

Stability and instability for subsonic travelling waves of the Nonlinear Schrödinger Equation in dimension one

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Abstract

We study the stability/instability of the subsonic travelling waves of the Nonlinear Schrödinger Equation in dimension one. Our aim is to propose several methods for showing instability (use of the Grillakis-Shatah-Strauss theory, proof of existence of an unstable eigenvalue via an Evans function) or stability. For the later, we show how to construct in a systematic way a Liapounov functional for which the travelling wave is a local minimizer. These approaches allow to give a complete stability/instability analysis in the energy space including the critical case of the kink solution. We also treat the case of a cusp in the energy-momentum diagram.

Key-words: travelling wave, Nonlinear Schrödinger Equation, Gross-Pitaevskii Equation, stability, Evans function, Liapounov functional.

MSC (2010): 35B35, 35C07, 35J20, 35Q40, 35Q55.

1 Introduction

This paper is a continuation of our previous work [16], where we consider the one dimensional Nonlinear Schrödinger Equation

$$i \frac{\partial \Psi}{\partial t} + \partial_x^2 \Psi + \Psi f(|\Psi|^2) = 0. \quad (\text{NLS})$$

This equation appears as a relevant model in condensed matter physics: Bose-Einstein condensation and superfluidity (see [60], [32], [37], [1]); Nonlinear Optics (see, for instance, the survey [44]). Several nonlinearities may be encountered in physical situations: $f(\varrho) = \pm \varrho$ gives rise to the focusing/defocusing cubic (NLS); $f(\varrho) = 1 - \varrho$ to the so called Gross-Pitaevskii equation; $f(\varrho) = -\varrho^2$ (see [46] for Bose-Einstein condensates); more generally a pure power; the “cubic-quintic” (NLS) (see [4]), where

$$f(\varrho) = -\alpha_1 + \alpha_3 \varrho - \alpha_5 \varrho^2$$

and α_1, α_3 and α_5 are positive constants such that f has two positive roots; and in Nonlinear Optics, we may take (see [44]):

$$f(\varrho) = -\alpha \varrho^\nu - \beta \varrho^{2\nu}, \quad f(\varrho) = -\frac{\varrho_0}{2} \left(\frac{1}{(1 + \frac{1}{\varrho_0})^\nu} - \frac{1}{(1 + \frac{\varrho}{\varrho_0})^\nu} \right), \quad f(\varrho) = -\alpha \varrho \left(1 + \gamma \tanh\left(\frac{\varrho^2 - \varrho_0^2}{\sigma^2}\right) \right), \quad (1)$$

where $\alpha, \beta, \gamma, \nu, \sigma > 0$ are given constants (the second one, for instance, takes into account saturation effects), etc. As a consequence, as in our work [16], we shall consider a rather general nonlinearity f , with

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f of class \mathcal{C}^2 . In the context of Bose-Einstein condensation or Nonlinear Optics, the natural condition at infinity appears to be

$$|\Psi|^2 \rightarrow r_0^2 \quad \text{as } |x| \rightarrow +\infty,$$

where $r_0 > 0$ is such that $f(r_0^2) = 0$.

For solutions Ψ of (NLS) which do not vanish, we may use the Madelung transform

$$\Psi = A \exp(i\phi)$$

and rewrite (NLS) as an hydrodynamical system with an additional quantum pressure

$$\left\{ \begin{array}{l} \partial_t A + 2\partial_x \phi \partial_x A + A \partial_x^2 \phi = 0 \\ \partial_t \phi + (\partial_x \phi)^2 - f(A^2) - \frac{\partial_x^2 A}{A} = 0 \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l} \partial_t \rho + 2\partial_x(\rho u) = 0 \\ \partial_t u + 2u\partial_x u - \partial_x(f(\rho)) - \partial_x\left(\frac{\partial_x^2(\sqrt{\rho})}{\sqrt{\rho}}\right) = 0, \end{array} \right. \quad (2)$$

with $(\rho, u) \equiv (A^2, \partial_x \phi)$. When neglecting the quantum pressure and linearizing this Euler system around the particular trivial solution $\Psi = r_0$ (or $(A, u) = (r_0, 0)$), we obtain the free wave equation

$$\left\{ \begin{array}{l} \partial_t \bar{A} + r_0 \partial_x \bar{U} = 0 \\ \partial_t \bar{U} - 2r_0 f'(r_0^2) \partial_x \bar{A} = 0 \end{array} \right.$$

with associated speed of sound

$$\mathbf{c}_s \equiv \sqrt{-2r_0^2 f'(r_0^2)} > 0$$

provided f satisfies the defocusing assumption $f'(r_0^2) < 0$ (that is the Euler system is hyperbolic in the region $\rho \simeq r_0^2$), which we will assume throughout the paper. Concerning the rigorous justification of the free wave regime for the Gross-Pitaevkii equation (in arbitrary dimension), see [9]. The speed of sound \mathbf{c}_s enters in a crucial way in the question of existence of travelling waves for (NLS) with modulus tending to r_0 at infinity (see, *e.g.*, [16]).

The Nonlinear Schrödinger equation formally preserves the energy

$$E(\psi) \equiv \int_{\mathbb{R}} |\partial_x \psi|^2 + F(|\psi|^2) dx,$$

where $F(\varrho) \equiv \int_{\varrho}^{r_0^2} f$. Since $F(\varrho) \sim \frac{\mathbf{c}_s^2}{8r_0^2}(\varrho - r_0^2)^2 \sim \frac{\mathbf{c}_s^2}{2}(\sqrt{\varrho} - r_0)^2$ when $\varrho \rightarrow r_0^2$, it follows that the natural energy space turns out to be the space

$$\mathcal{Z} \equiv \left\{ \psi \in L^\infty(\mathbb{R}), \partial_x \psi \in L^2(\mathbb{R}), |\psi| - r_0 \in L^2(\mathbb{R}) \right\} \subset \mathcal{C}_b(\mathbb{R}, \mathbb{C}),$$

endowed with the distance

$$d_{\mathcal{Z}}(\psi, \tilde{\psi}) \equiv \|\partial_x \psi - \partial_x \tilde{\psi}\|_{L^2(\mathbb{R})} + \||\psi| - |\tilde{\psi}|\|_{L^2(\mathbb{R})} + |\psi(0) - \tilde{\psi}(0)|.$$

The Cauchy problem has been shown to be locally well posed in the Zhidkov space $\{\psi \in L^\infty(\mathbb{R}), \partial_x \psi \in L^2(\mathbb{R})\}$ by P. Zhidkov [69] (see also the work by C. Gallo [24]). For global well-posedness results, see [25] and [29]. More precisely, the local well-posedness we shall use is the following.

Theorem 1 ([69], [24]) *Let $\Psi^{\text{in}} \in \mathcal{Z}$. Then, there exists $T_* > 0$ and a unique solution Ψ to (NLS) such that $\Psi|_{t=0} = \Psi^{\text{in}}$ and $\Psi - \Psi^{\text{in}} \in \mathcal{C}([0, T_*], H^1(\mathbb{R}))$. Moreover, $E(\Psi(t))$ does not depend on t .*

The other quantity formally conserved by the Schrödinger flow, due to the invariance by translation, is the momentum. The momentum is not easy to define in dimension one for maps that vanish somewhere (see [10], [11]). However, if ψ does not vanish, we may lift $\psi = Ae^{i\phi}$, and then the correct definition of the momentum is given by [45]:

$$P(\psi) \equiv \int_{\mathbb{R}} \langle i\psi | \partial_x \psi \rangle \left(1 - \frac{r_0^2}{|\psi|^2}\right) dx = \int_{\mathbb{R}} (A^2 - r_0^2) \partial_x \phi dx,$$

where $\langle \cdot | \cdot \rangle$ denotes the real scalar product in \mathbb{C} . We define

$$\mathcal{Z}_{\text{hy}} \equiv \{v \in \mathcal{Z}, \inf_{\mathbb{R}} |v| > 0\},$$

which is the open subset of \mathcal{Z} in which we have lifting and where the hydrodynamical formulation (2) of (NLS) is possible through the Madelung transform. It turns out that if the initial datum belongs to \mathcal{Z}_{hy} , the solution of (NLS) provided by Theorem 1 remains in \mathcal{Z}_{hy} for small times, and that the momentum is indeed conserved on this time interval (see [24]).

1.1 The travelling waves and energy-momentum diagrams

The travelling waves with speed of propagation c are special solutions of (NLS) of the form

$$\Psi(t, x) = U(x - ct).$$

The profile U has then to solve the ODE

$$\partial_x^2 U + Uf(|U|^2) = ic\partial_x U \tag{TW}_c$$

together with the condition $|U(x)| \rightarrow r_0$ as $x \rightarrow \pm\infty$. These particular solutions play an important role in the long time dynamics of (NLS) with nonzero condition at infinity. Possibly conjugating (TW_c) , we see that we may assume that $c \geq 0$ without loss of generality. Moreover, we shall restrict ourselves to travelling waves which belong to the energy space \mathcal{Z} (so that $|U| \rightarrow r_0$ at $\pm\infty$ by Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow C_0(\mathbb{R}, \mathbb{C}) \equiv \{h \in C(\mathbb{R}, \mathbb{C}), \lim_{\pm\infty} h = 0\}$). For travelling waves $U_c \in \mathcal{Z}$ that do not vanish in \mathbb{R} , hence that may be lifted $U_c = A_c e^{i\phi_c}$, the ODE (TW_c) can be transformed (see, *e.g.*, [16]) into the system

$$\partial_x \phi_c = \frac{c}{2} \cdot \frac{\eta_c}{\eta_c + r_0^2}, \quad 2\partial_x^2 \eta_c + \mathcal{V}'_c(\eta_c) = 0, \quad \text{with} \quad \eta_c \equiv A_c^2 - r_0^2,$$

and where the function \mathcal{V}_c is related to f by the formula

$$\mathcal{V}_c(\xi) \equiv c^2 \xi^2 - 4(r_0^2 + \xi)F(r_0^2 + \xi).$$

To a nontrivial travelling wave U_c is associated (see [16]) some $\xi_c \geq -r_0^2$ such that $\mathcal{V}_c(\xi_c) = 0 \neq \mathcal{V}'_c(\xi_c)$ and \mathcal{V}_c is negative between ξ_c and $-r_0^2$, and η_c varies between 0 and ξ_c , that is there holds $\{\inf_{\mathbb{R}} |U_c|, \sup_{\mathbb{R}} |U_c|\} = \{r_0, \sqrt{r_0^2 + \xi_c}\}$. Moreover, the only travelling wave solution (if it exists) that vanishes somewhere is for $c = 0$ and is called the kink: it is an odd solution (up to a space translation) and then $\xi_0 = 0$.

We have also seen in [16] that any travelling wave in \mathcal{Z} with speed $c > c_s$ is constant; and also that any nonconstant travelling wave in \mathcal{Z} of speed $c_* \in (0, c_s)$ belongs to a unique (up to the natural invariances: phase factor and translation) local branch $c \mapsto U_c$ defined for c close to c_* .

In [16], we have investigated the qualitative behaviours of the travelling waves for (NLS) with nonzero condition at infinity for a general nonlinearity f . A particular attention has been paid in [16] to the transonic limit, where we have an asymptotic behaviour governed by the Korteweg-de Vries or the generalized Korteweg-de Vries equation. In order to illustrate the very different situations we may encounter when we allow a general nonlinearity f , we give now some energy-momentum diagrams we have obtained (one is taken

from the appendix in [17], where we have performed numerical simulations in dimension two for the model cases we have studied in [16]).

- The Gross-Pitaevskii nonlinearity: $f(\varrho) = 1 - \varrho$ (see figure 1).

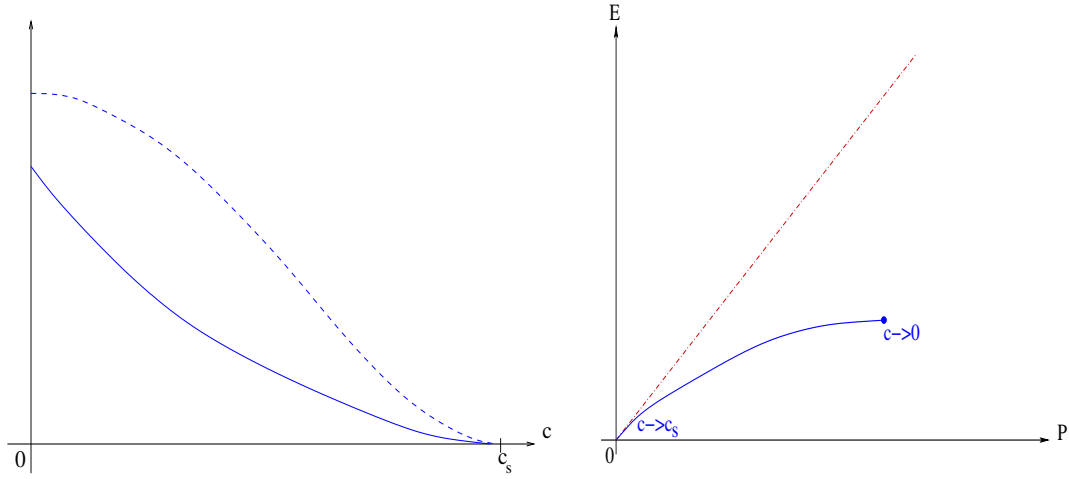


Figure 1: (a) Energy (dashed curve) and momentum (full curve) vs. speed; (b) (E, P) diagram

- A cubic-quintic-septic nonlinearity (I): $f(\varrho) = -(\varrho - 1) + \frac{3}{2}(\varrho - 1)^2 - \frac{3}{2}(\varrho - 1)^3$ (see figure 2).

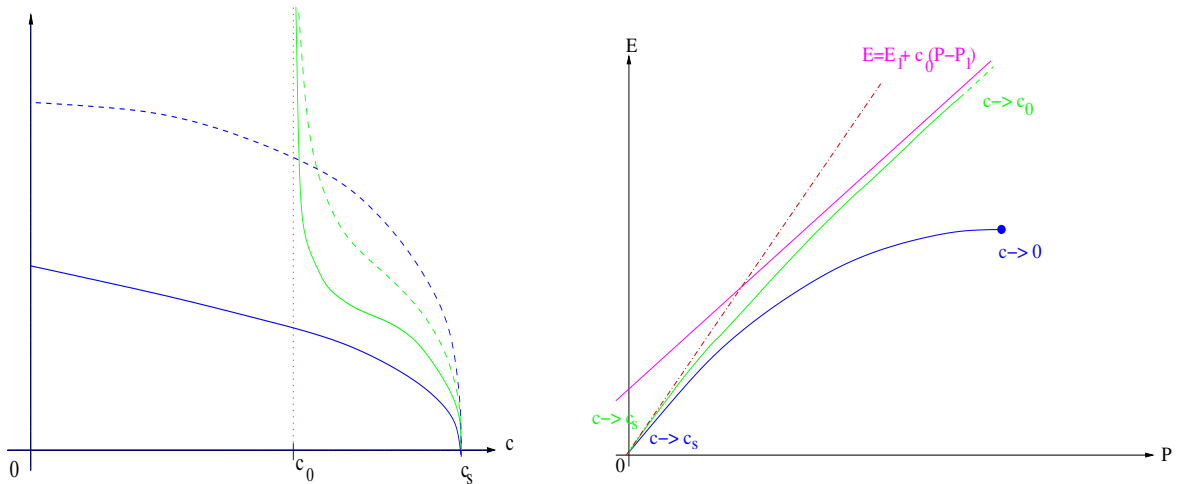


Figure 2: (a) Energy (dashed curve) and momentum (full curve) vs. speed; (b) (E, P) diagram

- A cubic-quintic-septic nonlinearity (II): $f(\varrho) \equiv -4(\varrho - 1) - 36(\varrho - 1)^3$ or $f(\varrho) \equiv -4(\varrho - 1) - 60(\varrho - 1)^3$. For these two nonlinearities, the graph of E and P vs. speed c is given in figure 3, but the (E, P) diagrams are respectively those in figure 4.

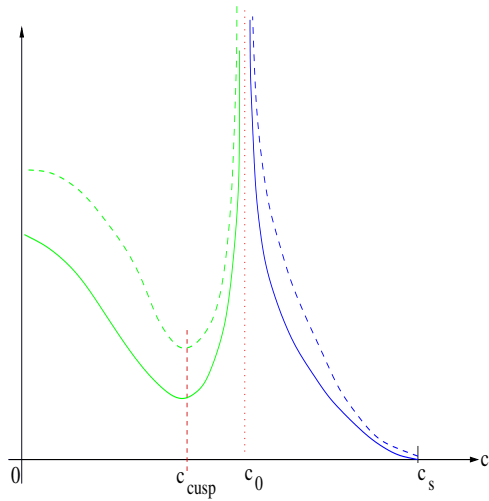


Figure 3: Energy (dashed curve) and momentum (full curve) vs. speed

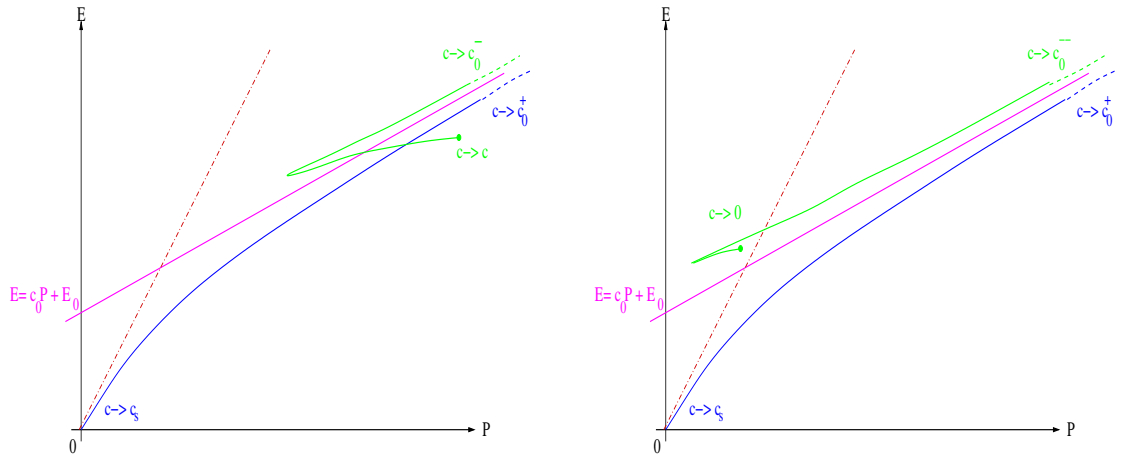


Figure 4: The two (E, P) diagrams

- A cubic-quintic-septic nonlinearity (III): $f(\varrho) \equiv -\frac{1}{2}(\varrho - 1) + \frac{3}{4}(\varrho - 1)^2 - 2(\varrho - 1)^3$ (see figure 5).

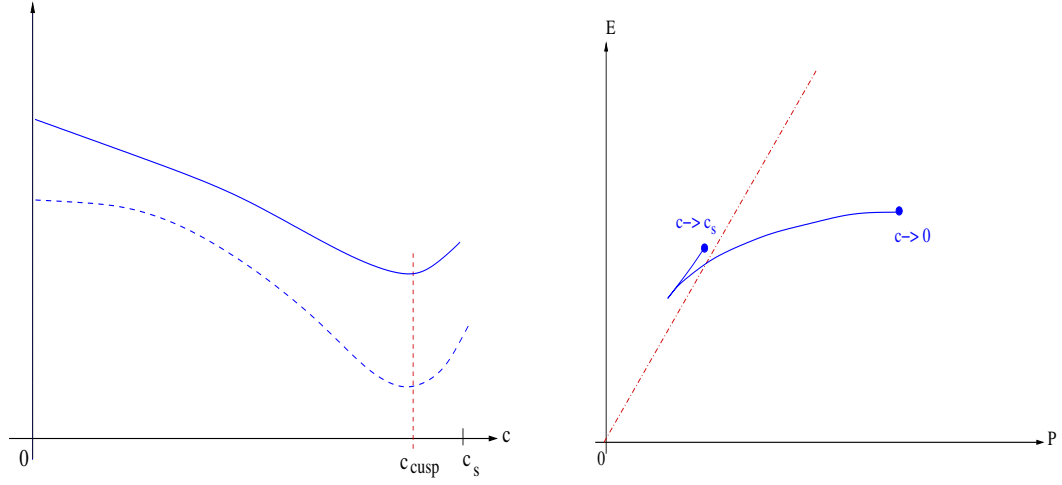


Figure 5: (a) Energy (dashed curve) and momentum (full curve) vs. speed; (b) (E, P) diagram

- A degenerate case: $f(\varrho) \equiv -2(\varrho - 1) + 3(\varrho - 1)^2 - 4(\varrho - 1)^3 + 5(\varrho - 1)^4 - 6(\varrho - 1)^5$ (see figure 6).

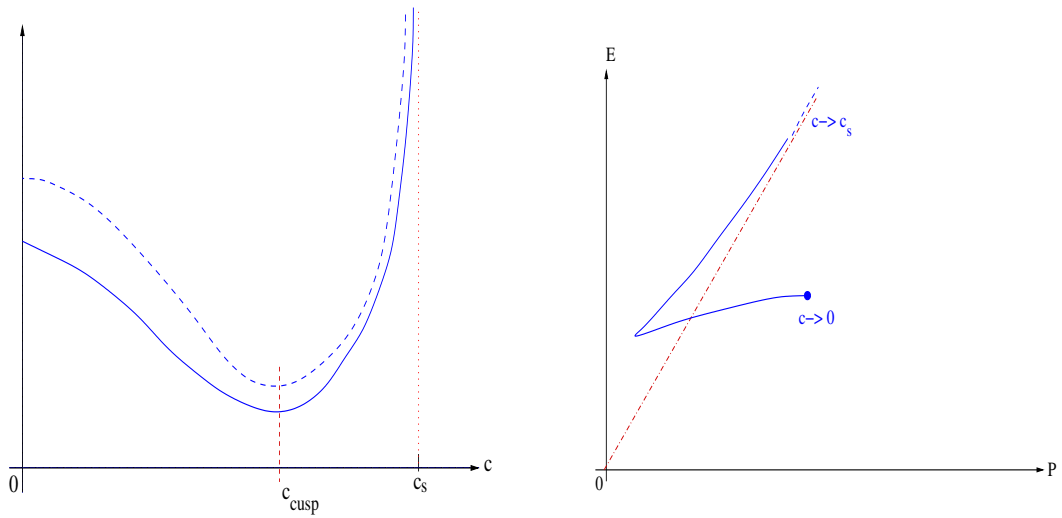


Figure 6: (a) Energy (dashed curve) and momentum (full curve) vs. speed; (b) (E, P) diagram

- A perturbation of the previous degenerate case: $f(\varrho) \equiv -2(\varrho - 1) + (3 - 10^{-3})(\varrho - 1)^2 - 4(\varrho - 1)^3 + 5(\varrho - 1)^4 - 6(\varrho - 1)^5$ (see figure 7).

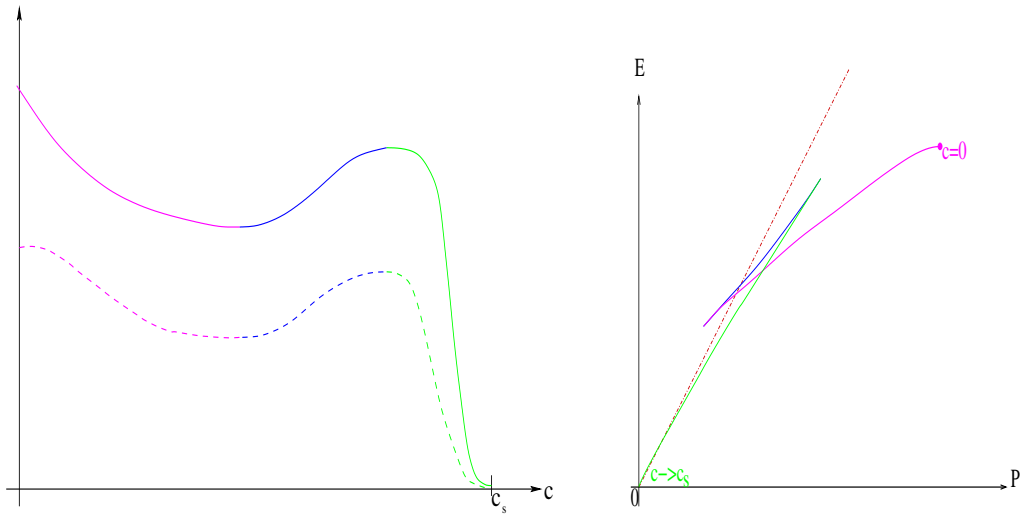


Figure 7: (a) Energy (*) and momentum (+) vs. speed; (b) qualitative (E, P) diagram

- A saturated (NLS): $f(\varrho) \equiv \exp\left(\frac{1-\varrho}{\varrho_0}\right) - 1$ with $\varrho_0 = 0.4$ (see figure 8).

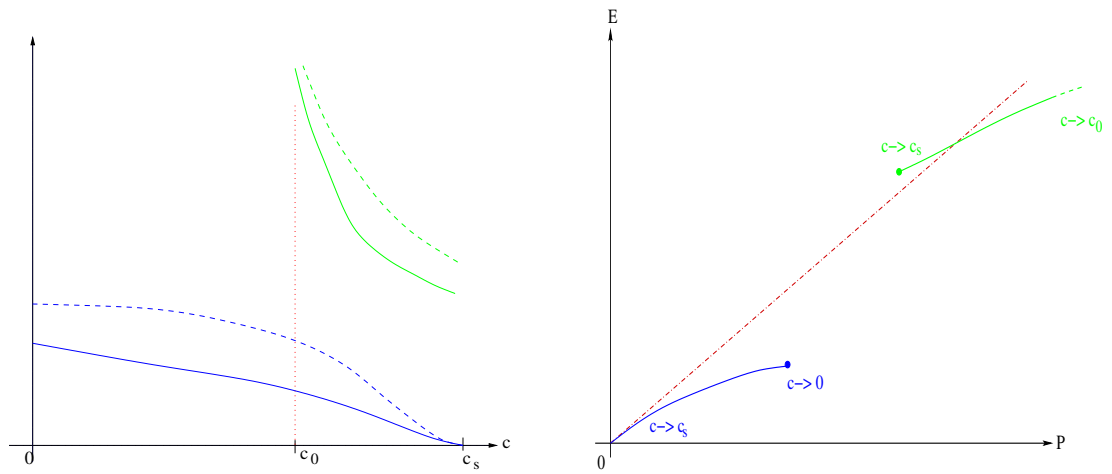


Figure 8: (a) Energy (dashed curve) and momentum (full curve) vs. speed; (b) (E, P) diagram

- Another saturated (NLS): $f(\varrho) \equiv \frac{\varrho_0}{2} \left(\frac{1}{(1 + \varrho/\varrho_0)^2} - \frac{1}{(1 + 1/\varrho_0)^2} \right)$, with $\varrho_0 = 0.08$ (see figure 9).

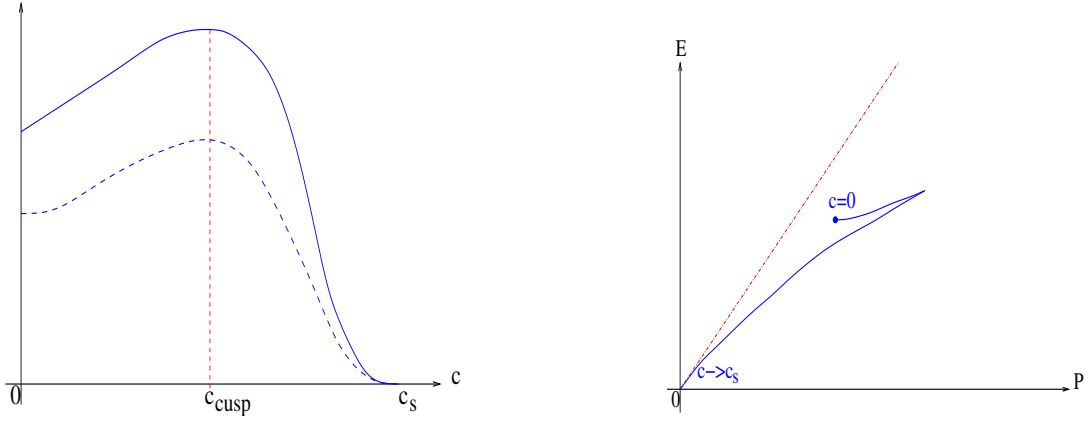


Figure 9: (a) Energy (dashed curve) and momentum (full curve), (b) (E, P) diagram

- The cubic-quintic nonlinearity: $f(\varrho) \equiv -(\varrho - 1) - 3(\varrho - 1)^2$ (see figure 10).

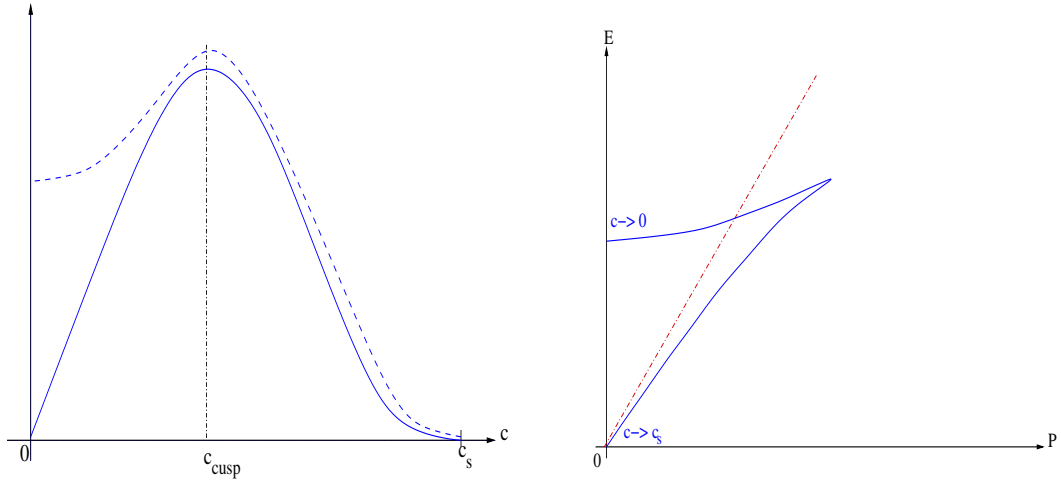


Figure 10: (a) Energy (dashed curve) and momentum (full curve), (b) (E, P) diagram

Through the study (in [16]) of these model cases, we have shown that if the energy-momentum diagram is well-known for the Gross-Pitaevskii equation, the qualitative properties of the travelling waves solutions can *not* be easily deduced from the global shape of the nonlinearity f . In particular, even if we restrict ourselves to smooth and decreasing nonlinearities (as is the Gross-Pitaevskii one), we see that we may have a great variety of behaviours: multiplicity of solutions; branches with diverging energy and momentum; nonexistence of travelling wave for some $c_0 \in (0, c_s)$; branches of solutions that cross; existence of sonic travelling wave; transonic limit governed by the (mKdV) or more generally by the (gKdV) solitary wave equation instead of the usual (KdV) one; existence of cusps...

We investigate now the behaviour at infinity of the nontrivial travelling waves, which depend whether $c = \mathbf{c}_s$ or not. We denote \mathbb{N} the set of nonnegative integers and \mathbb{N}^* the set of positive integers. We consider for $m \in \mathbb{N}$ the following assumption:

(\mathcal{A}_m) f is of class \mathcal{C}^{m+3} near r_0^2 . Moreover, for $1 \leq j < m+2$, we have $\frac{f^{(j)}(r_0^2)}{(j+1)!} r_0^{2j} = (-1)^{j+1} \frac{\mathbf{c}_s^2}{4}$ but $\frac{f^{(m+2)}(r_0^2)}{(m+3)!} r_0^{2(m+2)} \neq (-1)^{m+3} \frac{\mathbf{c}_s^2}{4}$ (note that for $j = 1$, equality always holds by definition of the speed of sound $\mathbf{c}_s = \sqrt{-2r_0^2 f'(r_0^2)}$).

Proposition 1 *Let $U_c \in \mathcal{Z}$ be a non constant travelling wave of speed $0 \leq c \leq \mathbf{c}_s$.*

(i) *If $c = 0$, then there exist $\phi_0 \in \mathbb{R}$ such that $e^{i\phi_0} U_0$ is a real-valued function and there exist two real constants $M_0 \neq 0$ (depending only on f and ξ_0) and x_0 such that, as $x \rightarrow \pm\infty$,*

$$e^{i\phi_0} U_0(x) \mp r_0 \sim M_0 \exp\left(-\mathbf{c}_s |x - x_0|\right) \quad \text{if } \xi_0 = -r_0^2, \quad e^{i\phi_0} U_0(x) - r_0 \sim M_0 \exp\left(-\mathbf{c}_s |x - x_0|\right) \quad \text{if } \xi_0 \neq -r_0^2.$$

(ii) *If $0 < c < \mathbf{c}_s$, then U_c does not vanish, hence can be lifted: $U_c = A_c e^{i\phi_c}$. Furthermore, there exist four real constants M_c, Θ_c (depending only on f, c and ξ_c), x_0 and ϕ_0 such that, as $x \rightarrow \pm\infty$,*

$$|U_c(x)|^2 - r_0^2 = \eta_c(x) \sim \frac{2r_0^2}{c} \partial_x \phi(x) \sim M_c \exp\left(-\sqrt{\mathbf{c}_s^2 - c^2} |x - x_0|\right),$$

and

$$\phi(x) - \phi_0 \mp \Theta_c \sim -\text{sgn}(x) \frac{cM_c}{2r_0^2 \sqrt{\mathbf{c}_s^2 - c^2}} \exp\left(-\sqrt{\mathbf{c}_s^2 - c^2} |x - x_0|\right).$$

(iii) *If $c = \mathbf{c}_s$ then $U_{\mathbf{c}_s}$ does not vanish, hence can be lifted: $U_{\mathbf{c}_s} = A_{\mathbf{c}_s} e^{i\phi_{\mathbf{c}_s}}$. We assume that there exists $m \in \mathbb{N}$ such that (\mathcal{A}_m) is verified and denote*

$$\Lambda_m \equiv \frac{4}{r_0^{2(m+1)}} \left[\frac{r_0^{2(m+2)}}{(m+3)!} f^{(m+2)}(r_0^2) + (-1)^{m+2} \frac{\mathbf{c}_s^2}{4} \right] \neq 0.$$

Then, we have, as $x \rightarrow \pm\infty$,

$$|U_{\mathbf{c}_s}(x)|^2 - r_0^2 = \eta_{\mathbf{c}_s}(x) \sim \frac{2r_0^2}{\mathbf{c}_s} \partial_x \phi(x) \sim \text{sgn}(\xi_{\mathbf{c}_s}) \left(\frac{4}{(m+1)^2 |\Lambda_m| x^2} \right)^{\frac{1}{m+1}}$$

and

$$\phi(x) \sim \frac{\mathbf{c}_s \text{sgn}(\xi_{\mathbf{c}_s})}{2r_0^2} \left(\frac{4}{(m+1)^2 |\Lambda_m|} \right)^{\frac{1}{m+1}} \begin{cases} \text{sgn}(x) \ln |x| & \text{if } m = 1 \\ \frac{m+1}{m-1} \text{sgn}(x) |x|^{\frac{m-1}{m+1}} & \text{if } m \geq 2, \end{cases}$$

and if $m = 0$, there exists $\Theta_{\mathbf{c}_s} \in \mathbb{R}$ and $\phi_0 \in \mathbb{R}$ such that

$$\phi(x) - \phi_0 \mp \Theta_{\mathbf{c}_s} \sim \text{sgn}(\xi_{\mathbf{c}_s}) \frac{2\mathbf{c}_s}{r_0^2 |\Lambda_0| x}.$$

In particular, since we impose $U_{\mathbf{c}_s} \in \mathcal{Z}$, we must have $m \in \{0, 1, 2\}$.

For the Gross-Pitaevkii nonlinearity ($f(\varrho) = 1 - \varrho$), we may compute explicitly the travelling waves for $0 < c < \mathbf{c}_s = \sqrt{2}$ (see [67], [10])

$$U_c(x) = \sqrt{\frac{2-c^2}{2}} \tanh\left(x \frac{\sqrt{2-c^2}}{2}\right) - i \frac{c}{\sqrt{2}},$$

up to the invariances of the problem: translations and multiplications by a phase factor. On this explicit formula, the decay of the phase and modulus can be checked. In particular, as $x \rightarrow \pm\infty$, we have

$$U_c(x) \rightarrow \pm \sqrt{1 - \frac{c^2}{\mathfrak{c}_s}} - i \frac{c}{\mathfrak{c}_s}.$$

Remark 1 In the above statements, the constants ϕ_0 and x_0 reflect the gauge and translation invariance. In the spirit of the model cases proposed in [16], for

$$f(\varrho) \equiv -2(\varrho - 1) + 3(\varrho - 1)^2 - 4(\varrho - 1)^3 + 5(\varrho - 1)^4 - 12(\varrho - 1)^5,$$

we obtain a smooth decreasing nonlinearity tending to $-\infty$ at $+\infty$ (thus qualitatively similar to the Gross-Pitaevskii nonlinearity) for which we have $r_0 = 1$, $\mathfrak{c}_s = 2$, and $\mathcal{V}_{\mathfrak{c}_s}(\xi) = -4\xi^4 - 8\xi^5$. For this nonlinearity f , there exists a nontrivial sonic travelling wave of infinite energy (corresponding to $\xi_{\mathfrak{c}_s} = -1/2$), since $m = 3$.

The aim of this paper is to investigate the stability of the travelling waves for the one dimensional (NLS). We recall the definition of orbital stability in a metric space $(\mathcal{X}, d_{\mathcal{X}})$ for which we have a local in time existence result.

Definition 1 Let $0 \leq c \leq \mathfrak{c}_s$ and $U_c \in \mathcal{Z}$ be a nontrivial travelling wave of speed c . We say that U_c is orbitally stable in $(\mathcal{X}, d_{\mathcal{X}})$, where $\mathcal{X} \subset \mathcal{Z}$, if for any $\epsilon > 0$, there exists $\delta > 0$ such that for any initial datum $\Psi^{\text{in}} \in \mathcal{X}$ such that $d_{\mathcal{X}}(\Psi^{\text{in}}, U_c) \leq \delta$, any solution Ψ to (NLS) with initial datum Ψ^{in} is global in \mathcal{X} and

$$\sup_{t \geq 0} \inf_{\substack{y \in \mathbb{R} \\ \theta \in \mathbb{R}}} d_{\mathcal{X}}(\Psi(t), e^{i\theta} U_c(\cdot - y)) \leq \epsilon.$$

In the sequel, U_c will always stands for a nontrivial travelling wave, and we freeze the translation invariance by imposing that $|U_c|$ is even. Moreover, the solutions of (NLS) we consider will always be those given by Theorem 1.

1.2 Stability and instability in the case $0 < c < \mathfrak{c}_s$

1.2.1 Stability for the hydrodynamical and the energy distances

The first stability result for the travelling waves for (NLS) with nonzero condition at infinity is due to Z. Lin [47]. The analysis relies on the hydrodynamical form of (NLS), which is valid for solutions that never vanish. The advantage is to work with a fixed functional space since $(\eta, u) = (A^2 - r_0^2, \partial_x \phi) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$, whereas the travelling waves have a limit $r_0 e^{\pm i\Theta_c}$ (up to a phase factor) at $\pm\infty$ depending on the speed c . The result of Z. Lin [47] establishes rigorously the stability criterion found in [12], [3].

Theorem 2 ([47]) Assume that $0 < c_* < \mathfrak{c}_s$ is such that there exists a nontrivial travelling wave U_{c_*} . Then, there exists some small $\sigma > 0$ such that U_{c_*} belongs to a locally unique continuous branch of nontrivial travelling waves U_c defined for $c_* - \sigma \leq c \leq c_* + \sigma$.

(i) Assume

$$\frac{dP(U_c)}{dc} \Big|_{c=c_*} < 0.$$

Then, $U_{c_*} = A_* e^{i\phi_*}$ is orbitally stable in the sense that for any $\epsilon > 0$, there exists $\delta > 0$ such that if $\Psi^{\text{in}} = A^{\text{in}} e^{i\phi^{\text{in}}} \in \mathcal{Z}$ verifies

$$\|A^{\text{in}} - A_*\|_{H^1(\mathbb{R})} + \|\partial_x \phi^{\text{in}} - \partial_x \phi_*\|_{L^2(\mathbb{R})} \leq \delta,$$

then the solution Ψ to (NLS) such that $\Psi|_{t=0} = \Psi^{\text{in}}$ never vanishes, can be lifted $\Psi = A e^{i\phi}$, and we have

$$\sup_{t \geq 0} \inf_{y \in \mathbb{R}} \left\{ \|A(t) - A_*(\cdot - y)\|_{H^1(\mathbb{R})} + \|\partial_x \phi(t) - \partial_x \phi_*(\cdot - y)\|_{L^2(\mathbb{R})} \right\} \leq \epsilon.$$

(ii) Assume

$$\frac{dP(U_c)}{dc} \Big|_{c=c_*} > 0.$$

Then, $U_{c_*} = A_* e^{i\phi_*}$ is orbitally unstable in the sense that there exists $\epsilon > 0$ such that, for any $\delta > 0$, there exists $\Psi^{\text{in}} = A^{\text{in}} e^{i\phi^{\text{in}}} \in \mathcal{Z}$ verifying

$$\|A^{\text{in}} - A_*\|_{H^1(\mathbb{R})} + \|\partial_x \phi^{\text{in}} - \partial_x \phi_*\|_{L^2(\mathbb{R})} \leq \delta,$$

but such that if Ψ denotes the solution to (NLS) with $\Psi|_{t=0} = \Psi^{\text{in}}$, then there exists $t > 0$ such that Ψ does not vanish on the time interval $[0, t]$ but

$$\inf_{y \in \mathbb{R}} \left\{ \|A(t) - A_*(\cdot - y)\|_{H^1(\mathbb{R})} + \|\partial_x \phi(t) - \partial_x \phi_*(\cdot - y)\|_{L^2(\mathbb{R})} \right\} \geq \epsilon.$$

By one dimensional Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow \mathcal{C}_0(\mathbb{R})$, it is clear that since U_{c_*} does not vanish in \mathbb{R} , by imposing $\|\Psi^{\text{in}} - |U_{c_*}|\|_{H^1(\mathbb{R})} = \|A^{\text{in}} - A_*\|_{H^1(\mathbb{R})}$ small, Ψ^{in} does not vanish in \mathbb{R} and thus can be lifted.

Remark 2 We point out that C. Gallo in [24] fills two gaps in the proof of [47]: the first one concerns the local in time existence for the hydrodynamical system (see (15) in section 3.3) and the second one is about the conservation of the energy and the momentum. Furthermore, we make two additional remarks on the proof of [47] in section 3.3.

Theorem 2 is stability or instability in the open set $\mathcal{Z}_{\text{hy}} \subset \mathcal{Z}$ for the hydrodynamical distance

$$d_{\text{hy}}(\psi, \tilde{\psi}) \equiv \|A - \tilde{A}\|_{H^1(\mathbb{R})} + \|\partial_x \phi - \partial_x \tilde{\phi}\|_{L^2(\mathbb{R})} + \left| \arg \left(\frac{\psi(0)}{\tilde{\psi}(0)} \right) \right|, \quad \psi = A e^{i\phi}, \quad \tilde{\psi} = \tilde{A} e^{i\tilde{\phi}},$$

which is not the energy distance. Here, $\arg : \mathbb{C}^* \rightarrow (-\pi, +\pi]$ is the principal argument. For the stability, it suffices to consider the phase $\theta \in \mathbb{R}$ such that $\arg \left(\frac{\Psi(t)}{e^{i\theta} U_{c_*}(\cdot - y)} \right)$ is zero at $x = 0$, where y is the translation parameter. For the instability, no matter what is the phase $\theta \in \mathbb{R}$. The result of [47] is based on the application of the Grillakis-Shatah-Strauss theory [35] (see also [13], [66]) to the hydrodynamical formulation of (NLS) (see section 3.3). One difficulty is to overcome the fact that the Hamiltonian operator ∂_x is not onto.

On the energy-momentum diagrams, the stability can be checked either on the graphs of E and P with respect to c , either on the concavity of the curve $P \mapsto E$. Indeed, we have seen in [16] that the so called Hamilton group relation

$$c = \frac{dE}{dP}, \quad \text{or} \quad \frac{dE(U_c)}{dc} = c \frac{dP(U_c)}{dc},$$

holds, where the derivative is computed on the local branch. Therefore,

$$\frac{d^2 E}{dP^2} = \frac{d}{dP} \frac{dE}{dP} = \frac{dc}{dP}.$$

This means that we have stability when $P \mapsto E$ is concave, that is $\frac{d^2 E}{dP^2} < 0$, and instability if $P \mapsto E$ is convex, i.e. $\frac{d^2 E}{dP^2} > 0$.

Actually, the proof of [35], [47] provides an explicit control, as shown in the following lemma.

Lemma 1 *Under the assumptions of Theorem 2 and in the case (i) of stability, we have, provided $d_{\text{hy}}(\Psi^{\text{in}}, U_{c_*})$ is small enough,*

$$\begin{aligned} \sup_{t \geq 0} \inf_{y \in \mathbb{R}} \left\{ \|A(t) - A_*(\cdot - y)\|_{H^1(\mathbb{R})} + \|\partial_x \phi(t) - \partial_x \phi_*(\cdot - y)\|_{L^2(\mathbb{R})} \right\} \\ \leq K \sqrt{|E(\Psi^{\text{in}}) - E(U_{c_*})| + |P(\Psi^{\text{in}}) - P(U_{c_*})|}, \end{aligned} \quad (3)$$

as well as the control

$$\sup_{t \geq 0} \inf_{\substack{y \in \mathbb{R} \\ \theta \in \mathbb{R}}} d_{\text{hy}}(\Psi(t), e^{i\theta} U_{c_*}(\cdot - y)) \leq K d_{\text{hy}}(\Psi^{\text{in}}, U_{c_*}). \quad (4)$$

Remark 3 The second estimate (4) is not a simple consequence of the control (3), but relies on a comparison to U_c for some c close to c_* instead on a comparison to U_{c_*} (this idea has also been used in [68]). It follows that, in the definition of stability for U_{c_*} , one can take $\delta = \mathcal{O}(\varepsilon)$.

Let us lay the emphasis on the fact that Theorem 2 of Z. Lin is given in the hydrodynamical distance d_{hy} , which is not the energy distance $d_{\mathcal{Z}}$. As a matter of fact, the Madelung transform

$$\mathcal{M} : (\mathcal{Z}_{\text{hy}}, d_{\mathcal{Z}}) \ni U \mapsto \left(\eta, u, \frac{U(0)}{|U(0)|} \right) \in H^1(\mathbb{R}, \mathbb{R}) \times L^2(\mathbb{R}, \mathbb{R}) \times \mathbb{S}^1,$$

where $U = Ae^{i\phi}$, $\eta = A^2 - r_0^2$ and $u = \partial_x \phi$ is not so well behaved.

