^y |u'|(t)\ dt \Big| \leq 2|x-y|M(x).$$

Therefore, since g_n is non decreasing,

$$\Lambda_n(x) \leq \int_0^1 \frac{g_n(2|x-y|M(x))}{|x-y|^{p+1}}\ dy.$$

This last integral can be estimated by direct computation:

$$\int_0^1 \frac{g_n(2|x-y|M(x))}{|x-y|^{p+1}}\ dy \leq 2\int_0^1 \frac{g_n(2tM(x))}{t^{p+1}}\ dt = 2^{p+1}M(x)^p \int_0^{2M(x)} \frac{g_n(\tau)}{\tau^{p+1}}\ d\tau.$$

In view of hypothesis (b) for (g_n), the last integral is bounded independently of x and n by a constant A, hence, for $x \in (0,1)$,

$$\Lambda_n(x) \leq 2^{p+1}AM(x)^p.$$

Since $p > 1$, we know from the theory of maximal functions that $M \in L^p((0,1),\mathbb{R})$, hence the functions Λ_n are dominated by $2^{p+1}AM(x)^p \in L^1((0,1),\mathbb{R})$.

Consequently, we may use Lebesgue's Theorem to deduce that for $u \in W^{1,p}((0,1), X)$,

$$G_n(u) = \frac{1}{2}\int_0^1 \Lambda_n(x)\ dx \to \frac{1}{2}\int_0^1 2|u'|_p(x)\ dx = E^p(u) \qquad \text{as} \ \ n \to +\infty.$$

Assume now $N \geq 1$ arbitrary. Then, using polar coordinates,

$$G_n(u) = \frac{1}{|\mathbb{S}^{N-1}|}\int_{\Omega\times\mathbb{S}^{N-1}} \int_{I_{x,\omega}} \frac{g_n(d(u(x),u(x+t\omega)))}{t^{p+1}}\ dt d\mathcal{H}^{N-1}(\omega) dx,$$

where $I_{x,t} \subset \mathbb{R}_+$ is open and contains the interval $(0, dist(x, \partial\Omega))$. If $u \in W^{1,p}(\Omega, X)$, then for almost every $x \in \Omega$ and $\omega \in \mathbb{S}^{N-1}$, $u \in W^{1,p}(I_{x,\omega}, X)$, so that, using the case $N = 1$,

$$\int_{I_{x,\omega}} \frac{g_n(d(u(x), u(x + t\omega)))}{t^{p+1}}\, dt$$

tends to ${}^{\omega}e(u)(x)$ (note that $t \geq 0$, hence there is no longer the factor 2 as in (27)) as $n \to +\infty$ and, $M(x,\omega)$ standing for the maximal function along the half-line $x + \mathbb{R}_+\omega$, is dominated by $C(p,\Omega)AM(x,\omega)^p \in L^1(\Omega \times \mathbb{S}^{N-1}, \mathbb{R})$ (since $p > 1$). Hence, by dominated convergence, we have

$$\lim_{n \to +\infty} G_n(u) = E^p(u).$$

Step 2: $(G_n(u))_{n\in\mathbb{N}}$ is bounded. We assume now $u \in L^p(\Omega, X)$ and the sequence $(G_n(u))_{n\in\mathbb{N}}$ bounded, and show that this implies $u \in W^{1,p}(\Omega, X)$. The proof relies on Lemma 4 in [15] and follows the lines of the proof of Theorem 3 in [15].

Let us first consider the extension of u by reflection across the boundary $U \in L^p(\Omega_{\delta_0}, X)$ (Ω is smooth). It follows then straightfowardly that

$$G_n(U, \Omega_{\delta_0}) \leq C(p,\Omega) G_n(u, \Omega).$$

This allows, for u, to reduce the problem to the following: $\Omega = B_2(0) \subset \mathbb{R}^N$, $(G_n(u, B_2(0)))_{n\in\mathbb{N}}$ is bounded and let us prove that $E^p(u, B_1(0)) < \infty$.