Lemma 2 (i) *The mapping $\mathcal{M} : (\mathcal{Z}_{\text{hy}}, d_{\mathcal{Z}}) \rightarrow H^1(\mathbb{R}, \mathbb{R}) \times L^2(\mathbb{R}, \mathbb{R}) \times \mathbb{S}^1$ is an homeomorphism.*

(ii) *There exists $\phi_* \in \mathcal{C}^2(\mathbb{R}, \mathbb{R})$ such that $\partial_x \phi_* \in L^2(\mathbb{R})$ and a sequence $(\phi_n)_{n \geq 1}$ of functions in $H^1(\mathbb{R}, \mathbb{R})$ such that, when $n \rightarrow +\infty$,*

$$0 < d_{\text{hy}}(e^{i\phi_*}, e^{i\phi_*} e^{i\phi_n}) \rightarrow 0 \quad \text{but} \quad \frac{d_{\mathcal{Z}}(e^{i\phi_*}, e^{i\phi_*} e^{i\phi_n})}{d_{\text{hy}}(e^{i\phi_*}, e^{i\phi_*} e^{i\phi_n})} \rightarrow +\infty.$$

Therefore, \mathcal{M}^{-1} is not locally Lipschitz continuous in general. However, for the stability issues, we compare the $d_{\mathcal{Z}}$ and the d_{hy} distances to some fixed travelling wave U_* , which enjoys some nice decay properties at infinity. Let us now stress the link between the two distances d_{hy} and $d_{\mathcal{Z}}$ in this case.

Lemma 3 *Let $0 \leq c_* \leq c_s$ and assume that $U_* \in \mathcal{Z}$ is a non constant travelling wave with speed c_* that does not vanish. If $c_* = c_s$, we further assume that assumption (\mathcal{A}_0) is verified. Then, there exists some constants K and $\delta > 0$, depending only on U_* , such that for any $\psi \in \mathcal{Z}$ verifying $d_{\mathcal{Z}}(\psi, U_*) \leq \delta$, we have*

$$\frac{1}{K} d_{\text{hy}}(\psi, U_*) \leq d_{\mathcal{Z}}(\psi, U_*) \leq K d_{\text{hy}}(\psi, U_*).$$

An immediate corollary of Lemma 3 is that Theorem 2 is also a stability/instability result in the energy distance. If one wishes only a stability/instability result, it is sufficient to invoke the fact that the mapping \mathcal{M} is an homeomorphism. However, the use of Lemma 3 provides a stronger explicit control similar to the one obtained in Remark 1 (see (3)). In particular, in the definition of stability for U_{c_*} in $(\mathcal{Z}, d_{\mathcal{Z}})$, one can take $\delta = \mathcal{O}(\varepsilon)$.

Corollary 1 *Assume that $0 < c_* < c_s$ is such that there exists a nontrivial travelling wave U_{c_*} . Then, there exists some small $\sigma > 0$ such that U_{c_*} belongs to a locally unique continuous branch of nontrivial travelling waves U_c defined for $c_* - \sigma \leq c \leq c_* + \sigma$.*

(i) *If $\frac{dP(U_c)}{dc} \Big|_{c=c_*} < 0$, then $U_{c_*} = A_* e^{i\phi_*}$ is orbitally stable in $(\mathcal{Z}, d_{\mathcal{Z}})$. Furthermore, if $\Psi(t)$ is the (global) solution to (NLS) with initial datum Ψ^{in} , then we have, for some constant K depending only on U_{c_*} and provided $d_{\mathcal{Z}}(\Psi^{\text{in}}, U_{c_*})$ is sufficiently small,*

$$\sup_{t \geq 0} \inf_{\substack{y \in \mathbb{R} \\ \theta \in \mathbb{R}}} d_{\mathcal{Z}}(\Psi(t), e^{i\theta} U_{c_*}(\cdot - y)) \leq K \sqrt{|E(\Psi^{\text{in}}) - E(U_{c_*})| + |P(\Psi^{\text{in}}) - P(U_{c_*})|},$$

as well as the control

$$\sup_{t \geq 0} \inf_{\substack{y \in \mathbb{R} \\ \theta \in \mathbb{R}}} d_{\mathcal{Z}}(\Psi(t), e^{i\theta} U_{c_*}(\cdot - y)) \leq K d_{\mathcal{Z}}(\Psi^{\text{in}}, U_{c_*}).$$

(ii) *If $\frac{dP(U_c)}{dc} \Big|_{c=c_*} > 0$, then $U_{c_*} = A_* e^{i\phi_*}$ is orbitally unstable in $(\mathcal{Z}, d_{\mathcal{Z}})$.*

For the Gross-Pitaevskii nonlinearity ($f(\varrho) = 1 - \varrho$), the stability (for the energy distance $d_{\mathcal{Z}}$) of the travelling waves with speed $0 < c < c_s$ was proved by F. Béthuel, P. Gravejat and J.-C. Saut in [10] through the variational characterization that these solutions are minimizers of the energy under the constraint of fixed momentum. However, in view of the energy momentum diagrams in section 1.1, this constraint minimization approach can not be used in the general setting we consider here. Indeed, this method provides only stability, but there may exist unstable travelling waves. Moreover, it follows from the proof of Theorem 2 that stable waves are local minimizers of the energy at fixed momentum but not necessarily global minimizers. Finally, we emphasize that the spectral methods allow to derive an explicit (Lipschitz) control in case of stability.

1.2.2 Stability via a Liapounov functional

Another way to prove the orbital stability is to find a Liapounov functional. By Liapounov functional, we mean a functional which is conserved by the (NLS) flow and for which the travelling wave U_c is a local minimum (for instance, a critical point with second derivative $\geq \delta \text{Id}$ for some $\delta > 0$). Such a Liapounov functional always exists in the Grillakis-Shatah-Strauss theory when $\frac{dP(U_c)}{dc}|_{c=c_*} < 0$, as shown by Theorem A in Appendix A. Its direct application to our problem leads us to define the functional in \mathcal{Z}_{hy}

$$\mathcal{L}(\psi) \equiv E(\psi) - c_* P(\psi) + \frac{M}{2} \left(P(\psi) - P(U_{c_*}) \right)^2,$$

where M is some positive parameter. It turns out that \mathcal{L} is such a Liapounov functional when M is sufficiently large. Since the proof relies on the Grillakis-Shatah-Strauss framework, we have to work in the hydrodynamical variables. However, by Lemma 3, we recover the case of the energy distance.

Theorem 3 *Assume that for some $c_* \in (0, c_s)$ and $\sigma > 0$ small, $(0, c_s) \supset [c_* - \sigma, c_* + \sigma] \ni c \mapsto U_c \in \mathcal{Z}$ is a continuous branch of nontrivial travelling waves with $\frac{dP(U_c)}{dc}|_{c=c_*} < 0$. If*

$$M > \frac{1}{\frac{dP(U_c)}{dc}|_{c=c_*}} > 0,$$

there exists $\epsilon > 0$ and K , depending only on U_{c_} , such that for any $\psi \in \mathcal{Z}$ with $\inf_{\substack{y \in \mathbb{R} \\ \theta \in \mathbb{R}}} d_{\text{hy}}(\psi, e^{i\theta} U_{c_*}(\cdot - y)) \leq \epsilon$,*

we have

$$\inf_{\substack{y \in \mathbb{R} \\ \theta \in \mathbb{R}}} d_{\text{hy}}^2(\psi, e^{i\theta} U_{c_*}(\cdot - y)) \leq K \left(\mathcal{L}(\psi) - \mathcal{L}(U_{c_*}) \right)$$

and analogously with d_{hy} replaced by $d_{\mathcal{Z}}$. Consequently, $U_{c_} = A_* e^{i\phi_*}$ is orbitally stable in $(\mathcal{Z}_{\text{hy}}, d_{\text{hy}})$ and in $(\mathcal{Z}, d_{\mathcal{Z}})$. Furthermore, if $\Psi(t)$ is the (global) solution to (NLS) with initial datum Ψ^{in} , then we have*

$$\sup_{t \geq 0} \inf_{\substack{y \in \mathbb{R} \\ \theta \in \mathbb{R}}} d_{\text{hy}}(\Psi(t), e^{i\theta} U_{c_*}(\cdot - y)) \leq K \sqrt{\mathcal{L}(\Psi^{\text{in}}) - \mathcal{L}(U_{c_*})} \leq K d_{\text{hy}}(\Psi^{\text{in}}, U_{c_*}).$$

provided $d_{\text{hy}}(\Psi^{\text{in}}, U_{c_})$ is sufficiently small, and analogously with d_{hy} replaced by $d_{\mathcal{Z}}$.*

For the travelling waves for (NLS) in dimension one, this type of Liapounov functional appears for the first time in the paper [3] by I. Barashenkov. However, in [3], the problem is treated directly on the wave function Ψ , whereas the correct proof holds on the hydrodynamical variables, in particular because of the gauge invariance $(\theta, \Psi) \mapsto e^{i\theta} \Psi$. For instance, the work [3] suggests that we have stability for H^1 perturbations, whereas it holds only for perturbations in the energy space. Finally, we fill some gaps in the proof of [3].

1.2.3 Instability via the existence of an unstable eigenvalue

In the Grillakis-Shatah-Strauss theory [35], the instability is not shown by proving the existence of a unstable eigenvalue for the linearized (NLS) and then a nonlinear instability result (see however [36] when the Hamiltonian skew-adjoint operator is onto). There exists, however, some general results that prove the existence of unstable eigenvalues. For the instability of bound states for (NLS) (and also for nonlinear Klein-Gordon equation), that is solutions of the form $e^{i\omega t}U_\omega(x)$, the work [34] by M. Grillakis shows that the condition $\frac{d}{d\omega} \left(\int_{\mathbb{R}^d} |U_\omega|^2 \right)_{|\omega=\omega_*} > 0$ is sufficient for the existence of such an unstable eigenvalue. However, the proof relies on the fact that the bound states are real valued functions (up to a phase factor) and it is not clear whether it extends to the case of travelling waves we are studying. Indeed, since we have to work in hydrodynamical variables in order to have a fixed functional space, the linearized operator does not have (for $c \neq 0$) the structure required for the application of [34]. Another general result is due to O. Lopes [49] but it assumes that the linearized equation can be solved using a semigroup. This is not the case for our problem once it is written in hydrodynamical variables (see below). Finally, the paper [48] by Z. Lin proposes an alternative approach for the existence of unstable eigenvalues. The method has the advantage of allowing pseudo-differential equations (like the Benjamin-Ono equation). However, the results are given for three model equations involving a scalar unknown, and it is not clear whether the proof can be extended to the case of systems.

The linearization of (NLS) near the travelling wave U_{c_*} in the frame moving with speed c_* is

$$i \frac{\partial \psi}{\partial t} - ic_* \partial_x \psi + \partial_x^2 \psi + \psi f(|U_{c_*}|^2) + 2 \langle \psi | U_{c_*} \rangle f'(|U_{c_*}|^2) U_{c_*} = 0, \quad (5)$$

and thus, searching for exponentially growing modes $\psi(t, x) = e^{\lambda t} w(x)$ leads to the eigenvalue problem

$$i \lambda w - ic_* \partial_x w + \partial_x^2 w + w f(|U_{c_*}|^2) + 2 \langle w | U_{c_*} \rangle f'(|U_{c_*}|^2) U_{c_*} = 0, \quad (6)$$

with $\text{Re}(\lambda) > 0$ and $w \neq 0$. For one dimensional problems, the linear instability is commonly shown through the use of Evans functions (see the classical paper [57] and also the review article [63]). For our problem, we look for an unstable eigenvalue for the equation written in hydrodynamical variables, namely we look for exponentially growing solutions $(\underline{\eta}, \underline{u})$ of the linear problem (written in the moving frame)

$$\begin{cases} \partial_t \underline{\eta} - c_* \partial_x \underline{\eta} + 2 \partial_x ((r_0^2 + \eta_*) \underline{u} + \underline{\eta} u_*) = 0 \\ \partial_t \underline{u} - c_* \partial_x \underline{u} + 2 \partial_x (u_* \underline{u}) - \partial_x (f'(r_0^2 + \eta_*) \underline{\eta}) - \partial_x \left\{ \frac{1}{2\sqrt{r_0^2 + \eta_*}} \partial_x^2 \left(\frac{\underline{\eta}}{\sqrt{r_0^2 + \eta_*}} \right) - \frac{\underline{\eta} \partial_x^2 (\sqrt{r_0^2 + \eta_*})}{2(r_0^2 + \eta_*)^{3/2}} \right\} = 0, \end{cases} \quad (7)$$

where (η_*, u_*) is the reference solution. The advantage is here again to work with a fixed functional space in variables (η, u) . Due to the term $\partial_x \left\{ \frac{1}{2\sqrt{r_0^2 + \eta_*}} \partial_x^2 \left(\frac{\underline{\eta}}{\sqrt{r_0^2 + \eta_*}} \right) \right\}$, this equation can not be solved using a semigroup, except in the trivial case where η_* is constant, hence the result of [49] does not apply. However, system (7) is a particular case of the Euler-Korteweg system for capillary fluids (see [5] for a survey on this model). We may then use a linear instability result already shown for the Euler-Korteweg system with the Evans function method, as in [70] by K. Zumbrun for a simplified system, and more recently in [6] by S. Benzoni-Gavage for the complete Euler-Korteweg system.

Theorem 4 *Assume that for some $c_* \in (0, c_s)$ and $\sigma > 0$ small, $(0, c_s) \supset [c_* - \sigma, c_* + \sigma] \ni c \mapsto U_c \in \mathcal{Z}$ is a continuous branch of nontrivial travelling waves with*

$$\frac{dP(U_c)}{dc} \Big|_{c=c_*} > 0.$$

Then, there exists exactly one unstable eigenvalue $\gamma_0 \in \{\text{Re} > 0\}$ for (6) and $\gamma_0 \in (0, +\infty)$, that is (NLS) is (spectrally) linearly unstable.

Once we have shown the existence of an unstable eigenvalue for the linearized (NLS) equation (5), we can prove a nonlinear instability result as in [38], [20]. Note that here, we no longer work in the hydrodynamical variables, where the high order derivatives involve nonlinear terms, but on the semilinear (NLS) equation.

Corollary 2 *Under the assumptions of Theorem 4, U_{c_*} is unstable in $U_{c_*} + H^1(\mathbb{R}, \mathbb{C})$ (endowed with the natural H^1 distance): there exists ϵ such that for any $\delta > 0$, there exists $\Psi^{\text{in}} \in U_{c_*} + H^1(\mathbb{R})$ such that $\|\Psi^{\text{in}} - U_{c_*}\|_{H^1(\mathbb{R})} \leq \delta$ but if $\Psi \in U_{c_*} + \mathcal{C}([0, T^*], H^1(\mathbb{R}))$ denotes the maximal solution of (NLS), then there exists $0 < t < T^*$ such that $\|\Psi(t) - U_{c_*}\|_{H^1(\mathbb{R})} \geq \epsilon$.*

Since the proof is very similar to the one in [38], [20], we omit it. We may actually prove a stronger instability result, since the above one is not proved by tracking the exponentially growing mode. In [21], a spectral mapping theorem is shown and used to show the nonlinear instability by tracking this exponentially growing mode, which is a natural mechanism of instability. In Appendix B, we show that this spectral mapping theorem holds for a wide class of Hamiltonian equation. The direct application of Corollary B.2 in Appendix B gives the following nonlinear instability result.

Corollary 3 *We make the assumptions of Theorem 4, so that there exists an unstable eigenmode $(\gamma_0, w) \in (0, +\infty) \times H^1(\mathbb{R})$, $\|w\|_{H^1} = 1$. There exists $M > 0$ such that for any solution $\psi \in \mathcal{C}(\mathbb{R}_+, H^1(\mathbb{R}, \mathbb{C}))$ of the linearized equation (5), we have the growth estimate of the semigroup*

$$\forall t \geq 0 \quad \|\psi(t)\|_{H^1(\mathbb{R})} \leq M e^{\gamma_0 t} \|\psi(0)\|_{H^1(\mathbb{R})}.$$

Moreover, U_{c_*} has also the following instability property: there exists $K > 0$, $\delta > 0$ and $\varepsilon_0 > 0$, such that for any $0 < \delta < \delta_0$, the solution $\Psi(t)$ to (NLS) with initial datum $\Psi^{\text{in}} = U_{c_*} + \delta w \in U_{c_*} + H^1(\mathbb{R})$ exists at least on $[0, \gamma_0^{-1} \ln(2\varepsilon_0/\delta)]$ and verifies

$$\|\Psi(t) - U_{c_*} - \delta e^{\gamma_0 t} w\|_{H^1(\mathbb{R})} \leq K \delta^2 e^{2\gamma_0 t}.$$

In particular, for $t = \gamma_0^{-1} \ln(2\varepsilon_0/\delta)$ and $\epsilon \equiv \frac{\varepsilon_0}{K}$, we have

$$\inf_{y \in \mathbb{R}} \|\Psi(t) - |U_{c_*}|(\cdot - y)\|_{L^2(\mathbb{R})} \geq \epsilon \quad \text{and} \quad \inf_{y \in \mathbb{R}} \|\Psi(t) - |U_{c_*}|(\cdot - y)\|_{L^\infty(\mathbb{R})} \geq \epsilon,$$

which implies

$$\inf_{\substack{y \in \mathbb{R} \\ \theta \in \mathbb{R}}} \|\Psi(t) - e^{i\theta} U_{c_*}(\cdot - y)\|_{H^1(\mathbb{R})} \geq \epsilon$$

as well as

$$\inf_{\substack{y \in \mathbb{R} \\ \theta \in \mathbb{R}}} d_{\text{hy}}(\Psi(t), e^{i\theta} U_{c_*}(\cdot - y)) \geq \epsilon \quad \text{and} \quad \inf_{\substack{y \in \mathbb{R} \\ \theta \in \mathbb{R}}} d_{\mathbb{Z}}(\Psi(t), e^{i\theta} U_{c_*}(\cdot - y)) \geq \epsilon.$$

With the above result, we then show the nonlinear instability also in the energy space, and thus recover the instability result of Z. Lin but this time by tracking the unstable growing mode.

1.2.4 Instability at a cusp

In this section, we investigate the question of stability in the degenerate case $\frac{dP}{dc} = 0$. In [35] (see also [36]), a stability result for the wave of speed c_* is shown when the action $c \mapsto S(c) = E(U_c) - cP(U_c)$ (on the local branch) is such that, for instance, $\frac{d^2 S}{dc^2} = -\frac{dP}{dc}$ is positive for $c \neq c_*$ but vanishes for $c = c_*$. In the energy-momentum diagrams of section 1.1, the situation is different since $\frac{dP}{dc}$ changes sign at the cusps, or, equivalently, the action $c \mapsto S(c) = E(U_c) - cP(U_c)$ (on the local branch) changes its concavity at the cusp. The work [19] by A. Comech and D. Pelinovsky shows that for the Nonlinear Schrödinger equation, a bound

state associated with a cusp in the energy-charge diagram is unstable. The proof relies on a careful analysis of the linearized equation, which is spectrally stable, but linearly unstable (with polynomial growth for the linear problem). A similar technique was used by A. Comech, S. Cuccagna and D. Pelinovsky in [18] for the generalized Korteweg-de Vries equation. Then, M. Ohta in [55] also proves the nonlinear instability of these “bound states” using a Liapounov functional as in [35]. However, in [55], it is assumed that $J = T'(0)$ and is onto, which are both not true here (and there are further restrictions due to the introduction of an intermediate Hilbert space). Recently, M. Maeda has extended in [50] the above instability result, removing some assumptions in [55]. We show the instability of travelling waves associated with a cusp in the energy-momentum diagram in the generic case where $\frac{d^2 P}{dc^2} \neq 0$. Our approach follows the lines of [50], but with some modifications since our problem does not fit exactly the general framework of this paper. In particular, we can not find naturally a space “Y”, and some functions appearing in the proof do not lie in the range of the skew-adjoint operator ∂_x involved in the Hamiltonian formalism. We overcome this difficulty using an approximation argument (similar to the one used in [47]).

Theorem 5 *Assume that for some $c_* \in (0, c_s)$ and $\sigma > 0$ small, $(0, c_s) \supset [c_* - \sigma, c_* + \sigma] \ni c \mapsto U_c \in \mathcal{Z}$ is a continuous branch of nontrivial travelling waves with*

$$\frac{dP(U_c)}{dc} \Big|_{c=c_*} = 0 \neq \frac{d^2 P(U_c)}{dc^2} \Big|_{c=c_*},$$

and assume in addition that f is of class \mathcal{C}^2 . Then, U_{c_} is orbitally unstable in $(\mathcal{Z}, d_{\mathcal{Z}})$.*

1.3 Stability in the case $c = 0$

1.3.1 Instability for the bubbles

When $c = 0$, we have two types of stationary waves: the bubbles, when $\xi_0 > -r_0^2$, are even functions (up to a translation) that do not vanish, and the kinks, when $\xi_0 = -r_0^2$, are odd functions (up to a translation). The instability of stationary bubbles has been shown by A. de Bouard [20] (and is true even in higher dimension). The proof in [20] relies on the proof of the existence of an unstable eigenvalue for the linearized (NLS), and then the proof of a nonlinear instability result. An alternative proof of the linear instability of the bubbles is given in [58] (Theorem 3.11 (ii)).

Theorem 6 ([20]) *Assume that there exists a bubble, that is a nontrivial stationary ($c = 0$) wave U_0 which does not vanish. Then, U_0 is (linearly and nonlinearly) unstable in $U_0 + H^1(\mathbb{R})$ (endowed with the natural H^1 metric), that is there exists ϵ such that for any $\delta > 0$, there exists $\Psi^{\text{in}} \in U_0 + H^1(\mathbb{R})$ such that $\|\Psi^{\text{in}} - U_0\|_{H^1(\mathbb{R})} \leq \delta$ but if $\Psi \in U_0 + \mathcal{C}([0, T^*], H^1(\mathbb{R}))$ denotes the maximal solution of (NLS), then there exists $0 < t < T^*$ such that $\|\Psi(t) - U_0\|_{H^1(\mathbb{R})} \geq \epsilon$.*

Actually, in the same way that Corollary 3 is a better instability result than Corollary 2, we have the following stronger instability result, which is a direct consequence of Corollary B.2 in Appendix B.

Proposition 2 *Assume that there exists a bubble, that is a nontrivial stationary ($c = 0$) wave U_0 which does not vanish. Then, U_0 is (nonlinearly) unstable in $U_0 + H^1(\mathbb{R})$, $(\mathcal{Z}, d_{\mathcal{Z}})$ and $(\mathcal{Z}_{\text{hy}}, d_{\text{hy}})$ in the same sense as in Corollary 3.*

Finally, we would like to emphasize that we may recover the instability result for bubbles from the proof of Theorem 2, relying on the hydrodynamical form of (NLS), which holds true here since bubbles do not vanish. Our result holds in the energy space and for the hydrodynamical distance.

Theorem 7 *Assume that there exists a bubble, that is a nontrivial stationary ($c = 0$) wave U_0 which does not vanish. Then, there exists some small $\sigma > 0$ such that U_0 belongs to a locally unique continuous branch of nontrivial travelling waves U_c defined for $0 \leq c \leq \sigma$. Then, $c \mapsto P(U_c)$ has a derivative at $c = 0$,*

$$\frac{dP(U_c)}{dc} \Big|_{c=0} > 0$$

and $U_0 = A_* e^{i\phi_*}$ is orbitally unstable for the distances d_Z and d_{hy} .

Proof. We give a proof based on the argument of Z. Lin [47], which is possible since U_0 is a bubble hence does not vanish and the spectral decomposition used in [47] still holds when $c = 0$. Moreover, it is clear that the mapping $c \mapsto (\eta_c, u_c) \in H^1 \times L^2$ is smooth up to $c = 0$, using the uniform exponential decay at infinity near $c = 0$ and arguing as in [16]. Therefore, it suffices to show that $\frac{dP(U_c)}{dc} \Big|_{c=0} > 0$. From the expression of the momentum given in subsection 1.2 in [16], we have, for $0 \leq c \leq \sigma$,

$$P(U_c) = c \operatorname{sgn}(\xi_c) \int_0^{\xi_c} \frac{\xi^2}{r_0^2 + \xi} \frac{d\xi}{\sqrt{-\mathcal{V}_c(\xi)}} = c \left| \int_0^{\xi_0} \frac{\xi^2}{r_0^2 + \xi} \frac{d\xi}{\sqrt{-\mathcal{V}_0(\xi)}} \right| + o(c)$$

since $\xi_0 > -r_0^2$. Indeed, we are allowed to pass to the limit in the integral once it is written with the change of variables $\xi = t\xi_c$:

$$\int_0^{\xi_c} \frac{\xi^2}{r_0^2 + \xi} \frac{d\xi}{\sqrt{-\mathcal{V}_c(\xi)}} = \int_0^1 \frac{\xi_c^3 t^2}{r_0^2 + t\xi_c} \frac{dt}{\sqrt{-\mathcal{V}_c(t\xi_c)}},$$

since $\xi_0 > -r_0^2$. Therefore,

$$\frac{dP(U_c)}{dc} \Big|_{c=0} = \left| \int_0^{\xi_0} \frac{\xi^2}{r_0^2 + \xi} \frac{d\xi}{\sqrt{-\mathcal{V}_0(\xi)}} \right| > 0$$

since $\xi_0 \neq 0$ (U_0 is not trivial). The conclusion follows then from the proof of Theorem 2. \square

When we know that $\frac{dP(U_c)}{dc} \Big|_{c=0} > 0$, we may also use the Evans function as in Theorem 4 to show the existence of an unstable eigenmode. However, due to the fact that the kink U_0 is real-valued, we can use the arguments in [20], [58].

1.3.2 Stability analysis for the kinks

We now turn to the case of the kinks ($\xi_0 = -r_0^2$ and U_0 is odd up to a translation). Since U_0 vanishes at the origin, the hydrodynamical form of (NLS) can not be used. The stability of the kink as attracted several recent works. L. Di Menza and C. Gallo [21] have investigated the linear stability through the Vakhitov-Kolokolov function VK, defined by

$$\text{VK}(\lambda) \equiv \int_{\mathbb{R}} \left([-\partial_x^2 - f(U_0^2) - \lambda]^{-1} (\partial_x U_0) \right) (\partial_x U_0) dx,$$

where U_0 is the kink, for $\lambda \in (\lambda_*, 0)$ for some $\lambda_* < 0$. They show that the Vakhitov-Kolokolov function VK has a limit VK_0 when $\lambda \rightarrow 0^-$. If $\text{VK}_0 > 0$, then the linearization of (NLS) around the kink has an unstable real positive eigenvalue. When $\text{VK}_0 < 0$, the linearization of (NLS) around the kink has a spectrum included in $i\mathbb{R}$ (spectral stability). Note that the approach of [47] (extending [35]) does not give directly unstable eigenvalues in the case $\frac{dP}{dc} > 0$. Recently, the link between the quantity $\frac{dP}{dc}$ and the sign of VK_0 has been given by D. Pelinovsky and P. Kevrekidis [58] (proof of Lemma 3.10 there, the factor $\sqrt{2}$ coming from the coefficients of the (NLS) equation in [58]):

$$2\sqrt{2}\text{VK}_0 = \lim_{c \rightarrow 0} \frac{dP(U_c)}{dc}; \tag{8}$$

and they also prove, in a different way from [21] that we have spectral stability when $\lim_{c \rightarrow 0} \frac{dP}{dc} < 0$ and existence of an unstable eigenvalue (in \mathbb{R}_+^*) if $\lim_{c \rightarrow 0} \frac{dP}{dc} > 0$. It is shown in [58] that the limit $\lim_{c \rightarrow 0} \frac{dP(U_c)}{dc}$

does exist. Actually, they prove that the function $[0, c_0) \ni c \mapsto P(U_c)$ is of class \mathcal{C}^1 and that the derivative at $c = 0$ is also given by (see (8))

$$\begin{aligned} \lim_{c \rightarrow 0} \frac{dP(U_c)}{dc} &= 2\sqrt{2}\text{VK}_0 = 2\sqrt{2} \lim_{\lambda \rightarrow 0^-} \int_{\mathbb{R}} \left([-\partial_x^2 - f(U_0^2) - \lambda]^{-1} (\partial_x U_0) \right) (\partial_x U_0) dx \\ &= 2\sqrt{2} \int_{\mathbb{R}} \text{Im} \left(\frac{\partial U_c}{\partial c} \Big|_{c=0} \right) \partial_x U_0 dx. \end{aligned} \quad (9)$$

Our next lemma gives an explicit formula of the expression (9), involving only the nonlinearity f .

Lemma 4 *Assume that U_0 is a kink. Then, there exists $c_0 \in (0, c_s)$ such that U_0 belongs to the (locally) unique branch $[0, c_0) \ni c \mapsto U_c \in \mathcal{Z}$. Moreover, $P(U_c) \rightarrow r_0^2 \pi$ as $c \rightarrow 0$ and the continuous extension $[0, c_0) \ni c \mapsto P(U_c)$ has a derivative at $c = 0$ given by*

$$\frac{dP(U_c)}{dc} \Big|_{c=0} = -\frac{8r_0^3}{3\sqrt{F(0)}} + \frac{1}{2} \int_0^{r_0^2} \frac{(\varrho - r_0^2)^2}{\varrho^{3/2}} \left(\frac{1}{\sqrt{F(\varrho)}} - \frac{1}{\sqrt{F(0)}} \right) d\varrho.$$

The advantage of the formula given in Lemma 4 compared to (9) is that it allows a direct computation of $\frac{dP(U_c)}{dc} \Big|_{c=0}$ when f is known, which does not require to compute numerically U_0 and $\frac{\partial U_c}{\partial c} \Big|_{c=0}$. For instance, it is quite well adapted to the stability analysis as in [22]. Let us observe that it may happen that a kink is unstable (see [43], [21]).

In the case of linear instability, [21] shows that then, nonlinear instability holds. Actually, C. Gallo and L. Di Menza prove in [21] a stronger result, where they show that the L^∞ norm (and not only the H^1 norm) does not remain small.

Theorem 8 ([21]) *Assume that there exists a kink, that is a nontrivial stationary ($c = 0$) wave U_0 vanishing somewhere, and satisfying $\frac{dP(U_c)}{dc} \Big|_{c=0} > 0$. Then, U_0 is (linearly and nonlinearly) unstable in the sense that there exists ϵ such that for any $\delta > 0$, there exists $\Psi^{\text{in}} \in U_0 + H^1(\mathbb{R})$ such that $\|\Psi^{\text{in}} - U_0\|_{H^1(\mathbb{R})} \leq \delta$ but if $\Psi \in U_0 + \mathcal{C}([0, T^*), H^1(\mathbb{R}))$ denotes the maximal solution of (NLS), then there exists $0 < t < T^*$ such that $\|\Psi(t) - U_0\|_{L^\infty(\mathbb{R})} \geq \epsilon$.*

The proof in [21] relies on the tracking of the exponentially growing eigenmode. One may actually improve slightly the result as this was done in Corollary 3. As a matter of fact, this was the result in Theorem 8 that has motivated us for Corollary 3.

We focus now on the nonlinear stability issue when there is linear (spectral) stability, that is when $\frac{dP(U_c)}{dc} \Big|_{c=0} < 0$. Concerning the Gross-Pitaevskii nonlinearity ($f(\varrho) = 1 - \varrho$), for which we have $\frac{dP(U_c)}{dc} \Big|_{c=0} < 0$, we quote two papers on this question. The first one is the work of P. Gérard and Z. Zhang [30] where the stability is shown by inverse scattering, hence in a space of functions sufficiently decaying at infinity. The analysis then relies on the integrability of the one-dimensional (GP) equation. The other work is by F. Béthuel, P. Gravejat, J.-C. Saut and D. Smets [11]. They prove the orbital stability of the kink of the Gross-Pitaevskii equation by showing that the kink is a global minimizer of the energy under the constraint that a variant of the momentum is fixed (recall that the definition of the momentum has to be clarified for an arbitrary function in the energy space), and that the corresponding minimizing sequences are compact (up to space translations and phase factors). In this approach, it is crucial (see [10], [11]) that $E_{\text{kink}} < c_s P_{\text{kink}} = c_s r_0^2 \pi$ in order to prevent the dichotomy case for the minimizing sequences. However, since the energy of the kink is equal to

$$E_{\text{kink}} = 4 \int_{-r_0^2}^0 \frac{F(r_0^2 + \xi)}{\sqrt{-\mathcal{V}_0(\xi)}} d\xi = 2 \int_{-r_0^2}^0 \sqrt{\frac{F(r_0^2 + \xi)}{r_0^2 + \xi}} d\xi = 2 \int_0^{r_0^2} \sqrt{\frac{F(\varrho)}{\varrho}} d\varrho,$$

whereas its momentum is always equal to $r_0^2\pi$, it is clear that the condition $E_{\text{kink}} < \mathfrak{c}_s P_{\text{kink}} = \mathfrak{c}_s r_0^2\pi$ does not hold in general, as shown in the following example.

Example. For $\kappa \geq 0$, consider

$$f(\varrho) \equiv 1 - \varrho + \kappa(1 - \varrho)^3,$$

which is smooth and decrease to $-\infty$ as the Gross-Pitaevskii nonlinearity. We have $r_0 = 1$, $\mathfrak{c}_s = \sqrt{2}$, $F(\varrho) = (1 - \varrho)^2/2 + \kappa(1 - \varrho)^4/4$ and

$$E_{\text{kink}} = 2 \int_0^{r_0^2} \sqrt{\frac{F(\varrho)}{\varrho}} d\varrho = 2 \int_0^{r_0^2} \sqrt{\frac{2(1 - \varrho)^2 + \kappa(1 - \varrho)^4}{4\varrho}} d\varrho > \mathfrak{c}_s r_0^2\pi = \pi\sqrt{2}$$

for κ large (the left-hand side tends to $+\infty$), and numerical computations show that it is the case for $\kappa \geq 14$. Furthermore, Lemma 4 gives

$$\sqrt{F(0)} \frac{dP(U_c)}{dc} \Big|_{c=0} = -\frac{8}{3} + \frac{1}{2} \int_0^1 \frac{(\varrho - 1)^2}{\varrho^{3/2}} \left(\sqrt{\frac{F(0)}{F(\varrho)}} - 1 \right) d\varrho. \quad (10)$$

Since $\frac{F(0)}{F(\varrho)} = \frac{2 + \kappa}{2(\varrho - 1)^2 + \kappa(\varrho - 1)^4}$, it can be easily checked that the right-hand side of (10) is a decreasing function of κ tending to

$$-\frac{8}{3} + \frac{1}{2} \int_0^1 \frac{(\varrho - 1)^2}{\varrho^{3/2}} \left(\frac{1}{(\varrho - 1)^2} - 1 \right) d\varrho = -1$$

when $\kappa \rightarrow +\infty$ (by monotone convergence). In particular, for any $\kappa \geq 0$, we have $\frac{dP(U_c)}{dc} \Big|_{c=0} < 0$, that is the kink is always (linearly) stable. The energy-momentum diagram for this type of nonlinearity with κ large is as in on the right of figure 4 (the left part correspond to κ smaller).

In comparison with the constraint minimization approach as in [10, 11], which allows to establish a global minimization result, the spectral methods as in [35, 47] allow to put forward locally minimizing properties, which turn out to be useful for the stability analysis in dimension 1.

In the stability analysis of the kink, one issue is the definition of the momentum P , which was up to now given only for maps in \mathcal{Z}_{hy} , that is for maps that never vanish, but the kink vanishes at the origin. In [11], the notion of momentum was extended to the whole energy space \mathcal{Z} , hence including maps vanishing somewhere, as a quantity defined mod 2π , and was called ‘‘untwisted momentum’’. This notion will be useful for our stability result.

Lemma 5 ([11]) *If $\psi \in \mathcal{Z}$, the limit*

$$\mathfrak{P}(\psi) \equiv \lim_{R \rightarrow +\infty} \left[\int_{-R}^{+R} \langle i\psi | \partial_x \psi \rangle dx - r_0^2 \left(\arg(\psi(+R)) - \arg(\psi(-R)) \right) \right]$$

exists in $\mathbb{R}/(2\pi r_0^2\mathbb{Z})$. The mapping $\mathfrak{P} : \mathcal{Z} \rightarrow \mathbb{R}/(2\pi r_0^2\mathbb{Z})$ is continuous and if $\psi \in \mathcal{Z}$ verifies $\inf_{\mathbb{R}} |\psi| > 0$ (i.e. $\psi \in \mathcal{Z}_{\text{hy}}$), then $\mathfrak{P}(\psi) = P(\psi) \bmod 2\pi r_0^2$. Finally, if $\Psi \in \mathcal{C}([0, T], \mathcal{Z})$ is a solution to (NLS), then $\mathfrak{P}(\Psi(t))$ does not depend on t .

Proof. For sake of completeness, we recall the proof of [11]. Let $\psi \in \mathcal{Z}$ and let us verify the Cauchy criterion. Since $|\psi| \rightarrow r_0 > 0$ at $\pm\infty$, we may lift $\psi = A_{\pm} e^{i\phi_{\pm}}$ in $(-\infty, -R_0)$ and in $(+R_0, +\infty)$ for some R_0 sufficiently

large. For $R' > R > R_0$, we thus have in $\mathbb{R}/(2\pi r_0^2\mathbb{Z})$

$$\begin{aligned}
& \left[\int_{-R'}^{+R'} \langle i\psi | \partial_x \psi \rangle dx - r_0^2 \left(\arg(\psi(+R')) - \arg(\psi(-R')) \right) \right] \\
& \quad - \left[\int_{-R}^{+R} \langle i\psi | \partial_x \psi \rangle dx - r_0^2 \left(\arg(\psi(+R)) - \arg(\psi(-R)) \right) \right] \\
& = \int_R^{R'} \langle i\psi | \partial_x \psi \rangle dx + \int_{-R'}^{-R} \langle i\psi | \partial_x \psi \rangle dx \\
& \quad - r_0^2 \left(\arg(\psi(R')) - \arg(\psi(R)) \right) + r_0^2 \left(\arg(\psi(-R')) - \arg(\psi(-R)) \right) \\
& = \int_R^{R'} A_+^2 \partial_x \phi_+ dx + \int_{-R'}^{-R} A_-^2 \partial_x \phi_- dx \\
& \quad - r_0^2 \left(\phi_+(R') - \phi_+(R) \right) + r_0^2 \left(\phi_-(-R') - \phi_-(-R) \right) \\
& = \int_R^{R'} (A_+^2 - r_0^2) \partial_x \phi_+ dx + \int_{-R'}^{-R} (A_-^2 - r_0^2) \partial_x \phi_- dx.
\end{aligned}$$

The absolute value of each term is $\leq K \int_{\pm x \geq \pm R} |\partial_x \psi|^2 + (|\psi| - r_0)^2 dx$ thus tends to zero if $R \rightarrow +\infty$. Thus, $\mathfrak{P}(\psi)$ is well-defined. The proof of the continuity follows the same lines, and allows to show that \mathfrak{P} is actually locally Lipschitz continuous. Let $\psi \in \mathcal{Z}$. If $\tilde{\psi} \in \mathcal{Z}$ and $d_{\mathcal{Z}}(\tilde{\psi}, \psi)$ is small enough, we have $\|\tilde{\psi} - \psi\|_{L^\infty}$ as small as we want. In particular, if $R_0 > 0$ is large enough so that $|\psi| \geq 3r_0/4$ for $|x| \geq R$, we have $|\tilde{\psi}| \geq r_0/2$ for $|x| \geq R_0$. As a consequence, writing $\psi = A_\pm e^{i\phi_\pm}$ and $\tilde{\psi} = \tilde{A}_\pm e^{i\tilde{\phi}_\pm}$ in $(-\infty, -R_0)$ and in $(+R_0, +\infty)$, we have, in $\mathbb{R}/(2\pi r_0^2\mathbb{Z})$ and for $R > R_0$,

$$\begin{aligned}
& \left[\int_{-R}^{+R} \langle i\psi | \partial_x \psi \rangle dx - r_0^2 \left(\arg(\psi(+R)) - \arg(\psi(-R)) \right) \right] \\
& \quad - \left[\int_{-R}^{+R} \langle i\tilde{\psi} | \partial_x \tilde{\psi} \rangle dx - r_0^2 \left(\arg(\tilde{\psi}(+R)) - \arg(\tilde{\psi}(-R)) \right) \right] \\
& = \int_{-R_0}^{+R_0} \langle i(\psi - \tilde{\psi}) | \partial_x \psi \rangle + \langle i\tilde{\psi} | \partial_x (\psi - \tilde{\psi}) \rangle dx \\
& \quad + \int_{R_0}^R A_+^2 \partial_x \phi_+ - \tilde{A}_+^2 \partial_x \tilde{\phi}_+ dx - r_0^2 \left(\phi_+(+R) - \tilde{\phi}_+(+R) \right) \\
& \quad + \int_{-R}^{-R_0} A_-^2 \partial_x \phi_- - \tilde{A}_-^2 \partial_x \tilde{\phi}_- dx + r_0^2 \left(\phi_-(-R) - \tilde{\phi}_-(-R) \right) \\
& = \int_{-R_0}^{+R_0} \langle i(\psi - \tilde{\psi}) | \partial_x \psi \rangle + \langle i\tilde{\psi} | \partial_x (\psi - \tilde{\psi}) \rangle dx \\
& \quad + r_0^2 \left(\phi_+(+R_0) - \tilde{\phi}_+(+R_0) \right) + r_0^2 \left(\phi_-(-R_0) - \tilde{\phi}_-(-R_0) \right) \\
& \quad + \int_{R_0}^R (A_+^2 - r_0^2) \partial_x \phi_+ - (\tilde{A}_+^2 - r_0^2) \partial_x \tilde{\phi}_+ dx + \int_{-R}^{-R_0} (A_-^2 - r_0^2) \partial_x \phi_- - (\tilde{A}_-^2 - r_0^2) \partial_x \tilde{\phi}_- dx.
\end{aligned}$$

We now estimate all the terms. For the last line, we use Cauchy-Schwarz to get $|\int_{R_0}^R (A_+^2 - r_0^2) \partial_x \phi_+ dx| \leq K(\psi) \|A_+ - r_0\|_{L^2(\mathbb{R})} \|A_+ \partial_x \phi_+\|_{L^2(\mathbb{R})} \leq K(\psi) d_{\mathcal{Z}}(\psi, \tilde{\psi})$, and similarly for the other terms. Moreover, using that $(\psi - \tilde{\psi})(x) = (\psi - \tilde{\psi})(0) + \int_0^x \partial_x (\psi - \tilde{\psi})$, we get by Cauchy-Schwarz $\|\psi - \tilde{\psi}\|_{C^0([-R_0, +R_0])} \leq |(\psi - \tilde{\psi})(0)| + \sqrt{R_0} \|\partial_x \psi - \partial_x \tilde{\psi}\|_{L^2(\mathbb{R})} \leq K(R_0) d_{\mathcal{Z}}(\psi, \tilde{\psi})$. Thus, the terms of the second line can be estimated by $K(\psi, R_0) d_{\mathcal{Z}}(\psi, \tilde{\psi})$, and for those of the first line, they can also be bounded by $\leq K(\psi, R_0) d_{\mathcal{Z}}(\psi, \tilde{\psi})$. Passing

to the limit as $R \rightarrow +\infty$ then gives

$$\left| \mathfrak{P}(\psi) - \mathfrak{P}(\tilde{\psi}) \pmod{2\pi r_0^2} \right| \leq K(\psi, R_0) d_{\mathcal{Z}}(\psi, \tilde{\psi}).$$

This completes the proof for the definition of \mathfrak{P} . To show that \mathfrak{P} is constant under the (NLS) flow, we use that $\Psi \in \Psi(0) + \mathcal{C}([0, T], H^1)$ and the approximation by smoother solutions (see Proposition 1 in [11]). \square

For the stability of the kink, we can no longer use the Grillakis-Shatah-Strauss theory applied to the hydrodynamical formulation of (NLS), since the kink vanishes at the origin. Therefore, it is natural to consider the Liapounov functional \mathcal{L} introduced in section 1.2.2, which becomes in the stationary case $c = 0$:

$$\mathcal{L}(\psi) = E(\psi) + \frac{M}{2} \left(P(\psi) - P(U_0) \right)^2.$$

Since the momentum P is not well-defined in \mathcal{Z} , we have to replace it by the untwisted momentum \mathfrak{P} , which is defined modulo $2\pi r_0^2$. Consequently, it is natural to define the functional in \mathcal{Z}

$$\mathcal{K}(\psi) \equiv E(\psi) + 2Mr_0^4 \sin^2 \left(\frac{\mathfrak{P}(\psi) - r_0^2 \pi}{2r_0^2} \right),$$

which is well-defined and continuous in \mathcal{Z} since \sin^2 is π -periodic. In addition, \mathcal{K} is conserved by the (NLS) flow as E and \mathfrak{P} .

Theorem 9 *Assume that there exists a kink, that is a nontrivial stationary ($c = 0$) wave U_0 which is odd. Assume also that*

$$\frac{dP(U_c)}{dc} \Big|_{c=0} < 0.$$

Then, there exists some small $\mu_ > 0$ such that U_0 is a local minimizer of \mathcal{K} . More precisely, denoting*

$$\mathcal{V}_{\mu_*} \equiv \{ \psi \in \mathcal{Z}, \inf_{\mathbb{R}} |\psi| < \mu_* \},$$

we have, for any $\psi \in \mathcal{V}_{\mu_} \setminus \{ e^{i\theta} U_0(\cdot - y), \theta \in \mathbb{R}, y \in \mathbb{R} \}$,*

$$\mathcal{K}(\psi) > \mathcal{K}(U_0) = E(U_0).$$

The crucial point in this result is to prove that the functional $\mathcal{K}(\psi)$ controls the infimum $\inf_{\mathbb{R}} |\psi|$. As we shall see in the proof (section 6.2), the key idea is to study the infimum of the functional \mathcal{K} at fixed $\inf_{\mathbb{R}} |\psi|$ (small), and then to prove (see Proposition 6.2 that for $\psi \in \mathcal{V}_{\mu_*}$, there holds, for some constant K depending only on f ,

$$\mathcal{K}(\psi) \geq \mathcal{K}(U_0) + \frac{(\inf_{\mathbb{R}} |\psi|)^2}{K}.$$

This will be achieved by a fine analysis of some minimizing sequences. From this locally minimizing property of the kink when $\frac{dP(U_c)}{dc} \Big|_{c=0} < 0$, we infer its orbital stability, provided we can prove some compactness on the minimizing sequences. A main step here is the control on $\inf_{\mathbb{R}} |\psi|$. Our method allows to infer a control on the distance of the solution to (NLS) to the orbit of the kink, but it is much weaker than those obtained by spectral methods in Lemma 1 or Corollary 1 for instance.

Theorem 10 *Assume that there exists a kink, that is an odd nontrivial stationary ($c = 0$) wave U_0 and that*

$$\frac{dP(U_c)}{dc} \Big|_{c=0} < 0.$$

Then, U_0 is orbitally stable in $(\mathcal{Z}, d_{\mathcal{Z}})$. Moreover, if $\Psi(t)$ is the (global) solution to (NLS) with initial datum Ψ^{in} , we have the control

$$\sup_{t \geq 0} \inf_{\substack{y \in \mathbb{R} \\ \theta \in \mathbb{R}}} d_{\mathcal{Z}}(\Psi(t), e^{i\theta} U_0(\cdot - y)) \leq K \sqrt[8]{\mathcal{K}(\Psi^{\text{in}}) - E(U_0)} \leq K \sqrt[4]{d_{\mathcal{Z}}(\Psi^{\text{in}}, U_0)}$$

provided that the right-hand side is sufficiently small.

This result settles the nonlinear stability under the condition $\frac{dP(U_c)}{dc} \Big|_{c=0} < 0$ for a general nonlinearity f . In particular, it may be applied to the nonlinearity f given in the example above. It shows that the stability of the kink holds with $\delta = \mathcal{O}(\varepsilon^4)$. We do not claim that the exponent $1/8$ is optimal.

For a complete study of the stability of the travelling waves, it would remain to investigate the case of the sonic ($c = c_s$) travelling waves (when they exist). The methods we have developed do not apply directly, and we give in section 7 some of the difficulties associated with this critical situation.

2 Decay at infinity (proof of Proposition 1)

For simplicity, we shall denote

$$\mathcal{V}(\xi) \equiv \mathcal{V}_{c_s}(\xi) = c_s^2 \xi^2 - 4(r_0^2 + \xi)F(r_0^2 + \xi).$$

We freeze the invariance by translation by imposing $|U_c|$ (hence also $\partial_x \phi$) even, so that we can use the formulas in [16]. In particular, it suffices to show the asymptotics for $x \rightarrow +\infty$: the case $x \rightarrow -\infty$ follows by symmetry. We start with the proof of case (iii). Under assumption (\mathcal{A}_m) and since $F' = -f$, we infer the Taylor expansion

$$\begin{aligned} \mathcal{V}(\xi) &= c_s^2 \xi^2 + 4(r_0^2 + \xi) \left(\frac{1}{2!} f'(r_0^2) \xi^2 + \dots + \frac{1}{(m+2)!} f^{(m+1)}(r_0^2) \xi^{m+2} + \frac{1}{(m+3)!} f^{(m+2)}(r_0^2) \xi^{m+3} + \mathcal{O}(\xi^{m+4}) \right) \\ &= \frac{4r_0^2}{(m+3)!} f^{(m+2)}(r_0^2) \xi^{m+3} + \frac{4}{(m+2)!} f^{(m+1)}(r_0^2) \xi^{m+3} + \mathcal{O}(\xi^{m+4}) \\ &= \frac{4}{r_0^{2(m+1)}} \left[\frac{r_0^{2(m+2)}}{(m+3)!} f^{(m+2)}(r_0^2) + (-1)^{m+2} \frac{c_s^2}{4} \right] \xi^{m+3} + \mathcal{O}(\xi^{m+4}) = \Lambda_m \xi^{m+3} + \mathcal{O}(\xi^{m+4}) \end{aligned}$$

since when (\mathcal{A}_m) holds, all the terms $\mathcal{O}(\xi^{m+2})$ cancel out. The coefficient Λ_m is not zero by assumption. Note that the existence of a nontrivial sonic wave, which depends on the global behaviour of \mathcal{V} , imposes that $\Lambda_m \xi^{m+3} < 0$ when ξ is small and has the sign of ξ_{c_s} . Therefore, from the formula (following from the Hamiltonian equation $2\partial_x^2 \eta_c + \mathcal{V}'(\eta_c) = 0$, see [16] for example)

$$x = -\text{sgn}(\xi_{c_s}) \int_{\xi_{c_s}}^{\eta_{c_s}(x)} \frac{d\xi}{\sqrt{-\mathcal{V}(\xi)}}$$

and since there holds, as $\eta \rightarrow 0$ (with the sign of ξ_{c_s}),

$$\begin{aligned} \int_{\xi_{c_s}}^{\eta} \frac{d\xi}{\sqrt{-\mathcal{V}(\xi)}} &= \int_{\xi_{c_s}}^{\eta} \frac{d\xi}{\sqrt{-\Lambda_m \xi^{m+3}}} + \int_{\xi_{c_s}}^{\eta} \frac{\mathcal{V}(\xi) - \Lambda_m \xi^{m+3}}{\sqrt{-\mathcal{V}(\xi)} \sqrt{-\Lambda_m \xi^{m+3}} \left[\sqrt{-\mathcal{V}(\xi)} + \sqrt{-\Lambda_m \xi^{m+3}} \right]} d\xi \\ &= -\frac{2 \text{sgn}(\xi_{c_s})}{m+1} \left(\frac{1}{\sqrt{-\Lambda_m \eta^{m+1}}} - \frac{1}{\sqrt{-\Lambda_m \xi_{c_s}^{m+1}}} \right) + \begin{cases} \mathcal{O}(1) & \text{if } m = 0 \\ \mathcal{O}(|\ln |\eta||) & \text{if } m = 1 \\ \mathcal{O}(\eta^{-\frac{m-1}{2}}) & \text{if } m \geq 2 \end{cases} \end{aligned}$$

(here, we use that the last integrand is $\mathcal{O}(\xi^{-(m+1)/2})$ as $\xi \rightarrow 0$), it follows that, as $x \rightarrow +\infty$,

$$\eta_{c_s}(x) = \operatorname{sgn}(\xi_{c_s}) \left(\frac{4}{(m+1)^2 |\Lambda_m|} \right)^{\frac{1}{m+1}} \frac{1}{x^{\frac{2}{m+1}}} + \begin{cases} \mathcal{O}\left(\frac{1}{x^3}\right) & \text{if } m = 0 \\ \mathcal{O}\left(\frac{\ln(x)}{x^2}\right) & \text{if } m = 1 \\ \mathcal{O}\left(\frac{1}{x^{\frac{4}{m+1}}}\right) & \text{if } m \geq 2. \end{cases}$$

This shows the asymptotics for the modulus, or η_{c_s} . The asymptotic expansion for $\partial_x \phi_{c_s}$ is easily deduced from the equation on the phase $2\partial_x \phi_{c_s} = c_s \eta_{c_s} / (r_0^2 + \eta_{c_s})$, and the phase ϕ_{c_s} is then computed by integration, which completes the proof of case (iii).

The proof of (ii) is easier. Indeed, in this case, the function \mathcal{V} has the expansion

$$\mathcal{V}(\xi) = c_s^2 \xi^2 - 4(r_0^2 + \xi)F(r_0^2 + \xi) = \mathcal{O}(\xi^3),$$

hence

$$\mathcal{V}_c(\xi) = \mathcal{V}(\xi) - (c_s^2 - c^2)\xi^2 = -(c_s^2 - c^2)\xi^2 + \mathcal{O}(\xi^3).$$

As a consequence, the result follows from the expansion, for $\eta \rightarrow 0$

$$\begin{aligned} \int_{\xi_c}^{\eta} \frac{d\xi}{\sqrt{-\mathcal{V}(\xi)}} &= \int_{\xi_c}^{\eta} \frac{d\xi}{\sqrt{(c_s^2 - c^2)\xi^2}} + \int_{\xi_c}^{\eta} \frac{\mathcal{V}(\xi)}{\sqrt{-\mathcal{V}_c(\xi)} \sqrt{(c_s^2 - c^2)\xi^2} \left[\sqrt{-\mathcal{V}_c(\xi)} + \sqrt{(c_s^2 - c^2)\xi^2} \right]} d\xi \\ &= \operatorname{sgn}(\xi_c) \frac{\ln(\eta/\xi_c)}{\sqrt{c_s^2 - c^2}} + \int_{\xi_c}^0 \frac{\mathcal{V}(\xi)}{\sqrt{-\mathcal{V}_c(\xi)} \sqrt{(c_s^2 - c^2)\xi^2} \left[\sqrt{-\mathcal{V}_c(\xi)} + \sqrt{(c_s^2 - c^2)\xi^2} \right]} d\xi + \mathcal{O}(\eta) \end{aligned}$$

since the integrand for the last integral is continuous at $\xi = 0$. This yields the desired expansion for the modulus:

$$\begin{aligned} \eta_c(x) &= \xi_c \exp \left(-x \sqrt{c_s^2 - c^2} - \int_{\xi_c}^0 \frac{\mathcal{V}(\xi)}{\sqrt{-\xi^2 \mathcal{V}_c(\xi)} \left[\sqrt{-\mathcal{V}_c(\xi)} + \sqrt{(c_s^2 - c^2)\xi^2} \right]} d\xi \right) + \mathcal{O} \left[\exp \left(-2x \sqrt{c_s^2 - c^2} \right) \right] \\ &= M_c \exp \left(-x \sqrt{c_s^2 - c^2} \right) + \mathcal{O} \left[\exp \left(-2x \sqrt{c_s^2 - c^2} \right) \right], \end{aligned}$$

with

$$M_c \equiv \xi_c \exp \left(- \int_{\xi_c}^0 \frac{\mathcal{V}(\xi)}{\sqrt{-\xi^2 \mathcal{V}_c(\xi)} \left[\sqrt{-\mathcal{V}_c(\xi)} + \sqrt{(c_s^2 - c^2)\xi^2} \right]} d\xi \right) \neq 0,$$

and hence for the phase by similar computations to those above. The proof of case (i) is similar, separating the case $\xi_0 = -r_0^2$ of the kink (even solution) from the case $\xi_0 \neq -r_0^2$ of the bubble (odd solution) and is omitted. \square

3 Stability results deduced from the hydrodynamical formulation of (NLS)

3.1 Proof of Lemma 2

(i) **The mapping \mathcal{M} is an homeomorphism.** Let $\psi = Ae^{i\phi}$, $(\psi_n = A_n e^{i\phi_n})_n \in \mathcal{Z}$ such that $\psi_n \rightarrow \psi$ for d_{hy} . Then, $A_n - A \rightarrow 0$ in H^1 , $\partial_x \phi_n \rightarrow \partial_x \phi$ in L^2 and we may assume (possibly adding some multiple of 2π

to ϕ_n , that $\phi_n(0) \rightarrow \phi(0)$. We write, using the embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ for the before last line,

$$\begin{aligned}
d_{\mathcal{Z}}(\psi_n, \psi) &= \|\partial_x \psi_n - \partial_x \psi\|_{L^2} + \||\psi_n| - |\psi|\|_{L^2} + |\psi_n(0) - \psi(0)| \\
&= \|\mathbf{e}^{i\phi_n} \partial_x A_n + iA_n \mathbf{e}^{i\phi_n} \partial_x \phi_n - \mathbf{e}^{i\phi} \partial_x A - iA \mathbf{e}^{i\phi} \partial_x \phi\|_{L^2} + \|A_n - A\|_{L^2} + |A_n(0) \mathbf{e}^{i\phi_n(0)} - A(0) \mathbf{e}^{i\phi(0)}| \\
&\leq \|(\mathbf{e}^{i\phi_n} - \mathbf{e}^{i\phi}) \partial_x A\|_{L^2} + \|\mathbf{e}^{i\phi} (\partial_x A_n - \partial_x A)\|_{L^2} + \|(A_n - A) \mathbf{e}^{i\phi_n} \partial_x \phi_n\|_{L^2} + \|A(\mathbf{e}^{i\phi_n} - \mathbf{e}^{i\phi}) \partial_x \phi\|_{L^2} \\
&\quad + \|A \mathbf{e}^{i\phi_n} (\partial_x \phi - \partial_x \phi_n)\|_{L^2} + \|A_n - A\|_{L^2} + |(A_n(0) - A(0)) \mathbf{e}^{i\phi_n(0)}| + |A(0) (\mathbf{e}^{i\phi_n(0)} - \mathbf{e}^{i\phi(0)})| \\
&\leq \|(\mathbf{e}^{i\phi_n} - \mathbf{e}^{i\phi}) \partial_x A\|_{L^2} + K \|A_n - A\|_{H^1} + \|(A_n - A)\|_{H^1} \|\partial_x \phi_n\|_{L^2} + \|A\|_{L^\infty} \|(\mathbf{e}^{i\phi_n} - \mathbf{e}^{i\phi}) \partial_x \phi\|_{L^2} \\
&\quad + \|A\|_{L^\infty} \|\partial_x \phi - \partial_x \phi_n\|_{L^2} + \|A\|_{L^\infty} |\mathbf{e}^{i\phi_n(0)} - \mathbf{e}^{i\phi(0)}| \\
&= \|(\mathbf{e}^{i\phi_n} - \mathbf{e}^{i\phi}) \partial_x A\|_{L^2} + \|A\|_{L^\infty} \|(\mathbf{e}^{i\phi_n} - \mathbf{e}^{i\phi}) \partial_x \phi\|_{L^2} + o_{n \rightarrow +\infty}(1),
\end{aligned} \tag{11}$$

from the convergences we have. Now observe that $\phi_n(x) = \phi_n(0) + \int_0^x \partial_x \phi_n(t) dt \rightarrow \phi(0) + \int_0^x \partial_x \phi(t) dt = \phi(0)$ pointwise, hence, by the Dominated Convergence Theorem, $\|(\mathbf{e}^{i\phi_n} - \mathbf{e}^{i\phi}) \partial_x A\|_{L^2} \rightarrow 0$ and similarly for the other term. Therefore, $d_{\mathcal{Z}}(\psi_n, \psi) \rightarrow 0$ as wished.

Let now $\psi = A \mathbf{e}^{i\phi}$, $(\psi_n = A_n \mathbf{e}^{i\phi_n})_n \in \mathcal{Z}$ such that $\psi_n \rightarrow \psi$ for $d_{\mathcal{Z}}$. Then, $A_n - A = |\psi_n| - |\psi| \rightarrow 0$ in L^2 , $\partial_x \psi_n \rightarrow \partial_x \psi$ in L^2 and $\psi_n(0) \rightarrow \psi(0)$. Since $|\cdot|$ is 1-Lipschitz continuous, we infer for the modulus

$$\|\partial_x A_n - \partial_x A\|_{L^2} = \|\partial_x |\psi_n| - \partial_x |\psi|\|_{L^2} \leq \|\partial_x \psi_n - \partial_x \psi\|_{L^2}.$$

Moreover, $\psi_n(0) \rightarrow \psi(0)$ and this implies $\arg(\psi_n(0)/\psi(0)) \rightarrow 0$. Therefore, it suffices to show that $\partial_x \phi_n \rightarrow \partial_x \phi$ in L^2 . We use the formula $A^2 \partial_x \phi = \langle i\psi | \partial_x \psi \rangle$, which yields

$$\partial_x \phi_n - \partial_x \phi = \frac{\langle i\psi_n | \partial_x \psi_n \rangle}{A_n^2} - \frac{\langle i\psi | \partial_x \psi \rangle}{A^2} = \langle i\psi_n | \partial_x \psi_n \rangle \left(\frac{1}{A_n^2} - \frac{1}{A^2} \right) + \frac{\langle i\psi_n | \partial_x (\psi - \psi_n) \rangle}{A^2} - \frac{\langle i(\psi - \psi_n) | \partial_x \psi \rangle}{A^2},$$

hence

$$\|\partial_x \phi_n - \partial_x \phi\|_{L^2} \leq \frac{\|\psi_n\|_{L^\infty} \|A\|_{L^\infty}}{(\inf_{\mathbb{R}} A^2)(\inf_{\mathbb{R}} A_n^2)} \|A_n - A\|_{L^2} + \frac{\|\psi_n\|_{L^\infty}}{\inf_{\mathbb{R}} A^2} \|\partial_x \psi - \partial_x \psi_n\|_{L^2} + \frac{1}{\inf_{\mathbb{R}} A^2} \|\psi_n - \psi\|_{L^2}. \tag{12}$$

The first two terms tend to zero as $n \rightarrow +\infty$. For the last term, we use here again the Dominated Convergence Theorem since $\psi_n(x) = \psi_n(0) + \int_0^x \partial_x \psi_n(t) dt \rightarrow \psi(0) + \int_0^x \partial_x \psi(t) dt = \psi(0)$ pointwise. This concludes for (i).

Proof of (ii). Let us define $\phi_* : \mathbb{R} \rightarrow \mathbb{R}$ by $\phi_*(x) \equiv \frac{1}{2}(\ln x)^2 \mathbf{1}_{x \geq 1}$. Then, straightforward computations give $\partial_x \phi_*(x) = \frac{\ln x}{x} \mathbf{1}_{x \geq 1} \in L^2(\mathbb{R})$ and, for $X \geq e$, by monotonicity of $\partial_x \phi_*$,

$$\int_X^{2X} (\partial_x \phi_*)^2 dx \geq X \frac{\ln^2(2X)}{(2X)^2} \geq \frac{(\ln X)^2}{4X}. \tag{13}$$

We now consider $\phi_n : \mathbb{R} \rightarrow \mathbb{R}$ defined by $\phi_n(x) = 0$ if $x \leq 0$ or $x \geq 3n\pi$, $\phi_n(x) = x/n$ if $0 \leq x \leq n\pi$, $\phi_n(x) = \pi$ if $n\pi \leq x \leq 2n\pi$ and $\phi_n(x) = 3\pi - x/n$ if $2n\pi \leq x \leq 3n\pi$. Then, we easily obtain

$$d_{\text{hy}}(\mathbf{e}^{i\phi_*}, \mathbf{e}^{i\phi_* + i\phi_n}) = \|\partial_x \phi_n\|_{L^2} = \sqrt{2 \times \frac{\pi n}{n^2}} = \sqrt{\frac{2\pi}{n}} \rightarrow 0.$$

Moreover,

$$d_{\mathcal{Z}}(\mathbf{e}^{i\phi_*}, \mathbf{e}^{i\phi_* + i\phi_n}) = \|\partial_x \phi_* \mathbf{e}^{i\phi_*} - (\partial_x \phi_* + \partial_x \phi_n) \mathbf{e}^{i\phi_* + i\phi_n}\|_{L^2} \geq \|\partial_x \phi_* (\mathbf{e}^{i\phi_n} - 1)\|_{L^2} - \|\partial_x \phi_n\|_{L^2},$$

and, by our choice of ϕ_n and using (13),

$$\|\partial_x \phi_* (\mathbf{e}^{i\phi_n} - 1)\|_{L^2}^2 \geq \int_{n\pi}^{2n\pi} 4(\partial_x \phi_*)^2 dx \geq \frac{(\ln X)^2}{X} \Big|_{X=n\pi} \sim \frac{(\ln n)^2}{n\pi}.$$

Since $\frac{\ln n}{\sqrt{n\pi}} \gg \sqrt{\frac{2\pi}{n}} = d_{\text{hy}}(\mathbf{e}^{i\phi_*}, \mathbf{e}^{i\phi_* + i\phi_n})$, it follows that, as wished

$$d_{\mathcal{Z}}(\mathbf{e}^{i\phi_*}, \mathbf{e}^{i\phi_* + i\phi_n}) \geq \frac{\ln n}{\sqrt{n\pi}}(1 + o(1)) \gg \sqrt{\frac{2\pi}{n}} = d_{\text{hy}}(\mathbf{e}^{i\phi_*}, \mathbf{e}^{i\phi_* + i\phi_n}).$$

We do not know whether the mapping \mathcal{M} is locally Lipschitz, but is probably not.

3.2 Proof of Lemma 3

Note first that since U_* does not vanish, if δ is sufficiently small and $d_{\mathcal{Z}}(\psi, U_*) \leq \delta$, then $\|\psi\| - \|U_*\|_{L^\infty} \leq (1/2)\inf_{\mathbb{R}} |U_*|$, hence $|\psi| \geq (1/2)\inf_{\mathbb{R}} |U_*| > 0$ in \mathbb{R} , thus ψ does not vanish, may be lifted $\psi = A \exp(i\phi)$, and we may further assume $\phi(0) - \phi_*(0) \in (-\pi, +\pi]$. In (11), we can easily check that the terms leading to the "o(1)" are indeed controlled by $K(\psi)d_{\text{hy}}(\psi_n, \psi)$. In other words, we have

$$d_{\mathcal{Z}}(\psi, U_*) \leq \|(\mathbf{e}^{i\phi} - \mathbf{e}^{i\phi_*})\partial_x A_*\|_{L^2} + \|A_*\|_{L^\infty} \|(\mathbf{e}^{i\phi} - \mathbf{e}^{i\phi_*})\partial_x \phi_*\|_{L^2} + K(U_*)d_{\text{hy}}(\psi, U_*),$$

provided $d_{\text{hy}}(\psi, U_*)$ is small enough. In order to bound the two remaining terms, we write, for $x \in \mathbb{R}$,

$$\phi(x) - \phi_*(x) = \phi(0) - \phi_*(0) + \int_0^x \partial_x \phi(y) - \partial_x \phi_*(y) dy,$$

which implies, using that $\mathbb{R} \ni \theta \mapsto \mathbf{e}^{i\theta}$ is 1-Lipschitz and the Cauchy-Schwarz inequality,

$$\left| 1 - \mathbf{e}^{i(\phi_*(x) - \phi(x))} \right| \leq \left| \phi(0) - \phi_*(0) \right| + \sqrt{|x|} \|u - u_*\|_{L^2}. \quad (14)$$

Consequently,

$$\|(\mathbf{e}^{i\phi} - \mathbf{e}^{i\phi_*})\partial_x A_*\|_{L^2} \leq \left| \phi(0) - \phi_*(0) \right| \|\partial_x A_*\|_{L^2} + \|u - u_*\|_{L^2} \|\sqrt{|x|}\partial_x A_*\|_{L^2}$$

and

$$\|(\mathbf{e}^{i\phi} - \mathbf{e}^{i\phi_*})\partial_x \phi_*\|_{L^2} \leq \left| \phi(0) - \phi_*(0) \right| \|\partial_x \phi_*\|_{L^2} + \|u - u_*\|_{L^2} \|\sqrt{|x|}\partial_x \phi_*\|_{L^2}.$$

Both terms are $\leq K(U_*)d_{\text{hy}}(\psi, U_*)$. Indeed, $U_* \in \mathcal{Z}$ is a travelling wave, hence A_* , $\partial_x A_*$, $\partial_x \phi_*$ are bounded functions which decay at infinity exponentially if $0 \leq c < \mathbf{c}_s$ (cf. Proposition 1 (i) or (ii)). If $c = \mathbf{c}_s$, since assumption (\mathcal{A}_0) is satisfied, we invoke Proposition 1 (iii), which ensures that $\partial_x \phi_*$ and $\partial_x A_*$ decay at the rate $\mathcal{O}(|x|^{-2})$ ($\partial_x A_*$ decays faster actually). Therefore, $\sqrt{|x|}\partial_x \phi_* \in L^2$. Gathering these estimates provides

$$d_{\mathcal{Z}}(\psi, U_*) \leq K(U_*)d_{\text{hy}}(\psi, U_*).$$

On the other hand, from (12) and the estimate $\|A - A_*\|_{H^1} \leq d_{\mathcal{Z}}(\psi, U_*)$ (see the proof of (i)), we infer

$$d_{\text{hy}}(\psi, U_*) \leq K(U_*)d_{\mathcal{Z}}(\psi, U_*) + \frac{1}{\inf_{\mathbb{R}} A^2} \|\psi - U_*\|_{L^2} \|\partial_x U_*\|_{L^2}.$$

Using here again the estimate $|\psi(x) - U_*(x)| \leq |\phi(0) - \phi_*(0)| + \sqrt{|x|} \|\partial_x \psi - U_*\|_{L^2}$, we deduce

$$d_{\text{hy}}(\psi, U_*) \leq K(U_*)d_{\mathcal{Z}}(\psi, U_*).$$

The proof is complete. □

3.3 Two remarks on the proof of Theorem 2

We would like to point out two minor points concerning the proof of Theorem 2 by Z. Lin. We recall that the proof of Z. Lin [47] relies on the Grillakis-Shatah-Strauss theory [35] once we have written (NLS) under the hydrodynamical form (2), denoting $\psi = Ae^{i\phi}$, $(\rho, u) \equiv (|\psi|^2 = A^2, \partial_x \phi)$:

$$\begin{cases} \partial_t \rho + 2\partial_x(\rho u) = 0 \\ \partial_t u + 2u\partial_x u - \partial_x(f(\rho)) - \partial_x\left(\frac{\partial_x^2(\sqrt{\rho})}{\sqrt{\rho}}\right) = 0, \end{cases}$$

or more precisely, with $\eta \equiv \rho - r_0^2 = |\psi|^2 - r_0^2$ and denoting $\frac{\delta E}{\delta \eta}$, $\frac{\delta E}{\delta u}$ the variational derivative,

$$\frac{\partial}{\partial t} \begin{pmatrix} \eta \\ u \end{pmatrix} = J \begin{pmatrix} \frac{\delta E}{\delta \eta} \\ \frac{\delta E}{\delta u} \end{pmatrix}, \quad J \equiv \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}. \quad (15)$$

The first remark is that the scalar product in the Hilbert space $X = H^1 \times L^2$ can not be $((\eta, u), (\tilde{\eta}, \tilde{u}))_{H^1 \times L^2} = \int_{\mathbb{R}} \eta \tilde{\eta} + u \tilde{u} dx$ as used in [47], but the natural one is $((\eta, u), (\tilde{\eta}, \tilde{u}))_{H^1 \times L^2} = \int_{\mathbb{R}} \eta \tilde{\eta} + \partial_x \eta \partial_x \tilde{\eta} + u \tilde{u} dx$. This requires to make some minor changes in the proof, especially not to identify $(H^1)^*$ with H^1 . For instance, a linear mapping B is associated with the momentum through the formula

$$P_{\text{hy}}(\eta, u) \equiv \int_{\mathbb{R}} \eta u dx = \frac{1}{2} (B(\eta, u), (\eta, u))_{H^1 \times L^2} \quad \text{with} \quad B \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

for the (non hilbertian) scalar product $((\eta, u), (\tilde{\eta}, \tilde{u}))_{H^1 \times L^2} = \int_{\mathbb{R}} \eta \tilde{\eta} + u \tilde{u} dx$. The correct definition is actually

$$P_{\text{hy}}(\eta, u) = \int_{\mathbb{R}} \eta u dx = \frac{1}{2} \langle B(\eta, u), (\eta, u) \rangle_{X^*, X} \quad \text{with} \quad B \equiv \begin{pmatrix} 0 & \iota^* \\ \iota & 0 \end{pmatrix},$$

where $\iota : H^1 \hookrightarrow L^2$ is the canonical injection. As already mentioned in 1.2, the two points in the proof of [47] that have been completed by C. Gallo in [24] are that: [47] uses a local in time existence for the hydrodynamical system (15) in $H^1 \times L^2$, and not only in $\{\rho \in L^\infty, \partial_x \rho \in L^2\} \times L^2$; and that the energy and the momentum are indeed conserved for the local solution if the initial datum does not vanish.

The second point is that in the proof of stability (theorem 3.5 in [35]), it is made use of the fact that if $\mathcal{U} \in X$ and $(\mathcal{U}_n)_{n \in \mathbb{N}} \in X$ is a sequence such that $E(\mathcal{U}_n) \rightarrow E(\mathcal{U})$ and $P_{\text{hy}}(\mathcal{U}_n) \rightarrow P_{\text{hy}}(\mathcal{U})$, then there exists a sequence $(\tilde{\mathcal{U}}_n)_{n \in \mathbb{N}} \in X$ such that $\mathcal{U}_n - \tilde{\mathcal{U}}_n \rightarrow 0$ in X , $E(\tilde{\mathcal{U}}_n) \rightarrow E(\mathcal{U})$ and $P_{\text{hy}}(\tilde{\mathcal{U}}_n) = P_{\text{hy}}(\mathcal{U})$. In the context of bound states, the existence of such a sequence $(\tilde{\mathcal{U}}_n)_{n \in \mathbb{N}} \in X$ follows by simple scaling in space, since then the momentum or charge is simply $\int_{\mathbb{R}^d} \mathcal{U}_n^2 dx$. However, for the one dimensional travelling waves for (NLS), the momentum P is scaling invariant. We do not know if the existence of such a sequence holds in a general framework, but for the problem we are studying, we can rely on the following lemma, which is an adaptation of Lemma 6 in [10] (see also Lemma in [11]).

Lemma 3.1 *There exists $\mathfrak{p}_0 > 0$ and $K > 0$, depending only on f such that for any $\mathfrak{p} \in (-\mathfrak{p}_0, +\mathfrak{p}_0)$ and $\mu \in \mathbb{R}$ with $|\mu| \leq |\mathfrak{p}|$, there exists $w = ae^{i\varphi} \in H^1([0, 1/(2|\mathfrak{p}|)], \mathbb{C})$ verifying*

$$w(0) = w\left(\frac{1}{2|\mathfrak{p}|}\right), \quad |w(0)| = r_0 + \mu, \quad \int_0^{1/(2|\mathfrak{p}|)} (a^2 - r_0^2) \partial_x \varphi dx = \mathfrak{p}$$

and

$$\int_0^{1/(2|\mathfrak{p}|)} |\partial_x w|^2 + F(|w|^2) dx \leq K|\mathfrak{p}|$$

Proof. If $\mathfrak{p} = 0$, we simply take $w = r_0$. We then assume $0 < \mathfrak{p} < \mathfrak{p}_0$, since the case $-\mathfrak{p}_0 < \mathfrak{p} < 0$ will follow by complex conjugation. We then define, for some small δ to be determined later,

$$w(x) \equiv \sqrt{r_0^2 - \delta + 2\mathfrak{p}\left(1 - |8\mathfrak{p}x - 1|\right)_+} \exp \left[i\left(1 - |4\mathfrak{p}x - 1|\right)_+ \right] = ae^{i\varphi}.$$

It is clear that $w \in H^1([0, 1/(2\mathfrak{p})], \mathbb{C})$ and that $w(0) = w(1/(2\mathfrak{p})) = \sqrt{r_0^2 - \delta}$, thus $|w(0)| = r_0 + \mu$ provided we choose $\delta = -\mu^2 - 2r_0\mu = \mathcal{O}(|\mu|)$. Moreover, using that the phase φ has compact support $[0, 1/(2\mathfrak{p})]$,

$$\begin{aligned} \int_0^{1/(2\mathfrak{p})} (a^2 - r_0^2) \partial_x \varphi \, dx &= \int_0^{1/(2\mathfrak{p})} \left\{ -\delta + 2\mathfrak{p}\left(1 - |8\mathfrak{p}x - 1|\right)_+ \right\} \partial_x \left(1 - |4\mathfrak{p}x - 1|\right)_+ \, dx \\ &= 2\mathfrak{p} \int_0^{1/(2\mathfrak{p})} \left(1 - |8\mathfrak{p}x - 1|\right)_+ \partial_x \left(1 - |4\mathfrak{p}x - 1|\right)_+ \, dx \\ &= 2\mathfrak{p} \int_0^{1/(2\mathfrak{p})} \left(1 - |8\mathfrak{p}x - 1|\right)_+ \partial_x \left(1 - |4\mathfrak{p}x - 1|\right)_+ \, dx. \end{aligned}$$

For the last integral, the first factor is equal to 0 if $x \geq 1/(4\mathfrak{p})$ and the second factor is equal to $4\mathfrak{p}$ when $x \leq 1/(4\mathfrak{p})$. Hence, direct computation gives

$$\int_0^{1/(2\mathfrak{p})} (a^2 - r_0^2) \partial_x \varphi \, dx = 2\mathfrak{p} \int_0^{1/(4\mathfrak{p})} \left(1 - |8\mathfrak{p}x - 1|\right)_+ \times 4\mathfrak{p} \, dx = \mathfrak{p}.$$

For the energy part, notice first that

$$|a^2 - r_0^2| = \left| -\delta + 2\mathfrak{p}\left(1 - |8\mathfrak{p}x - 1|\right)_+ \right| \leq |\delta| + 2\mathfrak{p}_0$$

is as small as we want if $|\delta|$ and \mathfrak{p}_0 are chosen sufficiently small. Therefore,

$$F(|w|^2) \leq K(a^2 - r_0^2)^2.$$

By simple computations, we have

$$\begin{aligned} &\int_0^{1/(2\mathfrak{p})} |\partial_x w|^2 + F(|w|^2) \, dx \\ &\leq K \int_0^{1/(2\mathfrak{p})} \mathfrak{p}^2 \left| \partial_x \left(1 - |8\mathfrak{p}x - 1|\right)_+ \right|^2 + \left| \partial_x \left(1 - |4\mathfrak{p}x - 1|\right)_+ \right|^2 + \left(-\delta + 2\mathfrak{p}\left(1 - |8\mathfrak{p}x - 1|\right)_+ \right)^2 \, dx \\ &\leq K\mathfrak{p}^3 + K\mathfrak{p} + K \frac{\delta^2 + \mathfrak{p}^2}{\mathfrak{p}} \leq K\mathfrak{p} \end{aligned}$$

since $\delta = \mathcal{O}(|\mu|) = \mathcal{O}(\mathfrak{p})$, which concludes the proof. \square

We then consider a sequence $\mathcal{U}_n = (\eta_n, u_n) \in X = H^1 \times L^2$ and show the existence of the desired sequence $\tilde{\mathcal{U}}_n = (\tilde{\eta}_n, \tilde{u}_n) \in X$. We recall that \mathcal{U}_n (resp. \mathcal{U}) is associated with a mapping $\psi_n \in \mathcal{Z}$ (resp. U_*) that does not vanish. We have $P_{\text{hy}}(\mathcal{U}_n) = P(\psi_n) \rightarrow P(U_*)$, thus for n large enough, $|P(\psi_n) - P(U_*)| \leq \mathfrak{p}_0$. For n fixed, we now pick $R_n > 0$ large enough so that

$$\int_{R_n}^{+\infty} |\partial_x \psi_n|^2 + (|\psi_n| - r_0)^2 \, dx \leq [P(\psi_n) - P(U_*)]^2.$$

In particular, by Sobolev embedding,

$$\left| |\psi_n|(R_n) - r_0 \right| \leq \| |\psi_n| - r_0 \|_{L^\infty([R_n, +\infty))} \leq \sqrt{\int_{R_n}^{+\infty} |\partial_x \psi_n|^2 + (|\psi_n| - r_0)^2 \, dx} \leq |P(\psi_n) - P(U_*)|.$$

We are now in position to apply (for n large) Lemma 3.1 with $(\mathbf{p}, \mu) = (P(U_*) - P(\psi_n), |\psi_n|(R_n) - r_0)$. This provides the mapping $w_n \in H^1([0, 1/(2|\mathbf{p}|)], \mathbb{C})$. Since $|\psi_n|(R_n) - r_0 \rightarrow 0$, for n large enough, there exists $\theta_n \in \mathbb{R}$ such that $\psi_n(R_n) = e^{i\theta_n} |\psi_n|(R_n) = e^{i\theta_n} (r_0 + \mu) = e^{i\theta_n} w_n(0)$. We then consider the mapping $\tilde{\psi}_n \in \mathcal{Z}$ defined by

$$\tilde{\psi}_n(x) \equiv \begin{cases} \psi_n(x) & \text{if } x \leq R_n \\ e^{i\theta_n} w_n(x - R_n) & \text{if } R_n \leq x \leq R_n + \frac{1}{2|P(\psi_n) - P(U_*)|} \\ \psi_n\left(x - \frac{1}{2|P(\psi_n) - P(U_*)|}\right) & \text{if } x \geq R_n + \frac{1}{2|P(\psi_n) - P(U_*)|}. \end{cases}$$

From the construction of w_n and the phase factor θ_n , $\tilde{\psi}_n$ is well-defined and continuous. It is clear that

$$P(\tilde{\psi}_n) = P(\psi_n) + \int_0^{1/(2\mathbf{p})} (a_n^2 - r_0^2) \partial_x \varphi_n \, dx = P(\psi_n) + \mathbf{p} = P(U_*)$$

for every (large) n , and that

$$E(\tilde{\psi}_n) = E(\psi_n) + \int_0^{1/(2\mathbf{p})} |\partial_x w|^2 + F(|w|^2) \, dx = E(U_*) + o(1) + \mathcal{O}(|\mathbf{p}|) = E(U_*) + o(1) + \mathcal{O}(|P(U_*) - P(\psi_n)|)$$

converges to $P(U_*)$ as $n \rightarrow +\infty$. Denoting $\tilde{\mathcal{U}}_n \in X$ the hydrodynamical expression of $\tilde{\psi}_n$, it remains to show that $\mathcal{U}_n - \tilde{\mathcal{U}}_n \rightarrow 0$ in $X = H^1 \times L^2$. We thus compute, with the definition of $\tilde{\psi}_n$,

$$\begin{aligned} \|\mathcal{U}_n - \tilde{\mathcal{U}}_n\|_X^2 &= \int_{R_n}^{+\infty} \left| \partial_x |\psi_n| - \partial_x |\tilde{\psi}_n| \right|^2 + (|\psi_n| - |\tilde{\psi}_n|)^2 + (u_n - \tilde{u}_n)^2 \, dx \\ &\leq 2 \int_{R_n}^{+\infty} |\partial_x |\psi_n||^2 + |\partial_x |\tilde{\psi}_n||^2 + (|\psi_n| - r_0)^2 + (|\tilde{\psi}_n| - r_0)^2 + u_n^2 + \tilde{u}_n^2 \, dx \\ &\leq 4K \int_{R_n}^{+\infty} |\partial_x \psi_n|^2 + (|\psi_n| - r_0)^2 \, dx + 2 \int_0^{1/(2|P(U_*) - P(\psi_n)|)} |\partial_x w_n|^2 + (|w_n|^2 - r_0^2)^2 \, dx \\ &\leq 4K[P(\psi_n) - P(U_*)]^2 + K|P(\psi_n) - P(U_*)| \rightarrow 0. \end{aligned}$$

For the before last inequality, we have used that for $|x| \geq R_n$, ψ_n has modulus uniformly close to r_0 , hence $|\partial_x |\psi_n||^2 + u_n^2 \leq K|\partial_x \psi_n|^2$. Note that the construction still holds for the energy distance, the computations being similar.

3.4 Proof of Lemma 1

Proof of estimate (3). Instead of concluding the stability proof as in [35], we can notice that we have actually the bound

$$E_{\text{hy}}(\mathcal{U}) - E_{\text{hy}}(\mathcal{U}_{c_*}) \geq \frac{1}{K} \inf_{y \in \mathbb{R}} \|\mathcal{U} - \mathcal{U}_{c_*}(\cdot - y)\|^2 \quad (16)$$

as soon as $P(\mathcal{U}_{c_*}) = P_{\text{hy}}(\mathcal{U}_{c_*}) = P_{\text{hy}}(\mathcal{U})$ and $\mathcal{U} \in \mathcal{O}_\varepsilon \equiv \left\{ \mathcal{V} \in X, \inf_{y \in \mathbb{R}} \|\mathcal{V} - \mathcal{U}_{c_*}(\cdot - y)\|_X < \varepsilon \right\}$ for some small ε . If Ψ^{in} does not have momentum equal to $P_{\text{hy}}(\mathcal{U}_{c_*})$, we use Lemma 3.1 to infer that there exists $\tilde{\Psi}(t)$, with momentum equal to $P_{\text{hy}}(\mathcal{U}_{c_*}) = P(\mathcal{U}_{c_*})$, and such that $E(\tilde{\Psi}(t)) - E(\Psi(t)) = \mathcal{O}(|P(\Psi(t)) - P(\mathcal{U}_{c_*})|)$ and $d_{\text{hy}}(\Psi(t), \tilde{\Psi}(t)) \leq \mathcal{O}(\sqrt{|P(\Psi(t)) - P(\mathcal{U}_{c_*})|})$. Therefore, for $t \geq 0$, denoting $\Psi_{\text{hy}}(t) \in X$ and $\tilde{\Psi}_{\text{hy}}(t) \in X$

the hydrodynamical variables for Ψ and $\tilde{\Psi}(t)$,

$$\begin{aligned} \inf_{y \in \mathbb{R}} \|\Psi_{\text{hy}}(t) - \mathcal{U}_{c_*}(\cdot - y)\| &\leq \inf_{y \in \mathbb{R}} \left[\|\tilde{\Psi}_{\text{hy}}(t) - \mathcal{U}_{c_*}(\cdot - y)\| + \|\Psi_{\text{hy}}(t) - \tilde{\Psi}_{\text{hy}}(t)\| \right] \\ &\leq \sqrt{K} \sqrt{E(\tilde{\Psi}(t)) - E(U_{c_*})} + \mathcal{O}(\sqrt{|P(\Psi(t)) - P(U_{c_*})|}) \\ &\leq K \left[\sqrt{|E(\Psi(t)) - E(U_{c_*})|} + |P(\Psi^{\text{in}}) - P(U_{c_*})| + \sqrt{|P(\Psi^{\text{in}}) - P(U_{c_*})|} \right], \end{aligned}$$

which yields (3).

The above estimate is optimal when $P(\Psi^{\text{in}}) = P(U_{c_*})$ since U_{c_*} is a critical point of the action $E - c_*P$. This bound shows that, in the definition of stability, one has to take $\delta = \mathcal{O}(\varepsilon^2)$ in general. The estimate (3) shows that one can actually take $\delta = \mathcal{O}(\varepsilon)$.

Proof of estimate (4). The point is to compare $\Psi(t)$ to U_c with $c \simeq c_*$ such that $P(U_c) = P(\Psi^{\text{in}})$ instead of comparing to U_{c_*} . In other words, we replace $\tilde{\Psi}(t)$ by U_c . Note first that since $\frac{dP}{dc}|_{c=c_*} < 0$, there exists, by the implicit function theorem, such a $c \simeq c_*$. We then proceed as follows. Let $\Psi^{\text{in}} \in \mathcal{Z}$ be close to U_{c_*} . Then, there exists $c = c(\Psi^{\text{in}}) \simeq c_*$ such that $P(U_c) = P(\Psi^{\text{in}})$. Moreover, since $\frac{dP}{dc}|_{c=c_*} \neq 0$, it follows

$$\|\mathcal{U}_c - \mathcal{U}_{c_*}\| \leq K|c - c_*| \leq K|P(U_c) - P(U_{c_*})| = K|P(\Psi^{\text{in}}) - P(U_{c_*})| \leq K\|\Psi_{\text{hy}}^{\text{in}} - \mathcal{U}_{c_*}\| \leq Kd_{\text{hy}}(\Psi^{\text{in}}, U_{c_*}). \quad (17)$$

From (16), it comes

$$E_{\text{hy}}(\mathcal{U}) - E_{\text{hy}}(\mathcal{U}_c) \geq \frac{1}{K} \inf_{y \in \mathbb{R}} \|\mathcal{U} - \mathcal{U}_c(\cdot - y)\|^2$$

as soon as $P_{\text{hy}}(\mathcal{U}) = P_{\text{hy}}(\mathcal{U}_c)$. The fact that the constant K can be taken uniform with respect to c for c close to c_* comes directly from the proof in [35]. Therefore, for $t \geq 0$,

$$\begin{aligned} \inf_{y \in \mathbb{R}} \|\Psi_{\text{hy}}(t) - \mathcal{U}_{c_*}(\cdot - y)\| &\leq \inf_{y \in \mathbb{R}} \left[\|\Psi_{\text{hy}}(t) - \mathcal{U}_c(\cdot - y)\| + \|\mathcal{U}_c(\cdot - y) - \mathcal{U}_{c_*}(\cdot - y)\| \right] \\ &\leq \sqrt{K} \sqrt{E(\Psi(t)) - E(U_c)} + \mathcal{O}(|P(\Psi^{\text{in}}) - P(U_{c_*})|). \end{aligned}$$

Using that $P(\Psi(t)) = P_{\text{hy}}(\Psi_{\text{hy}}(t)) = P_{\text{hy}}(\mathcal{U}_c)$ and that \mathcal{U}_c is a critical point of the action $E_{\text{hy}} - cP_{\text{hy}}$, we infer $E(\Psi(t)) - E(U_c) = [E_{\text{hy}} - cP_{\text{hy}}](\Psi_{\text{hy}}^{\text{in}}) - [E_{\text{hy}} - cP_{\text{hy}}](\mathcal{U}_c) = \mathcal{O}(\|\Psi_{\text{hy}}^{\text{in}} - \mathcal{U}_c\|^2)$. Consequently,

$$\inf_{y \in \mathbb{R}} \|\Psi_{\text{hy}}(t) - \mathcal{U}_{c_*}(\cdot - y)\| \leq K \left(\|\Psi_{\text{hy}}^{\text{in}} - \mathcal{U}_c\| + \|\Psi_{\text{hy}}^{\text{in}} - \mathcal{U}_{c_*}\| \right) \leq Kd_{\text{hy}}(\Psi^{\text{in}}, U_{c_*}) + K\|\mathcal{U}_c - \mathcal{U}_{c_*}\| \leq Kd_{\text{hy}}(\Psi^{\text{in}}, U_{c_*}),$$

by (17). This gives (4).

4 Instability result for cusps: proof of Theorem 5

In this section, we set $\mathcal{F}_c \equiv E_{\text{hy}} - cP_{\text{hy}}$ and we assume

$$-\frac{d^2 \mathcal{F}_c(U_c)}{dc^2} \Big|_{c=c_*} = \frac{dP(U_c)}{dc} \Big|_{c=c_*} = 0 \quad \text{and} \quad 0 \neq \ddot{P}_* \equiv \frac{d^2 P(U_c)}{dc^2} \Big|_{c=c_*} = -\frac{d^3 \mathcal{F}_c(U_c)}{dc^3} \Big|_{c=c_*}.$$

The approach is reminiscent to the proof of M. Maeda [50]. Several modifications are necessary since for the skew-adjoint operator $J = \partial_x$, we can not find the required Hilbert space Y . More degenerate cases can probably be considered as in [50].

We shall denote $\mathbb{I} : X \rightarrow X^*$ and $\mathbb{I}_{H^1} : H^1 \rightarrow (H^1)^*$ the Riesz isomorphisms and $\mathcal{U} = (\eta, u)^t \in X = H^1(\mathbb{R}, \mathbb{R}) \times L^2(\mathbb{R}, \mathbb{R})$ and $H \equiv L^2(\mathbb{R}, \mathbb{R}) \times L^2(\mathbb{R}, \mathbb{R})$, endowed with its canonical scalar product. They are the corresponding Hilbert spaces needed in [50]. We consider the symmetric matrix

$$\mathbb{B} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which is such that $\mathbb{B}^2 = \mathbb{I}_2$ and $2P_{\text{hy}}(\mathcal{U}) = (\mathbb{B}\mathcal{U}, \mathcal{U})_H$.

Our assumption $\frac{dP(U_c)}{dc} \Big|_{c=c_*} = 0 \neq \frac{d^2P(U_c)}{dc^2} \Big|_{c=c_*}$ will simplify a little the computations in [50]. The functions η_1 and η_2 used in [50] become now

$$\eta_1(\gamma) = \mathcal{F}_{c_*+\gamma}(\mathcal{U}_{c_*+\gamma}) - \mathcal{F}_{c_*}(\mathcal{U}_{c_*}) - \gamma \frac{d\mathcal{F}_c(U_c)}{dc} \Big|_{c=c_*} \sim -\frac{\gamma^3}{6} \ddot{P}_*$$

and

$$\eta_2(\gamma) = \frac{d\eta_1}{d\gamma} = -P(U_{c_*+\gamma}) + P(U_{c_*}) \sim -\frac{\gamma^2}{2} \ddot{P}_*.$$

In order to clarify the dualities used in [50], we provide some elements of the proof adapted to our context.

Lemma 4.1 *There exists $\gamma_0 > 0$ small and $\sigma : (-\gamma_0, +\gamma_0) \rightarrow \mathbb{R}$ with $\sigma(\gamma) \sim -\frac{\gamma^2 \ddot{P}_*}{2\|\mathcal{U}_*\|_H^2}$ and such that, for any $\gamma \in (-\gamma_0, +\gamma_0)$,*

$$P_{\text{hy}}(\mathcal{U}_{c_*+\gamma} + \sigma(\gamma)\mathbb{B}\mathcal{U}_{c_*+\gamma}) = P_{\text{hy}}(\mathcal{U}_*).$$

Proof. We have

$$P_{\text{hy}}(\mathcal{U}_{c_*+\gamma} + \sigma\mathbb{B}\mathcal{U}_{c_*+\gamma}) = \frac{1}{2}(\mathbb{B}\mathcal{U}_{c_*+\gamma} + \sigma\mathcal{U}_{c_*+\gamma}, \mathcal{U}_{c_*+\gamma} + \sigma\mathbb{B}\mathcal{U}_{c_*+\gamma})_H = P_{\text{hy}}(\mathcal{U}_{c_*+\gamma}) + \sigma\|\mathcal{U}_{c_*+\gamma}\|_H^2 + \sigma^2 P_{\text{hy}}(\mathcal{U}_{c_*+\gamma}).$$

Since $\|\mathcal{U}_*\|_H^2 \neq 0$, the conclusion follows from an easy implicit function argument near $\sigma = \gamma = 0$. In [50], the linear mapping B is seen from X to X^* , but here, there is no confusion to define $\mathcal{U}_{c_*+\gamma} + \sigma\mathbb{B}\mathcal{U}_{c_*+\gamma} \in H = L^2 \times L^2$. \square

We define, for $\gamma \in (-\gamma_0, +\gamma_0)$,

$$\mathcal{W}(\gamma) \equiv \mathcal{U}_{c_*+\gamma} + \sigma(\gamma)\mathbb{B}\mathcal{U}_{c_*+\gamma},$$

which then verifies by construction $P_{\text{hy}}(\mathcal{W}(\gamma)) = P_{\text{hy}}(\mathcal{U}_*)$.