Next, we follow word for word the lines of subsection 3.2 in [15], just replacing, for $k \in \mathbb{N}$, $\{10^{-(k+1)} < |g(x) - g(y)| < 10^{-k}\}$ by $A_k \equiv \{10^{-(k+1)} < d(u(x), u(y)) < 10^{-k}\}$. This implies that for every $s > 0$,

$$\iint_{\{d(u(x),u(y))>s\}} \frac{dxdy}{|x-y|^{p+N}} < \infty, \qquad \text{and} \qquad \liminf_{k\to+\infty} \iint_{A_k} \frac{10^{-p(k+1)}}{|x-y|^{p+N}}\, dxdy < \infty. \tag{28}$$

We set, for $k \in \mathbb{N}$ and $\omega \in \mathbb{S}^{N-1}$, $A_k(\omega) \equiv \{(x,h),\ 10^{-(k+1)} < d(u(x), u(x+h\omega)) < 10^{-k}\}$. From (28), passing to polar coordinates and using Fatou's Lemma, we deduce

$$\int_{\mathbb{S}^{N-1}} \liminf_{k\to+\infty} \iint_{A_k(\omega)} \frac{10^{-p(k+1)}}{h^{p+1}}\, dhdx\ d\mathcal{H}^{N-1}(\omega) \leq \liminf_{k\to+\infty} \iint_{A_k} \frac{10^{-p(k+1)}}{|x-y|^{p+N}}\, dxdy < \infty$$

and

$$\text{for every } s > 0, \qquad \int_{\mathbb{S}^{N-1}} \iint_{\{d(u(x),u(x+h\omega))>s\}} \frac{dxdh}{h^{p+1}}\, d\mathcal{H}^{N-1}(\omega) < \infty.$$

As a consequence, for every $s > 0$ and almost every $\omega \in \mathbb{S}^{N-1}$,

$$\iint_{\{d(u(x),u(x+h\omega))>s\}} \frac{dxdh}{h^{p+1}} < \infty, \qquad V(\omega) \equiv \liminf_{k\to+\infty} \iint_{A_k(\omega)} \frac{10^{-p(k+1)}}{h^{p+1}}\, dhdx < \infty \tag{29}$$

and $V \in L^1(\mathbb{S}^{N-1})$. In view of [12], section 1.10, we first show that, for these $\omega \in \mathbb{S}^{N-1}$,

$${}^{\omega}E(u) \leq V(\omega). \tag{30}$$

Pick then one such $\omega \in \mathbb{S}^{N-1}$. Up to a rotation, we may assume $\omega = (0, ..., 0, 1)$. We write $x = (x', x_N)$, and use once again Fatou's Lemma to infer from (29) that for every $s > 0$,

$$\int_{|x'|\leq 1} \iint_{\{d(u(x',x_N),u(x',x_N+h))>s\}} \frac{dx_N dh}{h^{p+1}}\, dx' < \infty$$

and

$$\int_{|x'|\leq 1} \liminf_{k\to+\infty} \iint_{A_k(\omega,x')} \frac{10^{-p(k+1)}}{h^{p+1}}\, dx_N dh\; dx' \leq V(\omega) < \infty, \tag{31}$$

where $A_k(\omega,x') \equiv \{(x_N,h),\ 10^{-(k+1)} < d(u(x',x_N),u(x',x_N+h) < 10^{-k}\}$. Therefore, for almost every $|x'|\leq 1$ and every $s>0$,

$$\iint_{\{d(u(x',x_N),u(x',x_N+h))>s\}} \frac{dx_N dh}{h^{p+1}} < \infty, \qquad V(\omega,x') \equiv \liminf_{k\to+\infty} \iint_{A_k(\omega,x')} \frac{10^{-p(k+1)}}{h^{p+1}}\, dx_N dh < \infty. \tag{32}$$

Now, we are reduced to one-dimensional problem. Let $\varepsilon>0$, x' as above and let us prove the estimate

$$\int_{|x_N|\leq\sqrt{1-|x'|^2}-\varepsilon} {}^{\omega}e_\varepsilon(u)(x',x_N)\, dx_N \leq C_p V(\omega,x'). \tag{33}$$