Lemma 4.2 *As $\gamma \rightarrow 0$, there holds $\mathcal{F}_{c_*}(\mathcal{W}(\gamma)) - \mathcal{F}_{c_*}(\mathcal{U}_{c_*}) \sim -\frac{\gamma^3}{6} \ddot{P}_*$.*

Proof. Using that $\mathcal{F}'_{c_*+\gamma}(\mathcal{U}_{c_*+\gamma}) = 0$, $P_{\text{hy}}(\mathcal{W}(\gamma)) = P_{\text{hy}}(\mathcal{U}_{c_*}) = -\frac{d\mathcal{F}_c(U_c)}{dc} \Big|_{c=c_*}$ and $\sigma(\gamma) = \mathcal{O}(\gamma^2)$, we have by Taylor expansion

$$\begin{aligned} \mathcal{F}_{c_*}(\mathcal{W}(\gamma)) - \mathcal{F}_{c_*}(\mathcal{U}_{c_*}) &= \mathcal{F}_{c_*+\gamma}(\mathcal{U}_{c_*+\gamma} + \sigma(\gamma)\mathbb{B}\mathcal{U}_{c_*+\gamma}) - \mathcal{F}_{c_*}(\mathcal{U}_{c_*}) + \gamma P_{\text{hy}}(\mathcal{W}(\gamma)) \\ &= \mathcal{F}_{c_*+\gamma}(\mathcal{U}_{c_*+\gamma}) - \mathcal{F}_{c_*}(\mathcal{U}_{c_*}) - \gamma \frac{d\mathcal{F}_c(U_c)}{dc} \Big|_{c=c_*} + \mathcal{O}(\gamma^4) \sim -\frac{\gamma^3}{6} \ddot{P}_*, \end{aligned}$$

as wished. \square

We recall that we have defined the tubular neighbourhood $\mathcal{O}_\varepsilon = \{\mathcal{V} \in X, \inf_{y \in \mathbb{R}} \|\mathcal{V} - \mathcal{U}_*(\cdot - y)\|_X < \varepsilon\}$.

Lemma 4.3 For $\varepsilon > 0$ small enough, there exist four \mathcal{C}^1 mappings $\bar{\gamma}, \alpha, \bar{y} : \mathcal{O}_\varepsilon \rightarrow \mathbb{R}$ and $\vartheta : \mathcal{O}_\varepsilon \rightarrow X$, satisfying, for $\mathcal{U} \in \mathcal{O}_\varepsilon$,

$$\mathcal{U}(\cdot - \bar{y}(\mathcal{U})) = \mathcal{W}(\bar{\gamma}(\mathcal{U})) + \vartheta(\mathcal{U}) + \alpha(\mathcal{U})\mathbb{B}\mathcal{U}_{c_* + \bar{\gamma}(\mathcal{U})}$$

and the orthogonality relations

$$(\vartheta(\mathcal{U}), \partial_x \mathcal{U}_{c_* + \bar{\gamma}(\mathcal{U})})_H = (\vartheta(\mathcal{U}), [\partial_c \mathcal{U}_c]_{|c=c_* + \bar{\gamma}(\mathcal{U})})_H = (\vartheta(\mathcal{U}), \mathbb{B}\mathcal{U}_{c_* + \bar{\gamma}(\mathcal{U})})_H = 0.$$

Finally, $\mathbb{I}^{-1}\bar{\gamma}' \in H^2 \times H^1$ and $\mathbb{I}_{H^1}^{-1} \frac{\partial \bar{\gamma}}{\partial \eta} \in H^4$.

Proof. We consider the mapping $G : X \times \mathbb{R} \times (-\gamma_0, +\gamma_0) \times \mathbb{R} \rightarrow \mathbb{R}^3$ defined by

$$G(\mathcal{U}, y, \gamma, \alpha) \equiv \begin{pmatrix} (\mathcal{U}(\cdot - y) - \mathcal{W}(\gamma) - \alpha\mathbb{B}\mathcal{U}_{c_* + \gamma}, \partial_x \mathcal{U}_{c_* + \gamma})_H \\ (\mathcal{U}(\cdot - y) - \mathcal{W}(\gamma) - \alpha\mathbb{B}\mathcal{U}_{c_* + \gamma}, [\partial_c \mathcal{U}_c]_{|c=c_* + \gamma})_H \\ (\mathcal{U}(\cdot - y) - \mathcal{W}(\gamma) - \alpha\mathbb{B}\mathcal{U}_{c_* + \gamma}, \mathbb{B}\mathcal{U}_{c_* + \gamma})_H \end{pmatrix}.$$

Then $G(\mathcal{U}_*, 0, 0, 0) = 0$ since $\mathcal{W}(0) = \mathcal{U}_*$. In order to show that G is of class \mathcal{C}^1 , we have to pay attention to the translation term $\mathcal{U}(\cdot - y)$, since differentiation in y requires $\mathcal{U} \in H^1 \times H^1$ whereas we only have $\mathcal{U} \in X = H^1 \times L^2$. It thus suffices to write

$$G(\mathcal{U}, y, \gamma, \alpha) = \begin{pmatrix} (\mathcal{U}, \partial_x \mathcal{U}_{c_* + \gamma}(\cdot + y))_H - (\mathcal{W}(\gamma) + \alpha\mathbb{B}\mathcal{U}_{c_* + \gamma}, \partial_x \mathcal{U}_{c_* + \gamma})_H \\ (\mathcal{U}, [\partial_c \mathcal{U}_c]_{|c=c_* + \gamma}(\cdot + y))_H - (\mathcal{W}(\gamma) + \alpha\mathbb{B}\mathcal{U}_{c_* + \gamma}, [\partial_c \mathcal{U}_c]_{|c=c_* + \gamma})_H \\ (\mathcal{U}, \mathbb{B}\mathcal{U}_{c_* + \gamma}(\cdot + y))_H - (\mathcal{W}(\gamma) + \alpha\mathbb{B}\mathcal{U}_{c_* + \gamma}, \mathbb{B}\mathcal{U}_{c_* + \gamma})_H \end{pmatrix}$$

to see that G is indeed of class \mathcal{C}^1 on $X \times \mathbb{R} \times (-\gamma_0, +\gamma_0) \times \mathbb{R}$ since $c \mapsto \mathcal{U}_c$ is smooth. Moreover, using that $\partial_\gamma \mathcal{W}|_{\gamma=0} = [\partial_c \mathcal{U}_c]_{|c=c_*}$, we infer

$$\frac{\partial G}{\partial(y, \gamma, \alpha)}(\mathcal{U}_*, 0, 0, 0) = \begin{pmatrix} (\mathcal{U}_*, \partial_x^2 \mathcal{U}_*)_H & -([\partial_c \mathcal{U}_c]_{|c=c_*}, \partial_x \mathcal{U}_*)_H & -(\mathbb{B}\mathcal{U}_*, \partial_x \mathcal{U}_*)_H \\ -(\mathcal{U}_*, \partial_x [\partial_c \mathcal{U}_c]_{|c=c_*})_H & -\|[\partial_c \mathcal{U}_c]_{|c=c_*}\|_H^2 & -(\mathbb{B}\mathcal{U}_*, [\partial_c \mathcal{U}_c]_{|c=c_*})_H \\ -(\mathcal{U}_*, \mathbb{B}\partial_x \mathcal{U}_*)_H & -([\partial_c \mathcal{U}_c]_{|c=c_*}, \mathbb{B}\mathcal{U}_*)_H & -\|\mathbb{B}\mathcal{U}_*\|_H^2 \end{pmatrix}.$$

Here, [50] uses assumption 2 (iii), that reads for us $([\partial_c \mathcal{U}_c]_{|c=c_*}, \partial_x \mathcal{U}_*)_H = -(\partial_x [\partial_c \mathcal{U}_c]_{|c=c_*}, \mathcal{U}_*)_H = 0$. It is indeed the case since \mathcal{U}_c is chosen even for any c (close to c_*). Furthermore, $(\mathcal{U}_*, \partial_x^2 \mathcal{U}_*)_H = -\|\partial_x \mathcal{U}_*\|_H^2$ by integration by parts, $(\mathbb{B}\mathcal{U}_*, \partial_x \mathcal{U}_*)_H = (\mathcal{U}_*, \mathbb{B}\partial_x \mathcal{U}_*)_H = 0$ since $\mathbb{B}\partial_x = J$ is skew-adjoint, and $(\mathbb{B}\mathcal{U}_*, [\partial_c \mathcal{U}_c]_{|c=c_*})_H = \partial_c [P_{\text{hy}}(\mathcal{U}_c)]_{|c=c_*} = 0$ by hypothesis. Therefore,

$$\frac{\partial G}{\partial(y, \gamma, \alpha)}(\mathcal{U}_*, 0, 0, 0) = \begin{pmatrix} -\|\partial_x \mathcal{U}_{c_*}\|_H^2 & 0 & 0 \\ 0 & -\|[\partial_c \mathcal{U}_c]_{|c=c_*}\|_H^2 & 0 \\ 0 & 0 & -\|\mathbb{B}\mathcal{U}_*\|_H^2 \end{pmatrix}$$

is invertible, thus the implicit function theorem provides three real-valued functions $\underline{y}, \underline{\gamma}$ and $\underline{\alpha}$, defined near \mathcal{U}_* (in X) and with $\underline{y}(\mathcal{U}_*) = \underline{\gamma}(\mathcal{U}_*) = \underline{\alpha}(\mathcal{U}_*) = 0$, such that $G(\mathcal{U}, \underline{y}(\mathcal{U}), \underline{\gamma}(\mathcal{U}), \underline{\alpha}(\mathcal{U})) = 0$. These functions are extended to \mathcal{O}_ε (for ε small enough) by the formulas $\bar{y}(\mathcal{U}) \equiv \underline{y}(\mathcal{U}(\cdot - y)) + y$, $\bar{\gamma}(\mathcal{U}) \equiv \underline{\gamma}(\mathcal{U}(\cdot - y))$ and $\alpha(\mathcal{U}) \equiv \underline{\alpha}(\mathcal{U}(\cdot - y))$ for any $y \in \mathbb{R}$ such that $\mathcal{U}(\cdot - y)$ lies in the neighbourhood of \mathcal{U}_* where $\underline{y}, \underline{\gamma}$ and $\underline{\alpha}$ are defined. Consequently, the mapping

$$\vartheta(\mathcal{U}) \equiv \mathcal{U}(\cdot - \bar{y}(\mathcal{U})) - \mathcal{W}(\bar{\gamma}(\mathcal{U})) - \bar{\alpha}(\mathcal{U})\mathbb{B}\mathcal{U}_{c_* + \bar{\gamma}(\mathcal{U})}$$

is orthogonal in H to $\partial_x \mathcal{U}_{c_* + \bar{\gamma}(\mathcal{U})}$, $[\partial_c \mathcal{U}_c]_{|c=c_* + \bar{\gamma}(\mathcal{U})}$ and $\mathbb{B}\mathcal{U}_{c_* + \bar{\gamma}(\mathcal{U})}$, as desired. Since f is assumed of class \mathcal{C}^2 , we have $\mathcal{U}_c \in H^4$ and the regularities $\mathbb{I}^{-1}\bar{\gamma}' \in H^2 \times H^1$ and $\mathbb{I}_{H^1}^{-1} \frac{\partial \bar{\gamma}}{\partial \eta} \in H^4$ follow easily. \square

Remark 4.1 We would like to point out that in [50] (Lemma 3), it is claimed that “ $w(u)$ ” is orthogonal to “ $\partial_\omega \phi_{\omega+\Lambda(u)}$ ” (we refer to the notations in [50]). However, since “ $T(\theta(u)) - \Psi(\Lambda(u))$ ” is already orthogonal to “ $\partial_\omega \phi_{\omega+\Lambda(u)}$ ” by construction, this is equivalent to “ $\langle B\phi_{\omega+\Lambda(u)}, \partial_\omega \phi_{\omega+\Lambda(u)} \rangle = 0$ ”, or “ $\partial_{\omega'}[Q(\phi_{\omega'})] = 0$ ”

at “ $\omega' = \omega + \Lambda(u)$ ”. We have not understood why this should happen since in general, for the function $\omega' \mapsto Q(\phi_{\omega'})$, the point ω is the only local critical point. For this reason, we have added a component to the original mapping G in [50]. Let us observe that then, Lemma 3 in [50] uses the assumption “ $d''(\omega) = 0$ ”. On the other hand, the derivative of G in [50] assumes “ $u \in D(T'(0))$ ”, for otherwise the expression “ $G_{1,1}(u, \theta, \Lambda) = \langle T'(0)T(\theta)u, T'(0)\phi_{\omega+\Lambda} \rangle$ ”, for instance, is meaningless. We have therefore given some details showing clearly the smoothness of G .

We now prove a lemma which shows that the quadratic functional associated with \mathcal{F}_*'' gives a good control on $\vartheta(\mathcal{U})$ thanks to the orthogonality conditions on this function. This result is in the spirit of Lemma 7 in [55].

Lemma 4.4 *There exist $0 < \gamma_1 \leq \gamma_0$ and $K_0 > 0$ such that if $\gamma \in (-\gamma_1, +\gamma_1)$ and if $\vartheta \in X$ verifies*

$$(\vartheta, \partial_x \mathcal{U}_{c_*+\gamma})_H = (\vartheta, [\partial_c \mathcal{U}_c]_{|c=c_*+\gamma})_H = (\vartheta, \mathbb{B}\mathcal{U}_{c_*+\gamma})_H = 0,$$

then $\langle \mathcal{F}_{c_*+\gamma}''(\mathcal{U}_{c_*+\gamma})\vartheta, \vartheta \rangle_{X^*, X} \geq K_0 \|\vartheta\|_X^2$.

Proof. As a first step, we prove that if $\vartheta \in X$ verifies $\vartheta \neq 0$,

$$(\vartheta, \partial_x \mathcal{U}_*)_H = (\vartheta, [\partial_c \mathcal{U}_c]_{|c=c_*})_H = (\vartheta, \mathbb{B}\mathcal{U}_*)_H = 0,$$

then $\langle \mathcal{F}_*''(\mathcal{U}_*)\vartheta, \vartheta \rangle_{X^*, X} > 0$. Indeed, assume that $\langle \mathcal{F}_*''(\mathcal{U}_*)\vartheta, \vartheta \rangle_{X^*, X} \leq 0$. Let $\chi \in X$ be a negative eigenvector of \mathcal{F}_*'' . We claim that can not have $((\vartheta, \chi)_H, ([\partial_c \mathcal{U}_c]_{|c=c_*}, \chi)_H) = (0, 0)$. For otherwise, $(\vartheta, \chi)_H = 0$ implies that ϑ is L^2 -orthogonal to χ , which is the eigenvector associated with the only negative eigenvalue $-\mu_0$ of \mathcal{F}_*'' seen as an unbounded operator on L^2 , thus $\langle \mathcal{F}_*''(\mathcal{U}_*)\vartheta, \vartheta \rangle_{X^*, X} \geq 0$, and since we assume equality, this means that ϑ belongs to the kernel of $\mathcal{F}_*''(\mathcal{U}_*)$, which is spanned by $\mathcal{U}_* = \partial_x \mathcal{U}_*$, but the condition $(\vartheta, \partial_x \mathcal{U}_*)_H = 0$ then implies $\vartheta = 0$, a contradiction. Therefore, there exists $(a, b) \in \mathbb{R}^2$ such that $(a, b) \neq (0, 0)$ and $(a[\partial_c \mathcal{U}_c]_{|c=c_*} + b\vartheta, \chi)_H = 0$. The nonzero vector $p \equiv a[\partial_c \mathcal{U}_c]_{|c=c_*} + b\vartheta$ then verifies $(p, \chi)_H = 0$ and $(p, \mathcal{J}\mathcal{U}_*)_H = a([\partial_c \mathcal{U}_c]_{|c=c_*}, \mathcal{J}\mathcal{U}_*)_H + b(\vartheta, \mathcal{J}\mathcal{U}_*)_H = 0$, so that $\langle \mathcal{F}_*''(\mathcal{U}_*)p, p \rangle_{X^*, X} > 0$. Here, we have used once again that $([\partial_c \mathcal{U}_c]_{|c=c_*}, \mathcal{J}\mathcal{U}_*)_H = 0$ since the left vector is an even function and the right vector an odd function. However, in view of the equality $\langle \mathcal{F}_*''(\mathcal{U}_*)[\partial_c \mathcal{U}_c]_{|c=c_*}, \phi \rangle_{X^*, X} = (\mathbb{B}\mathcal{U}_*, \phi)_H$, valid for any $\phi \in X$ (which follows from differentiation of $E'_{\text{hy}}(\mathcal{U}_c) = cP'_{\text{hy}}(\mathcal{U}_c) = c(\mathbb{B}\mathcal{U}_c, \cdot)_H$ at $c = c_*$), we have

$$\langle \mathcal{F}_*''(\mathcal{U}_*)[\partial_c \mathcal{U}_c]_{|c=c_*}, \vartheta \rangle_{X^*, X} = (\mathbb{B}[\partial_c \mathcal{U}_c]_{|c=c_*}, \vartheta)_H = 0.$$

As a consequence,

$$\begin{aligned} 0 < \langle \mathcal{F}_*''(\mathcal{U}_*)p, p \rangle_{X^*, X} &= a^2 \langle \mathcal{F}_*''(\mathcal{U}_*)[\partial_c \mathcal{U}_c]_{|c=c_*}, [\partial_c \mathcal{U}_c]_{|c=c_*} \rangle_{X^*, X} + b^2 \langle \mathcal{F}_*''(\mathcal{U}_*)\vartheta, \vartheta \rangle_{X^*, X} \\ &= a^2 (\mathbb{B}\mathcal{U}_*, [\partial_c \mathcal{U}_c]_{|c=c_*})_H + b^2 \langle \mathcal{F}_*''(\mathcal{U}_*)\vartheta, \vartheta \rangle_{X^*, X} = b^2 \langle \mathcal{F}_*''(\mathcal{U}_*)\vartheta, \vartheta \rangle_{X^*, X}, \end{aligned}$$

since $(\mathbb{B}\mathcal{U}_*, [\partial_c \mathcal{U}_c]_{|c=c_*})_H = \partial_c [P_{\text{hy}}(\mathcal{U}_c)]_{|c=c_*} = 0$ in our situation. We reach a contradiction since the right-hand side is supposed ≤ 0 .

We now prove the lemma by contradiction, and then assume that there exists sequences $(\vartheta_n)_{n \geq 1} \in X$ and $(\gamma_n)_{n \geq 1} \in (0, \gamma_0)$ such that $\gamma_n \rightarrow 0$, $\|\vartheta_n\|_X^2 = 1$ and

$$(\vartheta_n, \partial_x \mathcal{U}_{c_*+\gamma_n})_H = (\vartheta_n, [\partial_c \mathcal{U}_c]_{|c=c_*+\gamma_n})_H = (\vartheta_n, \mathbb{B}\mathcal{U}_{c_*+\gamma_n})_H = 0, \quad (18)$$

but $\langle \mathcal{F}_{c_*+\gamma_n}''(\mathcal{U}_{c_*+\gamma_n})\vartheta_n, \vartheta_n \rangle_{X^*, X} \rightarrow 0$. Possibly passing to a subsequence, we may assume the existence of some $\vartheta = (\zeta, v) \in X$ such that $\vartheta_n \equiv (\zeta_n, v_n) \rightharpoonup \vartheta$ in $X = H^1 \times L^2$. We then show the lower semicontinuity of $\langle \mathcal{F}_*''(\mathcal{U}_*)\vartheta, \vartheta \rangle_{X^*, X}$. This is roughly a verification of part of assumption (A3) in [55], used in Lemma 7 there. By compact Sobolev embedding, we may assume $\zeta_n \rightarrow \zeta$ in $L^\infty_{\text{loc}}(\mathbb{R})$. A straightforward computation gives

$$\begin{aligned} \langle \mathcal{F}_{c_*+\gamma}''(\mathcal{U}_{c_*+\gamma})\vartheta, \vartheta \rangle_{X^*, X} &= \int_{\mathbb{R}} \frac{(\partial_x \zeta)^2}{2(r_0^2 + \eta_{c_*+\gamma})} - \frac{\partial_x \zeta \partial_x \eta_{c_*+\gamma}}{(r_0^2 + \eta_{c_*+\gamma})^2} + \frac{\zeta^2 (\partial_x \eta_{c_*+\gamma})^2}{4(r_0^2 + \eta_{c_*+\gamma})^3} \\ &\quad + 2(r_0^2 + \eta_{c_*+\gamma})v^2 + 2(2u_{c_*+\gamma} - (c_* + \gamma))v\zeta - f'(r_0^2 + \eta_{c_*+\gamma})\zeta^2 \, dx. \end{aligned}$$

Since $r_0^2 + \eta_{c_* + \gamma_n}$ remains bounded away from zero uniformly and $\eta_{c_* + \gamma_n} \rightarrow \eta_{c_*}$ in $W^{1,\infty}(\mathbb{R}) \cap H^1(\mathbb{R})$ as $n \rightarrow +\infty$, the weak convergence $\zeta_n \rightharpoonup \zeta$ in H^1 implies

$$\begin{aligned} & \int_{\mathbb{R}} \frac{(\partial_x \zeta)^2}{2(r_0^2 + \eta_{c_*})} - \frac{\partial_x \zeta \partial_x \eta_{c_*}}{(r_0^2 + \eta_{c_*})^2} + \frac{\zeta^2 (\partial_x \eta_{c_*})^2}{4(r_0^2 + \eta_{c_*})^3} dx \\ & \leq \liminf_{n \rightarrow +\infty} \int_{\mathbb{R}} \frac{(\partial_x \zeta_n)^2}{2(r_0^2 + \eta_{c_* + \gamma_n})} - \frac{\partial_x \zeta_n \partial_x \eta_{c_* + \gamma_n}}{(r_0^2 + \eta_{c_* + \gamma_n})^2} + \frac{\zeta_n^2 (\partial_x \eta_{c_* + \gamma_n})^2}{4(r_0^2 + \eta_{c_* + \gamma_n})^3} dx. \end{aligned} \quad (19)$$

For the remaining terms, we write, for some $R > 0$ to be determined later,

$$\begin{aligned} & \int_{\mathbb{R}} 2(r_0^2 + \eta_{c_* + \gamma_n})v_n^2 + 2(2u_{c_* + \gamma_n} - (c_* + \gamma_n))v_n \zeta_n - f'(r_0^2 + \eta_{c_* + \gamma_n})\zeta_n^2 dx \\ & = \int_{\mathbb{R}} 2 \left[(r_0^2 + \eta_{c_* + \gamma_n})^{1/2} v_n + \frac{(2u_{c_* + \gamma_n} - (c_* + \gamma_n))\zeta_n}{2(r_0^2 + \eta_{c_* + \gamma_n})^{1/2}} \right]^2 dx \\ & \quad + \int_{|x| \leq R} + \int_{|x| \geq R} \frac{1}{2} \left(-\frac{(2u_{c_* + \gamma_n} - (c_* + \gamma_n))^2}{r_0^2 + \eta_{c_* + \gamma_n}} - 2f'(r_0^2 + \eta_{c_* + \gamma_n}) \right) \zeta_n^2 dx. \end{aligned}$$

For the first integral, we may use that $(\zeta_n, v_n) \rightharpoonup (\zeta, v)$ in $L^2 \times L^2$ and the fact that $(\eta_{c_* + \gamma_n}, u_{c_* + \gamma_n})$ converges to (η_*, u_*) uniformly to deduce

$$(r_0^2 + \eta_{c_* + \gamma_n})^{1/2} v_n + \frac{(2u_{c_* + \gamma_n} - (c_* + \gamma_n))\zeta_n}{2(r_0^2 + \eta_{c_* + \gamma_n})^{1/2}} \rightharpoonup (r_0^2 + \eta_*)^{1/2} v + \frac{(2u_* - c_*)\zeta}{2(r_0^2 + \eta_*)^{1/2}} \quad \text{in } L^2, \quad (20)$$

hence,

$$\begin{aligned} & \int_{\mathbb{R}} 2 \left[(r_0^2 + \eta_*)^{1/2} v + \frac{(2u_* - c_*)\zeta}{2(r_0^2 + \eta_*)^{1/2}} \right]^2 dx \\ & \leq \liminf_{n \rightarrow +\infty} \int_{\mathbb{R}} 2 \left[(r_0^2 + \eta_{c_* + \gamma_n})^{1/2} v_n + \frac{(2u_{c_* + \gamma_n} - (c_* + \gamma_n))\zeta_n}{2(r_0^2 + \eta_{c_* + \gamma_n})^{1/2}} \right]^2 dx. \end{aligned} \quad (21)$$

Since $\zeta_n \rightarrow \zeta$ in $L^\infty([-R, R])$ and $(u_{c_* + \gamma_n}, \eta_{c_* + \gamma_n}) \rightarrow (u_*, \eta_*)$ uniformly, it follows that

$$\begin{aligned} & \int_{|x| \leq R} \frac{1}{2} \left(-\frac{(2u_* - c_*)^2}{r_0^2 + \eta_*} - 2f'(r_0^2 + \eta_*) \right) \zeta^2 dx \\ & = \lim_{n \rightarrow +\infty} \int_{|x| \leq R} \frac{1}{2} \left(-\frac{(2u_{c_* + \gamma_n} - (c_* + \gamma_n))^2}{r_0^2 + \eta_{c_* + \gamma_n}} - 2f'(r_0^2 + \eta_{c_* + \gamma_n}) \right) \zeta_n^2 dx. \end{aligned}$$

For the last integral, we have to use the decay at infinity of $\eta_{c_* + \gamma}$ and $u_{c_* + \gamma}$ uniformly for $|\gamma|$ small. This gives

$$-\frac{(2u_{c_* + \gamma_n} - (c_* + \gamma_n))^2}{r_0^2 + \eta_{c_* + \gamma_n}} - 2f'(r_0^2 + \eta_{c_* + \gamma_n}) \rightarrow \frac{c_s^2 - c_*^2}{r_0^2}$$

as $|x| \rightarrow +\infty$, uniformly in n . Since $0 < c_* < c_s$, there exist some small $\delta > 0$ and some $R > 0$ large such that, for any n and any x with $|x| \geq R$,

$$-\frac{(2u_{c_* + \gamma_n} - (c_* + \gamma_n))^2}{r_0^2 + \eta_{c_* + \gamma_n}} - 2f'(r_0^2 + \eta_{c_* + \gamma_n}) \geq \delta.$$

In particular, since $\zeta_n \rightharpoonup \zeta$ in L^2 ,

$$\mathbf{1}_{|x| \geq R} \left(-\frac{(2u_{c_* + \gamma_n} - (c_* + \gamma_n))^2}{r_0^2 + \eta_{c_* + \gamma_n}} - 2f'(r_0^2 + \eta_{c_* + \gamma_n}) \right)^{1/2} \zeta_n \rightharpoonup \mathbf{1}_{|x| \geq R} \left(-\frac{(2u_* - c_*)^2}{r_0^2 + \eta_*} - 2f'(r_0^2 + \eta_*) \right)^{1/2} \zeta,$$

in L^2 , thus

$$\begin{aligned} & \int_{|x| \geq R} \frac{1}{2} \left(-\frac{(2u_* - c_*)^2}{r_0^2 + \eta_*} - 2f'(r_0^2 + \eta_*) \right) \zeta^2 dx \\ & \leq \lim_{n \rightarrow +\infty} \int_{|x| \geq R} \frac{1}{2} \left(-\frac{(2u_{c_*+\gamma_n} - (c_* + \gamma_n))^2}{r_0^2 + \eta_{c_*+\gamma_n}} - 2f'(r_0^2 + \eta_{c_*+\gamma_n}) \right) \zeta_n^2 dx. \end{aligned} \quad (22)$$

Combining these three \lim inequalities, we deduce

$$\langle \mathcal{F}_*''(\mathcal{U}_*)\vartheta, \vartheta \rangle_{X^*, X} \leq \lim_{n \rightarrow +\infty} \langle \mathcal{F}_{c_*+\gamma_n}''(\mathcal{U}_{c_*+\gamma_n})\vartheta_n, \vartheta_n \rangle_{X^*, X} = 0. \quad (23)$$

Turning back to our sequence (ϑ_n, γ_n) , we may pass to the limit in (18):

$$(\vartheta, \partial_x \mathcal{U}_*)_H = (\vartheta, [\partial_c \mathcal{U}_c]_{c=c_*})_H = (\vartheta, \mathbb{B}\mathcal{U}_*)_H = 0.$$

Comparing with (23), we deduce from our first claim that $\vartheta = 0$. This means that we must have equality in all the above \lim inequalities. In particular, the weak convergence (22) is actually strong, thus $\zeta_n \rightarrow \zeta = 0$ in $L^2(\mathbb{R})$ (the strong convergence in $\{|x| \leq R\}$ being already known since $\zeta_n \rightarrow \zeta$ in $L_{\text{loc}}^\infty(\mathbb{R})$). Going back to the equality in (19) thus provides $\partial_x \zeta_n \rightarrow \partial_x \zeta = 0$ in $L^2(\mathbb{R})$, since $r_0^2 + \eta_{c_*+\gamma_n}$ remains uniformly bounded away from zero and by weak convergence, $0 = \int_{\mathbb{R}} \frac{\zeta^2 (\partial_x \eta_{c_*})^2}{4(r_0^2 + \eta_{c_*})^3} dx = \lim_{n \rightarrow +\infty} \int_{\mathbb{R}} \frac{\partial_x \zeta_n \partial_x \eta_{c_*+\gamma_n}}{(r_0^2 + \eta_{c_*+\gamma_n})^2} dx$. Finally, the equality in (21) means that (20) is actually a strong convergence, that is $v_n \rightarrow v = 0$ in L^2 since $\zeta_n \rightarrow \zeta$ in L^2 . The contradiction then follows: $1 = \|\vartheta_n\|_X^2 = \|\zeta_n\|_{L^2}^2 + \|\partial_x \zeta_n\|_{L^2}^2 + \|v_n\|_{L^2}^2 \rightarrow 0$. \square

Remark 4.2 This result is also Lemma 7 in [50], and is said to be Lemma 7 in [55]. However, the hypothesis of Lemma 7 in [55] are not verified, and in particular assumption (B3) there. It is natural to believe that this assumption is verified in most physical situations, but it is not clear whether it always holds true in the general framework of [50] without further hypothesis.

The next lemma provides a control for $\alpha(\mathcal{U})$.

Lemma 4.5 *Assume $\varepsilon > 0$ small enough. Then, there exists $K > 0$ such that for any $\mathcal{U} \in \mathcal{O}_\varepsilon$ satisfying $P_{\text{hy}}(\mathcal{U}) = P_{\text{hy}}(\mathcal{U}_*)$, there holds*

$$|\alpha(\mathcal{U})| \leq K \left(\bar{\gamma}^2(\mathcal{U}) \|\vartheta(\mathcal{U})\|_X + \|\vartheta(\mathcal{U})\|_X^2 \right).$$

Proof. It is the same as in [50], Lemma 8, but give it for completeness. We expand and use that $\mathbb{B}^2 = \text{Id}_2$ and the definition $\mathcal{W}(\gamma) \equiv \mathcal{U}_{c_*+\gamma} + \sigma(\gamma)\mathbb{B}\mathcal{U}_{c_*+\gamma}$ for the second line:

$$\begin{aligned} P_{\text{hy}}(\mathcal{U}_*) &= P_{\text{hy}}(\mathcal{U}) = P_{\text{hy}}(\mathcal{U}(\cdot - \bar{y}(\mathcal{U}))) = P_{\text{hy}}\left(\mathcal{W}(\bar{\gamma}(\mathcal{U})) + \vartheta(\mathcal{U}) + \alpha(\mathcal{U})\mathbb{B}\mathcal{U}_{c_*+\bar{\gamma}(\mathcal{U})}\right) \\ &= P_{\text{hy}}(\mathcal{W}(\bar{\gamma}(\mathcal{U}))) + P_{\text{hy}}(\vartheta(\mathcal{U})) + \alpha^2(\mathcal{U})P_{\text{hy}}(\mathbb{B}\mathcal{U}_{c_*+\bar{\gamma}(\mathcal{U})}) \\ &\quad + \alpha(\mathcal{U})(\mathbb{B}\vartheta(\mathcal{U}), \mathbb{B}\mathcal{U}_{c_*+\bar{\gamma}(\mathcal{U})})_H + (\mathbb{B}\mathcal{U}_{c_*+\bar{\gamma}(\mathcal{U})}, \vartheta(\mathcal{U}))_H + \alpha(\mathcal{U})(\mathbb{B}\mathcal{U}_{c_*+\bar{\gamma}(\mathcal{U})}, \mathbb{B}\mathcal{U}_{c_*+\bar{\gamma}(\mathcal{U})})_H \\ &\quad + \sigma(\bar{\gamma}(\mathcal{U}))(\mathcal{U}_{c_*+\bar{\gamma}(\mathcal{U})}, \vartheta(\mathcal{U}))_H + \sigma(\bar{\gamma}(\mathcal{U}))\alpha(\mathcal{U})(\mathcal{U}_{c_*+\bar{\gamma}(\mathcal{U})}, \mathbb{B}\mathcal{U}_{c_*+\bar{\gamma}(\mathcal{U})})_H. \end{aligned}$$

Since $P_{\text{hy}}(\mathcal{W}(\bar{\gamma}(\mathcal{U}))) = P_{\text{hy}}(\mathcal{U}_*)$, we infer

$$-\alpha(\mathcal{U}) \left[\|\mathcal{U}_*\|_H^2 + o(1) \right] = \sigma(\bar{\gamma}(\mathcal{U}))(\mathcal{U}_{c_*+\bar{\gamma}(\mathcal{U})}, \vartheta(\mathcal{U}))_H + P_{\text{hy}}(\vartheta(\mathcal{U}))$$

and the conclusion follows since $\sigma(\gamma) = \mathcal{O}(\gamma^2)$ by Lemma 4.1. \square

Now, we give a lemma useful to estimate $\vartheta(\mathcal{U})$.

Lemma 4.6 *Assume $\varepsilon > 0$ small enough. Then, there exists $K > 0$ such that for any $\mathcal{U} \in \mathcal{O}_\varepsilon$ satisfying $P_{\text{hy}}(\mathcal{U}) = P_{\text{hy}}(\mathcal{U}_*)$ and $\mathcal{F}_*(\mathcal{U}) - \mathcal{F}_*(\mathcal{U}_*) < 0$, there holds*

$$\|\vartheta(\mathcal{U})\|_X^2 \leq K|\bar{\gamma}(\mathcal{U})|^3.$$

In particular, $|\alpha(\mathcal{U})| \leq K|\bar{\gamma}(\mathcal{U})|^3$.

Proof. It is the same as in [50], Lemma 9. Note first that the last assertion is a direct consequence of the first one and Lemma 4.5. Next, we argue by contradiction and assume that there exists a sequence $\mathcal{U}_n \rightarrow \mathcal{U}_*$ in X such that $\mathcal{F}_*(\mathcal{U}_n) - \mathcal{F}_*(\mathcal{U}_*) < 0$ and $\|\vartheta(\mathcal{U}_n)\|_X^2 \gg |\bar{\gamma}(\mathcal{U}_n)|^3$. For simplicity, we denote $\bar{\gamma}_n = \bar{\gamma}(\mathcal{U}_n)$, $\vartheta_n = \vartheta(\mathcal{U}_n)$, $\alpha_n = \alpha(\mathcal{U}_n)$. Then, by Lemma 4.5, we have $|\alpha_n| \leq K(\bar{\gamma}_n^2 \|\vartheta_n\|_X + \|\vartheta_n\|_X^2) \leq K(\|\vartheta_n\|_X^{7/3} + \|\vartheta_n\|_X^2) = \mathcal{O}(\|\vartheta_n\|_X^2)$. Therefore, by Taylor expansion and Lemma 4.3, it holds

$$\begin{aligned} \mathcal{F}_*(\mathcal{U}_n) - \mathcal{F}(\mathcal{U}_*) &= \mathcal{F}_*(\mathcal{U}_n(\cdot - \bar{y}_n)) - \mathcal{F}(\mathcal{U}_*) = \mathcal{F}_*(\mathcal{W}(\bar{\gamma}_n)) + \vartheta_n + \alpha_n \mathbb{B}\mathcal{U}_{c_* + \bar{\gamma}_n} - \mathcal{F}(\mathcal{U}_*) \\ &= \mathcal{F}_*(\mathcal{W}(\bar{\gamma}_n)) - \mathcal{F}(\mathcal{U}_*) + \langle \mathcal{F}'_*(\mathcal{W}(\bar{\gamma}_n)), \vartheta_n + \alpha_n \mathbb{B}\mathcal{U}_{c_* + \bar{\gamma}_n} \rangle_{X^*, X} \\ &\quad + \frac{1}{2} \langle \mathcal{F}''_*(\mathcal{W}(\bar{\gamma}_n)) \vartheta_n, \vartheta_n \rangle_{X^*, X} + o(\|\vartheta(\mathcal{U}_n)\|_X^2). \end{aligned} \tag{24}$$

However, by Lemma 4.2, $\mathcal{F}_{c_*}(\mathcal{W}(\gamma)) - \mathcal{F}_{c_*}(\mathcal{U}_{c_*}) = \mathcal{O}(|\gamma|^3)$, and since $\mathcal{F}'_*(\mathcal{W}(\gamma)) = \mathcal{F}'_*(\mathcal{W}(0)) + o(1) = \mathcal{F}'_*(\mathcal{U}_*) + o(1) = o(1)$, there holds $\langle \mathcal{F}'_*(\mathcal{W}(\bar{\gamma}_n)), \alpha_n \mathbb{B}\mathcal{U}_{c_* + \bar{\gamma}_n} \rangle_{X^*, X} = o(\|\vartheta_n\|_X^2)$. Furthermore, using $\mathcal{F}'_* = \mathcal{F}'_{c_* + \bar{\gamma}_n} + \bar{\gamma}_n \mathbb{B}$, the third orthogonality condition in Lemma 4.3 and that $\sigma(\gamma) = \mathcal{O}(\gamma^2)$, we deduce

$$\begin{aligned} \langle \mathcal{F}'_*(\mathcal{W}(\bar{\gamma}_n)), \vartheta_n \rangle_{X^*, X} &= \langle \mathcal{F}'_{c_* + \bar{\gamma}_n}(\mathcal{W}(\bar{\gamma}_n)), \vartheta_n \rangle_{X^*, X} + \bar{\gamma}_n \langle \mathbb{B}\mathcal{W}(\bar{\gamma}_n), \vartheta_n \rangle_H \\ &= \langle \mathcal{F}'_{c_* + \bar{\gamma}_n}(\mathcal{U}_{c_* + \bar{\gamma}_n}) + \sigma(\bar{\gamma}_n) \mathbb{B}\mathcal{U}_{c_* + \bar{\gamma}_n}, \vartheta_n \rangle_{X^*, X} + \bar{\gamma}_n \sigma(\bar{\gamma}_n) \langle \mathcal{U}_{c_* + \bar{\gamma}_n}, \vartheta_n \rangle_H \\ &= \mathcal{O}(\bar{\gamma}_n^2 \|\vartheta_n\|_X) + \mathcal{O}(|\bar{\gamma}_n|^3 \|\vartheta_n\|_X) = o(\|\vartheta_n\|_X^{7/3}) = o(\|\vartheta_n\|_X^2). \end{aligned}$$

For the last line, we have used another Taylor expansion with $\mathcal{F}'_{c_* + \bar{\gamma}_n}(\mathcal{U}_{c_* + \bar{\gamma}_n}) = 0$. Finally, Lemma 4.4 yields $\langle \mathcal{F}''_{c_* + \bar{\gamma}_n}(\mathcal{U}_{c_* + \bar{\gamma}_n}) \vartheta_n, \vartheta_n \rangle_{X^*, X} \geq K_0 \|\vartheta_n\|_X^2$. Reporting these expansions in (24) yields

$$\mathcal{F}_*(\mathcal{U}_n) - \mathcal{F}(\mathcal{U}_*) \geq \frac{K_0}{4} \|\vartheta_n\|_X^2 + o(\|\vartheta_n\|_X^2) \geq \frac{K_0}{8} \|\vartheta_n\|_X^2$$

for n sufficiently large, which contradicts our assumption. \square

We now need to find an extension of the functionals "A" and "P" used in [50] (and also in [55]). In these works, these functionals are built on what should be here " $J^{-1} \partial_c \mathcal{U}_c = \mathbb{B} \partial_x^{-1} \partial_c \mathcal{U}_c$ ", but unfortunately, $\partial_c \phi_c$ does not have vanishing integral over \mathbb{R} (for instance, $\partial_c \eta_c$ has constant sign). We rely instead on a construction of a suitable approximation of " $J^{-1} \partial_c \mathcal{U}_c$ ". A similar construction is used in [47].

Lemma 4.7 *For any $0 < \kappa < 1$, there exists a \mathcal{C}^2 mapping $\Upsilon_\kappa : (-\gamma_1, +\gamma_1) \rightarrow X$ such that, for any $\gamma \in (-\gamma_1, +\gamma_1)$, $\Upsilon_\kappa(\gamma) \in H^2 \times H^1$ is an odd function verifying*

$$\|J\Upsilon_\kappa(\gamma) - [\partial_c \mathcal{U}_c]_{|c=c_* + \gamma}\|_X \leq \kappa.$$

Proof. We fix an even function $\Theta_0 \in \mathcal{C}_c^\infty(\mathbb{R})$ such that $\int_{\mathbb{R}} \Theta_0 dx = 1$. For $T > 0$ to be fixed later, but independent of γ and κ , we set $t_\kappa \equiv T/\kappa^2 > 0$ and

$$\Upsilon_\kappa(\gamma)(x) \equiv \mathbb{B} \int_0^x \left[[\partial_c \mathcal{U}_c]_{|c=c_* + \gamma}(y) - \frac{1}{t_\kappa} \Theta_0\left(\frac{y}{t_\kappa}\right) \int_{\mathbb{R}} [\partial_c \mathcal{U}_c]_{|c=c_* + \gamma}(z) dz \right] dy.$$

It is clear that $\Upsilon_\kappa(\gamma) \in \mathcal{C}^1(\mathbb{R})$ and that, since $J = \partial_x \mathbb{B}$ and $\mathbb{B}^2 = \text{Id}_2$,

$$J\Upsilon_\kappa(\gamma) - [\partial_c \mathcal{U}_c]_{|c=c_* + \gamma} = \frac{1}{t_\kappa} \Theta_0\left(\frac{\cdot}{t_\kappa}\right) \int_{\mathbb{R}} [\partial_c \mathcal{U}_c]_{|c=c_* + \gamma}(z) dz$$

In particular,

$$\|J\Upsilon_\kappa(\gamma) - \partial_c[\mathcal{U}_c]_{|c=c_*+\gamma}\|_X^2 = \left[\frac{1}{t_\kappa} \|\Theta_0\|_{L^2}^2 + \frac{1}{t_\kappa^3} \|\partial_x \Theta_0\|_{L^2}^2 \right] \left(\int_{\mathbb{R}} [\partial_c \mathcal{U}_c]_{|c=c_*+\gamma}(z) dz \right)^2 \leq \kappa^2$$

if we choose $T = T(c_*, \mathcal{U}_*, \Theta_0) > 0$ sufficiently large and γ_1 smaller if necessary. Moreover, $\Upsilon_\kappa(\gamma)$ is odd since \mathcal{U}_c and Θ_0 are even. In addition, the even function $y \mapsto [\partial_c \mathcal{U}_c]_{|c=c_*+\gamma}(y) - \frac{1}{t_\kappa} \Theta_0\left(\frac{y}{t_\kappa}\right) \int_{\mathbb{R}} [\partial_c \mathcal{U}_c]_{|c=c_*+\gamma}(z) dz$ decays exponentially at infinity (since Θ_0 has compact support and $\partial_c \mathcal{U}_c$ decays exponentially), and has zero integral (since Θ_0 has integral equal to one), hence

$$\Upsilon_\kappa(\gamma)(x) = -\mathbb{B} \int_x^{+\infty} \left[[\partial_c \mathcal{U}_c]_{|c=c_*+\gamma}(y) - \frac{1}{t_\kappa} \Theta_0\left(\frac{y}{t_\kappa}\right) \int_{\mathbb{R}} [\partial_c \mathcal{U}_c]_{|c=c_*+\gamma}(z) dz \right] dy$$

and decays exponentially at infinity. It follows easily from these two equalities that $\gamma \mapsto \Upsilon_\kappa(\gamma) \in L^2 \times L^2$ is well defined and continuous, hence also $\gamma \mapsto \Upsilon_\kappa(\gamma) \in H^2 \times H^1$. By the same type of arguments,

$$\frac{\partial \Upsilon_\kappa}{\partial \gamma}(\gamma)(x) = \mathbb{B} \int_0^x \left[[\partial_c^2 \mathcal{U}_c]_{|c=c_*+\gamma}(y) - \frac{1}{t_\kappa} \Theta_0\left(\frac{y}{t_\kappa}\right) \int_{\mathbb{R}} [\partial_c^2 \mathcal{U}_c]_{|c=c_*+\gamma}(z) dz \right] dy$$

is well-defined and is a continuous function of γ with values into $H^2 \times H^1$, and similarly for the second derivative. \square

We now define, in the tubular neighbourhood \mathcal{O}_ε of \mathcal{U}_* , the functional (corresponding to "A" in [50])

$$\Omega_\kappa(\mathcal{U}) \equiv (\mathcal{U}(\cdot - \bar{y}(\mathcal{U})), \Upsilon_\kappa(\bar{\gamma}(\mathcal{U})))_H = (\mathcal{U}, \Upsilon_\kappa(\bar{\gamma}(\mathcal{U}))(\cdot + \bar{y}(\mathcal{U})))_H$$

depending on $\kappa \in (0, 1)$, which will be determined later. The first properties of Ω_κ are given below.