For that purpose, we argue as in the proof of theorem 4 in [15]. Let us define $J\in\mathbb{N}$ such that $(J-1)\varepsilon < \sqrt{1-|x'|^2}-\varepsilon \leq J\varepsilon$. Hence, we have

$$\int_{|x_N|\leq\sqrt{1-|x'|^2}-\varepsilon} {}^{\omega}e_\varepsilon(u)(x',x_N)\, dx_N \leq \sum_{j=-J}^{J}\int_{j\varepsilon}^{(j+1)\varepsilon} {}^{\omega}e_\varepsilon(u)(x',x_N)\, dx_N. \tag{34}$$

Let now $-J\leq j\leq J$, $x_N\in[j\varepsilon,(j+1)\varepsilon]$, and let $g(t)\equiv d(u(x',x_N),u(x',x_N+t)) : (-2\varepsilon,2\varepsilon)\to\mathbb{R}$. In view of (32), it follows from Theorem 4 in [15] that $g\in W^{1,p}((-2\varepsilon,2\varepsilon),\mathbb{R})$. In particular, g is continuous and then, Lemma 4 in [15] yields the estimate

$$\Big(\sup_{(-2\varepsilon,2\varepsilon)} g - \inf_{(-2\varepsilon,2\varepsilon)} g\Big)^p \leq C_p\varepsilon^{p-1}\liminf_{k\to+\infty}\iint_{A_k(\omega,x')\cap(j\varepsilon,(j+1)\varepsilon)\times(-2\varepsilon,2\varepsilon)} \frac{10^{-p(k+1)}}{h^{p+1}}\, dx_N dh,$$

where C_p depends only on p. Consequently, since $\inf_{(-2\varepsilon,2\varepsilon)} g = 0$ (g is continuous, non-negative and $g(0)=0$), we infer

$$\sup_{(-2\varepsilon,2\varepsilon)} g \leq C_p\varepsilon^{p-1}\liminf_{k\to+\infty}\iint_{A_k(\omega,x')\cap(j\varepsilon,(j+1)\varepsilon)\times(-2\varepsilon,2\varepsilon)} \frac{10^{-p(k+1)}}{h^{p+1}}\, dx_N dh.$$

Thus,

$$\begin{aligned}
&\int_{j\varepsilon}^{(j+1)\varepsilon} {}^{\omega}e_\varepsilon(u)(x',x_N)\, dx_N \\
&\quad = \frac{1}{2\varepsilon^p}\int_{j\varepsilon}^{(j+1)\varepsilon} d^p(u(x',x_N),u(x',x_N+\varepsilon)) + d^p(u(x',x_N),u(x',x_N-\varepsilon))\, dx_N \\
&\quad \leq C_p \liminf_{k\to+\infty}\iint_{A_k(\omega,x')\cap(j\varepsilon,(j+1)\varepsilon)\times(-2\varepsilon,2\varepsilon)} \frac{10^{-p(k+1)}}{h^{p+1}}\, dx_N dh.
\end{aligned}$$

We now sum this inequality for $-J\leq j\leq J$, and use the fact that, for any sequences (a_n) and (b_n) bounded from below, $\liminf a_n + \liminf b_n \leq \liminf(a_n+b_n)$. By (34), this leads us to

$$\int_{|x_N|\leq\sqrt{1-|x'|^2}-\varepsilon} {}^{\omega}e_\varepsilon(u)(x',x_N)\, dx_N \leq C_p\liminf_{k\to+\infty}\sum_{j=-J}^{J}\iint_{A_k(\omega,x')\cap(j\varepsilon,(j+1)\varepsilon)\times(-2\varepsilon,2\varepsilon)} \frac{10^{-p(k+1)}}{h^{p+1}}\, dx_N dh$$

and (33) follows. Integrating then (33) in x' and using (31) yields (30).

To conclude the proof, we integrate (30) for $\omega\in\mathbb{S}^{N-1}$ and apply [12], section 1.10, to deduce that $E^p(u) < +\infty$, *i.e.* $u\in W^{1,p}(\Omega,X)$. This finishes the proof of theorem 5. $\square$

3 Traces of maps in $W^{s,p}(\Omega, X)$

Let Ω be a smooth bounded domain in $\mathbb{R}^N$ (with lipschitz boundary is enough), and let $1 \leq p < \infty$, $0 < s \leq 1$. Then, arguing as in Lemma 4, there exists $\delta_0 > 0$, depending only on Ω, and a constant C depending on Ω, p and s such that for any $u \in W^{s,p}(\Omega, X)$, the extension U of u by reflection across the boundary on Ω_{δ_0} is in $W^{s,p}(\Omega_{\delta_0}, X)$ and for any $\xi \in X$,

$$|U|_{W^{s,p}(\Omega_{\delta_0},X)} \leq C(1 + |d(u,\xi)|_{L^p(\Omega)} + |u|_{W^{s,p}(\Omega,X)}).$$

We now introduce the quantity

$$\|u\|^p_{W^{s,p}(\Omega,X)} \equiv \sum_{i=1}^N \int_\Omega \int_0^{\delta_0} \frac{d(U(x), U(x+t\vec{e}_i))^p}{t^{sp+1}}\, dxdt,$$

where $(\vec{e}_1, \ldots, \vec{e}_N)$ is the canonical basis of $\mathbb{R}^N$ and U is the reflection of u across the boundary. We expect, as it is well-known for real-valued Sobolev maps, that the "semi-norm" $||u||_{W^{s,p}(\Omega,X)}$ is "equivalent" to $|u|_{W^{s,p}(\Omega,X)}$.

Lemma 9 *Let $1 \leq p < \infty$, $0 < s < 1$ and $u \in L^p(\Omega, X)$, where X is a length space. Then, there exists a constant $C > 0$, depending on p, s and Ω, such that for any $\xi \in X$,*

$$\|u\|_{W^{s,p}(\Omega,X)} \leq C(|d(u,\xi)|_{L^p} + |u|_{W^{s,p}}) \quad \text{and} \quad |u|_{W^{s,p}(\Omega,X)} \leq C(|d(u,\xi)|_{L^p} + \|u\|_{W^{s,p}}).$$

3.1 Proof of Lemma 9

First, note that for any $\xi \in X$,

$$\int_{\{|x-y|\geq\delta_0\}} \frac{d(u(x),u(y))^p}{|x-y|^{sp+N}}\, dxdy \leq C_{\delta_0} |d(u,\xi)|^p_{L^p(\Omega)},$$

hence, it suffices to show that

$$C_1\|u\|^p_{W^{s,p}} \leq \Lambda(u) \equiv \int_{\{|x-y|<\delta_0\}} \frac{d(u(x),u(y))^p}{|x-y|^{sp+N}}\, dxdy \leq C_2\|u\|^p_{W^{s,p}}. \tag{35}$$

Proof of the right-hand side of (35). For a.e. x and $y \in \Omega$ such that $|x-y| < \delta_0$ and $0 \leq i \leq N$, we define $z_i \equiv (x_1, \ldots, x_i, y_{i+1}, \ldots, y_N)$. Since $|x-y| < \delta_0$, we have $z_i \in \Omega_{\delta_0}$. Moreover, $z_0 = y$ and $z_N = x$, hence, by the triangle inequality,

$$d(u(x), u(y)) = d(U(x), U(y)) \leq \sum_{i=1}^N d(U(z_i), U(z_{i-1})).$$

Noticing that $d(U(z_i), U(z_{i-1}))$ does not depend on $y_1, \ldots, y_{i-1}, x_{i+1}, \ldots, x_N$, we infer

$$\Lambda(u) \leq \sum_{i=1}^N \int G_i(x_1, \ldots, x_{i-1}, y_{i+1}, \ldots, y_N) d(U(z_i), U(z_{i-1}))^p dx_1 \ldots dx_i dy_i \ldots dy_N, \tag{36}$$

where

$$G_i(x_1, \ldots, x_{i-1}, y_{i+1}, \ldots, y_N) \equiv \int_{\{|x-y|<\delta_0\}} \frac{dy_1 \ldots dy_{i-1} dx_{i+1} \ldots dx_N}{|x-y|^{sp+N}}.$$