Lemma 4.8 *For any $0 < \kappa < 1$, $\Omega_\kappa : \mathcal{O}_\varepsilon \rightarrow \mathbb{R}$ is of class \mathcal{C}^1 . In addition, there exists some bounded mapping $\mathcal{N}_{\bar{\gamma}} : \mathcal{O}_\varepsilon \rightarrow X$ such that if $\Psi_{\text{hy}} \in \mathcal{C}^1([0, T], X)$ is a solution to (15) that remains in \mathcal{O}_ε , then*

$$\frac{d}{dt} \Omega_\kappa(\Psi_{\text{hy}}(t)) = \Xi_\kappa(\Psi_{\text{hy}}(t)),$$

where $\Xi_\kappa : \mathcal{O}_\varepsilon \rightarrow \mathbb{R}$ is defined by

$$\Xi_\kappa(\mathcal{U}) \equiv - \left\langle \mathcal{F}'_{c_*+\bar{\gamma}(\mathcal{U})}(\mathcal{U}), \left\{ J\Upsilon_\kappa(\bar{\gamma}(\mathcal{U}))(\cdot + \bar{y}(\mathcal{U})) + (\mathcal{U}, \partial_\gamma \Upsilon_\kappa(\bar{\gamma}(\mathcal{U}))(\cdot + \bar{y}(\mathcal{U})))_H \mathcal{N}_{\bar{\gamma}}(\mathcal{U}) \right\} \right\rangle_{X^*, X}.$$

Proof. The fact that Ω_κ is of class \mathcal{C}^1 follows directly from the second expression and the fact that \bar{y} and $\bar{\gamma}$ are \mathcal{C}^1 (in [50] formula (3.11), the same remark as for the smoothness of G after Lemma 4.3 holds, since it requires " $u \in D(T'(0))$ "). If $\Psi_{\text{hy}} = (\eta, u) \in \mathcal{C}^1([0, T], X)$ is a solution to (15) that remains in \mathcal{O}_ε , we therefore have, denoting $\bar{\gamma}(t) = \bar{\gamma}(\Psi_{\text{hy}}(t))$ and $\bar{y}(t) = \bar{y}(\Psi_{\text{hy}}(t))$,

$$\begin{aligned} \frac{d}{dt} \Omega_\kappa(\Psi_{\text{hy}}(t)) &= (\partial_t \Psi_{\text{hy}}(t), \Upsilon_\kappa(\bar{\gamma}(t))(\cdot + \bar{y}(t)))_H \\ &\quad + (\Psi_{\text{hy}}(t), \partial_x \Upsilon_\kappa(\bar{\gamma}(t))(\cdot + \bar{y}(t)))_H \langle \bar{y}'(\Psi_{\text{hy}}(t)), \partial_t \Psi_{\text{hy}}(t) \rangle_{X^*, X} \\ &\quad + (\Psi_{\text{hy}}(t), \partial_\gamma \Upsilon_\kappa(\bar{\gamma}(t))(\cdot + \bar{y}(t)))_H \langle \bar{\gamma}'(\Psi_{\text{hy}}(t)), \partial_t \Psi_{\text{hy}}(t) \rangle_{X^*, X}. \end{aligned} \tag{25}$$

We now observe that the invariance of Ω_κ by translation provides by differentiation the equality, for $\mathcal{U} \in \mathcal{O}_\varepsilon$,

$$\begin{aligned} 0 &= \frac{d}{dy} \Omega_\kappa(\mathcal{U}(\cdot - y))|_{y=0} = (\mathcal{U}, \partial_x \Upsilon_\kappa(\bar{\gamma}(\mathcal{U}))(\cdot + \bar{y}(\mathcal{U})))_H = (\mathbb{B}\mathcal{U}, J\Upsilon_\kappa(\bar{\gamma}(\mathcal{U}))(\cdot + \bar{y}(\mathcal{U})))_H \\ &= \langle P'_{\text{hy}}(\mathcal{U}), J\Upsilon_\kappa(\bar{\gamma}(\mathcal{U}))(\cdot + \bar{y}(\mathcal{U})) \rangle_{X^*, X}. \end{aligned} \tag{26}$$

In particular, the second term in (25) vanishes. In addition, since $\Psi_{\text{hy}} = (\eta, u) \in \mathcal{C}^1([0, T], X)$ a solution to (15) that remains in \mathcal{O}_ε , there holds, denoting by $\frac{\delta E_{\text{hy}}}{\delta \Psi}$ the variational derivative,

$$\begin{aligned} (\partial_t \Psi_{\text{hy}}(t), \Upsilon_\kappa(\bar{\gamma}(t))(\cdot + \bar{y}(t)))_H &= \left(J \frac{\delta E_{\text{hy}}}{\delta \Psi_{\text{hy}}}(\Psi_{\text{hy}}(t)), \Upsilon_\kappa(\bar{\gamma}(t))(\cdot + \bar{y}(t)) \right)_H \\ &= - \left(\frac{\delta E_{\text{hy}}}{\delta \Psi_{\text{hy}}}(\Psi_{\text{hy}}(t)), J \Upsilon_\kappa(\bar{\gamma}(t))(\cdot + \bar{y}(t)) \right)_H = - \left\langle E'_{\text{hy}}(\Psi_{\text{hy}}(t)), J \Upsilon_\kappa(\bar{\gamma}(t))(\cdot + \bar{y}(t)) \right\rangle_{X^*, X} \\ &= - \langle \mathcal{F}'_{c_* + \bar{\gamma}(t)}(\Psi_{\text{hy}}(t)), J \Upsilon_\kappa(\bar{\gamma}(t))(\cdot + \bar{y}(t)) \rangle_{X^*, X} - (c_* + \bar{\gamma}(t)) \left\langle P'_{\text{hy}}(\Psi_{\text{hy}}(t)), J \Upsilon_\kappa(\bar{\gamma}(t))(\cdot + \bar{y}(t)) \right\rangle_{X^*, X} \\ &= - \langle \mathcal{F}'_{c_* + \bar{\gamma}(t)}(\Psi_{\text{hy}}(t)), J \Upsilon_\kappa(\bar{\gamma}(t))(\cdot + \bar{y}(t)) \rangle_{X^*, X}, \end{aligned}$$

by (26). In addition, since $\frac{\delta P_{\text{hy}}}{\delta \Psi_{\text{hy}}} = \mathbb{B} \Psi_{\text{hy}}$ and $J\mathbb{B} = \partial_x$,

$$\begin{aligned} \langle \bar{\gamma}'(\Psi_{\text{hy}}(t)), \partial_t \Psi_{\text{hy}}(t) \rangle_{X^*, X} &= \left\langle \bar{\gamma}'(\Psi_{\text{hy}}(t)), J \frac{\delta E_{\text{hy}}}{\delta \Psi_{\text{hy}}}(\Psi_{\text{hy}}(t)) \right\rangle_{X^*, X} \\ &= \left\langle \bar{\gamma}'(\Psi_{\text{hy}}(t)), J \frac{\delta \mathcal{F}_{c_* + \bar{\gamma}(\mathcal{U})}}{\delta \Psi_{\text{hy}}}(\Psi_{\text{hy}}(t)) \right\rangle_{X^*, X} + (c_* + \bar{\gamma}(\mathcal{U})) \left\langle \bar{\gamma}'(\Psi_{\text{hy}}(t)), \partial_x \Psi_{\text{hy}}(t) \right\rangle_{X^*, X}. \end{aligned}$$

The second term vanishes since $\bar{\gamma}$ is invariant by translation (by definition, see the proof of Lemma 4.3). As a consequence,

$$\begin{aligned} \langle \bar{\gamma}'(\Psi_{\text{hy}}(t)), \partial_t \Psi_{\text{hy}}(t) \rangle_{X^*, X} &= \left(J \frac{\delta \mathcal{F}_{c_* + \bar{\gamma}(t)}}{\delta \Psi_{\text{hy}}}(\Psi_{\text{hy}}(t)), \mathbb{I}^{-1} \bar{\gamma}'(\Psi_{\text{hy}}(t)) \right)_X \\ &= - \left(\frac{\delta \mathcal{F}_{c_* + \bar{\gamma}(t)}}{\delta \Psi_{\text{hy}}}(\Psi_{\text{hy}}(t)), J \mathbb{I}^{-1} \bar{\gamma}'(\Psi_{\text{hy}}(t)) \right)_X \\ &= - \left(\frac{\delta \mathcal{F}_{c_* + \bar{\gamma}(t)}}{\delta \Psi_{\text{hy}}}(\Psi_{\text{hy}}(t)), J \mathbb{I}^{-1} \bar{\gamma}'(\Psi_{\text{hy}}(t)) \right)_H \\ &\quad - \left(\partial_x \frac{\delta \mathcal{F}_{c_* + \bar{\gamma}(t)}}{\delta \eta}(\Psi_{\text{hy}}(t)), \partial_x J \mathbb{I}_{H^1}^{-1} \frac{\partial \bar{\gamma}}{\partial \eta}(\Psi_{\text{hy}}(t)) \right)_{L^2}. \end{aligned}$$

The first term is simply $-\langle \mathcal{F}'_{c_* + \bar{\gamma}(\mathcal{U})}(\Psi_{\text{hy}}(t)), J \mathbb{I}^{-1} \bar{\gamma}'(\Psi_{\text{hy}}(t)) \rangle_{X^*, X}$. We then define $\mathcal{N}_{\bar{\gamma}} : \mathcal{O}_\varepsilon \rightarrow X$ by $\mathcal{N}_{\bar{\gamma}}(\mathcal{U}) \equiv J \mathbb{I}^{-1} \bar{\gamma}'(\mathcal{U}) - (\partial_x^2 J \mathbb{I}_{H^1}^{-1} \frac{\delta \bar{\gamma}}{\delta \eta}(\mathcal{U}), 0) \in X = H^1 \times L^2$ (see the regularity shown for $\bar{\gamma}'$ in Lemma 4.3), so that integration by parts yields

$$\langle \bar{\gamma}'(\Psi_{\text{hy}}(t)), \partial_t \Psi_{\text{hy}}(t) \rangle_{X^*, X} = - \langle \mathcal{F}'_{c_* + \bar{\gamma}(\mathcal{U})}(\Psi_{\text{hy}}(t)), \mathcal{N}_{\bar{\gamma}}(\Psi_{\text{hy}}(t)) \rangle_{X^*, X}.$$

Inserting these relations into (25) then gives

$$\begin{aligned} \frac{d}{dt} \Omega_\kappa(\Psi_{\text{hy}}(t)) &= - \left\langle \mathcal{F}'_{c_* + \bar{\gamma}(t)}(\Psi_{\text{hy}}(t)), \right. \\ &\quad \left. \left\{ J \Upsilon_\kappa(\bar{\gamma}(t))(\cdot + \bar{y}(t)) + (\Psi_{\text{hy}}(t), \partial_\gamma \Upsilon_\kappa(\bar{\gamma}(t))(\cdot + \bar{y}(t)))_H \mathcal{N}_{\bar{\gamma}}(\Psi_{\text{hy}}(t)) \right\} \right\rangle_{X^*, X}, \end{aligned}$$

which is the desired equality.

If $\Psi_{\text{hy}} \in \mathcal{C}^0([0, T], X)$ is just a continuous in time solution to (15) that remains in \mathcal{O}_ε , then the integrated relation

$$\Omega_\kappa(\Psi_{\text{hy}}(t)) = \Omega_\kappa(\Psi_{\text{hy}}^{\text{in}}) + \int_0^t \Xi_\kappa(\Psi_{\text{hy}}(\tau)) \, d\tau$$

holds, as can be seen by using the continuity of the flow and the approximation of such a solution by smoother ones (see [24]). \square

We now compute the asymptotics of $\Xi_\kappa(\mathcal{W}(\gamma))$ for $\gamma \rightarrow 0$ and small κ .

Lemma 4.9 *We have*

$$\Xi_\kappa(\mathcal{W}(\gamma)) = -\frac{\gamma^2 \ddot{P}_*}{2} + o_{(\gamma, \kappa) \rightarrow (0, 0)}(\gamma^2).$$

Proof. The proof follows the one of Lemma 5 in [50]. As a first step, notice that $\bar{\gamma}(\mathcal{W}(\gamma)) = \gamma$, $\bar{y}(\mathcal{W}(\gamma)) = 0$, as it can be seen from the equality $G(\mathcal{W}(\gamma), 0, 0, 0) = 0$ and the local uniqueness of the solution to $G = 0$. Therefore, since $\mathcal{F}'_{c_* + \gamma}(\mathcal{U}_{c_* + \gamma}) = 0$ and $\sigma(\gamma) \sim -\gamma^2 \ddot{P}_*/(2\|\mathcal{U}_*\|_H^2)$,

$$\begin{aligned} \mathcal{F}'_{c_* + \bar{\gamma}(\mathcal{W}(\gamma))}(\mathcal{W}(\gamma)) &= \mathcal{F}'_{c_* + \gamma}(\mathcal{U}_{c_* + \gamma} + \sigma(\gamma)\mathbb{B}\mathcal{U}_{c_* + \gamma}) = \sigma(\gamma)\mathcal{F}''_{c_* + \gamma}(\mathcal{U}_{c_* + \gamma})[\mathbb{B}\mathcal{U}_{c_* + \gamma}] + o_{\gamma \rightarrow 0}(\gamma^2) \\ &= -\frac{\gamma^2 \ddot{P}_*}{2\|\mathcal{U}_*\|_H^2} \mathcal{F}''_*(\mathcal{U}_*)[\mathbb{B}\mathcal{U}_*] + o_{\gamma \rightarrow 0}(\gamma^2). \end{aligned}$$

In addition, since \mathcal{U}_c is even and $\Upsilon_\kappa(\gamma)$ is odd, we deduce

$$(\mathcal{W}(\gamma)(\cdot + \bar{y}(\mathcal{W}(\gamma))), \partial_\gamma \Upsilon_\kappa(\gamma))_H = (\mathcal{U}_{c_* + \gamma} + \sigma(\gamma)\mathbb{B}\mathcal{U}_{c_* + \gamma}, \partial_\gamma \Upsilon_\kappa(\gamma))_H = 0.$$

Consequently,

$$\Xi_\kappa(\mathcal{W}(\gamma)) = \frac{\gamma^2 \ddot{P}_*}{2\|\mathcal{U}_*\|_H^2} \langle \mathcal{F}''_*(\mathcal{U}_*)[\mathbb{B}\mathcal{U}_*], J\Upsilon_\kappa(\gamma) \rangle_{X^*, X} + o_{\gamma \rightarrow 0}(\gamma^2),$$

where “ $o_{\gamma \rightarrow 0}(\gamma^2)$ ” does not depend on κ . Moreover, Lemma 4.7 provides $\|J\Upsilon_\kappa(\gamma) - [\partial_c \mathcal{U}_c]_{|c=c_* + \gamma}\|_X \leq \kappa$ independently of $\gamma \in (-\gamma_1, +\gamma_1)$, hence

$$\begin{aligned} \Xi_\kappa(\mathcal{W}(\gamma)) &= \frac{\gamma^2 \ddot{P}_*}{2\|\mathcal{U}_*\|_H^2} \langle \mathcal{F}''_*(\mathcal{U}_*)[\mathbb{B}\mathcal{U}_*], [\partial_c \mathcal{U}_c]_{|c=c_* + \gamma} \rangle_{X^*, X} + o_{(\gamma, \kappa) \rightarrow (0, 0)}(\gamma^2) \\ &= \frac{\gamma^2 \ddot{P}_*}{2\|\mathcal{U}_*\|_H^2} \langle \mathcal{F}''_*(\mathcal{U}_*)[\mathbb{B}\mathcal{U}_*], [\partial_c \mathcal{U}_c]_{|c=c_*} \rangle_{X^*, X} + o_{(\gamma, \kappa) \rightarrow (0, 0)}(\gamma^2). \end{aligned}$$

Finally, using once again the equality (for $\phi \in X$) $\langle \mathcal{F}''_*(\mathcal{U}_*)[\partial_c \mathcal{U}_c]_{|c=c_*}, \phi \rangle_{X^*, X} = (\mathbb{B}\mathcal{U}_*, \phi)_H$ and that \mathcal{F}''_* is self-adjoint, we infer

$$\langle \mathcal{F}''_*(\mathcal{U}_*)[\mathbb{B}\mathcal{U}_*], [\partial_c \mathcal{U}_c]_{|c=c_*} \rangle_{X^*, X} = \langle \mathcal{F}''_*(\mathcal{U}_*)([\partial_c \mathcal{U}_c]_{|c=c_*}), \mathbb{B}\mathcal{U}_* \rangle_{X^*, X} = \|\mathbb{B}\mathcal{U}_*\|_H^2 = \|\mathcal{U}_*\|_H^2,$$

and reporting this into the previous expression gives the result. \square

We now compute the asymptotics of Ξ_κ for more general functions.

Lemma 4.10 *Let $\varepsilon > 0$ be small enough. If $\mathcal{U} \in \mathcal{O}_\varepsilon$ satisfies $P_{\text{hy}}(\mathcal{U}) = P_{\text{hy}}(\mathcal{U}_*)$ and $\mathcal{F}_*(\mathcal{U}) - \mathcal{F}_*(\mathcal{U}_*) < 0$, then, we have*

$$\Xi_\kappa(\mathcal{U}) = -\frac{\bar{\gamma}^2(\mathcal{U}) \ddot{P}_*}{2} + o(\bar{\gamma}^2(\mathcal{U}))$$

uniformly for $0 < \kappa \leq |\bar{\gamma}(\mathcal{U})|^3$.

Proof. First, we may apply Lemma 4.6 and infer that $\|\vartheta(\mathcal{U})\|_X^2 + |\alpha(\mathcal{U})| = \mathcal{O}(|\bar{\gamma}^3(\mathcal{U})|)$. Then, we write $\Xi_\kappa(\mathcal{U}) = \Xi_\kappa(\mathcal{U}(\cdot - \bar{y}(\mathcal{U}))) = \Xi_\kappa(\mathcal{W}(\bar{\gamma}(\mathcal{U})) + \vartheta(\mathcal{U}) + \alpha(\mathcal{U})\mathbb{B}\mathcal{U}_{c_* + \bar{\gamma}(\mathcal{U})}) = \Xi_\kappa(\mathcal{W}(\bar{\gamma}(\mathcal{U})) + \vartheta(\mathcal{U})) + \mathcal{O}(|\bar{\gamma}(\mathcal{U})|^3)$ and, recalling the expression

$$\Xi_\kappa(\mathcal{U}) = -\left\langle \mathcal{F}'_{c_* + \bar{\gamma}(\mathcal{U})}(\mathcal{U}), \left\{ J\Upsilon_\kappa(\bar{\gamma}(\mathcal{U}))(\cdot + \bar{y}(\mathcal{U})) + (\mathcal{U}, \partial_\gamma \Upsilon_\kappa(\bar{\gamma}(\mathcal{U}))(\cdot + \bar{y}(\mathcal{U})))_H \mathcal{N}_{\bar{\gamma}}(\mathcal{U}) \right\} \right\rangle_{X^*, X},$$

we wish to make a Taylor expansion. First, note that

$$\begin{aligned} \mathcal{F}'_{c_* + \bar{\gamma}(\mathcal{U})}(\mathcal{U}) &= \mathcal{F}'_{c_* + \bar{\gamma}(\mathcal{U})}(\mathcal{W}(\bar{\gamma}(\mathcal{U}))) + \mathcal{F}''_{c_* + \bar{\gamma}(\mathcal{U})}(\mathcal{W}(\bar{\gamma}(\mathcal{U})))[\vartheta(\mathcal{U})] + \mathcal{O}(|\bar{\gamma}(\mathcal{U})|^3) \\ &= \mathcal{F}'_{c_* + \bar{\gamma}(\mathcal{U})}(\mathcal{W}(\bar{\gamma}(\mathcal{U}))) + \mathcal{F}''_{c_* + \bar{\gamma}(\mathcal{U})}(\mathcal{U}_{c_* + \bar{\gamma}(\mathcal{U})})[\vartheta(\mathcal{U})] + \mathcal{O}(|\bar{\gamma}(\mathcal{U})|^3) \end{aligned}$$

hence, since $\mathcal{F}'_{c_*+\bar{\gamma}}(\mathcal{U}_{c_*+\bar{\gamma}}) = 0$ and (Lemma 4.1) $\sigma(\gamma) = \mathcal{O}(\gamma^2)$, we have $\mathcal{W}(\bar{\gamma}) = \mathcal{U}_{c_*+\bar{\gamma}} + \mathcal{O}(\gamma^2)$, thus

$$\begin{aligned} \Xi_\kappa(\mathcal{U}) - \Xi_\kappa(\mathcal{W}(\bar{\gamma}(\mathcal{U}))) &= \mathcal{O}(|\bar{\gamma}(\mathcal{U})|^3) \\ &\quad - \left\langle \mathcal{F}''_{c_*+\bar{\gamma}(\mathcal{U})}(\mathcal{U}_{c_*+\bar{\gamma}(\mathcal{U})})[\vartheta(\mathcal{U})], \left\{ J\Upsilon_\kappa(\bar{\gamma}(\mathcal{U}))(\cdot + \bar{y}(\mathcal{U})) + (\mathcal{U}, \partial_\gamma \Upsilon_\kappa(\bar{\gamma}(\mathcal{U}))(\cdot + \bar{y}(\mathcal{U})))_{H\mathcal{N}_{\bar{\gamma}}(\mathcal{U})} \right\} \right\rangle_{X^*, X}. \end{aligned}$$

Now, in the bracket term, we may replace \mathcal{U} by $\mathcal{W}(\bar{\gamma}(\mathcal{U})) + \mathcal{O}(\|\vartheta(\mathcal{U})\|_X)$ (since $\|\vartheta(\mathcal{U})\|_X^2 = \mathcal{O}(|\bar{\gamma}(\mathcal{U})|^3)$). By the computations of Lemma 4.9 and the equalities $\bar{\gamma}(\mathcal{W}(\gamma)) = \gamma$, $\bar{y}(\mathcal{W}(\gamma)) = 0$, this gives

$$\begin{aligned} \Xi_\kappa(\mathcal{U}) - \Xi_\kappa(\mathcal{W}(\bar{\gamma}(\mathcal{U}))) &= \mathcal{O}(|\bar{\gamma}(\mathcal{U})|^3) - \left\langle \mathcal{F}''_{c_*+\bar{\gamma}(\mathcal{U})}(\mathcal{U}_{c_*+\bar{\gamma}(\mathcal{U})})[\vartheta(\mathcal{U})], J\Upsilon_\kappa(\bar{\gamma}(\mathcal{U})) \right\rangle_{X^*, X} \\ &= \mathcal{O}(|\bar{\gamma}(\mathcal{U})|^3) - \left\langle \mathcal{F}''_{c_*+\bar{\gamma}(\mathcal{U})}(\mathcal{U}_{c_*+\bar{\gamma}(\mathcal{U})})[\vartheta(\mathcal{U})], \{\partial_c \mathcal{U}_c\}_{|c=c_*+\bar{\gamma}(\mathcal{U})} \right\rangle_{X^*, X} + \mathcal{O}(\kappa) \\ &= \mathcal{O}(|\bar{\gamma}(\mathcal{U})|^3) - \left\langle \mathcal{F}''_{c_*+\bar{\gamma}(\mathcal{U})}(\mathcal{U}_{c_*+\bar{\gamma}(\mathcal{U})})[\{\partial_c \mathcal{U}_c\}_{|c=c_*+\bar{\gamma}(\mathcal{U})}], \vartheta(\mathcal{U}) \right\rangle_{X^*, X} + \mathcal{O}(\kappa) \end{aligned}$$

using Lemma 4.7 and the self-adjointness of $\mathcal{F}''_{c_*+\bar{\gamma}(\mathcal{U})}$. Choosing $0 < \kappa \leq |\bar{\gamma}(\mathcal{U})|^3$ and from the equality (for $\phi \in X$) $\langle \mathcal{F}''_c(\mathcal{U}_c)[\partial_c \mathcal{U}_c], \phi \rangle_{X^*, X} = (\mathbb{B}\mathcal{U}_c, \phi)_H$, we infer

$$\Xi_\kappa(\mathcal{U}) - \Xi_\kappa(\mathcal{W}(\bar{\gamma}(\mathcal{U}))) = \mathcal{O}(|\bar{\gamma}(\mathcal{U})|^3) - (\mathbb{B}\mathcal{U}_{c_*+\bar{\gamma}(\mathcal{U})}, \vartheta(\mathcal{U}))_H = \mathcal{O}(|\bar{\gamma}(\mathcal{U})|^3),$$

by the orthogonality condition in Lemma 4.4. Inserting the expansion of $\Xi_\kappa(\mathcal{W}(\gamma))$ given in Lemma 4.9 yields the conclusion. \square

Proof of Theorem 5. We have to show that there exists $\varepsilon > 0$ such that, for any $\delta > 0$, we can choose an initial datum at distance $\leq \delta$ from \mathcal{U}_* but that escape from \mathcal{O}_ε . Since $\mathcal{W}(\gamma) \rightarrow \mathcal{U}_*$ in X , we shall take the initial datum to be $\mathcal{W}(\gamma)$ for some small γ , and denote $\Psi_{\text{hy}}(t)$ the corresponding solution. In view of Lemma 4.2, we have $\mathcal{F}_*(\mathcal{W}(\gamma)) - \mathcal{F}_*(\mathcal{U}_*) \sim -\gamma^3 \ddot{P}_*/6$, hence we can choose γ with the sign of $\ddot{P}_* \neq 0$ so that

$$\mathcal{F}_*(\mathcal{W}(\gamma)) - \mathcal{F}_*(\mathcal{U}_*) \sim -|\gamma|^3 |\ddot{P}_*|/6 < 0.$$

We now assume that $\Psi_{\text{hy}}(t)$ is globally defined and remains in \mathcal{O}_ε , where ε is as in Lemma 4.6. By conservation of energy and momentum and the construction of $\mathcal{W}(\gamma)$, we deduce $P_{\text{hy}}(\Psi_{\text{hy}}(t)) = P_{\text{hy}}(\mathcal{W}(\gamma)) = P_{\text{hy}}(\mathcal{U}_*)$, and $\mathcal{F}_*(\Psi_{\text{hy}}(t)) - \mathcal{F}_*(\mathcal{U}_*) = \mathcal{F}_*(\mathcal{W}(\gamma)) - \mathcal{F}_*(\mathcal{U}_*) < 0$. The first step is to have a control on $\bar{\gamma}(t) \equiv \bar{\gamma}(\Psi_{\text{hy}}(t))$. We denote $\alpha(t) = \alpha(\Psi_{\text{hy}}(t))$, $\bar{y}(t) = \bar{y}(\Psi_{\text{hy}}(t))$ and $\vartheta(t) = \vartheta(\Psi_{\text{hy}}(t))$. Applying Lemma 4.6, we obtain $\|\vartheta(t)\|_X^2 + |\alpha(t)| = \mathcal{O}(|\bar{\gamma}^3(t)|)$. In addition, Lemma 4.2 and Taylor expansion gives

$$\begin{aligned} \mathcal{F}_*(\Psi_{\text{hy}}(t)) - \mathcal{F}_*(\mathcal{U}_*) &= \mathcal{F}_*(\mathcal{W}(\bar{\gamma}(t)) + \vartheta(t) + \alpha(t)\mathbb{B}\mathcal{U}_{c_*+\bar{\gamma}(t)}) - \mathcal{F}_*(\mathcal{U}_*) \\ &= \mathcal{F}_*(\mathcal{W}(\bar{\gamma}(t))) - \mathcal{F}_*(\mathcal{U}_*) + \langle \mathcal{F}'_*(\mathcal{W}(\bar{\gamma}(t))), \vartheta(t) \rangle_{X^*, X} + \frac{1}{2} \langle \mathcal{F}''_*(\mathcal{W}(\bar{\gamma}(t)))\vartheta(t), \vartheta(t) \rangle_{X^*, X} + o(|\bar{\gamma}^3(t)|) \\ &= -\frac{\bar{\gamma}^3(t)\ddot{P}_*}{6} + \langle \mathcal{F}'_*(\mathcal{W}(\bar{\gamma}(t))), \vartheta(t) \rangle_{X^*, X} + \frac{1}{2} \langle \mathcal{F}''_*(\mathcal{W}(\bar{\gamma}(t)))\vartheta(t), \vartheta(t) \rangle_{X^*, X} + o(|\bar{\gamma}^3(t)|), \end{aligned}$$

where we have used that $\mathcal{F}'_*(\mathcal{W}(\bar{\gamma}(t))) = o(1)$ (for the terms involving $\alpha(t)$) and Lemma 4.2. Furthermore, by the orthogonality relations in Lemma 4.3 and using that $\sigma(\gamma) = \mathcal{O}(\gamma^2)$ and $\mathcal{F}'_c(\mathcal{U}_c) = 0$, it holds

$$\begin{aligned} \langle \mathcal{F}'_*(\mathcal{W}(\bar{\gamma}(t))), \vartheta(t) \rangle_{X^*, X} &= \langle \mathcal{F}'_{c_*+\bar{\gamma}(t)}(\mathcal{W}(\bar{\gamma}(t))), \vartheta(t) \rangle_{X^*, X} + \bar{\gamma}(t)(\mathbb{B}\mathcal{W}(\bar{\gamma}(t)), \vartheta(t))_H \\ &= \langle \mathcal{F}'_{c_*+\bar{\gamma}(t)}(\mathcal{U}_{c_*+\bar{\gamma}(t)} + \sigma(\bar{\gamma}(t)))\mathbb{B}\mathcal{U}_{c_*+\bar{\gamma}(t)}, \vartheta(t) \rangle_{X^*, X} = \mathcal{O}(|\bar{\gamma}^{7/2}(t)|). \end{aligned}$$

In addition, by Lemma 4.4, the before last term is $\geq K_0\|\vartheta(t)\|_X^2/2$. As a consequence, by conservation of $\mathcal{F}_*(\Psi_{\text{hy}}(t))$, we infer, for small γ ,

$$0 > -|\gamma|^3 |\ddot{P}_*|/3 > \mathcal{F}_*(\mathcal{W}(\gamma)) - \mathcal{F}_*(\mathcal{U}_*) = \mathcal{F}_*(\Psi_{\text{hy}}(t)) - \mathcal{F}_*(\mathcal{U}_*) \geq -\frac{\bar{\gamma}^3(t)\ddot{P}_*}{6} + o(|\bar{\gamma}^3(t)|),$$

In particular, this forces $\bar{\gamma}(t)$ to always be of the sign of \ddot{P}_* and to satisfy $|\bar{\gamma}(t)| \geq |\gamma|/2$ (provided ε and γ are small enough).

Since now, we have a good upper bound for $|\bar{\gamma}(t)|$, we can choose $\kappa = \kappa(\gamma) \equiv \gamma^3/8$, which is such that, for any $t \geq 0$, $\kappa \leq |\bar{\gamma}(t)|^3$. In particular, we can apply Lemma 4.10 and get

$$\Xi_\kappa(\Psi_{\text{hy}}(t)) = -\frac{\bar{\gamma}(t)^2 \ddot{P}_*}{2} + o(\bar{\gamma}(t)^2).$$

With this choice $\kappa = \kappa(\gamma)$, we deduce from Lemma 4.8 that

$$\frac{d}{dt} \Omega_{\kappa(\gamma)}(\Psi_{\text{hy}}(t)) = \Xi_\kappa(\Psi_{\text{hy}}(t)) = -\frac{\bar{\gamma}(t)^2 \ddot{P}_*}{2} + o(\bar{\gamma}(t)^2).$$

Since $|\bar{\gamma}(t)| \geq |\gamma|/2$, it follows that, when $\ddot{P}_* < 0$ (the case $\ddot{P}_* > 0$ is analogous),

$$\frac{d}{dt} \Omega_{\kappa(\gamma)}(\Psi_{\text{hy}}(t)) \geq -\frac{\gamma^2 \ddot{P}_*}{8} > 0,$$

hence $\Omega_{\kappa(\gamma)}(\Psi_{\text{hy}}(t))$ is unbounded as t goes to $+\infty$. However, by definition of Ω_κ , we have by Cauchy-Schwarz $|\Omega_{\kappa(\gamma)}(\mathcal{U})| \leq \|\mathcal{U}\|_H \|\Upsilon_{\kappa(\gamma)}\|_H \leq C(\gamma)$ for $\mathcal{U} \in \mathcal{O}_\varepsilon$. We have reached a contradiction. The proof of Theorem 5 is complete. \square

5 The linear instability ($0 < \mathbf{c}_* < \mathbf{c}_s$)

5.1 Proof of Theorem 4

Existence of at least one unstable eigenvalue. The proof of the existence of at least one unstable eigenvalue relies on Evans function technique, as in [70], [6]. We shall actually use Theorem 1 in [6] when observing (see *e.g.* [5]) that the Euler-Korteweg system

$$\begin{cases} \partial_t \rho + 2\partial_x(\rho u) = 0 \\ \partial_t u + 2u\partial_x u - \partial_x(f(\rho)) - \partial_x\left(K(\rho)\partial_x^2 \rho + \frac{1}{2}K'(\rho)(\partial_x \rho)^2\right) = 0, \end{cases} \quad (\text{EK})$$

where $K : (0, +\infty) \rightarrow (0, +\infty)$ is the (smooth enough) capillarity, reduces to (2) (where, we recall, $\Psi = Ae^{i\phi}$, $\rho = A^2$ and $u = \partial_x \phi$), namely

$$\begin{cases} \partial_t \rho + 2\partial_x(\rho u) = 0 \\ \partial_t u + 2u\partial_x u - \partial_x(f(\rho)) - \partial_x\left(\frac{\partial_x^2(\sqrt{\rho})}{\sqrt{\rho}}\right) = 0, \end{cases}$$

for the capillarity $K(\rho) = 1/(2\rho)$, as can be shown by straightforward computations. The associated eigenvalue problem in the moving frame is

$$\begin{cases} \lambda \zeta - c_* \partial_x \zeta + 2\partial_x((r_0^2 + \eta_*)v + \zeta u_*) = 0 \\ \lambda v - c_* \partial_x v + 2\partial_x(u_* v) - \partial_x(f'(r_0^2 + \eta_*)\zeta) - \partial_x\left\{\frac{1}{2\sqrt{r_0^2 + \eta_*}}\partial_x^2\left(\frac{\zeta}{\sqrt{r_0^2 + \eta_*}}\right) - \frac{\zeta\partial_x^2(\sqrt{r_0^2 + \eta_*})}{2(r_0^2 + \eta_*)^{3/2}}\right\} = 0. \end{cases} \quad (27)$$

The link with the original eigenvalue problem (6) is done through the formula

$$w = U_* \left(\frac{\zeta}{2} + i \int_{-\infty}^x v \right), \quad (28)$$

since this corresponds to $\Psi = U_{c_*} + \psi = U_{c_*} + e^{\lambda t} w(x) = (A_{c_*} + e^{\lambda t} \zeta(x)) \exp(i\phi_{c_*} + ie^{\lambda t} \int_{-\infty}^x v)$. Notice indeed that the second equation in (27) gives $\int_{\mathbb{R}} v \, dx = 0$. It then follows from Theorem 1 in [6] that under the assumption $\left. \frac{dP(U_c)}{dc} \right|_{c=c_*} > 0$, there exists at least one unstable eigenvalue $\gamma_0 \in (0, +\infty)$.

Existence of at most one unstable eigenvalue. The fact that there exists at most one unstable eigenvalue follows from arguments as in [7] (Appendix B) and is a direct consequence of Theorem 3.1 in [57], that we recall now.

Theorem 11 ([57]) *Let \mathcal{J} and \mathcal{L} be two operators on a real Hilbert space X , with \mathcal{L} self-adjoint and \mathcal{J} skew-symmetric. Then, the number of eigenvalues, counting algebraic multiplicities, of $[\mathcal{J}\mathcal{L}]_{\mathbb{C}}$ in the right-half plane $\{\operatorname{Re} > 0\}$ is less than or equal to the number of negative eigenvalues of \mathcal{L} , counting multiplicities.*

In order to apply this result to our problem, let us write the eigenvalue problem (27) under the form

$$\lambda \begin{pmatrix} \zeta \\ v \end{pmatrix} = -\partial_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathcal{L} \begin{pmatrix} \zeta \\ v \end{pmatrix},$$

where \mathcal{M} is the self-adjoint Sturm-Liouville operator

$$\mathcal{M} \equiv -f'(r_0^2 + \eta_*) - \frac{1}{2\sqrt{r_0^2 + \eta_*}} \partial_x^2 \left(\frac{\cdot}{\sqrt{r_0^2 + \eta_*}} \right) + \frac{\partial_x^2(\sqrt{r_0^2 + \eta_*})}{2(r_0^2 + \eta_*)^{3/2}}$$

(which is bounded from below) on $\mathcal{H} \equiv L^2 \times L^2$ and with

$$\mathcal{L} \equiv \begin{pmatrix} \mathcal{M} & 2u_* - c_* \\ 2u_* - c_* & 2(r_0^2 + \eta_*) \end{pmatrix}.$$

We are in the setting of Theorem 11 with $\mathcal{J} = -\partial_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ skew-symmetric and \mathcal{L} self-adjoint. We thus show that \mathcal{L} has at most one negative eigenvalue. Since $r_0^2 + \eta_*$ remains bounded away from zero, it is clear that, for $\sigma < 0$ and (ζ, v) given, $\mathcal{L}(\zeta, v)^t = \sigma(\zeta, v)^t$ if and only if

$$\mathcal{M}^\dagger \zeta - \frac{(c_* - 2u_*)^2}{2(r_0^2 + \eta_*)} \cdot \frac{\sigma}{2(r_0^2 + \eta_*) - \sigma} \zeta = \sigma \zeta, \quad \text{with} \quad \mathcal{M}^\dagger \equiv \mathcal{M} - \frac{(c_* - 2u_*)^2}{2(r_0^2 + \eta_*)}, \quad (29)$$

since we may express v in terms of ζ with the second equation. We observe that the translation invariance shows that $\partial_x(\eta_*, u_*)^t$ belongs to the kernel of \mathcal{L} , that is, using once again the relation $2u_c = 2\partial_x \phi_c = c \frac{\eta_c}{\eta_c + r_0^2}$, $\mathcal{M}^\dagger \partial_x \eta_* = 0$. Furthermore, \mathcal{M}^\dagger has the same continuous spectrum as its constant coefficient limit as $x \rightarrow \pm\infty$, namely

$$-\frac{1}{2r_0^2} \partial_x^2 + \frac{c_s^2 - c_*^2}{2r_0^2},$$

that is $\sigma_{\text{ess}}(\mathcal{M}^\dagger) = [c_s^2 - c_*^2, +\infty) \subset (0, +\infty)$, since $0 < c_* < c_s$. Since $\partial_x \eta_*$ has exactly one zero (at $x = 0$), it follows from standard Sturm-Liouville theory that \mathcal{M}^\dagger has precisely one negative eigenvalue $\mu < 0$ and that the second eigenvalue is 0. Taking the scalar product with (29) yields

$$\langle \mathcal{M}^\dagger \zeta, \zeta \rangle_{L^2} - \int_{\mathbb{R}} \frac{\sigma(c_* - 2u_*)^2 \zeta^2}{2(r_0^2 + \eta_*)[2(r_0^2 + \eta_*) - \sigma]} \, dx = \sigma \|\zeta\|_{L^2}^2.$$

Now, for $s \leq 0$, we consider the self-adjoint operator

$$\mathcal{M}_s^\dagger \equiv \mathcal{M}^\dagger - \frac{(c_* - 2u_*)^2}{2(r_0^2 + \eta_*)} \cdot \frac{s}{2(r_0^2 + \eta_*) - s}.$$

Clearly, $\mathcal{M}_{s=0}^\dagger = \mathcal{M}^\dagger$, $\sigma_{\text{ess}}(\mathcal{M}_s^\dagger) \subset [c_s^2 - c_*^2, +\infty) \subset (0, +\infty)$, and $\mathbb{R}_- \ni s \mapsto \mathcal{M}_s^\dagger$ is decreasing. Let us assume now that the self-adjoint operator \mathcal{L} has at least two negative eigenvalues. Then, we denote $\sigma_1 < \sigma_2 < 0$ the two smallest eigenvalues of \mathcal{L} (necessarily simple), and ζ_1, ζ_2 two associated eigenvectors. Since \mathcal{L} is self-adjoint, $\langle \zeta_1, \zeta_2 \rangle_{L^2} = 0$. Furthermore, $\langle \mathcal{M}_{s=\sigma_1}^\dagger \zeta_1, \zeta_1 \rangle_{L^2} = \sigma_1 \|\zeta_2\|_{L^2}^2 < 0$, hence, by monotonicity, $\langle \mathcal{M}_s^\dagger \zeta_2, \zeta_2 \rangle_{L^2} < 0$ for any $\sigma_1 \leq s \leq 0$. Therefore, \mathcal{M}_s^\dagger has at least one negative eigenvalue for $\sigma_1 \leq s \leq 0$. We denote $\lambda_{\min}(s)$ the smallest eigenvalue of \mathcal{M}_s^\dagger . Then, $\lambda_{\min}(s=0) = \mu < 0$ and λ_{\min} decreases in $[\sigma_1, 0]$. Moreover, we may choose a positive eigenvector ζ_s for the eigenvalue $\lambda_1(s)$, with $\zeta_1 = \zeta_{\sigma_1}$. Since $\sigma(\mathcal{M}^\dagger) \cap \mathbb{R}_+ = \{\mu, 0\}$, it follows from the monotonicity that for any $\sigma_1 \leq s < 0$, we have $\sigma(\mathcal{M}_s^\dagger) \cap \mathbb{R}_- = \{\lambda_{\min}(s)\}$. When $s = \sigma_2 \in (\sigma_1, 0)$, we then have $\sigma_2 \in \sigma(\mathcal{M}_{s=\sigma_2}^\dagger) \cap \mathbb{R}_-$, and thus $\sigma_2 = \lambda_{\min}(\sigma_2)$, which implies that we may choose $\zeta_2 > 0$ without loss of generality. Similarly, if $s = \sigma_2$, we see that we may choose $\zeta_2 > 0$. We obtain a contradiction since then $\langle \zeta_1, \zeta_2 \rangle_{L^2} > 0$ and thus ζ_1 and ζ_2 cannot be orthogonal in L^2 . We have thus shown that \mathcal{L} has at most one negative eigenvalue, and then Theorem 11 shows that \mathcal{JL} has at most one eigenvalue in $\{\text{Re} > 0\}$, as wished.

5.2 Resolvent and semigroup estimates (proof of Corollary 3)

In this section, we drop the “*” for the travelling wave we are considering. When linearizing the (NLS) equation in the moving frame with speed c , we obtain

$$i \frac{\partial \psi}{\partial t} - ic \partial_x \psi + \partial_x^2 \psi + \psi f(|U|^2) + 2 \langle \psi, U \rangle f'(|U|^2) U = 0, \quad (30)$$

or

$$\begin{aligned} \frac{\partial}{\partial t} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} &= \begin{pmatrix} c \partial_x - 2f'(|U|^2) U_1 U_2 & -\partial_x^2 - f(|U|^2) - 2f'(|U|^2) U_2^2 \\ \partial_x^2 + f(|U|^2) + 2f'(|U|^2) U_1^2 & c \partial_x + 2f'(|U|^2) U_1 U_2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -\partial_x^2 - f(|U|^2) - 2f'(|U|^2) U_1^2 & -c \partial_x - 2f'(|U|^2) U_1 U_2 \\ c \partial_x - 2f'(|U|^2) U_1 U_2 & -\partial_x^2 - f(|U|^2) - 2f'(|U|^2) U_2^2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \end{aligned}$$

We wish to show that this linear equation can be solved using a continuous semigroup. In order to handle later the nonlinear terms, we work in $H^1(\mathbb{R}, \mathbb{C}^2)$ instead of $L^2(\mathbb{R}, \mathbb{C}^2)$. Therefore, we consider the unbounded operator $\mathcal{A} : D(\mathcal{A}) = H^3(\mathbb{R}, \mathbb{C}^2) \subset H^1(\mathbb{R}, \mathbb{C}^2) \rightarrow H^1(\mathbb{R}, \mathbb{C}^2)$ on $H^1(\mathbb{R}, \mathbb{C}^2)$ defined by

$$\mathcal{A} \equiv \begin{pmatrix} c \partial_x - 2f'(|U|^2) U_1 U_2 & -\partial_x^2 - f(|U|^2) - 2f'(|U|^2) U_2^2 \\ \partial_x^2 + f(|U|^2) + 2f'(|U|^2) U_1^2 & c \partial_x + 2f'(|U|^2) U_1 U_2 \end{pmatrix}.$$

It follows easily that for $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in H^1(\mathbb{R}, \mathbb{C}^2)$,

$$\begin{aligned} \text{Re} \left(\langle \mathcal{A} \psi | \psi \rangle_{H^1(\mathbb{R}, \mathbb{C}^2)} \right) &= \text{Re} \left(\langle -2f'(|U|^2) U_1 U_2 \psi_1, \psi_1 \rangle_{H^1(\mathbb{R}, \mathbb{C})} + \langle -2f'(|U|^2) U_1 U_2 \psi_2, \psi_2 \rangle_{H^1(\mathbb{R}, \mathbb{C})} \right. \\ &\quad \left. + \langle [f(|U|^2) + 2f'(|U|^2) U_1^2] \psi_1, \psi_2 \rangle_{H^1(\mathbb{R}, \mathbb{C})} - \langle [f(|U|^2) + 2f'(|U|^2) U_1^2] \psi_2, \psi_1 \rangle_{H^1(\mathbb{R}, \mathbb{C})} \right) \\ &\leq K \|\psi\|_{H^1(\mathbb{R}, \mathbb{C}^2)}^2. \end{aligned}$$

Moreover, the spectrum of \mathcal{A} is included in the half-space $\{\text{Re} \leq \sigma_0\}$, hence \mathcal{A} generates a continuous semigroup $e^{t\mathcal{A}}$ on $H^1(\mathbb{R}, \mathbb{C}^2)$.

In order to estimate the growth of the semigroup $e^{t\mathcal{A}}$ on $H^1(\mathbb{R}, \mathbb{C}^2)$, we could try to use the same approach as [21], which relies on the proof of the spectral mapping theorem in [31]. However, our situation is slightly different since in these studies, the reference solutions is real-valued (it is a bound state in [31] and the kink in [21]). Therefore, $U_2 = 0$ and \mathcal{A} has no diagonal term, and the system is much more decoupled than in our situation. As a matter of fact, it is not very clear whether the arguments of [31] carry over to our problem. We thus have chosen to use the general approach given in Appendix B. We thus verify the assumptions of Theorem B.4 (see also Corollary B.2) there, which are easy: \mathcal{A} generates a semigroup in $H^1(\mathbb{R}, \mathbb{C}^2)$ and the spectrum

of \mathcal{A} is of the form $i\mathbb{R} \cup \{-\gamma_0, +\gamma_0\}$, where $i\mathbb{R}$ is the essential spectrum and $\pm\gamma_0$ two simple eigenvalues. Moreover, the eigenvector associated with γ_0 belongs to $H^3(\mathbb{R}, \mathbb{C}^2) = D(J)$. Therefore, Theorem B.4 in Appendix B applies and the growth estimate for the linearized problem follows. For the nonlinear instability result, we argue as for Corollary B.2 in Appendix B, since the manifold $\mathfrak{M} = \{|U_*|(\cdot - y), y \in \mathbb{R}\}$ is transverse to the curve $\sigma \mapsto |U_* + \sigma w|$ in $r_0 + H^1(\mathbb{R})$. Indeed, it follows from (28) that $|U_* + \sigma w| = A_* + \sigma\zeta + \mathcal{O}_{H^1}(\sigma^2)$. Assume that $\zeta = \alpha \partial_x |U_*|$, with $\alpha \in \mathbb{R}$. Then, integration of the first equation of (27) provides

$$\lambda(|U_*| - r_0) - c_* \partial_x |U_*| + 2\left((r_0^2 + \eta_*)v + u_* \zeta\right) = 0,$$

hence, using that $|U_*| = \sqrt{r_0^2 + \eta_*}$ and the equality $2u_* = c\eta_*/(r_0^2 + \eta_*)$, we infer

$$v + \alpha \left\{ \lambda \frac{|U_*| - r_0}{r_0^2 + \eta_*} + \frac{c_* r_0^2}{4(r_0^2 + \eta_*)^{3/2}} \partial_x \eta_* \right\} = 0.$$

Since $\int_{\mathbb{R}} v = 0$ and $|U_*| - r_0$ has constant sign in \mathbb{R} , integrating over \mathbb{R} then implies $\alpha = 0$, which in turn yields $\zeta = v = 0$ and $w_* = 0$, a contradiction. Consequently, $\zeta \notin \mathbb{R} \partial_x |U_*|$ and the manifold $\mathfrak{M} = \{|U_*|(\cdot - y), y \in \mathbb{R}\}$ is indeed transverse to the curve $\sigma \mapsto |U_* + \sigma w|$ in $r_0 + H^1(\mathbb{R})$.