By equivalence of norms $|\cdot|_\infty$ and $|\cdot|_2$, there exist positive constants K_1, K_2 such that

$$K_1\tilde{G}_i \leq G_i \leq K_2\tilde{G}_i, \tag{37}$$

where

$$\tilde{G}_i(x_1,\dots,x_{i-1},y_{i+1},\dots,y_N) \equiv \int_{\{|x-y|_\infty<\delta_0\}} \frac{dy_1\dots dy_{i-1}dx_{i+1}\dots dx_N}{|x-y|_1^{sp+N}}.$$

Using $\{|x-y|_\infty<\delta_0\} = (x_1-\delta_0, x_1+\delta_0)\times\dots\times(x_{i-1}-\delta_0, x_{i-1}+\delta_0)\times(y_{i+1}-\delta_0, y_{i+1}+\delta_0)\times(y_N-\delta_0, y_N+\delta_0)$, Fubini's theorem and the inequality

$$\int_0^{\delta_0}\frac{dt}{(\alpha+t)^\sigma} \leq \frac{1}{(\sigma-1)\alpha^{\sigma-1}},$$

valid for $\alpha>0$ and $\sigma>1$, and (37), we deduce

$$\frac{C_1(s,p,N,\delta_0)}{|x_i-y_i|^{sp+1}} \leq K_1\tilde{G}_i \leq G_i \leq K_2\tilde{G}_i \leq \frac{C_2(s,p,N,\delta_0)}{|x_i-y_i|^{sp+1}}. \tag{38}$$

Inserting (38) in (36) yields

$$\Lambda(u) \leq C_2\sum_{i=1}^N\int\frac{d(U(z_i),U(z_{i-1}))^p}{|x_i-y_i|^{sp+1}}\,dx_1\dots dx_i dy_i\dots dy_N. \tag{39}$$

Using the change of variables $x_{i+1}\equiv y_{i+1}$, $\dots$, $x_N\equiv y_N$ and $y_i-x_i\equiv t\in(-\delta,\delta)$, we see that the right-hand sides of (39) and (35) (since z_{i-1} becomes $x+t\vec{e}_i$ and z_i becomes x) are equal.

Proof of the left-hand side of (35). To prove the left-hand side of (35), it suffices to show

$$\int_\Omega\int_0^{\delta_0}\frac{d(U(x),U(x+t\vec{e}_1))^p}{t^{sp+1}}\,dxdt \leq C_1\Lambda(u).$$

First, note that, by (38),

$$\frac{1}{t^{sp+1}} \leq \frac{G_1}{C_1} = C_1'\int_{B_{\delta_0}^{N-1}}\frac{d\xi}{|(t,\xi)|^{sp+N}},$$

where $B_{\delta_0}^{N-1} = \{\xi\in\mathbb{R}^{N-1},\ |\xi|<\delta_0\}$, hence

$$\int_\Omega\int_0^{\delta_0}\frac{d(U(x),U(x+t\vec{e}_1))^p}{t^{sp+1}}\,dxdt \leq C_1\int_\Omega\int_0^{\delta_0}\int_{B_{\delta_0}^{N-1}}\frac{d(U(x),U(x+t\vec{e}_1))^p}{|(t/2,\xi)|^{sp+N}}\,dxdtd\xi$$

By the triangle inequality, for any $\xi\in B_{\delta_0}^{N-1}$,

$$d(U(x),U(x+t\vec{e}_1)) \leq d\Big(U(x),U(x+(0,\xi)+\frac{t}{2}\vec{e}_1)\Big) + d\Big(U(x+(0,\xi)+\frac{t}{2}\vec{e}_1),U(x+t\vec{e}_1)\Big),$$

thus

$$\begin{aligned}\int_\Omega\int_0^{\delta_0}\frac{d(U(x),U(x+t\vec{e}_1))^p}{t^{sp+1}}\,dxdt &\leq C_1\int_\Omega\int_0^{\delta_0}\int_{B_{\delta_0}^{N-1}}\frac{d\Big(U(x),U(x+(0,\xi)+\frac{t}{2}\vec{e}_1)\Big)^p}{|(t/2,\xi)|^{sp+N}}\,dxdtd\xi \\ &\quad + C_1\int_\Omega\int_0^{\delta_0}\int_{B_{\delta_0}^{N-1}}\frac{d\Big(U(x+(0,\xi)+\frac{t}{2}\vec{e}_1),U(x+t\vec{e}_1)\Big)^p}{|(t/2,\xi)|^{sp+N}}\,dxdtd\xi.\end{aligned}$$

Setting $y \equiv x+(0,\xi)+\frac{t}{2}\vec{e}_1$ in the first integral and $x' \equiv x+t\vec{e}_1$, $y' \equiv x+(0,\xi)+\frac{t}{2}\vec{e}_1 = x'+(0,\xi)-\frac{t}{2}\vec{e}_1$ in the second integral, so that $|x-y| = |x'-y'| = |(t/2,\xi)|$, we conclude

$$\int_\Omega \int_0^{\delta_0} \frac{d(U(x),U(x+t\vec{e}_1))^p}{t^{sp+1}}\, dxdt \le C_1\Lambda(u),$$

and the proof of the left-hand side inequality in (35) is complete. □

3.2 Proof of Proposition 7

By standard localization, we can reduce the study to the case where, locally, Ω is the half-space $\mathbb{R}^N_+ = \mathbb{R}^{N-1}\times(0,+\infty)$. Suppose the result proved for $N=2$, namely $\Omega = (a,b)\times(0,1)$ and

$$\|u\|_{W^{s-\frac{1}{p},p}((a,b),X)} \le C_1|u|_{W^{s-\frac{1}{p},p}((a,b),X)} \le C|u|_{W^{s,p}((a,b)\times(0,1),X)}.$$

Then, the general result follows using Lemma 9 and integrating this inequality. We assume then $N=2$, $\Omega=(0,1)\times(0,1)$, and the trace is on $(0,1)\times\{0\}$. Let $u\in W^{s,p}(\Omega,X)$, with $0<s<1<sp$ (if $s=p=1$, there is nothing to prove, the trace is only in $L^1(\partial\Omega,X)$). We extend u by reflection across the boundary as a map, still denoted u, on $(-1,2)\times(0,1)$.

The case $s=1$. If $f\in W^{1,p}(\Omega,\mathbb{R})$, then the standard estimate for traces (see, *e.g.* [4], Chapter 5, Theorem 3) starts with the inequality, for x, $t\in(0,1)$,

$$\Big|f(x,0)-f(x+t,0)\Big| \le \Big|f(x,0)-f(x,t)\Big| + \Big|f(x,t)-f(x+t,t)\Big| + \Big|f(x+t,t)-f(x+t,0)\Big|, \quad (40)$$

which implies

$$\frac{1}{t}\Big|f(x,0)-f(x+t,0)\Big| \le \frac{1}{t}\int_0^t |\partial_2 f|(x,\tau)\, d\tau + \frac{1}{t}\int_0^t |\partial_1 f|(x+\tau,t)\, d\tau + \frac{1}{t}\int_0^t |\partial_1 f|(x+t,\tau)\, d\tau. \quad (41)$$

The classical estimate for traces, namely $|\mathrm{tr}(f)|_{1-\frac{1}{p},p} \le C|\nabla f|_{L^p}$ comes then from the majorization of the L^p norm of the right-hand side of the above inequality (see for instance the proof of Theorem 3, Chapter 5, in [4]):

$$||\text{rhs of (41)}||_{L^p(x,t)} \le C|\nabla f|_{L^p}. \quad (42)$$

Turning back to the estimate for u, for a.e. $x\in(0,1)$, we apply (41) to $f\equiv d(u(\cdot),u(x,0)) \in W^{1,p}(\Omega,\mathbb{R})$. Then, we use $|\nabla f| \le C|\nabla u|_1$ and obtain

$$\begin{aligned}\frac{1}{t}d(u(x+t,0),u(x,0)) &= \frac{1}{t}\Big|f(x,0)-f(x+t,0)\Big| \\ &\le \frac{1}{t}\int_0^t |\nabla u|_1(x,\tau)\, d\tau + \frac{1}{t}\int_0^t |\nabla u|_1(x+\tau,t)\, d\tau + \frac{1}{t}\int_0^t |\nabla u|_1(x+t,\tau)\, d\tau. \quad (43)\end{aligned}$$