6 Stability analysis for the kink ($c = 0$)

6.1 Proof of Lemma 4

Let us recall that the momentum $P(U_c)$, for $c > 0$, has the expression

$$P(U_c) = c \int_{\xi_c}^0 \frac{\xi^2}{r_0^2 + \xi} \frac{d\xi}{\sqrt{-\mathcal{V}_c(\xi)}},$$

since $\text{sgn}(\xi_c) = -1$. Therefore, we decompose $P(U_c)$ with two integrals:

$$P(U_c) = c \int_{\xi_c}^0 \frac{\xi^2}{r_0^2 + \xi} \frac{d\xi}{\sqrt{-\mathcal{V}'_c(\xi_c)(\xi - \xi_c)}} + c \int_{\xi_c}^0 \frac{\xi^2}{r_0^2 + \xi} \left(\frac{1}{\sqrt{-\mathcal{V}_c(\xi)}} - \frac{1}{\sqrt{-\mathcal{V}'_c(\xi_c)(\xi - \xi_c)}} \right) d\xi. \quad (31)$$

Using the change of variables $\xi = t\xi_c$, the second integral in (31) is equal to

$$\begin{aligned} & \xi_c^3 \int_1^0 \frac{t^2}{r_0^2 + t\xi_c} \left(\frac{1}{\sqrt{-\mathcal{V}_c(t\xi_c)}} - \frac{1}{\sqrt{-\xi_c \mathcal{V}'_c(\xi_c)(t-1)}} \right) dt \\ &= (-r_0^2)^3 \int_1^0 \frac{t^2}{r_0^2 - tr_0^2} \left(\frac{1}{\sqrt{-\mathcal{V}_0(-tr_0^2)}} - \frac{1}{\sqrt{-4r_0^2 F(0)(t-1)}} \right) dt + o_{c \rightarrow 0}(1) \\ &= \int_{-r_0^2}^0 \frac{\xi^2}{r_0^2 + \xi} \left(\frac{1}{\sqrt{-\mathcal{V}_0(\xi)}} - \frac{1}{\sqrt{4F(0)(\xi + r_0^2)}} \right) d\xi + o_{c \rightarrow 0}(1). \end{aligned}$$

The passage to the limit $c \rightarrow 0$ being justified by the dominated convergence theorem since the absolute value of the integrand is $\leq Kt$ for $0 \leq t \leq 1/2$ for small c and for $1/2 \leq t \leq 1$, since $\xi_c > -r_0^2$, $r_0^2 + t\xi_c \geq r_0^2(1-t)$ and hence is equal to

$$\begin{aligned} &= \left| \frac{t^2}{r_0^2 + t\xi_c} \cdot \frac{\mathcal{V}_c(t\xi_c) - \xi_c \mathcal{V}'_c(\xi_c)(t-1)}{\sqrt{-\mathcal{V}_c(t\xi_c)} \sqrt{-\xi_c \mathcal{V}'_c(\xi_c)(t-1)} \left[\sqrt{-\mathcal{V}_c(t\xi_c)} + \sqrt{-\xi_c \mathcal{V}'_c(\xi_c)(t-1)} \right]} \right| \\ &\leq K \frac{(1-t)^2}{(1-t)\sqrt{1-t}\sqrt{1-t}\sqrt{1-t}} = \frac{K}{\sqrt{1-t}} \in L^1((1/2, 1)). \end{aligned}$$

Furthermore, letting $\xi = \xi_c + (r_0^2 + \xi_c)t^2$, $t \geq 0$, the first integral in (31) is equal to

$$\begin{aligned} & \frac{1}{\sqrt{r_0^2 + \xi_c}} \int_0^{\sqrt{-\xi_c/(r_0^2 + \xi_c)}} \frac{(\xi_c + (r_0^2 + \xi_c)t^2)^2}{1 + t^2} \cdot \frac{2dt}{\sqrt{-\mathcal{V}'_c(\xi_c)}} \\ &= \frac{2}{\sqrt{r_0^2 + \xi_c} \sqrt{-\mathcal{V}'_c(\xi_c)}} \left\{ r_0^4 \left[\frac{\pi}{2} - \arctan \left(\sqrt{\frac{r_0^2 + \xi_c}{-\xi_c}} \right) \right] - 2r_0^2(r_0^2 + \xi_c) \sqrt{\frac{-\xi_c}{r_0^2 + \xi_c}} \right. \\ & \quad \left. + (r_0^2 + \xi_c)^2 \left[\sqrt{\frac{-\xi_c}{r_0^2 + \xi_c}} + \frac{1}{3} \sqrt{\frac{-\xi_c}{r_0^2 + \xi_c}}^3 \right] \right\}. \end{aligned}$$

by direct computation. Since $\xi_c \simeq -r_0^2$ is a simple zero of $\mathcal{V}_c(\xi) = c^2\xi^2 - 4(r_0^2 + \xi)F(r_0^2 + \xi)$, we have

$$\xi_c = -r_0^2 + \frac{c^2 r_0^4}{4F(0)} + \frac{c^4 r_0^6}{4F(0)} \left(\frac{r_0^2 f(0)}{F(0)} - 2 \right) + o_{c \rightarrow 0}(c^4) = -r_0^2 + \frac{c^2 r_0^4}{4F(0)} + \mathcal{O}_{c \rightarrow 0}(c^4),$$

thus

$$-\mathcal{V}'_c(\xi_c) = 4F(0) + \mathcal{O}_{c \rightarrow 0}(c^2)$$

and

$$\frac{2}{\sqrt{r_0^2 + \xi_c} \sqrt{-\mathcal{V}'_c(\xi_c)}} = \frac{2}{r_0^2 c} + \mathcal{O}_{c \rightarrow 0}(c).$$

As a consequence, the first integral in (31) is equal to

$$\frac{r_0^2 \pi}{c} + \left\{ -\frac{r_0^3}{\sqrt{F(0)}} - \frac{2r_0^3}{\sqrt{F(0)}} + \frac{r_0^3}{3\sqrt{F(0)}} \right\} + \mathcal{O}_{c \rightarrow 0}(c) = \frac{r_0^2 \pi}{c} - \frac{8r_0^3}{3\sqrt{F(0)}} + \mathcal{O}_{c \rightarrow 0}(c).$$

Gathering these two relations, we obtain

$$P(U_c) = r_0^2 \pi + c \left\{ -\frac{8r_0^3}{3\sqrt{F(0)}} + \int_{-r_0^2}^0 \frac{\xi^2}{r_0^2 + \xi} \left(\frac{1}{\sqrt{-\mathcal{V}_0(\xi)}} - \frac{1}{\sqrt{4F(0)(\xi + r_0^2)}} \right) d\xi \right\} + o_{c \rightarrow 0}(c),$$

as wished.

6.2 Proof of Theorem 9

Since we have a kink solution U_0 for $c = 0$, this implies that $\mathcal{V}_0(\xi) = -4(r_0^2 + \xi)F(r_0^2 + \xi)$ is negative in $(-r_0^2, 0)$ and that $-r_0^2$ is a simple zero of \mathcal{V}_0 , that is $F(0) > 0$. Then, $F > 0$ in $[0, r_0^2)$ and $F(\varrho) \simeq \frac{c^2}{4r_0^2}(\varrho - r_0^2)^2$ for $\varrho \rightarrow r_0^2$, it follows that there exists $K_0 > 0$ such that

$$F(\varrho) \geq \frac{1}{K_0}(\varrho - r_0^2)^2$$

We consider for $\mu \geq 0$ the quantity

$$\mathcal{H}_{\min}(\mu) \equiv \inf \left\{ \mathcal{H}(u), u \in \mathcal{Z}, \inf_{\mathbb{R}} |u| = \mu \right\}.$$

The study of $\mathcal{H}_{\min}(0)$ is easy.

Proposition 6.1 *There holds*

$$\mathcal{H}_{\min}(0) = E(U_0).$$

More precisely, for any $U \in \mathcal{Z}$,

$$E(U) \geq 4 \int_{\inf_{\mathbb{R}} |U|}^{r_0} \sqrt{F(s^2)} ds \quad \text{and} \quad E(U_0) = 4 \int_0^{r_0} \sqrt{F(s^2)} ds.$$

Finally, if $U \in \mathcal{Z}$, $\inf_{\mathbb{R}} |U| = 0$ and $\mathcal{H}(U) = E(U_0)$, then there exists $y \in \mathbb{R}$ and $\theta \in \mathbb{R}$ such that $U = e^{i\theta} U_0(\cdot - y)$.

Proof. Taking U_0 as a comparison map, we see that $\mathcal{K}_{\min}(0) \leq E(U_0)$. Moreover, if $U \in \mathcal{Z}$ and $\inf_{\mathbb{R}} |U| = \mu \geq 0$, we may assume, up to a translation, that $\mu = |U|(0)$. Then, denoting

$$G(r) \equiv 2 \int_{r_0}^r \sqrt{F(s^2)} ds,$$

we have the inequalities

$$\begin{aligned} \int_0^{+\infty} |\partial_x U|^2 + F(|U|^2) dx &\geq \int_0^{+\infty} |\partial_x |U||^2 + F(|U|^2) dx \geq 2 \int_0^{+\infty} |\sqrt{F(|U|^2)} \partial_x |U|| dx \\ &= \int_0^{+\infty} |\partial_x [G(|U|)]| dx \geq \left| \int_0^{+\infty} \partial_x [G(|U|)] dx \right| = \left| G(|U|(+\infty)) - G(|U|(0)) \right| \\ &= \left| G(r_0) - G(\mu) \right| = 2 \int_{\mu}^{r_0} \sqrt{F(s^2)} ds. \end{aligned}$$

Arguing similarly in $(-\infty, 0)$, we get

$$E(U) \geq 4 \int_{\mu}^{r_0} \sqrt{F(s^2)} ds.$$

For the kink U_0 , which is real-valued, we have the first integral $|\partial_x U_0|^2 = F(U_0^2)$, hence, using the change of variables $s = U_0(x)$,

$$E(U_0) = 4 \int_0^{+\infty} F(U_0^2) dx = 4 \int_0^{r_0} \sqrt{F(s^2)} ds.$$

If $\mu = 0$, we have then $E(U) \geq E(U_0)$, hence $\mathcal{K}(U) \geq E(U) \geq E(U_0)$ as wished.

Assume finally that $U \in \mathcal{Z}$ verifies $\inf_{\mathbb{R}} |U| = 0$ and $\mathcal{K}(U) = E(U_0)$. Then $\mu = 0$ and all the above inequalities are equalities. In particular, we must have $|\partial_x U| = |\partial_x |U||$ and equality in $|\partial_x |U||^2 + F(|U|^2) \geq 2|\sqrt{F(|U|^2)} \partial_x |U||$, which means that $|\partial_x |U|| = \sqrt{F(|U|^2)}$. Combining this ode with the condition $|U|(0) = 0$, we see that $|U| = |U_0|$, since $|U_0|$ solves $\partial_x U_0 = \sqrt{F(U_0^2)}$. Finally, the fact that $|\partial_x U| = |\partial_x |U||$ implies that the phase is constant in $(-\infty, 0)$ and in $(0, +\infty)$: there exist two constants $\theta_{\pm} \in \mathbb{R}$ satisfying $U(x) = e^{i\theta_{\pm}} |U_0|(x)$ for $\pm x \geq 0$. Therefore, $\mathfrak{P}(U) = r_0^2(\theta_+ - \theta_-) \bmod 2\pi r_0^2$, and then

$$E(U_0) = \mathcal{K}(u) = E(U_0) + 2Mr_0^4 \sin^2 \left(\frac{\theta_+ - \theta_- - \pi}{2} \right)$$

implies $\theta_+ - \theta_- = \pi \bmod 2\pi$, that is $U = e^{i\theta_+} U_0$ in \mathbb{R} , which is the desired result. \square

We recall the expansion $P(U_s) = r_0^2 \pi + s \dot{P}_0 + o(s)$ as $s \rightarrow 0$, where $\dot{P}_0 \equiv \frac{dP(U_s)}{ds} \Big|_{s=0}$. From the Hamilton group relation $\frac{dE(U_s)}{ds} = s \frac{dP(U_s)}{ds}$, we also infer by integration $E(U_s) = E(U_0) + \frac{s^2}{2} \dot{P}_0 + o(s^2)$. As a first step, we define the small parameter $\mu_* > 0$. The key point is to prove the following result.

Proposition 6.2 *There exist some constant $K > 0$ and a small $\mu_* > 0$ such that, for any $0 < \mu \leq \mu_*$,*

$$\mathcal{K}_{\min}(\mu) = \inf \left\{ \mathcal{K}(U), U \in \mathcal{Z}, \inf_{\mathbb{R}} |u| = \mu \right\} \geq E(U_0) + \frac{\mu^2}{K}.$$

Proof. Notice first that for $c > 0$ small, there exists U_c travelling wave of speed c and that $\inf_{\mathbb{R}} |U_c| = \sqrt{r_0^2 + \xi_c}$ with ξ_c a smooth function in c such that $\xi_c = -r_0^2 + \frac{c^2 r_0^4}{4F(0)} + \mathcal{O}(c^4)$, hence $\inf_{\mathbb{R}} |U_c| = \frac{cr_0^2}{2\sqrt{F(0)}} + \mathcal{O}(c^2)$ and

is smooth. Therefore, there exists, for $0 \leq \mu \leq \mu_*$ small, a unique σ_μ , with $\sigma_\mu = \frac{2\mu\sqrt{F(0)}}{r_0^2} + \mathcal{O}(\mu^2)$, such that $\mu = \inf_{\mathbb{R}} |U_{\sigma_\mu}|$. In particular, taking U_{σ_μ} as a comparison map in $\mathcal{K}_{\min}(\mu)$, we have

$$\begin{aligned} \mathcal{K}_{\min}(\mu) &\leq \mathcal{K}(U_{\sigma_\mu}) = E(U_{\sigma_\mu}) + 2Mr_0^4 \sin^2 \left(\frac{P(U_{\sigma_\mu}) - r_0^2 \pi}{2r_0^2} \right) \\ &= E(U_0) + \frac{\sigma_\mu^2}{2} \dot{P}_0 + o(\sigma_\mu^2) + 2Mr_0^4 \sin^2 \left(\frac{\sigma_\mu \dot{P}_0 + o(\sigma_\mu)}{2r_0^2} \right) \\ &= E(U_0) + \frac{\sigma_\mu^2}{2} \left(\dot{P}_0 + M\dot{P}_0^2 \right) + o(\sigma_\mu^2). \end{aligned}$$

In particular, it follows that, for some positive constant K and for μ_* small enough,

$$\mathcal{K}_{\min}(\mu) \leq E(U_0) + K\mu^2 \leq \frac{11}{10}E(U_0). \quad (32)$$

Consider now c small, a bounded open interval (x_-, x_+) and η a solution to the Newton equation

$$2\partial_x^2 \eta + \mathcal{V}'_c(\eta) = 0$$

in (x_-, x_+) , with $\partial_x \eta(x_+) \leq 0 \leq \partial_x \eta(x_-)$, $\eta(x_+) \leq -r_0^2 + \mu_*^2$ and $\eta(x_-) \leq -r_0^2 + \mu_*^2$. As $c \rightarrow 0$, \mathcal{V}_c converges to \mathcal{V}_0 in $\mathcal{C}^1([-r_0^2, 0])$. Moreover, \mathcal{V}_0 is negative in $(-r_0^2, 0)$ and has a simple zero at $-r_0^2$. Therefore, if c and $\mu_* > 0$ are sufficiently small, we must have $\int_{x_-}^{x_+} F(r_0^2 + \eta) dx \geq \frac{1}{2} \int_{\mathbb{R}} F(U_0^2) dx$. Consequently, if $v = Ae^{i\varphi}$ solves (TW_c) on a bounded interval (x_-, x_+) , satisfies $2\partial_x \varphi = \frac{c\eta}{r_0^2 + \eta}$ ($\eta \equiv A^2 - r_0^2$) and if $|v|$ is $\leq \mu_*$ at x_+ and at x_- , with $\partial_x |v|(x_+) \leq 0 \leq \partial_x |v|(x_-)$, then

$$\int_{x_-}^{x_+} |\partial_x v|^2 + F(|v|^2) dx \geq \frac{1}{2}E(U_0). \quad (33)$$

Here, we use that the Newton equation on the modulus $|V|$ actually holds true in (x_-, x_+) . Since $F > 0$ in $[0, r_0^2)$ and $F(\varrho) \simeq r_0^2(\varrho - r_0^2)^2$ when $\varrho \rightarrow r_0^2$, there exists $K > 0$ and $\kappa > 0$ such that $F(\varrho) \geq (\varrho - r_0^2)^2/K$ for $0 \leq \varrho \leq r_0^2(1 + \kappa)^2$. Hence, if $\inf_{\mathbb{R}} |v| \geq \mu > 0$, then

$$|P(v)| \leq \frac{K}{\mu}E(v). \quad (34)$$

Moreover, arguing as in the proof of Proposition 6.1, we show that there exists $\varkappa > 0$ such that if $U \in \mathcal{Z}$ and $|U|$ takes values $\leq \mu_*$ and $\geq r_0(1 + \kappa)$, then

$$E(U) \geq E(U_0)(1 + \varkappa).$$

In particular, since $\mathcal{K}_{\min}(\mu) \leq E(U_0) + \mathcal{O}(\mu^2)$, we may choose μ_* sufficiently small so that if $U \in \mathcal{Z}$ and $\mathcal{K}(U) \leq \mathcal{K}_{\min}(\mu) + \mu_*$, then $|U| \leq r_0(1 + \kappa)$. This means that for the mappings we are considering, $F(\varrho) \geq (\varrho - r_0^2)^2/K$.

Step 1: Construction of a suitable minimizing sequence. There exists a sequence $(V_n)_{n \geq 0}$ in \mathcal{Z} such that $\inf_{\mathbb{R}} |V_n| = \mu = |V_n|(0)$, $V_n = A_n e^{i\phi_n}$, $P(V_n) \in [0, \pi r_0^2]$,

$$2A_n^2 \partial_x \phi_n = c_n(A_n^2 - r_0^2), \quad c_n \equiv Mr_0^2 \sin \left(\frac{r_0^2 \pi - P(V_n)}{2r_0^2} \right) \geq 0$$

and

$$\lim_{n \rightarrow +\infty} \mathcal{K}(V_n) = \mathcal{K}_{\min}(\mu).$$

Since $\mu > 0$, the maps V we consider may be lifted $V = Ae^{i\phi}$. Therefore (with $u = \partial_x \phi$),

$$\begin{aligned} \mathcal{H}_{\min}(\mu) = \inf \left\{ \int_{\mathbb{R}} (\partial_x A)^2 + F(A^2) dx \right. \\ \left. + \inf \left\{ \int_{\mathbb{R}} A^2 u^2 dx + 2Mr_0^4 \sin^2 \left(\frac{\int_{\mathbb{R}} (A^2 - r_0^2) u dx - r_0^2 \pi}{2r_0^2} \right), u \in L^2(\mathbb{R}, \mathbb{R}) \right\}, \right. \\ \left. A \in r_0 + H^1(\mathbb{R}, \mathbb{R}), \inf_{\mathbb{R}} A = \mu \right\}. \end{aligned} \quad (35)$$

The infimum in u may be written

$$\inf_{p \in \mathbb{R}} \inf \left\{ \int_{\mathbb{R}} A^2 u^2 dx + 2Mr_0^4 \sin^2 \left(\frac{p - r_0^2 \pi}{2r_0^2} \right), u \in L^2(\mathbb{R}, \mathbb{R}) \text{ s.t. } \int_{\mathbb{R}} (A^2 - r_0^2) u dx = p \right\}.$$

For each $p \in \mathbb{R}$, we minimize in u a quadratic functional on an affine hyperplane, with minimizer given by

$$u_p = p \left(\int_{\mathbb{R}} \frac{(A^2 - r_0^2)^2}{A^2} dx \right)^{-1} \frac{A^2 - r_0^2}{A^2}.$$

As a consequence, the infimum in u in (35) is

$$\inf_{p \in \mathbb{R}} \left[\int_{\mathbb{R}} A^2 u_p^2 dx + 2Mr_0^4 \sin^2 \left(\frac{p - r_0^2 \pi}{2r_0^2} \right) \right] = \inf_{p \in \mathbb{R}} \left[p^2 \left(\int_{\mathbb{R}} \frac{(A^2 - r_0^2)^2}{A^2} dx \right)^{-1} + 2Mr_0^4 \sin^2 \left(\frac{p - \pi r_0^2}{2r_0^2} \right) \right].$$

It is clear that this last infimum is achieved only for p inside $[-\pi r_0^2, +\pi r_0^2]$. Indeed, the second term is $2\pi r_0^2$ -periodic and if $p > \pi r_0^2$, then $p - 2\pi r_0^2$ is a better competitor. Moreover, the function $p \mapsto \sin^2 \left(\frac{p - \pi r_0^2}{2r_0^2} \right)$ is continuous and even, hence we may consider some $p \in [0, \pi r_0^2]$ (depending on A), which is a minimizer for this last infimum. The corresponding u_p is then a minimizer for the infimum in u in (35). Writing that

$$\frac{d}{dp} \left[p^2 \left(\int_{\mathbb{R}} \frac{(A^2 - r_0^2)^2}{A^2} dx \right)^{-1} + 2Mr_0^4 \sin^2 \left(\frac{p - \pi r_0^2}{2r_0^2} \right) \right] = 2p \left(\int_{\mathbb{R}} \frac{(A^2 - r_0^2)^2}{A^2} dx \right)^{-1} + 2Mr_0^2 \sin \left(\frac{p - \pi r_0^2}{2r_0^2} \right) = 0,$$

we deduce the relations

$$2A^2 u = c(A^2 - r_0^2), \quad c \equiv Mr_0^2 \sin \left(\frac{p - \pi r_0^2}{2r_0^2} \right).$$

We conclude by considering a minimizing sequence (A_n) in (35), and translating in space so that $\inf_{\mathbb{R}} A_n = \mu = |A_n|(0)$.

Since $F \geq 0$ in \mathbb{R}_+ , we have

$$\int_{\mathbb{R}} |\partial_x V_n|^2 dx \leq \mathcal{H}(V_n) \leq \frac{12}{10} E(U_0)$$

for n large. Therefore, by compact Sobolev embedding $H^1([-R, +R]) \hookrightarrow L^\infty([-R, +R])$, we may assume, up to a possible subsequence, that there exists $V \in H_{\text{loc}}^1(\mathbb{R})$ such that for any $R > 0$, $V_n \rightharpoonup V$ in $H^1([-R, +R])$ and $V_n \rightarrow V$ uniformly on $[-R, +R]$. Moreover, by lower semicontinuity and Fatou's lemma, $E(V) \leq \liminf_{n \rightarrow +\infty} E(V_n)$. Since $|V_n| \geq \mu > 0$ in \mathbb{R} , $|V| \geq \mu > 0$ in \mathbb{R} and we may lift $V = Ae^{i\phi}$. Furthermore, $\inf_{\mathbb{R}} A_n = \mu = |V_n|(0)$, hence $\inf_{\mathbb{R}} A = \mu = |V|(0)$. We also know that $P(V_n) \in [0, r_0^2 \pi]$ for all n , hence we may assume, up to another subsequence, that $P(V_n)$ converges to some $P_\infty \in [0, r_0^2 \pi]$. We also set

$$c \equiv \lim_{n \rightarrow +\infty} c_n = Mr_0^2 \sin \left(\frac{P_\infty - \pi r_0^2}{2r_0^2} \right).$$

In view of Step 1, and the convergence $A_n \rightarrow A$ uniformly on any compact interval $[-R, +R]$, it follows that

$$2A^2 \partial_x \phi = c(A^2 - r_0^2) \quad \text{and} \quad \partial_x \phi_n \rightarrow \partial_x \phi \quad \text{in} \quad L_{\text{loc}}^\infty(\mathbb{R}). \quad (36)$$

Note that

$$\int_{\mathbb{R}} |\partial_x V|^2 + \frac{1}{K} (|V|^2 - r_0^2)^2 dx \leq E(V) < +\infty$$

hence $|V| \rightarrow r_0$ at $\pm\infty$. In particular, there exist $-\infty < R_- \leq 0 \leq R_+ < +\infty$ such that $|V| > \mu$ in $(-\infty, R_-)$ and in $(R_+, +\infty)$ and $|V|(R_{\pm}) = \mu$.

Step 2. There exist $-\infty < z_- \leq 0 \leq z_+ < +\infty$ such that

$$A(x) = A_c(x - R_+ + z_+) \quad \text{for } x \geq R_+ \quad \text{and} \quad A(x) = A_c(x - R_- + z_-) \quad \text{for } x \leq R_-.$$

We work for $x \geq R_+$, the other case being similar. We consider $\chi \in \mathcal{C}_c^1((R_+, +\infty), \mathbb{C})$, $t \in \mathbb{R}$ small such that $V_n^t \equiv v_n + t\chi$ verifies $|V_n^t| > \mu$ in $(R_+, +\infty)$. This is possible since $\inf_{\text{Supp}(\chi)} |V_n| > \mu$. Then, $|V_n^t| \geq \mu$ in \mathbb{R} and $|V_n^t|(0) = \mu$, hence V_n^t is then a comparison map for $\mathcal{K}_{\min}(\mu)$, and in view of the equality $P(V_n^t) = P(V_n) + 2t \int_{R_+}^{+\infty} \langle i\partial_x V_n | \chi \rangle dx + \mathcal{O}(t^2)$, it follows that

$$\begin{aligned} \mathcal{K}_{\min}(\mu) \leq \mathcal{K}(V_n^t) &= \mathcal{K}_{\min}(\mu) + o_{n \rightarrow +\infty}(1) + 2t \int_{R_+}^{+\infty} \langle \partial_x V_n, \partial_x \chi \rangle dx + t^2 \int_{R_+}^{+\infty} |\partial_x \chi|^2 dx \\ &\quad - 2t \int_{R_+}^{+\infty} f(|V_n|^2) \langle V_n, \chi \rangle dx + Mt \sin\left(\frac{P(V_n) - \pi r_0^2}{r_0^2}\right) \int_{R_+}^{+\infty} \langle i\partial_x V_n, \chi \rangle dx + \mathcal{O}_{t \rightarrow 0}(t^2). \end{aligned}$$

Letting $n \rightarrow +\infty$ and using the weak and strong convergences for V_n , we infer

$$0 \leq 2t \int_{R_+}^{+\infty} \langle \partial_x V, \partial_x \chi \rangle dx - 2t \int_{R_+}^{+\infty} f(|V|^2) \langle V, \chi \rangle dx - Mt \sin\left(\frac{r_0^2 \pi - P_{\infty}}{2r_0^2}\right) \int_{R_+}^{+\infty} \langle i\partial_x V, \chi \rangle dx + \mathcal{O}_{t \rightarrow 0}(t^2).$$

Dividing by $t \neq 0$ and letting $t \rightarrow 0^+$ and then $t \rightarrow 0^-$, we deduce that V solves (TW_c) in $(R_+, +\infty)$ and V has finite energy. Moreover, $|V|(R_+) = \mu$ is small, thus $V = e^{i\theta_+} U_c(\cdot - R_+ + z_+)$ in $(R_+, +\infty)$ for some constants z_+ and θ_+ , and the speed c is such that $\inf_{\mathbb{R}} A_c = \sqrt{r_0^2 + \xi_c} \leq \mu$, hence $c \leq \sigma(\mu) \leq K\mu$. Since $|V|$ has finite energy in \mathbb{R} and solves (TW_c) in $(R_+, +\infty)$, V is \mathcal{C}^1 in $[R_+, +\infty)$. Moreover, $|V|$ reaches a minimum at $x = R_+$, thus we must have $\partial_x^+ |V|(R_+) \geq 0$, which imposes $z_+ \geq 0$. Note that A_c being even, it is possible to translate V so that $R \equiv R_+ = -R_-$ and $z \equiv z_+ = -z_-$. Observe that $\mu = A_c(z) \geq A_0(z)$, hence $z \leq K\mu$. This yields

$$\int_{|x| \geq R} |\partial_x V|^2 + F(|V|^2) dx = \int_{|x| \geq z} |\partial_x U_c|^2 + F(|U_c|^2) dx \geq E(U_0) - K\mu. \quad (37)$$

In particular, we deduce from (32)

$$2RF(\mu^2) \leq \int_{|x| \leq R} |\partial_x V|^2 + F(|V|^2) dx \leq K\mu,$$

hence $R \leq K\mu$ for μ small ($F(0) > 0$).

Step 3. We prove that $A = \mu$ in $(R_-, R_+) = (-R, +R)$.

Indeed, if it is not the case, there exists a bounded interval (x_-, x_+) such that $A = |V| > \mu$ in (x_-, x_+) and $|V|(x_{\pm}) = \mu$, with $\partial_x |V|(x_+) \leq 0 \leq \partial_x |V|(x_-)$. Therefore, we can make perturbations of the amplitude A_n localized in (x_-, x_+) . Hence, arguing as in Step 2, we see that then, V solves (TW_c) in (x_-, x_+) , with $2A^2 \partial_x \phi = c(A^2 - r_0^2)$ and $|V|(x_{\pm}) = \mu$, $\partial_x |V|(x_+) \leq 0 \leq \partial_x |V|(x_-)$. We then are in position to apply (33), yielding

$$\int_{x_-}^{x_+} |\partial_x V|^2 + F(|V|^2) dx \geq \frac{1}{2} E(U_0),$$

but the combination with (37) provides

$$\begin{aligned} \frac{11}{10}E(U_0) &\geq \mathcal{H}_{\min}(\mu) \geq \int_{x_-}^{x_+} |\partial_x V|^2 + F(|V|^2) dx + \int_{|x| \geq R} |\partial_x V|^2 + F(|V|^2) dx \\ &\geq \frac{1}{2}E(U_0) + E(U_0) - K\mu_* = \frac{3}{2}E(U_0) - K\mu_*, \end{aligned}$$

which is not possible if μ_* is sufficiently small.

Step 4. We have $R = 0$ or ($z = 0$ and $c = \sigma_\mu$).

Indeed, assume $R > 0$, and consider $\zeta \in \mathcal{C}_c^1((0, +\infty), \mathbb{R})$, $\zeta \geq 0$, $t \geq 0$ and $V_n^t \equiv (A_n + t\zeta)e^{i\phi_n}$, so that $|V_n^t| = A_n + t\zeta \geq \mu$ in \mathbb{R} . Since $R > 0$, we actually have $\inf_{\mathbb{R}} |V_n^t| = \mu$ and V_n^t is a comparison map for $\mathcal{H}_{\min}(\mu)$. Arguing as before, we thus have

$$\begin{aligned} \mathcal{H}_{\min}(\mu) &\leq \mathcal{H}(V_n^t) = \mathcal{H}_{\min}(\mu) + o_{n \rightarrow +\infty}(1) + 2t \int_0^{+\infty} \partial_x A_n \partial_x \zeta dx + t^2 \int_0^{+\infty} (\partial_x \zeta)^2 dx \\ &\quad + 2t \int_0^{+\infty} A_n \zeta (\partial_x \phi_n)^2 dx + t^2 \int_0^{+\infty} \zeta^2 (\partial_x \phi_n)^2 dx - 2t \int_0^{+\infty} f(A_n^2) A_n \zeta dx \\ &\quad + Mr_0^2 t \sin\left(\frac{P(V_n) - r_0^2 \pi}{r_0^2}\right) \int_0^R 2A_n \zeta \partial_x \phi_n dx + \mathcal{O}_{t \rightarrow 0}(t^2). \end{aligned}$$

By (36), we may pass to the limit as $n \rightarrow +\infty$ in all the terms and deduce

$$0 \leq 2t \int_0^{+\infty} \partial_x A \partial_x \zeta dx + 2t \int_0^{+\infty} A \zeta (\partial_x \phi)^2 dx - 2t \int_0^{+\infty} f(A^2) A \zeta dx - 2ct \int_0^{+\infty} A \zeta \partial_x \phi dx + \mathcal{O}_{t \rightarrow 0}(t^2).$$

At this stage, we see the relevance of taking a minimizing sequence as chosen in Step 1, since it allows to pass to the limit in the nonlinear terms involving $\partial_x \phi_n$. As a consequence, using (36),

$$-\partial_x^2 A - Af(A^2) + \frac{c^2}{4} \frac{(A^2 - r_0^2)^2}{A^3} \geq 0$$

in the distributional sense in $(0, +\infty)$. The term $-Af(A^2) + \frac{c^2}{4} \frac{(A^2 - r_0^2)^2}{A^3}$ is continuous in \mathbb{R} . However, since $A(x) = \mu$ for $0 \leq x \leq R$ and $A(x) = A_c(x - R + z)$ for $x \geq R$, we infer $-\partial_x^2 A = -\partial_x A_c(z) \delta_{x=R}$ plus a piecewise continuous function in the distributional sense in $(0, +\infty)$. Since $\partial_x A_c(z) \geq 0$ (recall that $z \geq 0$), this forces to have $\partial_x A_c(z) = 0$, that is $z = 0$. Consequently, $\mu = |V|(R) = A(R) = A_c(z) = A_c(0)$ and then $c = \sigma_\mu$.

In the next step, we take into account the loss in the weak convergence $V_n \rightharpoonup V$.

Step 5. There exists $K > 0$ such that

$$E_\sharp \geq \frac{P_\sharp}{K}, \quad \text{where} \quad E_\sharp \equiv \liminf_{n \rightarrow +\infty} E(V_n) - E(V) \geq 0, \quad P_\sharp \equiv \lim_{n \rightarrow +\infty} P(V_n) - P(V) = P_\infty - P(V).$$

Let $\epsilon > 0$ be fixed but small, and pick some $X > 0$ large so that

$$\left| E(V) - \int_{|x| \leq X} |\partial_x V|^2 + F(|V|^2) dx \right| \leq \epsilon \quad \left| P(V) - \int_{|x| \leq X} (A^2 - r_0^2) u dx \right| \leq \epsilon.$$

We claim that there exists some small $\bar{\mu} > 0$, independent of ϵ such that $|V_n| \geq \bar{\mu}$ for $|x| \geq X$ and n large. Indeed, otherwise, we may argue as in Step 3 and show, as in the beginning of the proof there, that

$\int_{|x| \geq X} |\partial_x V_n|^2 + F(|V_n|^2) dx \geq \frac{1}{2} E(U_0)$. This is not possible since

$$\frac{12}{10} E(U_0) \geq \liminf_{n \rightarrow +\infty} E(V_n) \geq \frac{1}{2} E(U_0) + \int_{|x| \leq X} |\partial_x V|^2 + F(|V|^2) dx \geq \frac{1}{2} E(U_0) + E(V) - \epsilon,$$

and $E(V)$ is close to $E(U_0)$ as $\mu \rightarrow 0$. Therefore, as for (34),

$$\left| \int_{|x| \geq X} (A_n^2 - r_0^2) u_n dx \right| \leq \frac{K}{\bar{\mu}} \int_{|x| \geq X} |\partial_x V_n|^2 + F(|V_n|^2) dx.$$

Consequently,

$$\begin{aligned} E(V_n) - E(V) &\geq \int_{|x| \leq X} |\partial_x V_n|^2 + F(|V_n|^2) dx - \int_{|x| \leq X} |\partial_x V|^2 + F(|V|^2) dx + \int_{|x| \geq X} |\partial_x V_n|^2 + F(|V_n|^2) dx - \epsilon \\ &\geq \int_{|x| \leq X} |\partial_x V_n|^2 + F(|V_n|^2) dx - \int_{|x| \leq X} |\partial_x V|^2 + F(|V|^2) dx + \frac{\bar{\mu}}{K} \left| \int_{|x| \geq X} (A_n^2 - r_0^2) u_n dx \right| - \epsilon. \end{aligned}$$

Passing to the liminf and using the weak convergence in $[-X, +X]$, we infer

$$\liminf_{n \rightarrow +\infty} E(V_n) - E(V) \geq \frac{\bar{\mu}}{K} \liminf_{n \rightarrow +\infty} \left| P(V_n) - \int_{|x| \leq X} (A_n^2 - r_0^2) u_n dx \right| - \epsilon.$$

However, (36) implies

$$\int_{|x| \leq X} (A_n^2 - r_0^2) u_n dx \rightarrow \int_{|x| \leq X} (A^2 - r_0^2) u dx,$$

so that

$$E_{\sharp} \geq \frac{\bar{\mu}}{K} \left| P_{\infty} - \int_{|x| \leq X} (A^2 - r_0^2) u dx \right| - \epsilon \geq \frac{\bar{\mu}}{K} \left| P_{\infty} - P(V) \right| - \left(1 + \frac{\bar{\mu}}{K}\right) \epsilon = \frac{\bar{\mu}}{K} |P_{\sharp}| - \left(1 + \frac{\bar{\mu}}{K}\right) \epsilon.$$

Letting $\epsilon \rightarrow 0$, the conclusion follows.

Step 6. There exists $K > 0$ such that, if $R > 0$, then

$$\mathcal{H}_{\min}(\mu) \geq E(U_0) + \frac{\mu^2}{K}.$$

We recall the expansion $P(U_s) = r_0^2 \pi + s \dot{P}_0 + o(s)$ as $s \rightarrow 0$, where $\dot{P}_0 \equiv \frac{dP(U_s)}{ds} \Big|_{s=0}$. From the Hamilton group relation $\frac{dE(U_s)}{ds} = s \frac{dP(U_s)}{ds}$, we also infer by integration $E(U_s) = E(U_0) + \frac{s^2}{2} \dot{P}_0 + o(s^2)$. On the other hand, by definition of c_n ,

$$\begin{aligned} 2Mr_0^4 \sin^2 \left(\frac{P(V_n) - r_0^2 \pi}{2r_0^2} \right) &= Mr_0^4 \left[1 - \cos \left(\frac{P(V_n) - r_0^2 \pi}{r_0^2} \right) \right] \\ &= Mr_0^4 \left[1 - \sqrt{1 - \sin^2 \left(\frac{P(V_n) - r_0^2 \pi}{r_0^2} \right)} \right] = Mr_0^4 \left[1 - \sqrt{1 - \frac{c_n^2}{M^2}} \right] \end{aligned}$$

for n large. Here, we have used that $Mc_n = \sin((r_0^2 \pi - P(V_n))/r_0^2) \rightarrow Mc \in [0, K\mu_*]$ (cf. Step 2), thus $\cos((r_0^2 \pi - P(V_n))/r_0^2) \geq 0$, for otherwise, we would have, by Proposition 6.1

$$\begin{aligned} \mathcal{H}(V_n) = E(V_n) + 2Mr_0^4 \sin^2 \left(\frac{P(V_n) - r_0^2 \pi}{2r_0^2} \right) &\geq E(U_0) - K\mu + Mr_0^4 \left[1 + \sqrt{1 - \frac{c_n^2}{M^2}} \right] \\ &\geq E(U_0) - K\mu_* + 2Mr_0^4 + \mathcal{O}(\mu_*^2), \end{aligned}$$

but this contradicts (32) if μ_* is sufficiently small.

We assume $R > 0$, so that, by Step 4, $z = 0$ and $c = \sigma_\mu$. We recall the expansion $\sigma_\mu = \frac{2\mu\sqrt{F(0)}}{r_0^2} + \mathcal{O}(\mu^2) \sim \frac{2\mu\sqrt{F(0)}}{r_0^2}$. By definition of E_\sharp , one has

$$E_\sharp + E(V) + Mr_0^4 \left[1 - \sqrt{1 - \frac{c^2}{M^2}} \right] \leq \liminf_{n \rightarrow +\infty} E(V_n) + \lim_{n \rightarrow +\infty} 2Mr_0^4 \sin^2 \left(\frac{P(V_n) - r_0^2 \pi}{2r_0^2} \right) = \liminf_{n \rightarrow +\infty} \mathcal{K}(V_n) = \mathcal{K}_{\min}(\mu)$$

since (V_n) is minimizing for $\mathcal{K}_{\min}(\mu)$. Moreover, from the expression of V , it holds (for $R > 0$)

$$E(V) = E(U_{\sigma_\mu}) + 2R \left[\frac{\sigma_\mu^2 (r_0^2 - \mu^2)^2}{4\mu^2} + F(\mu^2) \right] \quad \text{and} \quad P(V) = P(U_{\sigma_\mu}) + R\sigma_\mu \frac{(r_0^2 - \mu^2)^2}{\mu^2}.$$

Furthermore, $P_\sharp = P_\infty - P(V)$ and $c = Mr_0^2 \sin((r_0^2 \pi - P_\infty)/r_0^2)$ with $P_\infty \in [0, r_0^2 \pi]$ and $\cos((r_0^2 \pi - P_\infty)/r_0^2) \geq 0$, thus

$$P_\sharp = P_\infty - P(V) = r_0^2 \pi - r_0^2 \arcsin \left(\frac{c}{Mr_0^2} \right) - P(U_{\sigma_\mu}) - R\sigma_\mu \frac{(r_0^2 - \mu^2)^2}{\mu^2}.$$

Combining this with the expansion of $E(U_\sigma)$ and $P(U_\sigma)$ gives

$$\begin{aligned} \mathcal{K}_{\min}(\mu) &\geq E(U_0) + E_\sharp + \frac{\sigma_\mu^2}{2} \dot{P}_0 + o(\sigma_\mu^2) + M \left[1 - \sqrt{1 - \frac{\sigma_\mu^2}{M^2}} \right] + 2R \left[\frac{\sigma_\mu^2 (r_0^2 - \mu^2)^2}{4\mu^2} + F(\mu^2) \right] \\ &\geq E(U_0) + \frac{|P_\sharp|}{K} + \frac{\sigma_\mu^2}{2} \left[\dot{P}_0 + \frac{1}{M} \right] + o(\mu^2) + 4RF(0) \\ &\geq E(U_0) + \frac{1}{K} \left| r_0^2 \arcsin(\sigma_\mu / (Mr_0^2)) + \sigma_\mu \dot{P}_0 + R\sigma \frac{(r_0^2 - \mu^2)^2}{\mu^2} + o(\sigma_\mu) \right| + \frac{\sigma_\mu^2}{2} \left[\dot{P}_0 + \frac{1}{M} \right] + o(\mu^2) + 4RF(0) \\ &\geq E(U_0) + \frac{1}{K} \left| \frac{\sigma_\mu}{M} + \sigma_\mu \dot{P}_0 + R\sigma_\mu \frac{(r_0^2 - \mu^2)^2}{\mu^2} + o(\sigma_\mu) \right| + \frac{\sigma_\mu^2}{2} \left[\dot{P}_0 + \frac{1}{M} \right] + o(\mu^2) + 4RF(0) \\ &\geq E(U_0) + \frac{\sigma_\mu}{K} \left| \dot{P}_0 + \frac{1}{M} + R \frac{(r_0^2 - \mu^2)^2}{\mu^2} \right| + \frac{\sigma_\mu^2}{2} \left[\dot{P}_0 + \frac{1}{M} \right] + 4RF(0) + o(\mu^2). \end{aligned}$$

The right-hand side is a continuous piecewise affine function of R (the "o" does not depend on R). Since $\frac{\sigma_\mu (r_0^2 - \mu^2)^2}{K\mu^2} \simeq \frac{1}{\mu} \gg 4F(0)$ and $\dot{P}_0 + \frac{1}{M} < 0$ (since $M > -\dot{P}_0^{-1}$ by hypothesis), it follows that the right-hand side is a function of R which is decreasing in $[0, R_0(\mu)]$ and increasing in $[R_0(\mu), +\infty)$, with

$$R_0(\mu) \equiv - \left(\dot{P}_0 + \frac{1}{M} \right) \frac{\mu^2}{(r_0^2 - \mu^2)^2} \sim - \left(\dot{P}_0 + \frac{1}{M} \right) \frac{\mu^2}{r_0^4} > 0.$$

Therefore, using once again that $\sigma_\mu^2 \sim \frac{4\mu^2 F(0)}{r_0^4}$,

$$\begin{aligned} \mathcal{K}_{\min}(\mu) &\geq E(U_0) + \frac{\sigma_\mu^2}{2} \left[\dot{P}_0 + \frac{1}{M} \right] + 4R_0(\mu)F(0) + o(\mu^2) \\ &= E(U_0) + \left[\dot{P}_0 + \frac{1}{M} \right] \frac{2\mu^2 F(0)}{r_0^4} - \left[\dot{P}_0 + \frac{1}{M} \right] \frac{4\mu^2 F(0)}{r_0^4} + o(\mu^2) \\ &= E(U_0) - \mu^2 \frac{2F(0)}{r_0^4} \left[\dot{P}_0 + \frac{1}{M} \right] + o(\mu^2). \end{aligned}$$

In view of our hypothesis $\dot{P}_0 + \frac{1}{M} < 0$, we infer that

$$\mathcal{K}_{\min}(\mu) \geq E(U_0) + \frac{\mu^2}{K}$$

for μ_* sufficiently small and some positive constant K , as wished. If the assumption $\dot{P}_0 + \frac{1}{M} < 0$ is not verified, but if $\dot{P}_0 + \frac{1}{M} > 0$ for instance, then the function of R above is increasing in $[0, +\infty)$, with minimum value achieved at $R = 0$ and equal to

$$E(U_0) + \frac{\sigma_\mu^2}{2} \left[\dot{P}_0 + \frac{1}{M} \right] + o(\mu^2) = E(U_0) + \frac{2\mu^2 F(0)}{r_0^4} \left[\dot{P}_0 + \frac{1}{M} \right] + o(\mu^2) \geq E(U_0) + \frac{\mu}{K}.$$

We then would have concluded a stronger estimate, which is actually in contradiction with (32), hence we are necessarily in the case $R > 0$. The assumption $\dot{P}_0 + \frac{1}{M} < 0$ is however crucial for the last step.

Step 7. We assume $\dot{P}_0 + \frac{1}{M} < 0$. Then, for μ_* sufficiently small, the case $R = 0$ does not occur.