Estimating the L^p norm of the right-hand side of (43) using (42), we obtain the desired estimate

$$||\mathrm{tr}(u)||^p_{W^{1-\frac{1}{p},p}} = \Big|\Big|\frac{1}{t}d(\mathrm{tr}(u)(x+t),\mathrm{tr}(u)(x))\Big|\Big|^p_{L^p(x,t)} \le C_pE^p(u).$$

The case $0<s<1<sp$. We first define the trace of $u\in W^{s,p}(\Omega,X)$ as a map in $L^p(\partial\Omega,X)$ exactly as in [12], section 1.12, using the Sobolev inequality, for $v\in W^{s,p}((0,1),X)$,

$$d(v(x),v(y)) \le C|v|_{W^{s,p}}|x-y|^{s-\frac{1}{p}}. \quad (44)$$

Next, the analogue of (43) writes

$$d(u(x+t,0),u(x,0)) \leq d(u(x,t),u(x,0)) + d(u(x+t,t),u(x,t)) + d(u(x+t,t),u(x+t,0)). \quad (45)$$

Dividing by t^s and taking the $L^p(x,t)$ norm yields

$$\|u\|^p_{W^{s-\frac{1}{p},p}} \leq \int_\Omega \frac{2}{t^{sp}} d(u(x,0),u(x,t))^p \; dtdx + \int_\Omega \frac{1}{t^{sp}} d(u(x+t,t),u(x,t))^p \; dtdx. \quad (46)$$

To estimate the second integral in (46), we write the scaled version of (44)

$$d(u(x+t,t),u(x,t))^p \leq Ct^{sp-1} \int_0^t \int_0^t d(u(x+y,t),u(x+y+z,t))^p \; \frac{dydz}{z^{sp+1}},$$

thus,

$$\begin{aligned}\int_\Omega \frac{1}{t^{sp}} d(u(x+t,t),&u(x,t))^p \; dtdx \\ &\leq C \int_0^1 \frac{1}{t} \int_0^t \int_0^t \int_0^1 d(u(x+y,t),u(x+y+z,t))^p \; \frac{dxdz}{z^{sp+1}} dydt \\ &\leq C \int_0^1 \frac{1}{t} \int_0^t \Big(\int_0^1 \int_0^2 d(u(x,t),u(x+z,t))^p \; \frac{dxdz}{z^{sp+1}} \Big) dydt = C\|u\|^p_{W^{s,p}}. \quad (47)\end{aligned}$$

For the first integral in (46), we first choose a $1 \leq q < p$ such that $sq > 1$ (which is possibe since $sp > 1$ and $p > 1$ since $0 < s < 1$). Then, we also write the scaled version of (44)

$$d(u(x,0),u(x,t))^q \leq C(q)t^{sq} \int_0^1 \int_0^t d(u(x,ty),u(x,ty+z))^q \; \frac{dzdy}{z^{sq+1}}.$$

We divide by t^{sq} and take the $L^{\frac{p}{q}}(x,t)$ norm, then use successively the triangle inequality for integrals (second line), $\int_0^1 y^{-q/p} \; dy < \infty$ since $p > q > 0$ and set $\tau = ty \in [0,y] \subset [0,1]$ (fifth line), and finally Hölder inequality with the measure $\frac{dz}{z}$, to infer