We argue in a similar way, but since $R = 0$, the expressions for $E(V)$ and $P(V)$ are given by

$$E(V) = E(U_c) - 4 \int_0^z F(|U_c|^2) dx \quad \text{and} \quad P(V) = P(U_c) - 2 \int_0^z \frac{c}{2} \frac{(r_0^2 - A_c^2)^2}{A_c^2} dx.$$

Here, we have used that $|\partial_x U_c|^2 = F(|U_c|^2)$ since U_c solves (TW $_c$). Combining this here again with the expansion of $E(U_c)$ and $P(U_c)$ gives, using that $0 \leq c \leq K\mu$,

$$\begin{aligned} \mathcal{H}_{\min}(\mu) &\geq E(U_0) + E_\# + \frac{c^2}{2} \dot{P}_0 + o(c^2) + M \left[1 - \sqrt{1 - \frac{c^2}{M^2}} \right] - 4 \int_0^z F(|U_c|^2) dx \\ &\geq E(U_0) + \frac{|P_\#|}{K} + \frac{c^2}{2} \left[\dot{P}_0 + \frac{1}{M} \right] - 4zF(0) + o(\mu^2) \\ &\geq E(U_0) + \frac{1}{K} \left| \arcsin(c/M) + c\dot{P}_0 - c \int_0^z \frac{(r_0^2 - A_c^2)^2}{A_c^2} dx + o(c) \right| + \frac{c^2}{2} \left[\dot{P}_0 + \frac{1}{M} \right] + o(\mu^2) - 4zF(0). \end{aligned}$$

Following the lines of the proof of Lemma 4, we have

$$c \int_0^z \frac{(r_0^2 - A_c^2)^2}{A_c^2} dx = 2 \arctan \left(\sqrt{\frac{\mu^2}{r_0^2 + \xi_c}} \right) + \mathcal{O}(\mu^2). \quad (38)$$

Indeed, noticing that $A_c = \mathcal{O}(\mu)$ in $[0, z]$ with $z \leq K\mu$, we write, expanding the square,

$$\int_0^z \frac{(r_0^2 - A_c^2)^2}{A_c^2} dx = \int_0^z \frac{r_0^4}{A_c^2} - 2 + A_c^2 dx = \int_0^z \frac{r_0^4}{A_c^2} dx + \mathcal{O}(\mu).$$

Then, using the change of variable $\xi = \eta_c(x)$,

$$\begin{aligned} \int_0^z \frac{(r_0^2 - A_c^2)^2}{A_c^2} dx &= \int_{\xi_c}^{\mu^2 - r_0^2} \frac{r_0^4}{(r_0^2 + \xi) \sqrt{-\mathcal{V}_c(\xi)}} d\xi + \mathcal{O}(\mu) \\ &= \int_{\xi_c}^{\mu^2 - r_0^2} \frac{r_0^4}{(r_0^2 + \xi) \sqrt{-\mathcal{V}'_c(\xi_c)(\xi - \xi_c)}} d\xi \\ &\quad + \int_{\xi_c}^{\mu^2 - r_0^2} \frac{r_0^4}{(r_0^2 + \xi)} \left(\frac{1}{\sqrt{-\mathcal{V}_c(\xi)}} - \frac{1}{\sqrt{-\mathcal{V}'_c(\xi_c)(\xi - \xi_c)}} \right) d\xi + \mathcal{O}(\mu) \\ &= \frac{2}{c} \arctan \left(\sqrt{\frac{\mu^2}{r_0^2 + \xi_c}} - 1 \right) + \mathcal{O}(\mu), \end{aligned} \quad (39)$$

by computations similar to those for the proof of Lemma 4. This proves (38). Therefore,

$$\mathcal{H}_{\min}(\mu) \geq E(U_0) + \frac{1}{K} \left| c \left[\dot{P}_0 + \frac{1}{M} \right] - 2 \arctan \left(\sqrt{\frac{\mu^2}{r_0^2 + \xi_c}} - 1 \right) + o(c) \right| + \frac{c^2}{2} \left[\dot{P}_0 + \frac{1}{M} \right] + o(\mu^2) - 4zF(0). \quad (40)$$

By (32), the left-hand side is $\leq E(U_0) + K\mu^2$. Since $\dot{P}_0 + \frac{1}{M} < 0$, $c \leq K\mu$, $z \leq K\mu$ and $F(0) > 0$, this implies

$$\left| c \left[\dot{P}_0 + \frac{1}{M} \right] - 2 \arctan \left(\sqrt{\frac{\mu^2}{r_0^2 + \xi_c}} \right) + o(c) \right| \leq K\mu,$$

thus

$$\arctan \left(\sqrt{\frac{\mu^2}{r_0^2 + \xi_c}} - 1 \right) \leq K\mu$$

and finally, for μ_* small enough,

$$0 \leq \frac{\mu^2}{r_0^2 + \xi_c} - 1 \leq K\mu^2.$$

Combining this with the equality $r_0^2 + \xi_c = \frac{c^2 r_0^4}{4F(0)} + \mathcal{O}(c^4)$ seen during the proof of Lemma 4, we infer

$$c = \frac{2\sqrt{F(0)}}{r_0^2} \mu + \mathcal{O}(\mu^2).$$

In particular, going back to (39) and since for $0 \leq x \leq z$,

$$r_0^2 + \xi_c = A_c^2(0) \leq A_c^2(x) \leq A_c^2(z) = \mu^2,$$

this implies

$$\frac{z r_0^4}{\mu^2} \leq \int_0^z \frac{r_0^4}{A_c^2} \leq \frac{2}{c} \arctan \left(\sqrt{\frac{\mu^2}{r_0^2 + \xi_c}} - 1 \right) + \mathcal{O}(\mu) \leq \frac{K\mu}{c} + K\mu \leq K,$$

which provides (since $c \approx \mu$)

$$z \leq K\mu^2.$$

Inserting this into (40) and keeping in mind that the left-hand side is $\leq E(U_0) + K\mu^2$, we deduce

$$c \left[\dot{P}_0 + \frac{1}{M} \right] - 2 \arctan \left(\sqrt{\frac{\mu^2}{r_0^2 + \xi_c}} - 1 \right) = o(\mu).$$

However, since $\arctan \left(\sqrt{\frac{\mu^2}{r_0^2 + \xi_c}} - 1 \right) \geq 0$, this gives

$$o(\mu) \leq c \left[\dot{P}_0 + \frac{1}{M} \right] \sim \frac{2\mu\sqrt{F(0)}}{r_0^2} \left[\dot{P}_0 + \frac{1}{M} \right],$$

yielding a contradiction for small μ since we have $\dot{P}_0 + \frac{1}{M} < 0$ by assumption. Therefore, the case $R = 0$ does not occur for sufficiently small μ_* . If we had $\dot{P}_0 + \frac{1}{M} > 0$, we would not have been able to show that $\mathcal{K}_{\min}(\mu)$ gives a control on μ .

The proof of Proposition 6.2 is complete. \square

Proof of Theorem 9. Let $U \in \mathcal{V}_{\mu_*}$. If $\mu \equiv \inf_{\mathbb{R}} |U| > 0$, then Proposition 6.2 gives $\mathcal{K}(U) \geq E(U_0) + \mu^2/K > E(U_0) = \mathcal{K}(U_0)$. If $\inf_{\mathbb{R}} |U| = 0$, we deduce from Proposition 6.1 that $\mathcal{K}(U) \geq E(U_0) + 2Mr_0^4 \sin^2(\mathfrak{P}(U) - \pi r_0^2) r_0^2$. Hence $\mathcal{K}(U) > E(U_0)$ except if $\mathcal{K}(U) = E(U_0)$. From the study of the equality case in Proposition 6.1, it follows that $U \in \{e^{i\theta} U_0(\cdot - y), y \in \mathbb{R}, \theta \in \mathbb{R}\}$, as claimed.

6.3 Proof of Theorem 10

As a first step, we shall need a quantified version of Proposition 6.1.

Proposition 6.3 *There exists $\epsilon_0 > 0$ and $K > 0$, depending only on f such that, for any $U \in \mathcal{Z}$ verifying*

$$\mathcal{K}(U) - E(U_0) \leq \epsilon_0 \quad \text{and} \quad \inf_{\mathbb{R}} |U| \leq \epsilon_0,$$

there holds

$$\inf_{\substack{y \in \mathbb{R} \\ \theta \in \mathbb{R}}} d_{\mathcal{Z}}(U, e^{i\theta} U_0(\cdot - y)) \leq K \left(\mathcal{K}(U) - E(U_0) + \inf_{\mathbb{R}} |U| \right)^{1/4}.$$

Proof. First, we translate the problem in space so that $\mu \equiv \inf_{\mathbb{R}} |U| = |U|(0)$ and shall choose the phase factor later. We follow the lines of the proof of Proposition 6.1 and actually get (writing $U = Ae^{i\phi}$ locally in $\{|U| > 0\}$)

$$\begin{aligned} \int_0^{+\infty} |\partial_x U|^2 + F(|U|^2) dx &= \int_0^{+\infty} \mathbf{1}_{|U|>0} A^2 (\partial_x \phi)^2 dx + \int_0^{+\infty} |\partial_x |U||^2 + F(|U|^2) dx \\ &= \int_0^{+\infty} \mathbf{1}_{|U|>0} A^2 (\partial_x \phi)^2 dx + \int_0^{+\infty} \left[\sqrt{F(|U|^2)} - |\partial_x |U|| \right]^2 dx \\ &\quad + 2 \int_0^{+\infty} |\sqrt{F(|U|^2)} \partial_x |U|| dx \\ &\geq \int_0^{+\infty} \mathbf{1}_{|U|>0} A^2 (\partial_x \phi)^2 dx + \int_0^{+\infty} \left[\sqrt{F(|U|^2)} - |\partial_x |U|| \right]^2 dx + 2 \int_{\mu}^{r_0} \sqrt{F(s^2)} ds. \end{aligned}$$

Arguing similarly in $(-\infty, 0)$, we get

$$E(U) \geq E(U_0) + \int_{\mathbb{R}} \mathbf{1}_{|U|>0} A^2 (\partial_x \phi)^2 dx + \int_{\mathbb{R}} \left[\sqrt{F(|U|^2)} - |\partial_x |U|| \right]^2 dx - 4 \int_0^{\mu} \sqrt{F(s^2)} ds. \quad (41)$$

The gradient of the phase is controlled using (41). We shall now estimate the modulus part. Let us denote $A \equiv |U|$ and

$$h \equiv \partial_x A - \sqrt{F(A^2)},$$

for which we have, by (41),

$$\|h\|_{L^2(\mathbb{R})}^2 \leq E(U) - E(U_0) + 4 \int_0^{\mu} \sqrt{F(s^2)} ds \leq E(U) - E(U_0) + K\mu. \quad (42)$$

Recall that U_0 verifies $(\partial_x U_0)^2 = F(U_0^2)$ in \mathbb{R} , hence $\partial_x U_0 = \sqrt{F(U_0^2)}$ in \mathbb{R}_+ . Setting $\Theta \equiv A - |U_0|$, we infer

$$\partial_x \Theta = \sqrt{F(A^2)} - \sqrt{F(U_0^2)} + h \quad \text{in } \mathbb{R}_+.$$

We set, for $x \geq 0$,

$$G(x, \theta) \equiv \sqrt{F((U_0(x) + \theta)^2)} - \sqrt{F(U_0^2(x))} + \frac{U_0(x) f(U_0^2(x)) \theta}{\sqrt{F(U_0^2(x))}}$$

Since U_0 verifies $\partial_x^2 U_0 + U_0(x) f(U_0^2(x)) = 0$ and $\partial_x U_0 = \sqrt{F(U_0^2(x))}$ in \mathbb{R}_+ , it follows that

$$G(x, \theta) = \sqrt{F((U_0(x) + \theta)^2)} - \sqrt{F(U_0^2(x))} - \frac{\partial_x^2 U_0(x)}{\partial_x U_0(x)} \theta.$$

Moreover, by Taylor expansion, we infer the existence of $K > 0$ and $\theta_0 > 0$ such that, for $|\theta| \leq \theta_0$, $x \in \mathbb{R}_+$,

$$|G(x, \theta)| \leq K\theta^2.$$

The estimate is clearly uniform in view of the exponential decay of $\partial_x U_0$ at infinity. Therefore,

$$\partial_x \Theta = \frac{\partial_x^2 U_0(x)}{\partial_x U_0(x)} \Theta + G(x, \Theta) + h(x). \quad (43)$$

We view this ode as a linear ode with source term $G(x, \Theta(x)) + h(x)$. Since $\partial_x U_0$ solves the homogeneous equation, we infer from Duhamel's formula and the fact that $\Theta(0) = A(0) - U_0(0) = |U(0)| = \mu$ that for $x \geq 0$,

$$\Theta(x) = \mu + \partial_x U_0(x) \int_0^x \frac{G(z, \Theta(z)) + h(z)}{\partial_x U_0(z)} dz. \quad (44)$$

We shall prove that this equation implies that if μ and $\|h\|_{L^2(\mathbb{R}_+)}$ are sufficiently small, then

$$\|\Theta\|_{L^2(\mathbb{R}_+)} \leq K \left(\|h\|_{L^2(\mathbb{R}_+)} + \mu \right). \quad (45)$$

We assume $\mu < \theta_0/2$. Note that since U_0 is a kink, we have the decays given in Proposition 1. Hence, there exists two positive constants K_1 and K_2 such that

$$\forall x \in \mathbb{R}_+, \quad \frac{e^{-c_s x}}{K_1} \leq \partial_x U_0(x) \leq K_2 e^{-c_s x}.$$

In particular, if $|\Theta(x)| \leq \theta_0$ in the interval $[0, R]$, then (44) implies, for $x \in [0, R]$,

$$\begin{aligned} |\Theta(x)| &\leq \mu + K_1 K_2 e^{-c_s x} \int_0^x e^{c_s z} \left[K \|\Theta\|_{L^\infty([0, R])} |\Theta(z)| + |h|(z) \right] dz \\ &\leq \mu + \frac{K K_1 K_2}{c_s} \|\Theta\|_{L^\infty([0, R])}^2 + \frac{K_1 K_2}{\sqrt{2c_s}} \|h\|_{L^2(\mathbb{R}_+)} \end{aligned}$$

by Cauchy-Schwarz. We thus choose $\|h\|_{L^2(\mathbb{R}_+)} + \mu$ sufficiently small so that

$$4 \left(\mu + \frac{K_1 K_2}{\sqrt{2c_s}} \|h\|_{L^2(\mathbb{R}_+)} \right) \leq \tilde{\theta}_0 \equiv \min \left\{ \theta_0, \frac{c_s}{2K K_1 K_2} \right\}.$$

Then, we consider the set \mathcal{R} of all $R > 0$ such that $|\Theta(x)| \leq \tilde{\theta}_0$ in the interval $[0, R]$. Since $\Theta \in H^1(\mathbb{R}, \mathbb{C})$ is continuous by Sobolev embedding and $|\Theta(0)| = \mu < \tilde{\theta}_0$, $\mathcal{R} \neq \emptyset$ and is closed in \mathbb{R}_+^* . Moreover, the above estimate shows that for $R \in \mathcal{R}$,

$$\|\Theta\|_{L^\infty([0, R])} \leq \mu + \frac{K K_1 K_2}{c_s} \|\Theta\|_{L^\infty([0, R])}^2 + \frac{K_1 K_2}{\sqrt{2c_s}} \|h\|_{L^2(\mathbb{R}_+)},$$

which gives

$$\|\Theta\|_{L^\infty([0, R])} \left(1 - \frac{K K_1 K_2}{c_s} \|\Theta\|_{L^\infty([0, R])} \right) \leq \mu + \frac{K_1 K_2}{\sqrt{2c_s}} \|h\|_{L^2(\mathbb{R}_+)},$$

and then

$$\|\Theta\|_{L^\infty([0, R])} \leq 2 \left[\mu + \frac{K_1 K_2}{\sqrt{2c_s}} \|h\|_{L^2(\mathbb{R}_+)} \right] \leq \frac{\tilde{\theta}_0}{2} < \tilde{\theta}_0. \quad (46)$$

Consequently, \mathcal{R} is open in \mathbb{R}_+^* . By connexity, $\mathcal{R} = \mathbb{R}_+^*$, proving (45). In what follows, we assume $\|h\|_{L^2(\mathbb{R}_+)} + \mu$ sufficiently small so that $\|\Theta\|_{L^\infty} \leq \tilde{\theta}_0$, thus $|G(x, \Theta)| \leq K\Theta^2$. In particular,

$$|\Theta(x)| \leq \mu + K_1 K_2 \int_0^x e^{-c_s(x-z)} \left[K \|\Theta\|_{L^\infty([0, R])} |\Theta(z)| + |h|(z) \right] dz.$$

For $R > 0$ to be determined later, we then deduce from classical convolution estimates that

$$\|\Theta\|_{L^2([0, R])} \leq \mu \sqrt{R} + K_3 \|\Theta\|_{L^\infty(\mathbb{R}_+)} \|\Theta\|_{L^2([0, R])} + K_3 \|h\|_{L^2(\mathbb{R}_+)}.$$

Imposing $\|h\|_{L^2(\mathbb{R}_+)} + \mu$ smaller if necessary, we may assume that

$$K_3\|\Theta\|_{L^\infty(\mathbb{R}_+)} \leq K_3K(\|h\|_{L^2(\mathbb{R}_+)} + \mu) \leq \frac{1}{2},$$

so that we get

$$\|\Theta\|_{L^2([0,R])} \leq K_4\left(\mu\sqrt{R} + \|h\|_{L^2(\mathbb{R}_+)}\right).$$

Reporting this into (43) provides

$$\|\partial_x\Theta\|_{L^2([0,R])}^2 \leq K_5\left(\mu^2R + \|h\|_{L^2(\mathbb{R}_+)}^2\right).$$

Arguing similarly in $[-R, 0]$ and using (42), we obtain an H^1 estimate for Θ in $[-R, R]$:

$$\|\Theta\|_{H^1([-R,R])}^2 \leq K_6\left(E(U) - E(U_0) + \mu^2R + \mu\right). \quad (47)$$

We now turn to the estimate in $\{|x| \geq R\}$. For that purpose, we write

$$\begin{aligned} \int_{|x| \geq R} (\partial_x|U|)^2 + \frac{1}{K}(|U|^2 - r_0^2)^2 dx &\leq E(U) - E(U_0) + \int_{|x| \geq R} (\partial_x U_0)^2 + F(U_0^2) dx \\ &\quad - \int_{|x| \leq R} (\partial_x|U|)^2 + F(|U|^2) dx + \int_{|x| \leq R} (\partial_x U_0)^2 + F(U_0^2) dx. \end{aligned} \quad (48)$$

Since U_0 decays exponentially (see Proposition 1), it follows that

$$\int_{|x| \geq R} |\partial_x U_0|^2 + F(U_0^2) dx \leq Ke^{-c_s R}.$$

Furthermore, by integration by parts,

$$\begin{aligned} & - \int_{|x| \leq R} (\partial_x|U|)^2 + F(|U|^2) dx + \int_{|x| \leq R} (\partial_x U_0)^2 + F(U_0^2) dx \\ &= - \int_{|x| \leq R} 2\partial_x U_0 \partial_x \Theta - 2U_0 f(U_0^2) \Theta dx - \int_{|x| \leq R} (\partial_x \Theta)^2 + F([U_0 + \Theta]^2) - F(U_0^2) - 2U_0 F'(U_0^2) \Theta dx \\ &\leq \int_{|x| \leq R} 2\Theta[\partial_x^2 U_0 + U_0 f(U_0^2)] dx - 2\Theta(+R)\partial_x U_0(+R) + 2\Theta(-R)\partial_x U_0(-R) + K\|\Theta\|_{H^1([-R,+R])}^2 \\ &\leq Ke^{-c_s R} + K\left(E(U) - E(U_0) + \mu^2R + \mu\right). \end{aligned}$$

For the before last line, we have used that $\theta \mapsto F([U_0 + \theta]^2) - F(U_0^2) - 2U_0 F'(U_0^2)\theta$ is $\mathcal{O}(\theta^2)$ as $\theta \rightarrow 0$ and for the last line, that U_0 solves $\partial_x^2 U_0 + U_0 f(U_0^2) = 0$, the exponential decay of $\partial_x U_0$ and the uniform bound on Θ . Reporting these estimates into (48) provides

$$\begin{aligned} \|\Theta\|_{H^1(\{|x| \geq R\})}^2 &= \int_{|x| \geq R} (\partial_x|U| - \partial_x|U_0|)^2 + (|U| - |U_0|)^2 dx \\ &\leq 2 \int_{|x| \geq R} (\partial_x|U|)^2 + (\partial_x|U_0|)^2 + (|U| - r_0)^2 + (|U_0| - r_0)^2 dx \\ &\leq K\left[E(U) - E(U_0) + e^{-c_s R} + \mu^2R + \mu\right]. \end{aligned}$$

Combining this with (47), we deduce that for any $R > 0$, there holds

$$\|\Theta\|_{H^1(\mathbb{R})}^2 \leq K\left[E(U) - E(U_0) + e^{-c_s R} + \mu^2R + \mu\right].$$

We then choose $R = \mu^{-1}$ if $\mu > 0$ or $R \rightarrow +\infty$ if $\mu = 0$, and get

$$\|\Theta\|_{H^1(\mathbb{R})} \leq K \sqrt{E(U) - E(U_0) + \mu}.$$

Notice that if $f' < 0$ everywhere, then we may give a quick proof of the above estimate, since, using here again integration by parts and that $\partial_x^2 U_0 + U_0 f(U_0^2) = 0$, we may deduce that

$$E(U) - E(U_0) \geq -4\mu \partial_x U_0(0) + \int_0^{+\infty} (\partial_x \Theta)^2 dx + \int_{\mathbb{R}} F((U_0 + \Theta)^2) - F(U_0^2) - 2U_0 \Theta F'(U_0^2) dx,$$

and since $f' < 0$, $F((U_0 + \theta)^2) - F(U_0^2) - 2U_0 \theta F'(U_0^2) \geq \theta^2/K$ by Taylor expansion, providing the desired H^1 bound on Θ .

Observe now that

$$\mathcal{K}(U) - E(U_0) \geq E(U) - E(U_0) \geq \int_{\mathbb{R}} \mathbf{1}_{|U|>0} [A \partial_x \phi]^2 dx,$$

hence

$$\begin{aligned} \|\partial_x U - \partial_x U_0\|_{L^2(\mathbb{R})} &= \|\partial_x(|U_0| + \Theta) e^{i\phi} \mathbf{1}_{|U|>0} + i \mathbf{1}_{|U|>0} A \partial_x \phi e^{i\phi} - \partial_x U_0\|_{L^2(\mathbb{R})} \\ &\leq \|e^{i\phi} \mathbf{1}_{|U|>0} \partial_x |U_0| - \partial_x U_0\|_{L^2(\mathbb{R})} + \|\mathbf{1}_{|U|>0} A \partial_x \phi\|_{L^2(\mathbb{R})} + \|\Theta\|_{L^2(\mathbb{R})} \\ &\leq \|e^{i\phi} \mathbf{1}_{|U|>0} \partial_x |U_0| - \partial_x U_0\|_{L^2(\mathbb{R})} + K \left[\mathcal{K}(U) - E(U_0) + \mu \right]^{1/2}. \end{aligned} \quad (49)$$

We distinguish now the cases $\mu = 0$ and $\mu > 0$, and begin with the assumption $\mu > 0$. Then, we have a global lifting $U = Ae^{i\phi}$ and

$$\begin{aligned} d_{\mathcal{Z}}(U, U_0) &= \|\partial_x U - \partial_x U_0\|_{L^2(\mathbb{R})} + \||U| - |U_0|\|_{L^2(\mathbb{R})} + |U(0) - U_0(0)| \\ &= \|\partial_x U - \partial_x U_0\|_{L^2(\mathbb{R})} + \|\Theta\|_{L^2(\mathbb{R})} + \mu \\ &\leq \|e^{i\phi} \partial_x |U_0| - \partial_x U_0\|_{L^2(\mathbb{R})} + K \left[\mathcal{K}(U) - E(U_0) + \mu \right]^{1/2}. \end{aligned}$$

Now, we notice that

$$\begin{aligned} \|e^{i\phi} \partial_x |U_0| - \partial_x U_0\|_{L^2(\mathbb{R})}^2 &= 2 \int_{\mathbb{R}} \left[(\partial_x U_0)^2 - \partial_x U_0 \partial_x |U_0| \cos(\phi) \right] dx \\ &= 2 \int_0^{+\infty} (\partial_x U_0)^2 (1 - \cos(\phi)) dx + 2 \int_{-\infty}^0 (\partial_x U_0)^2 (1 + \cos(\phi)) dx \end{aligned} \quad (50)$$

and that

$$\mathcal{K}(U) - E(U_0) \geq 2Mr_0^4 \sin^2 \left(\frac{\mathfrak{P}(U) - r_0^2 \pi}{2r_0^2} \right) \geq \frac{1}{K} (P(U) - r_0^2 \pi \bmod 2\pi r_0^2)^2. \quad (51)$$

We define $\delta = (\mathcal{K}(U) - E(U_0) + \mu)^{1/4}$. By Cauchy-Schwarz, we have

$$\begin{aligned} \left| \int_{|x| \geq \delta} (A^2 - r_0^2) \partial_x \phi dx \right| &\leq \frac{K}{\inf_{|x| \geq \delta} A} \left(\int_{|x| \geq \delta} (A^2 - r_0^2)^2 dx \right)^{1/2} \left(\int_{|x| \geq \delta} (A \partial_x \phi)^2 dx \right)^{1/2} \\ &\leq \frac{K}{\inf_{|x| \geq \delta} A} \left(E(U) - E(U_0) + \mu \right)^{1/2}. \end{aligned}$$

Inserting this into (51) gives

$$\begin{aligned} \left| \int_{|x| \leq \delta} (A^2 - r_0^2) \partial_x \phi dx - r_0^2 \pi \bmod 2\pi r_0^2 \right| &\leq K \left[(\mathcal{K}(U) - E(U_0))^{1/2} + \frac{1}{\inf_{|x| \geq \delta} A} \left(E(U) - E(U_0) + \mu \right)^{1/2} \right] \\ &\leq \frac{K}{\inf_{|x| \geq \delta} A} \left(\mathcal{K}(U) - E(U_0) + \mu \right)^{1/2}. \end{aligned}$$

In addition, by Cauchy-Schwarz,

$$\left| \int_{|x| \leq \delta} A^2 \partial_x \phi \, dx \right| \leq \sqrt{2\delta} \left(\sup_{|x| \leq \delta} A \right) \left(\mathcal{H}(U) - E(U_0) + \mu \right)^{1/2}.$$

Consequently,

$$\begin{aligned} r_0^2 \left| \phi(+\delta) - \phi(-\delta) - \pi \bmod 2\pi \right| &\leq \left| \int_{|x| \leq \delta} (A^2 - r_0^2) \partial_x \phi \, dx - r_0^2 \pi \bmod 2\pi r_0^2 \right| \\ &\quad + \sqrt{2\delta} \left(\sup_{|x| \leq \delta} A \right) \left(\mathcal{H}(U) - E(U_0) + \mu \right)^{1/2} \\ &\leq \left[\frac{K}{\inf_{|x| \geq \delta} A} + \sqrt{2\delta} \left(\sup_{|x| \leq \delta} A \right) \right] \left(\mathcal{H}(U) - E(U_0) + \mu \right)^{1/2}. \end{aligned} \quad (52)$$

From our choice $\delta = (\mathcal{H}(U) - E(U_0) + \mu)^{1/4} \ll 1$ and since $\|\Theta\|_{L^\infty(\mathbb{R})} \leq K(\mathcal{H}(U) - E(U_0) + \mu)^{1/2} = \mathcal{O}(\delta^2)$, we infer $\inf_{|x| \geq \delta} A \geq \inf_{|x| \geq \delta} |U_0| - \|\Theta\|_{L^\infty(\mathbb{R})} \geq \delta/K$. Similarly, $\sup_{|x| \leq \delta} A \leq \sup_{|x| \leq \delta} |U_0| + \|\Theta\|_{L^\infty(\mathbb{R})} \leq K\delta$. Reporting this into (52) yields

$$\left| \phi(+\delta) - \phi(-\delta) - \pi \bmod 2\pi \right| \leq K\delta.$$

We now freeze the gauge invariance by imposing $\phi(+\delta) = 0$. Note that then $\phi(-\delta) = \pi + \mathcal{O}(\delta)$. Furthermore, since $\phi(+\delta) = 0$,

$$\int_{|x| \geq \delta} (\partial_x \phi)^2 \, dx \leq \frac{K}{(\inf_{|x| \geq \delta} A)^2} \int_{|x| \geq \delta} A^2 (\partial_x \phi)^2 \, dx \leq \frac{K}{\delta^2} \delta^4 = K\delta^2,$$

which implies, for $x \geq \delta$,

$$|1 - \cos(\phi(x))| \leq |1 - \cos(\phi(0))| + \left| \int_{\delta}^x \partial_x \phi \sin(\phi) \right| \leq K\delta\sqrt{x}$$

and similarly, since $\cos(\phi(-\delta)) = \cos(\pi + \mathcal{O}(\delta)) = -1 + \mathcal{O}(\delta^2)$, for $x \leq -\delta$,

$$|1 + \cos(\phi(x))| \leq K\delta\sqrt{|x|}.$$

We turn back to (50) and infer

$$\begin{aligned} \|\mathbf{e}^{i\phi} \partial_x |U_0| - \partial_x U_0\|_{L^2(\mathbb{R})}^2 &\leq K\delta + 2 \int_{\delta}^{+\infty} (\partial_x U_0)^2 (1 - \cos(\phi)) \, dx + 2 \int_{-\infty}^{-\delta} (\partial_x U_0)^2 (1 + \cos(\phi)) \, dx \\ &\leq K\delta + K\delta \int_{\mathbb{R}} (\partial_x U_0)^2 \sqrt{|x|} \, dx = K\delta. \end{aligned}$$

Inserting these estimates in (49), it follows that

$$d_{\mathcal{Z}}(U, U_0) \leq K\delta.$$

We now turn to the case $\mu = 0$. Without loss of generality, we may assume that $|U| > 0$ in $(-\infty, 0)$ (since $|U| \rightarrow r_0 > 0$ at $\pm\infty$), and let $\ell \geq 0$ be such that $|U|(\ell) = 0$ and $|U| > 0$ in $(\ell, +\infty)$. We first estimate ℓ by writing that

$$|U_0|(\ell) = |U|(\ell) + \Theta(\ell) = \Theta(\ell) \leq \|\Theta\|_{L^\infty(\mathbb{R})} \leq K \left(\mathcal{H}(U) - E(U_0) + \mu \right)^{1/2} = K\delta^2,$$

thus

$$\ell \leq K\delta^2.$$

Moreover, we have two local liftings $U = Ae^{i\phi_+}$ in $[\ell, +\infty)$ and $U = Ae^{i\phi_-}$ in $(-\infty, 0)$. Going back to (49), we then deduce

$$d_{\mathcal{Z}}(U, U_0) \leq \|e^{i\phi_-} \partial_x |U_0| - \partial_x U_0\|_{L^2(-\infty, 0)} + \|e^{i\phi_+} \partial_x |U_0| - \partial_x U_0\|_{L^2(\ell, +\infty)} + K\delta + K \left[\mathcal{K}(U) - E(U_0) \right]^{1/2}.$$

Arguing as for the case $\mu > 0$, we obtain $|U| = A \geq \delta/K$ in $[\ell + \delta, +\infty)$ and in $(-\infty, -\delta)$. By definition of \mathfrak{P} , we have

$$\mathfrak{P}(U) = \int_{-\delta}^{\ell+\delta} \langle iU | \partial_x U \rangle + \int_{\ell+\delta}^{+\infty} (A^2 - r_0^2) \partial_x \phi_+ dx + r_0^2 \phi_+(\ell + \delta) + \int_{-\infty}^{-\delta} (A^2 - r_0^2) \partial_x \phi_- dx - r_0^2 \phi_+(-\delta)$$

in $\mathbb{R}/(2\pi r_0^2 \mathbb{Z})$, hence the same arguments as in the case $\mu > 0$ provide

$$\left| \phi_+(\ell + \delta) - \phi_+(-\delta) - \pi \bmod 2\pi \right| \leq K\delta,$$

since the integral $\int_{-\delta}^{\ell+\delta} \langle iU | \partial_x U \rangle$ is bounded by $K\sqrt{\delta}$ by Cauchy-Schwarz. Imposing $\phi_+(\ell + \delta)$ for the gauge invariance, we infer $1 - \cos(\phi_+(\ell + \delta)) = 0$ and $\phi_+(-\delta) = \pi + \mathcal{O}(\sqrt{\delta}) \bmod 2\pi$, hence $1 + \cos(\phi_-(-\delta)) = \mathcal{O}(\delta)$. Therefore, we conclude as before that

$$d_{\mathcal{Z}}(U, U_0) \leq K\delta,$$

which finishes the proof of the Proposition. \square

In order to prove Theorem 10, we use Proposition 6.2, which provides

$$\mathcal{K}(U) \geq E(U_0) + \frac{1}{K} \left(\inf_{\mathbb{R}} |U| \right)^2,$$

thus

$$\mu = \inf_{\mathbb{R}} |U| \leq K \sqrt{\mathcal{K}(U) - E(U_0)}.$$

Inserting this bound in Proposition 6.3 then gives

$$d_{\mathcal{Z}}(U, U_0) \leq K \left[\mathcal{K}(U) - E(U_0) + K \sqrt{\mathcal{K}(U) - E(U_0)} \right]^{1/4} \leq K \sqrt[8]{\mathcal{K}(U) - E(U_0)},$$

and the proof is complete.

7 About the stability analysis for the sonic waves ($c = \mathbf{c}_s$)

We have left aside in our study the case of the sonic waves ($c = \mathbf{c}_s$), but would like to say a few words on the difficulties associated with this critical case.

We note that if there exists a sonic nontrivial travelling wave, it does not vanish, hence we may use the hydrodynamical formulation (15) of (NLS) as in [47]. The point is that the Sturm-Liouville operator (see section 4 in [47])

$$L \equiv -\frac{\partial}{\partial x} \left(\frac{1}{4(r_0^2 - \eta)} \frac{\partial}{\partial x} \right) + q(x),$$

with

$$q(x) \equiv \frac{(\partial_x \eta)^2}{4(r_0^2 - \eta)^3} - \frac{\partial}{\partial x} \left(\frac{\partial_x \eta}{4(r_0^2 - \eta)^2} \right) - \frac{1}{2} f'(r_0^2 - \eta) - \frac{c^2 r_0^4}{4(r_0^2 - \eta)^3}$$

has, by Weyl's theorem, essential spectrum $\sigma_{\text{ess}}(L) = [0, +\infty)$ when $c = \mathbf{c}_s$. Indeed, we know from Proposition 1 that $\eta_{\mathbf{c}_s}$ and its derivatives tend to zero at infinity, hence, as $x \rightarrow \pm\infty$, $q(x) \rightarrow -\frac{1}{2} f'(r_0^2) - \frac{c^2}{4r_0^2} = 0$ since $c^2 = \mathbf{c}_s^2 = -2r_0^2 f'(r_0^2)$. Therefore, there does not exist $\delta > 0$ such that $\langle Hp, p \rangle \geq \delta \|p\|^2$ for any p orthogonal to

the subspace spanned by the negative and the zero eigenvalue, and thus the Grillakis-Shatah-Strauss theory does not apply.

In the case $\frac{dP}{dc}|_{c=c_s} < 0$, where it is natural to expect stability, a natural thing would be to try to work with the functional

$$\mathcal{L}(\psi) \equiv E(\psi) - c_s P(\psi) + \frac{M}{2} \left(P(\psi) - P(U_{c_s}) \right)^2$$

and to follow the lines of the proof of Theorem 9. Indeed, the spectral analysis shall not give positive definiteness of the Hessian due to presence of essential spectrum down to 0. Therefore, we may study \mathcal{L} at fixed $\mu = \inf_{\mathbb{R}} |\psi|$ close to $\inf_{\mathbb{R}} |U_{c_s}|$. When $0 < c_* < c_s$ and $\frac{dP}{dc}|_{c=c_*} \neq 0$, the infimum of $|U_c|$ contains a neighborhood of $\inf_{\mathbb{R}} |U_{c_*}|$ for c close to c_* . For $c_* = c_s$, this is no longer the case: we have only a one sided neighborhood of $\inf_{\mathbb{R}} |U_{c_s}|$. It is plausible that the study for μ in this one sided neighborhood of $\inf_{\mathbb{R}} |U_{c_s}|$ can be done as in the proof of Theorem 9, but for the remaining values of μ , we have to find a sharp ansatz, which is not very easy to find.

Furthermore, for the linear instability which is expected if $\frac{dP}{dc}|_{c=c_s} > 0$, let us mention the following point. For the eigenvalue problem studied in [6], the characteristic equation for the constant coefficient limit at infinity, namely

$$r^4 - (c_s^2 - c_*^2)r^2 - 2c_*\lambda r + \lambda^2 = 0.$$

becomes, when $c_* = c_s$,

$$r^4 - 2c_s\lambda r + \lambda^2 = 0. \tag{53}$$

The behaviour of the roots for small λ is then different from the case $0 < c_* < c_s$. Indeed, there exists a root $\sim \lambda/(2c_s)$ for $\lambda \rightarrow 0$, and for the three other roots, we use the variable $r = \sqrt[3]{\lambda}z$, which transforms $r^4 - 2c_s\lambda r + \lambda^2 = 0$ into $z^4 - 2c_s z + \lambda^{2/3} = 0$. This last equation has, for $\lambda \rightarrow 0$, three roots $\sim j^k \sqrt[3]{2c_s}$, where $j = e^{2i\pi/3}$ and $k = 0, 1, 2$. In particular, (53) has three roots $\sim j^k \sqrt[3]{2c_s\lambda}$, $k = 0, 1, 2$. The value $\lambda = 0$ is then a branching point, and we shall have a smooth problem not in λ but in $\sqrt[3]{\lambda}$. Since analyticity is not necessary for our purpose, we may define an Evans function \mathbb{D} in \mathbb{R}_+ , smooth, and such that, for $\lambda > 0$, $\mathbb{D}(\sqrt[3]{\lambda}) = 0$ if and only if λ is an unstable eigenvalue for (27). another difficulty comes from the fact that it will be difficult to find an analytic extension of the Evans function \mathbb{D} near 0 since, by Proposition 1, for $c_* = c_s$, u_* and η_* decay only at an algebraic rate and not an exponential rate. Consequently, we can not use the Gap Lemma of [30] and [40]. Finally, as a straightforward computation shows, the stable and unstable subspaces for the eigenvalue problem are transverse for $\lambda > 0$ but their continuous extensions at $\lambda = 0$ have a nontrivial intersection. Therefore, both stability and instability requires some further analysis, and the situation is then much more delicate than the one studied in subsection 5.1.

Appendix A: construction of a Liapounov functional in the stable case in the Grillakis-Shatah-Strauss framework

We work with the notations of [35], and recall them briefly. We consider a Hamiltonian equation in a real Hilbert space \mathcal{X} , with scalar product $(\cdot, \cdot)_{\mathcal{X}}$, under the form

$$\frac{\partial u}{\partial t} = JE'(u), \tag{H}$$

where $J : \mathcal{X}^* \rightarrow \mathcal{X}$ is a closed linear operator with dense domain and skew-symmetric. Assume that T is a \mathcal{C}_0 -group of unitary operators in \mathcal{X} generated by $T'(0)$, which is skew-adjoint and with dense domain, and that E is invariant by T , that is $E(T(s)u) = E(u)$ for any $s \in \mathbb{R}$, $u \in \mathcal{X}$. Assume moreover that $T(s)J = JT(-s)^*$ for any $s \in \mathbb{R}$ and that there exists $B : \mathcal{X} \rightarrow \mathcal{X}^*$ linear and bounded such that $B^* = B$ and JB is an extension of $T'(0)$. We then set

$$Q(u) \equiv \frac{1}{2} \langle Bu, u \rangle_{\mathcal{X}^*, \mathcal{X}}.$$

The basic assumptions of [35] are the following ones.

Assumption 1 (existence of solutions): For any $r > 0$ there exists $t_* > 0$, depending only on r , such that for any $u^{\text{in}} \in \mathcal{X}$, there exists a $u \in \mathcal{C}((-t_*, t_*), \mathcal{X})$ with $u(0) = u^{\text{in}}$ solution of (\mathcal{H}) in the sense that for any $\varphi \in D(J) \subset \mathcal{X}^*$,

$$\frac{d}{dt} \langle u(t), \varphi \rangle_{\mathcal{X}^*, \mathcal{X}} = - \langle E'(u(t)), J\varphi \rangle_{\mathcal{X}^*, \mathcal{X}} \quad \text{in } \mathcal{D}'((-t_*, t_*)),$$

and verifying $E(u(t)) = E(u^{\text{in}})$ and $Q(u(t)) = Q(u^{\text{in}})$ for $t \in (-t_*, t_*)$.

Assumption 2 (existence of “bound states”): There exists an interval $\Omega \subset \mathbb{R}$, not reduced to a singleton, and a mapping $\Omega \ni \omega \mapsto \phi_\omega \in \mathcal{X}$ of class \mathcal{C}^1 such that, for any $\omega \in \Omega$,

$$E'(\phi_\omega) = \omega Q'(\phi_\omega), \quad \phi_\omega \in D(T'(0)^3) \cap D(JIT'(0)^2), \quad T'(0)\phi_\omega \neq 0.$$

Assumption 3 (spectral decomposition): For each $\omega \in \Omega$, the operator $H_\omega \equiv E''(\phi_\omega) - \omega Q''(\phi_\omega) : \mathcal{X} \rightarrow \mathcal{X}^*$ has its kernel spanned by $T'(0)\phi_\omega$, has one negative simple eigenvalue and the rest of its spectrum is positive and bounded away from zero.

Under assumption 2, we consider some $\omega_* \in \Omega$ and the associated bound state ϕ_{ω_*} , and then define, for $M > 0$, the functional

$$\mathcal{L}_{\omega_*}(u) \equiv E(u) - \omega_* Q(u) + \frac{M}{2} \left(Q(u) - Q(\phi_{\omega_*}) \right)^2.$$

It is clear that ϕ_{ω_*} is a critical point of \mathcal{L}_{ω_*} : $\mathcal{L}'(\phi_{\omega_*}) = E'(\phi_{\omega_*}) - \omega_* Q'(\phi_{\omega_*}) = 0$. We denote by

$$\Lambda \equiv \mathcal{L}''_{\omega_*}(\phi_{\omega_*}) = H_{\omega_*} + M \langle Q'(\phi_{\omega_*}), \cdot \rangle_{\mathcal{X}^*, \mathcal{X}} Q'(\phi_{\omega_*})$$

its second derivative, which is a self-adjoint operator. The main result of this appendix is the following.

Theorem A *We make assumptions 2 and 3 and suppose moreover that the operator $\langle Q'(\phi_{\omega_*}), \cdot \rangle_{\mathcal{X}^*, \mathcal{X}} Q'(\phi_{\omega_*})$ is a compact perturbation of H_{ω_*} . If $\frac{dQ(\phi_\omega)}{d\omega} \Big|_{\omega=\omega_*} < 0$ and*

$$M > \frac{1}{\frac{dQ(\phi_\omega)}{d\omega} \Big|_{\omega=\omega_*}},$$

there exists $\delta > 0$ such that

$$\forall v \in X, \quad \text{s.t.} \quad (v, T'(0)\phi_{\omega_*})_{\mathcal{X}} = 0, \quad \langle \Lambda v, v \rangle \geq \delta \|v\|^2.$$

In particular, for any $u \in X$ with $\inf_{s \in \mathbb{R}} \|u - T(s)\phi_{\omega_}\|^2 \leq \epsilon$, we have*

$$\inf_{s \in \mathbb{R}} \|u - T(s)\phi_{\omega_*}\|^2 \leq \frac{2}{\delta} \left(\mathcal{L}(u) - \mathcal{L}(\phi_{\omega_*}) \right).$$

Therefore, when assumption 1 is moreover verified, the (global) solution $u(t)$ to (\mathcal{H}) with initial datum u^{in} verifies

$$\sup_{t \in \mathbb{R}} \inf_{s \in \mathbb{R}} \|u(t) - T(s)\phi_{\omega_*}\|^2 \leq \frac{2}{\delta} \left(\mathcal{L}(u^{\text{in}}) - \mathcal{L}(\phi_{\omega_*}) \right) \leq K \|u^{\text{in}} - \phi_{\omega_*}\|^2,$$

provided the right-hand side is sufficiently small.

We point out that the condition that the operator $\langle Q'(\phi_{\omega_*}), \cdot \rangle_{\mathcal{X}^*, \mathcal{X}} Q'(\phi_{\omega_*})$ is a compact perturbation of H_{ω_*} is not very restrictive, since in many cases coming from PDE's, it involves less derivatives than H_{ω_*} .

and $Q'(\phi_{\omega_*})$ tends to zero at spatial infinity.

This type of Liapounov functional has been used by I. Barashenkov in [3] to prove that the travelling waves of (NLS) in dimension one are stable when $\frac{dP}{dc} < 0$. The proof follows basically the one in [3], but some points have to be clarified. The interest of this type of Liapounov functional is that the saddle point ϕ_{ω_*} is now a non degenerate local minimum for \mathcal{L}_{ω_*} . This is a great advantage for numerical simulation of the ‘‘bound states’’, since a gradient flow method on \mathcal{L}_{ω_*} can be used. This approach has been used, with a very similar functional, in [56] by N. Papanicolaou and P. Spathis for the numerical simulation of the travelling waves for a planar ferromagnets model. In the same spirit, in [17], we also use a gradient flow method on this type of functional for the numerical simulation of the travelling waves for (NLS) in two dimensions.