$$\begin{aligned}\Big(\int_0^1 d(u(x,0),u(x,t))^p \; \frac{dt}{t^{sp}} \Big)^{\frac{q}{p}} &\leq C \Big\| \int_0^1 \int_0^1 d(u(x,ty),u(x,ty+z))^q \; \frac{dzdy}{z^{sq+1}} \Big\|_{L_t^{\frac{p}{q}}(0,1)} \\ &\leq C \int_0^1 \Big\| \int_0^1 d(u(x,ty),u(x,ty+z))^q \; \frac{dz}{z^{sq+1}} \Big\|_{L_t^{\frac{p}{q}}(0,1)} \; dy \\ &= C \int_0^1 \Big(\int_0^1 \Big[\int_0^1 d(u(x,ty),u(x,ty+z))^q \; \frac{dz}{z^{sq+1}} \Big]^{\frac{p}{q}} dt \Big)^{\frac{q}{p}} \; dy \\ &\leq C \int_0^1 \Big(\int_0^1 \Big[\int_0^1 d(u(x,\tau),u(x,\tau+z))^q \; \frac{dz}{z^{sq+1}} \Big]^{\frac{p}{q}} d\tau \Big)^{\frac{q}{p}} \; \frac{dy}{y^{\frac{q}{p}}} \\ &= C' \Big(\int_0^1 \Big[\int_0^1 \Big(\frac{d(u(x,\tau),u(x,\tau+z))}{z^s} \Big)^q \; \frac{dz}{z} \Big]^{\frac{p}{q}} d\tau \Big)^{\frac{q}{p}} \\ &\leq C \Big(\int_0^1 \int_0^1 \frac{d(u(x,\tau),u(x,\tau+z))^p}{z^{sp+1}} \; dzd\tau \Big)^{\frac{q}{p}}. \quad (48)\end{aligned}$$

Combining (47) and (48) with (46), we obtain $\|u\|_{W^{s-\frac{1}{p},p}} \leq C\|u\|_{W^{s,p}}$. $\square$

Remark 15 To define traces from $W^{s,p}(\Omega,X)$ into $W^{s-\frac{1}{p},p}(\partial\Omega,X)$ for $1<s<2$, one has to impose some smoothness to X to define ∇u. If $X=\mathcal{E}$ is a smooth riemannian manifold complete, but possibly non-compact, and isometrically embedded in $\mathbb{R}^q$, then we set for $s>1$ and $1\le p\le\infty$ $W^{s,p}(\Omega,\mathcal{E})\equiv W^{s,p}(\Omega,\mathbb{R}^q)\cap L^p(\Omega,\mathcal{E})$. Assume for instance $s-\frac{1}{p}<1<s<2$, then

$$\mathrm{tr}: W^{s,p}(\Omega,\mathcal{E})\to W^{s-\frac{1}{p},p}(\partial\Omega,\mathcal{E}).$$

The case $s-\frac{1}{p}\ge 1$ follows direcly from standard trace theory and the fact that the trace of a map in $W^{1,p}(\Omega,\mathcal{E})$ is in $L^p(\partial\Omega,\mathcal{E})$ (from [12], section 1.12).

Proof of Remark 15. We just point out the modifications to make in the proof of Proposition 7. Inequality (45) becomes

$$\|u\|^p_{W^{s-\frac{1}{p},p}}\le 2\int_0^1\int_0^1 d(u(x,h),u(x,t+h))^p\ \frac{dtdx}{t^{sp}}+\int_0^1\int_0^1 d(u(x+t,t+h),u(x,t+h))^p\ \frac{dtdx}{t^{sp}}.$$

For the second integral, we write as before the Sobolev embedding into Hölder continuous functions

$$\begin{aligned}&d(u(x+t,t+h),u(x,t+h))^p\\&\quad\le Ct^{sp-1}\Big(\int_0^t|\partial_1 u|_p(x+y,t+h)\ dy+\int_0^t\int_0^t|\partial_1 u(x+y,t+h)-\partial_1 u(x+y+z,t+h)|^p\ \frac{dydz}{z^{sp}}\Big).\end{aligned}$$

For the first integral, we write

$$\begin{aligned}&d(u(x,t+h),u(x,h))\\&\quad\le Ct^{s-1}\Big(\int_0^t|\partial_1 u|_1(x+y,t+h)\ dy+\int_0^t\int_0^t|\partial_1 u(x+y,t+h)-\partial_1 u(x+y+z,t+h)|\ \frac{dydz}{z^{s}}\Big),\end{aligned}$$

and argue as for the case $0<s<1$. □

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