Proof of Theorem A. Recall that the spectrum of H_{ω_*} is, by assumption 3, such that $-\lambda_*^2 \in \sigma(H_{\omega_*})$, $0 \in \sigma(H_{\omega_*})$ and $\sigma(H_{\omega_*}) \setminus \{-\lambda_*^2, 0\} \subset [\delta, +\infty)$ for some $\delta > 0$. Since we assume that $\langle Q'(\phi_{\omega_*}), \cdot \rangle_{\mathcal{X}^*, \mathcal{X}}$ is a compact perturbation of H_{ω_*} , the essential spectrum of Λ is the same as the one of H_{ω_*} , hence is included in $[\delta, +\infty)$. Furthermore, $0 \in \sigma(H_{\omega_*})$ and $\ker(H_{\omega_*}) = \mathbb{R}T'(0)\phi_{\omega_*}$ by assumption 3. Since $Q'(\phi_{\omega_*}) = B\phi_{\omega_*}$ and JB is an extension of $T'(0)$, we have $\langle Q'(\phi_{\omega_*}), T'(0)\phi_{\omega_*} \rangle_{\mathcal{X}^*, \mathcal{X}} = \langle B\phi_{\omega_*}, JB\phi_{\omega_*} \rangle_{\mathcal{X}^*, \mathcal{X}} = 0$, hence $\Lambda(T'(0)\phi_{\omega_*}) = 0$. Noticing that $\langle Q'(\phi_{\omega_*}), \cdot \rangle_{\mathcal{X}^*, \mathcal{X}}$ is a nonnegative operator, we infer that $\ker(\Lambda) = \ker(H_{\omega_*}) = \mathbb{R}T'(0)\phi_{\omega_*}$ is one-dimensional. Therefore, it suffices to show that Λ has no eigenvalues in $(-\infty, 0)$. As we have seen that $\langle Q'(\phi_{\omega_*}), \cdot \rangle_{\mathcal{X}^*, \mathcal{X}}$ is a nonnegative operator, we deduce that $\sigma(\Lambda) \subset [-\lambda_*^2, +\infty)$. Let us first show that $-\lambda_*^2 \notin \sigma(\Lambda)$ by contradiction. If $-\lambda_*^2$ is an eigenvalue of Λ , then there exists $v \in X$, $v \neq 0$, such that $0 = (\Lambda + \lambda_*^2)v = (H + \lambda_*^2)v + M\langle Q'(\phi_{\omega_*}), v \rangle_{\mathcal{X}^*, \mathcal{X}}Q'(\phi_{\omega_*})$. Taking the duality product with v yields $0 = \langle (H_{\omega_*} + \lambda_*^2)v, v \rangle_{\mathcal{X}^*, \mathbb{B}X} + M\langle Q'(\phi_{\omega_*}), v \rangle_{\mathcal{X}^*, \mathcal{X}}^2$. Since the two terms in the sum are nonnegative, this implies $\langle Q'(\phi_{\omega_*}), v \rangle_{\mathcal{X}^*, \mathcal{X}} = 0$ and $\langle (H_{\omega_*} + \lambda_*^2)v, v \rangle_{\mathcal{X}^*, \mathcal{X}} = 0$, which in turn implies $v \in \ker(H_{\omega_*} + \lambda_*^2) = \mathbb{R}\chi$ (here, χ is a negative eigenvector of H_{ω_*} for the eigenvalue $-\lambda_*^2 < 0$). As a consequence, we must have $\langle Q'(\phi_{\omega_*}), \chi \rangle_{\mathcal{X}^*, \mathcal{X}} = 0$. On the other hand, differentiating the equality $E'(\phi_\omega) - \omega Q'(\phi_\omega) = 0$ at $\omega = \omega_*$ yields $Q'(\phi_{\omega_*}) = H_{\omega_*}\phi'$, where $\phi' \equiv \frac{d\phi}{d\omega}|_{\omega=\omega_*}$. Thus we must have $0 = \langle H_{\omega_*}\phi', \chi \rangle_{\mathcal{X}^*, \mathcal{X}} = \langle H_{\omega_*}\chi, \phi' \rangle_{\mathcal{X}^*, \mathcal{X}} = -\lambda_*^2\langle \chi, \phi' \rangle$. Therefore, ϕ' is orthogonal to χ and this gives $\langle H_{\omega_*}\phi', \phi' \rangle_{\mathcal{X}^*, \mathcal{X}} \geq 0$. However, this is not possible if $\frac{dQ(\phi_\omega)}{d\omega}|_{\omega=\omega_*} < 0$, since we have $\frac{dQ(\phi_\omega)}{d\omega}|_{\omega=\omega_*} = -\langle H_{\omega_*}\phi', \phi' \rangle_{\mathcal{X}^*, \mathcal{X}}$. As a consequence, if λ is a negative element of the spectrum of Λ , then $-\lambda_*^2 < \lambda < 0$ and λ is an eigenvalue: there exists $v \in X$ such that $v \neq 0$ and

$$\lambda v = \Lambda v = H_{\omega_*}v + M\langle Q'(\phi_{\omega_*}), v \rangle_{\mathcal{X}^*, \mathcal{X}}Q'(\phi_{\omega_*}).$$

Since $-\lambda_*^2 < \lambda < 0$, we then infer

$$v = -M\langle Q'(\phi_{\omega_*}), v \rangle_{\mathcal{X}^*, \mathcal{X}}(H_{\omega_*} - \lambda)^{-1}Q'(\phi_{\omega_*}). \quad (\text{A.1})$$

Since $v \neq 0$, we can not have $\langle Q'(\phi_{\omega_*}), v \rangle_{\mathcal{X}^*, \mathcal{X}} = 0$. Then, taking the scalar product of (A.1) with $\mathbb{I}^{-1}Q'(\phi_{\omega_*})$ (here, $\mathbb{I} : \mathcal{X} \rightarrow \mathcal{X}^*$ is the usual Riesz isomorphism) gives

$$g(\lambda) = 0, \quad \text{where} \quad g(t) \equiv 1 + M((H_{\omega_*} - t)^{-1}Q'(\phi_{\omega_*}), \mathbb{I}^{-1}Q'(\phi_{\omega_*}))_{\mathcal{X}}, \quad -\lambda_*^2 < t < 0.$$

It is clear that g is smooth in $(-\lambda_*^2, 0)$ and that

$$g'(t) = M((H_{\omega_*} - t)^{-2}Q'(\phi_{\omega_*}), \mathbb{I}^{-1}Q'(\phi_{\omega_*}))_{\mathcal{X}} = M\|(H_{\omega_*} - t)^{-1}Q'(\phi_{\omega_*})\|_{\mathcal{X}}^2 > 0.$$

We now study the limit of g at 0^- . Let us recall that $H_{\omega_*}\phi' = Q'(\phi_{\omega_*})$ and that we have already seen that $\langle Q'(\phi_{\omega_*}), T'(0)\phi_{\omega_*} \rangle_{\mathcal{X}^*, \mathcal{X}} = 0$, i.e. $\mathbb{I}^{-1}Q'(\phi_{\omega_*})$ is orthogonal to $\ker(H_{\omega_*})$. Therefore, as $t \rightarrow 0^-$,

$$((H_{\omega_*} - t)^{-1}Q'(\phi_{\omega_*}), \mathbb{I}^{-1}Q'(\phi_{\omega_*})) \rightarrow (\phi', \mathbb{I}^{-1}Q'(\phi_{\omega_*})) = \langle Q'(\phi_{\omega_*}), \phi' \rangle_{\mathcal{X}^*, \mathcal{X}} = \frac{dQ(\phi_\omega)}{d\omega}|_{\omega=\omega_*}$$

and it thus comes

$$g(t) \rightarrow 1 + M \frac{dQ(\phi_\omega)}{d\omega} \Big|_{\omega=\omega_*} \quad \text{as } t \rightarrow 0^-.$$

Since $\frac{dQ(\phi_\omega)}{d\omega} \Big|_{\omega=\omega_*} < 0$ by hypothesis, it follows that if $M > -1/(\frac{dQ(\phi_\omega)}{d\omega} \Big|_{\omega=\omega_*}) > 0$, the function g increases in $(-\lambda_*^2, 0)$ and tends to some negative limit at 0^- . In particular, g is negative, hence we can not have $g(\lambda) = 0$ with $\lambda \in (-\lambda_*^2, 0)$. We have therefore shown that the spectrum of Λ consists in a simple eigenvalue 0 with eigenspace spanned by $T'(0)\phi_{\omega_*}$ and the rest of the spectrum is positive and bounded away from 0. This concludes the proof. \square

We would like to point out that in the proof of [3], the fact that $-\lambda_*^2 \notin \sigma(\Lambda)$ was not shown, the kernel of Λ was not studied and the essential spectrum was not considered. Moreover, the functional spaces are not given, hence we do not know for which perturbations stability holds.

Appendix B: from linear to nonlinear instability

We still consider in this appendix an abstract Hamiltonian equation in the framework of [35]

$$\frac{\partial u}{\partial t} = JE'(u) \tag{H}$$

on the real Hilbert space \mathcal{X} , with scalar product $(\cdot, \cdot)_{\mathcal{X}}$. Here $E : \mathcal{X} \rightarrow \mathbb{R}$ is of class \mathcal{C}^2 and $J : \mathcal{X}^* \rightarrow \mathcal{X}$ is a closed linear operator with dense domain and skew-symmetric in the sense that $(u, Jw)_{\mathcal{X}} = -\langle w, Ju \rangle_{\mathcal{X}^*, \mathcal{X}}$ for $u \in \mathcal{X}$, $w \in \mathcal{X}^*$.

We assume that there exists a \mathcal{C}_0 -group T of unitary operators in \mathcal{X} generated by $T'(0)$, which is skew-adjoint and with dense domain, and that E is invariant by T , that is $E(T(\omega)u) = E(u)$ for any $\omega \in \mathbb{R}$, $u \in \mathcal{X}$. Assume moreover that $T(\omega)J = JT(-\omega)^*$ for any $\omega \in \mathbb{R}$ and that there exists $B : \mathcal{X} \rightarrow \mathcal{X}^*$ linear and bounded such that $B^* = B$ and JB is an extension of $T'(0)$. We then set

$$Q(u) \equiv \frac{1}{2} \langle Bu, u \rangle_{\mathcal{X}^*, \mathcal{X}},$$

which is invariant by the flow (H) (see [35]). By ‘‘bound state’’, we mean a particular solution U of (H) of the form $U(t) = T(\omega t)\phi$ for some $\omega \in \mathbb{R}$ and where $\phi \in \mathcal{X}$, $\phi \neq 0$. In other words, $E'(\phi) = \omega Q'(\phi)$.

There exists an open interval $\Omega \subset \mathbb{R}$, not reduced to a singleton, and a mapping $\Omega \ni \omega \mapsto \phi_\omega \in \mathcal{X}$ of class \mathcal{C}^1 such that, for any $\omega \in \Omega$,

$$E'(\phi_\omega) = \omega Q'(\phi_\omega), \quad \phi_\omega \in D(T'(0)^3) \cap D(JIT'(0)^2), \quad T'(0)\phi_\omega \neq 0.$$

The solution $U(t) = T(\omega t)\phi$ is said to be stable in \mathcal{X} if for any $\varepsilon > 0$, there exists $\delta > 0$ such that any solution to (H) with initial datum $u^{\text{in}} \in B_{\mathcal{X}}(\phi, \delta)$ is global in time and remains in $B_{\mathcal{X}}(\phi, \varepsilon)$ for $t \geq 0$. Otherwise, it is said unstable. This supposes some knowledge on the Cauchy problem for (H) (at least existence of solutions). If we are given some Banach space $\mathcal{Y} \supset \mathcal{X}$ with continuous imbedding $\mathcal{X} \hookrightarrow \mathcal{Y}$, we may also say that the solution $U(t) = T(\omega t)\phi$ is said to be stable from \mathcal{X} to \mathcal{Y} if for any $\varepsilon > 0$, there exists $\delta > 0$ such that any solution to (H) with initial datum $u^{\text{in}} \in B_{\mathcal{X}}(\phi, \delta)$ remains in $B_{\mathcal{Y}}(\phi, \varepsilon)$ for $t \geq 0$. Clearly, a solution stable in \mathcal{X} is precisely a solution able from \mathcal{X} to \mathcal{X} , and is also stable from \mathcal{X} to \mathcal{Y} , hence instability from \mathcal{X} to \mathcal{Y} is a stronger statement than instability in \mathcal{X} .

In our framework, the notion of orbital stability is more relevant. Let us consider \mathbb{G} a group and $\mathbb{T} : \mathbb{R} \times \mathbb{G} \rightarrow \mathcal{G}\mathcal{L}_c(\mathcal{X})$ a unitary representation of $\mathbb{R} \times \mathbb{G}$ on \mathcal{X} , extending $T : \mathbb{R} \rightarrow \mathcal{X}$ and leaving E and Q invariant. Then, $U(t) = T(\omega t)\phi$ is said to be orbitally stable in \mathcal{X} (for the group \mathbb{G}) if for any $\varepsilon > 0$, there exists $\delta > 0$ such that any solution to (H) with initial datum $u^{\text{in}} \in B(\phi, \delta)$ is global in time and remains in $\cup_{(\omega, g) \in \mathbb{R} \times \mathbb{G}} B(\mathbb{T}(\omega, g)\phi, \varepsilon)$ for $t \geq 0$. We may also define orbital instability from \mathcal{X} to $\mathcal{Y} \supset \mathcal{X}$ in a natural way.

In [35, 36], a general framework for the stability analysis for the “bound state” has been given. In particular, the nonlinear orbital instability is proved in [35] through the construction of a Liapounov type functional. However, this method does not give a clear understanding neither of how we get farther from the “bound state”, nor on which timescale it occurs.

The need for allowing an additional group of invariances \mathbb{G} can be seen in the case of bound state solutions, that is $U(t) = e^{i\omega t} \phi_\omega$ to the Nonlinear Schrödinger equation

$$i\partial_t \Psi + \Delta \Psi + \Psi f(|\Psi|^2) = 0, \quad (\text{NLS})$$

or the Nonlinear Klein-Gordon equation in \mathbb{R}^d

$$\partial_t^2 \Psi = \Delta \Psi + \Psi f(|\Psi|^2), \quad (\text{NLKG})$$

since then, the invariance by translation in space must be taken into account in the definition of orbital stability, and we are in a case where $\mathbb{G} = \mathbb{R}^d$ acts naturally by translation. The translations are taken into account in [15]. In [35] and [36], the notion of orbital stability is for \mathbb{G} trivial. It is clear from the definition that orbital stability for $\mathbb{G} = \{0\}$ implies orbital stability for arbitrary \mathbb{G} . For the instability in the Nonlinear Schrödinger equation or the Nonlinear Klein-Gordon equation, [35] and [65] work with radial H^1 functions. The fact that this also implies the orbital instability with the action of $\mathbb{G} = \mathbb{R}^d$ by translations follows immediately from the fact that for any $\theta \in [0, 2\pi]$ the manifold $\mathfrak{M}_\theta \equiv \{e^{i\theta} \phi(\cdot - y), y \in \mathbb{R}^d\}$ is orthogonal to $H_{\text{rad}}^1(\mathbb{R}^d)$.

For the stability analysis of a “bound state” $U(t) = T(\omega_* t)(\phi_{\omega_*})$, it is natural to consider the linearization of (\mathcal{H}) near ϕ . More precisely, we linearize according to the ansatz $u(t) = T(\omega_* t)(\phi_{\omega_*} + v(t))$, so that the “bound state” becomes stationary. The linearized problem then becomes

$$\frac{\partial v}{\partial t} = J(E''(\phi) - \omega Q''(\phi))v = \mathcal{J}\mathcal{L}v, \quad (\mathcal{H}_{\text{lin}})$$

where, $\mathbb{I} : \mathcal{X} \rightarrow \mathcal{X}^*$ denoting the Riesz isomorphism, $\mathcal{J} \equiv \mathbb{I}\mathbb{I} : \mathcal{X} \rightarrow \mathcal{X}$ is skew-adjoint.

The purpose of this appendix is to give a general result, for Hamiltonian equations, showing that linear instability implies nonlinear (orbital) instability. By linear instability, we mean that the complexification of $[\mathcal{J}\mathcal{L}]_{\mathbb{C}}$ has at least one eigenvalue in the right-half space $\{\text{Re} > 0\}$. The argument follows ideas from the works of F. Rousset and N. Tzvetkov [61, 62].

Showing the existence of an unstable eigenvalue can be done through various techniques: [36] (in the framework of [35] when J is onto), [34] (assuming a special structure of the Hamiltonian equation); for uses of the Vakhitov-Kolokolov function, see [20], [21] or [58]. When J is not onto, we quote [49]. For one dimensional partial differential equations, one may also use Evans’ function (see the survey [63]) as in [57], [26], [40], [70]. The paper [48] proposes another approach which allows to treat pseudo-differential equations, such as the BBM equation, the Benjamin-Ono equation, regularized Boussinesq equations, the Intermediate Long Wave equation...

In order to pass from linear to nonlinear instability, the following result is standard. We refer to the paper [38] by D. Henry, J. Perez and W. Wreszinski. It can also be found in [34] and [64].

Theorem B.1 ([38], [34], [64]) *We assume that \mathcal{A} generates a continuous semigroup on X and that $\sigma(\mathcal{A})$ meets the right-half space $\{\text{Re} > 0\}$. We assume moreover that $F : X \rightarrow X$ is locally Lipschitz continuous and verifies, for some $\alpha > 0$, $\|F(v)\|_X = \mathcal{O}(\|v\|_X^{1+\alpha})$ as $v \rightarrow 0$. Then, the solution $\phi = 0$ is unstable for the equation $\partial_t v = \mathcal{A}v + F(v)$.*

In [64], it is claimed that an orbital instability result can also be established. Theorem B.1 shows nonlinear instability without assuming that the equation is Hamiltonian. However, if $(\mathcal{H}_{\text{lin}})$ can be solved using a semigroup, it does not give the growth of its norm. Moreover, it does not say that if the initial datum is in a most unstable direction, that is an eigendirection of \mathcal{A} corresponding to an eigenvalue of maximal positive real part (plus the complex conjugate if necessary), then you can track the exponential growth of the

solution. In particular, it does not explain the mechanism of instability and does not give any information on the timescale on which you see the instability. For instance, some strong instability results are shown by proving blow-up in finite time (see [8]), but the instability due to an exponentially growing mode holds on a much smaller timescale. We wish to provide here some results clarifying the instability mechanism by tracking the exponentially growing mode.

B.1 A spectral mapping theorem for linearized Hamiltonian equations

When we want to prove a nonlinear instability result from a linear instability one, one needs some information on the growth of the semigroup \mathcal{JL} , when such a semigroup $e^{t\mathcal{JL}}$ exists, which we shall assume in this appendix. The growth estimate on $e^{t\mathcal{JL}}$ relies classically on the following spectral mapping result due to J. Prüss [59], which generalizes the work of L. Gearhart [27].

Theorem B.2 ([59]) *Let X be a complex Hilbert space and \mathcal{A} an unbounded operator on X which generates a continuous semigroup $e^{t\mathcal{A}}$ on X . For $t \in (0, +\infty)$, we have*

$$\sigma(e^{t\mathcal{A}}) \setminus \{0\} = \left\{ e^{\lambda t}, \text{ either } \left(\lambda + \frac{2i\pi}{t}\mathbb{Z} \right) \cap \sigma(\mathcal{A}) \neq \emptyset, \text{ or } \sup_{k \in \mathbb{Z}} \left\| \left(\mathcal{A} - \lambda - \frac{2i\pi k}{t} \right)^{-1} \right\|_{\mathcal{L}_c(X)} = +\infty \right\}.$$

The following result is an immediate corollary.

Corollary B.1 *Let X be a complex Hilbert space and \mathcal{A} an unbounded operator on X which generates a continuous semigroup $e^{t\mathcal{A}}$ on X . Assume that for any $\gamma \in \mathbb{R}^*$, we have*

$$\limsup_{|\tau| \rightarrow +\infty} \left\| \left(\mathcal{A} - \gamma - i\tau \right)^{-1} \right\|_{\mathcal{L}_c(X)} < +\infty,$$

and that there exists $\vartheta_0 \in [0, +\infty)$ such that $\sigma_{\text{ess}}(\mathcal{A}) = \{i\vartheta, \vartheta \in \mathbb{R}, |\vartheta| \geq \vartheta_0\}$. Then, for any $t \in (0, +\infty)$, the spectral mapping holds: $\sigma(e^{t\mathcal{A}}) \setminus \{0\} = e^{t\sigma(\mathcal{A})}$.

Proof. Since $\sigma_{\text{ess}}(\mathcal{A}) = \{i\vartheta, \vartheta \in \mathbb{R}, |\vartheta| \geq \vartheta_0\}$, we have $\mathbb{S}^1 \subset e^{t\sigma(\mathcal{A})} \subset \sigma(e^{t\mathcal{A}})$. If $\lambda \in \mathbb{C}$ does not have modulus one, then note that when $(\lambda + \frac{2i\pi}{t}\mathbb{Z}) \cap \sigma(\mathcal{A}) = \emptyset$, the supremum for $k \in \mathbb{Z}$ in Theorem B.2 can be $+\infty$ only when $|k| \rightarrow +\infty$, and we conclude with our hypothesis. \square

The fact that we exclude 0 in the spectral mapping theorem just comes from the fact that we consider a semigroup and not a group. However, in most Hamiltonian PDE's, we have time reversibility and we have actually a continuous group and not only a semigroup. In most cases, we work with $A : D(A) \subset Y \rightarrow Y$ where Y is a real Hilbert space, thus for applying Theorem B.2 or Corollary B.1 we have to consider, as usual, the complexified operator $A_{\mathbb{C}} : D(A_{\mathbb{C}}) \equiv D(A) \oplus iD(A) \subset Y_{\mathbb{C}} \equiv Y \oplus iY \rightarrow Y_{\mathbb{C}}$ defined by $A_{\mathbb{C}}(u + iv) = Au + iAv$.

It seems that the first time Theorem B.2 is used to prove a growth estimate on a semigroup was by T. Kapitula and B. Sandstede in [40]. Later, the work [31] by F. Gesztesy *et al.* also uses this result for bound states for (NLS). The bounds on the resolvent in [40] were proved using the particular structure of the linearized operator. In [31], the computations are more involved and rely on suitable kernel estimates of some Hilbert-Schmidt operators. The same type of estimates have also been used in [21].

The main objective of this appendix is to provide a generalization of these results to a wide class of Hamiltonian equations. Indeed, the approaches in [40] and [31] seem specific to the problem. In addition, it is not clear whether the computations in [31] and [21] can be extended to other types of equations. In particular, in [16] and in the present paper, we have a situation similar to the one studied in [21], namely travelling wave solutions to a Nonlinear Schrödinger equation with nonzero condition at infinity, but for nonzero propagation speeds, the travelling wave is not real-valued (as it is the case in [21] for stationary waves or for bound state solutions), and the bloc diagonal structure of the linearized Hamiltonian disappears. An additional difficulty is that in [16] and the present work, the limits of the travelling waves at $+\infty$ and $-\infty$ differ.

The proof we give is based on ideas from [61, 62] and make very few spectral assumptions on \mathcal{L} .

Assumption (A): The spectrum of \mathcal{L} consists in a finite number (possibly zero) of nonpositive eigenvalues $-\mu_1, \dots, -\mu_q$ in $(-\infty, 0]$, each one with finite multiplicity, and the rest of the spectrum is positive and bounded away from 0. Furthermore that for any $1 \leq k \leq q$, we have $\ker(\mathcal{L} + \mu_k) \subset D(\mathcal{J})$ and $\mathcal{J}[\ker(\mathcal{L} + \mu_k)] \subset D(\mathcal{L})$. Finally, there exists $\vartheta_0 \in [0, +\infty)$ such that $\sigma_{\text{ess}}(\mathcal{J}\mathcal{L}) = \{i\vartheta, \vartheta \in \mathbb{R}, |\vartheta| \geq \vartheta_0\}$.

The first hypothesis on the location of the spectrum of \mathcal{L} is quite weak, since it is verified when \mathcal{L} is bounded from below and has essential spectrum positive and bounded away from zero. Indeed, if $\delta > 0$ is such that $\sigma_{\text{ess}}(\mathcal{L}) \subset [2\delta, +\infty)$, then the eigenvalues of \mathcal{L} in $(-\infty, \delta]$ are isolated, of finite multiplicity, and are bounded from below by assumption. The second hypothesis $\ker(\mathcal{L} + \mu_k) \subset D(\mathcal{J})$ is a regularity assumption on the eigenvectors.

Let us recall that Theorem 11 ensures that the number of eigenvalues (with algebraic multiplicities) of $\mathcal{J}\mathcal{L}$ in the right-half space $\{\text{Re} > 0\}$ is less than or equal to the number of negative eigenvalues of \mathcal{L} , hence is finite under assumption (A). Let us now state our main result, the proof of which is given in section B.3.1.

Theorem B.3 *We make assumption (A) and suppose that $\mathcal{J}\mathcal{L}$ generates a continuous semigroup. Then, for any $t \in (0, +\infty)$, the spectral mapping holds: $\sigma(\mathbf{e}^{t[\mathcal{J}\mathcal{L}]_{\mathbb{C}}}) \setminus \{0\} = \mathbf{e}^{t\sigma([\mathcal{J}\mathcal{L}]_{\mathbb{C}})}$. Furthermore, denoting*

$$\gamma_0 \equiv \sup \left\{ \text{Re}(\lambda), \lambda \in \sigma([\mathcal{J}\mathcal{L}]_{\mathbb{C}}) \cap \{\text{Re} \geq 0\} \right\} \in [0, +\infty),$$

for any $\beta > 0$, there exists $M(\beta) > 0$ such that, for any $t \geq 0$, we have

$$\|\mathbf{e}^{t\mathcal{J}\mathcal{L}}\|_{\mathcal{L}_c(\mathcal{X})} \leq M(\beta)\mathbf{e}^{(\gamma_0 + \beta)t}.$$

Assume in addition $\gamma_0 > 0$ and denote

$$m \equiv \max \left\{ \text{algebraic multiplicity of } \lambda, \lambda \in \sigma([\mathcal{J}\mathcal{L}]_{\mathbb{C}}) \text{ s.t. } \text{Re } \lambda = \gamma_0 \right\} \in \mathbb{N}^*.$$

Then, there exists $M_0 > 0$ such that, for any $t \geq 0$, we have

$$\|\mathbf{e}^{t\mathcal{J}\mathcal{L}}\|_{\mathcal{L}_c(\mathcal{X})} \leq M_0(1+t)^{m-1}\mathbf{e}^{\gamma_0 t}.$$

In particular, Theorem B.3 provides a very simple proof of the spectral mapping theorem used in [31] and [21]. Indeed, the self-adjoint operator \mathcal{L} involved in these papers is block diagonal:

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}_1 & 0 \\ 0 & \mathcal{L}_2 \end{pmatrix},$$

and both $\mathcal{L}_1, \mathcal{L}_2$ have at most two nonnegative eigenvalues. More generally, if \mathcal{L}_1 and \mathcal{L}_2 are closed self-adjoint operators on X verifying assumption (A) and if $\mathcal{N} : X \rightarrow X$ is a linear bounded operator which is compact with respect to \mathcal{L}_1 and \mathcal{L}_2 , then the self-adjoint operator

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}_1 & \mathcal{N} \\ \mathcal{N}^* & \mathcal{L}_2 \end{pmatrix}$$

also verifies assumption (A). Indeed, \mathcal{L} is bounded from below (since \mathcal{N} is bounded) and its essential spectrum is $\sigma_{\text{ess}}(\mathcal{L}_1) \cup \sigma_{\text{ess}}(\mathcal{L}_2) \subset [\delta, +\infty)$ for some positive δ , since \mathcal{N} is compact with respect to \mathcal{L}_1 and \mathcal{L}_2 . In [40] (section 7.1 there) and [28] (see Proposition 10 there), a spectral mapping theorem is used for such an operator. In [40], the specific algebra of the problem was used, and for [28], the proof relies on the arguments in [31], but here again, in both cases, we may use Theorem B.3 to show the same result.

B.2 Passing from linear to nonlinear instability

B.2.1 Semilinear type models

We start with a classical result for “semilinear” equations, proved in section B.3.3.

Theorem B.4 *Let X be a real Hilbert space, and consider an evolution equation of the form*

$$\frac{dv}{dt} = \mathcal{A}v + \Phi(v),$$

where $\Phi : X \rightarrow X$ is a locally Lipschitz mapping satisfying $\Phi(v) = \mathcal{O}(\|v\|_X^2)$ as $v \rightarrow 0$ and \mathcal{A} is a linear operator which generates a semigroup. We assume that $\mathcal{A}_\mathbb{C} : D(\mathcal{A}_\mathbb{C}) \subset X_\mathbb{C} \rightarrow X_\mathbb{C}$ has an unstable eigenvalue in the right half plane $\{\text{Re} > 0\}$ and a finite number of eigenvalues in $\{\text{Re} > 0\}$. We denote

$$\gamma_0 \equiv \sup \left\{ \text{Re}(\mu), \mu \in \sigma([\mathcal{J}\mathcal{L}]_\mathbb{C}) \cap \{\text{Re} > 0\} \right\} \in (0, +\infty)$$

and fix $\lambda \in \sigma(\mathcal{A}_\mathbb{C})$ with $\text{Re}(\lambda) = \gamma_0$ and an associated eigenvector $w_\mathbb{C} \in D(\mathcal{A}_\mathbb{C})$ such that $\|\text{Re}(w_\mathbb{C})\|_X = 1$. Assume furthermore that there exists $0 \leq \beta < \gamma_0$ and $M_0 > 0$ such that

$$\|e^{t\mathcal{A}}\|_{\mathcal{L}_c(X)} \leq M_0 e^{(\gamma_0 + \beta)t}.$$

Then, 0 is an unstable solution. More precisely, there exist $K > 0$, $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that, for any $0 < \delta < \delta_0$, the solution v with initial datum $v^{\text{in}} = \delta \text{Re}(w_\mathbb{C}) \in D(\mathcal{A})$ exists at least on $[0, \ln(2\varepsilon_0/\delta)/\gamma_0]$ and verifies, for $0 \leq t \leq \ln(2\varepsilon_0/\delta)/\gamma_0$,

$$\|v(t) - \delta \text{Re}(e^{t\lambda} w_\mathbb{C})\|_X \leq K \delta^2 e^{2t\gamma_0} \quad \text{and} \quad \|v(t)\|_X \geq \delta e^{t\gamma_0} - K \delta^2 e^{2t\gamma_0},$$

In particular, for $0 < \varepsilon < \varepsilon_0$, we see the instability for $t = \frac{1}{\gamma_0} \ln\left(\frac{2\varepsilon}{\delta}\right)$. If Y is a Banach space containing X and with continuous imbedding $X \hookrightarrow Y$, the trivial solution 0 is also unstable from X to Y .

Let us observe that it is always possible to choose the (complex) eigenvector w so that $\text{Re}(w_\mathbb{C}) \neq 0$ since for any $\theta \in \mathbb{R}$, $e^{i\theta} w$ is also an eigenvector. The following corollary deals with the orbital instability. We recall that under assumption (A), $[\mathcal{J}\mathcal{L}]_\mathbb{C}$ has a finite number of eigenvalues in $\{\text{Re} > 0\}$.

Corollary B.2 *We make assumption (A) and suppose that $\mathcal{J}\mathcal{L}$ generates a continuous semigroup. Let \mathcal{Y} be a Banach space containing \mathcal{X} and with continuous imbedding $\mathcal{X} \hookrightarrow \mathcal{Y}$. Assume moreover that $[\mathcal{J}\mathcal{L}]_\mathbb{C}$ has at least one eigenvalue in $\{\text{Re} > 0\}$ and choose $\lambda \in \mathbb{C}$ with*

$$\text{Re}(\lambda) = \gamma_0 \equiv \max \left\{ \text{Re}(\mu), \mu \in \sigma([\mathcal{J}\mathcal{L}]_\mathbb{C}) \cap \{\text{Re} > 0\} \right\} \in (0, +\infty)$$

and $w_\mathbb{C} \in D(\mathcal{A}_\mathbb{C})$ an associated eigenvector such that $\|\text{Re}(w_\mathbb{C})\|_\mathcal{X} = 1$. We assume moreover that $\mathfrak{M} \equiv \{\mathbb{T}(\omega, g)\phi_{\omega_*}, \omega \in \mathbb{R}, g \in \mathbb{G}\}$ is a \mathcal{C}^1 submanifold of \mathcal{X} . We finally suppose that the equation (\mathcal{H}) is semilinear in the sense that there exists $\Phi : \mathcal{X} \rightarrow \mathcal{X}$ locally Lipschitz continuous such that $\Phi(v) = \mathcal{O}(\|v\|_\mathcal{X}^2)$ as $v \rightarrow 0$ and

$$J(E' - \omega_* Q')(\phi_{\omega_*} + v) = J(E'' - \omega_* Q'')(\phi_{\omega_*})[v] + \Phi(v).$$

Then, there exists $K > 0$, $\varepsilon_0 > 0$ and $\delta_0 > 0$, depending only on $\text{Re}(w_\mathbb{C})$ and \mathfrak{M} , with the following properties. For any $0 < \delta < \delta_0$, the solution u to (\mathcal{H}) with initial datum $u^{\text{in}} = \phi_{\omega_*} + \delta \text{Re}(w_\mathbb{C}) \in D(\mathcal{A})$ exists at least on $[0, \ln(2\varepsilon_0/\delta)/\gamma_0]$ and verifies, for $0 \leq t \leq \ln(2\varepsilon_0/\delta)/\gamma_0$,

$$\text{dist}_\mathcal{Y}(u(t), \mathfrak{M}) \geq \frac{\delta}{K} e^{t\gamma_0} - K \delta^2 e^{2t\gamma_0}.$$

In particular, the “bound state” solution $T(\omega_* t)\phi_{\omega_*}$ is nonlinearly orbitally unstable from \mathcal{X} to \mathcal{Y} and, for $0 < \varepsilon < \varepsilon_0/K$, we see the instability for $t = \frac{1}{\gamma_0} \ln\left(\frac{2K\varepsilon}{\delta}\right)$.

In [38], a similar assertion is made for the orbital instability in the remark after Theorem 2 there, but with $\mathcal{Y} = \mathcal{X}$. For applications to PDE’s, the space \mathcal{X} may be a Sobolev space H^s , and \mathcal{Y} a space like L^2 or L^∞ for instance. The framework of [35] is the single energy space (for instance H^1), but an instability result established by tracking exponentially growing modes allows to prove instability from the regular space \mathcal{X} (H^1) to the nonregular space \mathcal{Y} (L^2 or L^∞). Here, we may obtain instability in L^2 .

Remark B.1 In the framework of [35], where a Lyapounov type functional is used, it follows that the instability is seen for a time at most equal to $K\frac{\varepsilon}{\delta^2}$, where K is some positive constant. This timescale is much larger than the natural one $\frac{1}{\gamma_0} \ln\left(\frac{2K\varepsilon}{\delta}\right)$.

B.2.2 Some applications

We may apply our result to the Nonlinear Schrödinger Equation

$$i\partial_t\Psi + \Delta\Psi + \Psi f(|\Psi|^2) = 0, \quad (\text{NLS})$$

or the Nonlinear Klein-Gordon equation

$$\partial_t^2\Psi = \Delta\Psi + \Psi f(|\Psi|^2) \quad (\text{NLKG})$$

in \mathbb{R}^d . We shall consider a nonlinearity f at least \mathcal{C}^1 , so that we are in the framework of [35].

- A bound state solutions for these two equations is a particular solution of the form $U(t) = e^{i\omega t}\phi_\omega$. The instability is in general linked to the fact that

$$\frac{d}{d\omega} \int_{\mathbb{R}^d} |\phi_\omega|^2 dx < 0 \quad \text{for (NLS),} \quad \text{resp.} \quad \frac{d}{d\omega} \left(\omega \int_{\mathbb{R}^d} |\phi_\omega|^2 dx \right) < 0 \quad \text{for (NLKG).} \quad (\text{B.1})$$

The existence of at least one unstable eigenvalue has been shown under assumption (B.1) by [34] for radial bound states with an arbitrary number of nodes and in [36] for radial ground states. Corollary B.2 may be applied with $\mathcal{X} = H^s(\mathbb{R}^d)$, where $s \in \mathbb{N}$, $s > d/2$ and assuming that the nonlinearity verifies $f \in \mathcal{C}^{s+2}$, and $\mathcal{Y} = L^2(\mathbb{R}^d)$ or $L^\infty(\mathbb{R}^d)$. The result in [54] shows the instability of linearly unstable bound states for (NLS) (in dimension $d = 2$) with $f(\varrho) = \varrho^{\frac{p-1}{2}}$ by showing the exponential growth of an unstable eigenmode. Our result gives a simple proof of this result, but restricted to the sufficiently smooth cases, namely p an odd integer or $p > 5 + 2s > 5 + d$. For non smooth nonlinearities, the situation is more delicate (see [54]). An alternative approach is to combine Strichartz estimates with the growth estimate on the semigroup $e^{t\mathcal{J}\mathcal{L}}$ given in Theorem B.3, as in [28].

- Corollary B.2 also applies to the Discrete Nonlinear Schrödinger Equation

$$\forall n \in \mathbb{Z}, \quad i\partial_t\Psi_n + \varepsilon(\Psi_{n+1} - 2\Psi_n + \Psi_{n-1}) + \Psi_n f(|\Psi_n|^2) = 0, \quad (\text{DNLS})$$

as studied in [53] with the saturated nonlinearity $f(\varrho) = \frac{\beta}{1+\varrho}$, $\beta > 0$ (existence of travelling wave solution) and in [23] (defocusing cubic (DNLS), *i.e.* $f(\varrho) = -\beta\varrho$ for some $\beta > 0$). The numerical analysis in the paper [23] shows the existence of linearly unstable bound state solutions. The travelling wave solutions numerically obtained in [53] are linearly stable, but it may happen that for other nonlinearities f , some are linearly unstable.

B.2.3 Quasilinear PDE's

For quasilinear problems, we shall not make restrictions on the smoothness of the nonlinearity. The result relies on the strategy of E. Grenier [33] and the works [61, 62]. We consider the evolution equation

$$\frac{du}{dt} = J(L_0 u + \nabla F(u)) \quad (\text{E})$$

for $u : \mathbb{R}^d \rightarrow \mathbb{R}^\nu$, where $F \in C^\infty(\mathbb{R}^\nu, \mathbb{R})$, with the following hypothesis. The operator J is a Fourier multiplier, skew-symmetric on L^2 , into and with domain containing H^1 . There exists $\sigma > 0$ such that the operator L_0 is a Fourier multiplier with domain containing $H^{2\sigma}$, symmetric and having a self-adjoint realization on $L^2(\mathbb{R}^d, \mathbb{R}^\nu)$. Moreover, for some $C > 0$, L_0 verifies:

$$\frac{1}{C} \|u\|_{H^\sigma}^2 \leq (L_0 u, u)_{L^2} \leq C \|u\|_{H^\sigma}^2.$$

The framework proposed in [62] was for L_0 coercive in H^1 , that is $\sigma = 1$. For the examples below, we shall have $\sigma = 1/2$ or $\sigma = 2$, which requires very few modifications to the proof of [62]. We still assume that for some group \mathbb{G} , there exists a unitary representation of \mathbb{G} on \mathcal{X} , $\mathbb{T} : \mathbb{G} \rightarrow \mathcal{GL}_c(\mathcal{X})$, leaving the equation (E) invariant.

We consider a stationary solution of the evolution equation (E), that is some $Q \in H^\infty(\mathbb{R}^d, \mathbb{R}^\nu)$ such that $L_0 Q + \nabla F(Q) = 0$. We are interested in the stability of this solution. We assume that the commutator $[J, \nabla^2 F(Q)]$ is bounded in L^2 , which is the case when J is bounded in L^2 or when $d = 1$ and $J = \partial_x$. We suppose that for the problem

$$\frac{\partial u}{\partial t} = J(L_0 u + \nabla F(u^a + u) - \nabla F(u^a) + G),$$

where u^a is smooth, bounded as well as its derivatives and $G \in \mathcal{C}(\mathbb{R}, H^s)$ for every s , we have local well-posedness for s large enough: there exists a time $T > 0$ and a unique solution in $\mathcal{C}([0, T], H^s)$. We moreover assume that for some continuous non-decreasing function $\kappa : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $\kappa(0) = 0$, the tame estimate

$$|(\partial_x^\alpha J \{ \nabla^2 F(w + v)[v] \}, \partial_x^\alpha v)_{L^2}| \leq \kappa(\|w\|_{W^{s+1, \infty}} + \|v\|_{H^s}) \|v\|_{H^s}^2,$$

with $|\alpha| \leq s$ holds true. In order to control high order derivatives, we finally require that for s large enough, there exists a self-adjoint operator \mathbb{M}_s and C_s such that

$$|(\mathbb{M}_s u, v)_{L^2}| \leq C_s \|u\|_{H^s} \|v\|_{H^s}, \quad (\mathbb{M}_s u, u)_{L^2} \geq \|u\|_{H^s}^2 - C_s \|u\|_{H^{s-\min(\sigma, 1)}}^2$$

and

$$\text{Re}(J L u, \mathbb{M}_s u)_{L^2} \leq C_s \|u\|_{H^s} \|u\|_{H^{s-\min(\sigma, 1)}}$$

(for a criterion which ensures the existence of such a multiplier, see Lemma 5.1 in [62]).

Adapting the strategy of [61, 62], we may deduce the following result. Since the proof is very similar, we omit it.

Theorem B.5 *We make the above assumptions and moreover that $L_0 + \nabla^2 F(Q)$ verifies hypothesis (A) in L^2 . We assume furthermore that $[J(L_0 + \nabla^2 F(Q))]_{\mathbb{C}}$ has an unstable eigenvalue in the right half-plane $\{\text{Re} > 0\}$, denote*

$$\gamma_0 \equiv \sup \left\{ \text{Re}(\lambda), \lambda \in \sigma([J(L_0 + \nabla^2 F(Q))]_{\mathbb{C}}) \cap \{\text{Re} > 0\} \right\} \in (0, +\infty)$$

and fix $\lambda \in \sigma([J(L_0 + \nabla^2 F(Q))]_{\mathbb{C}})$ with $\text{Re}(\lambda) = \gamma_0$ and an associated eigenvector $w_{\mathbb{C}} \in D([J(L_0 + \nabla^2 F(Q))]_{\mathbb{C}})$ such that $\|w_{\mathbb{C}}\|_{H^s} = 1$. There exists $s_0 \in \mathbb{N}$ such that, if $s \geq s_0$, Q is nonlinearly unstable from H^s

to L^2 and to L^∞ : there exists $K > 0$, $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that, for any $0 < \delta < \delta_0$, the H^s solution u to (E) with initial datum $u^{\text{in}} = Q + \delta \text{Re}(w_{\mathbb{C}}) \in H^s$ exists at least on $[0, \ln(2\varepsilon_0/\delta)/\gamma_0]$ and verifies, for $0 \leq t \leq \ln(2\varepsilon_0/\delta)/\gamma_0$,

$$\|u(t) - Q - \delta \text{Re}(e^{t\lambda} w_{\mathbb{C}})\|_{H^s} \leq K\delta^2 e^{2t\gamma_0},$$

hence

$$\|u(t) - Q\|_{L^2} \geq \delta e^{t\gamma_0} - K\delta^2 e^{2t\gamma_0} \quad \text{and} \quad \|u(t) - Q\|_{L^\infty} \geq \delta e^{t\gamma_0} - K\delta^2 e^{2t\gamma_0}.$$

If, in addition, $\mathfrak{M} \equiv \{\mathbb{T}(g)Q, g \in \mathbb{G}\}$ is a \mathcal{C}^1 submanifold of H^s , then we also have

$$\text{dist}_{L^2}(u(t), \mathfrak{M}) \geq K\delta e^{t\gamma_0} - K\delta^2 e^{2t\gamma_0} \quad \text{and} \quad \text{dist}_{L^\infty}(u(t), \mathfrak{M}) \geq K\delta e^{t\gamma_0} - K\delta^2 e^{2t\gamma_0}.$$

In particular, for $0 < \varepsilon < \varepsilon_0/K$, we see the nonlinear orbital instability for $t = \frac{1}{\gamma_0} \ln\left(\frac{2K\varepsilon}{\delta}\right)$.

B.2.4 Some applications to nonlinear dispersive wave equations

Some model quasilinear equations are given by wave equations (in one space dimension) such as the generalized Korteweg-de Vries equation

$$\partial_t u + \partial_x(f(u)) + \partial_x^3 u = 0, \quad (\text{gKdV})$$

the generalized regularized Korteweg-de Vries equation, also called Benjamin-Bona-Mahony equation or Peregrine equation when $f(u) = u^2/2$,

$$\partial_t u + \partial_x u + \partial_x(f(u)) - \partial_t \partial_x^2 u = 0, \quad (\text{gBBM})$$

the generalized regularized Boussinesq equation

$$\partial_t^2 u - \partial_x^2 u - \partial_x^2(f(u)) - \partial_t^2 \partial_x^2 u = 0. \quad (\text{grBsqs})$$

Each of these equations admit nontrivial solitary wave solutions $u(t, x) = U_c(x - ct)$ for c in $(0, +\infty)$, $(1, +\infty)$ and $(-\infty, -1) \cup (1, +\infty)$ respectively. For these solitary wave solutions, the momentum is, respectively,

$$P(U_c) = \int_{\mathbb{R}} U_c^2 dx = \|U_c\|_{L^2}^2, \quad P(U_c) = \int_{\mathbb{R}} U_c^2 + (\partial_x U_c)^2 dx, \quad P(U_c) = c \int_{\mathbb{R}} U_c^2 + (\partial_x U_c)^2 dx.$$

The existence of exactly one unstable eigenvalue has been shown with the use of an Evans function by R. Pego and M. Weinstein in [57] for these three equations under the condition $\frac{dP(U_c)}{dc} < 0$. The paper [49] by O. Lopes also gives a linear instability result. Equations (gBBM) and (grBsqs) turn out to be semilinear due to the regularization effect. Indeed, they may be written

$$\partial_t u + (1 - \partial_x^2)^{-1} \partial_x u + (1 - \partial_x^2)^{-1} \partial_x(f(u)) = 0, \quad \partial_t^2 u - (1 - \partial_x^2)^{-1} \partial_x^2 u - (1 - \partial_x^2)^{-1} \partial_x^2(f(u)) = 0.$$

Therefore, Corollary B.2 applies to these two models and this shows the nonlinear instability when linear instability holds.

In [48], some generalizations of the equations (gKdV), (gBBM) and (gBBM) have been proposed that take into account pseudodifferential operators. These are respectively

$$\partial_t u + \partial_x(f(u)) - \partial_x \mathcal{M}u = 0, \quad (\text{I})$$

$$\partial_t u + \partial_x u + \partial_x(f(u)) + \partial_t \mathcal{M}u = 0 \quad (\text{II})$$

and

$$\partial_t^2 u - \partial_x^2 u - \partial_x^2(f(u)) + \partial_t^2 \mathcal{M}u = 0. \quad (\text{III})$$

Here, \mathcal{M} is a Fourier multiplier of symbol $\hat{\mathcal{M}}$: $\widehat{\mathcal{M}w} = \hat{\mathcal{M}}\hat{w}$ (here, $\hat{\cdot}$ denotes Fourier transform). We assume $\hat{\mathcal{M}} \geq 0$ (otherwise, see [48]). When $\mathcal{M} = -\partial_x^2$, these equations reduce to (gKdV), (gBBM) and (gBBM)

respectively. The Benjamin-Ono equation ($\hat{\mathcal{M}} = |\xi|$), the Smith equation ($\hat{\mathcal{M}} = \sqrt{1 + \xi^2} - 1$) and the Intermediate Long Wave (or Whitham) equation ($\hat{\mathcal{M}} = \xi/\tanh(\xi H) - 1/H$, for some constant $H > 0$) are common models of dispersive wave equations that are of type (I). We refer to [48] for references on these models and the existence of solitary waves. The associated momentum is

$$P_I(U_c) = \int_{\mathbb{R}} U_c^2 dx = \|U_c\|_{L^2}^2 \quad P_{II}(U_c) = \|(1 + \mathcal{M})^{1/2} U_c\|_{L^2}^2 \quad P_{III}(U_c) = c\|(1 + \mathcal{M})^{1/2} U_c\|_{L^2}^2.$$

For these models, Evans' function type arguments do not work since we no longer have a differential equation (it is nonlocal). The paper [48] by Z. Lin proposes another approach than the Evans' function technique for establishing the existence of unstable eigenvalues. However, it is not completely clear whether this method extends easily to the case of systems such as the Euler-Korteweg system (EK) (given at the beginning of section 5.1).

Theorem B.6 ([48]) *We consider one of the equations (I), (II) or (III) with f of class \mathcal{C}^1 satisfying $f(0) = f'(0) = 0$ and $|f(u)| \gg |u|$ for $|u| \rightarrow +\infty$. We assume moreover that $\hat{\mathcal{M}}$ is even, nonnegative, and verifies, for some $m \geq 1$, $0 < \lim_{\xi \rightarrow +\infty} \frac{\hat{\mathcal{M}}(\xi)}{\xi^m} \leq \lim_{\xi \rightarrow +\infty} \frac{\hat{\mathcal{M}}(\xi)}{\xi^m} < \infty$. Assume that $c \mapsto \phi_c = U_c(x - ct)$ is a \mathcal{C}^1 branch of travelling wave solution to (I), (II) or (III) with $U_c \in H^{m/2}(\mathbb{R})$ defined near c_* and suppose that the linearized operator \mathcal{L} has exactly one negative eigenvalue, that $\ker \mathcal{L}$ is spanned by $\partial_x U_{c_*}$ and that $\frac{dP(U_c)}{dc} \Big|_{c=c_*} < 0$. Then, U_{c_*} is linearly unstable.*

It is not easy to determine whether the hypothesis of Theorem B.6 hold true when \mathcal{M} is not a (differential) Sturm-Liouville operator. See however [2] on this question. It is clear that if the assumptions of Theorem B.6 are verified, then assumption (A) is also satisfied. As for the (gBBM) and the (grBsQ) equations, the equations (II) and (II) turn out to be semilinear, thus we may prove nonlinear orbital instability by applying Corollary B.2.

The Kawahara equation (or fifth order (KdV) equation)

$$\partial_t u + \partial_x(f(u)) + \alpha \partial_x^3 u + \beta \partial_x^5 u = 0, \quad (\text{K})$$

with $\alpha, \beta \neq 0$ two real constants, is another relevant dispersive model. For this equation, it may happen that the linearized equation around the solitary wave has more than one negative eigenvalues, in which case the works [35, 36, 49, 48] do not give a clear necessary and sufficient condition for stability. The paper [14] by T. Bridges and G. Derks gives a sufficient condition for linear instability for solitary wave solutions, but also for other types of travelling solutions. This condition is probably not necessary since it may happen that there exist at least two unstable eigenvalues, or two complex conjugate eigenvalues.

Instead of stating a general result for nonlinear orbital instability, we shall consider several model cases on which we will verify the hypothesis of Theorem B.5, in particular the question of the existence of the multiplier \mathbb{M}_s .

Proposition B.1 *We consider the equation (I), namely*

$$\partial_t u + \partial_x(f(u)) - \partial_x \mathcal{M} u = 0$$

with f of class \mathcal{C}^1 satisfying $f(0) = f'(0) = 0$ and $|f(u)| \gg |u|$ for $|u| \rightarrow +\infty$. We assume that $\hat{\mathcal{M}}$ is one of the following functions:

$$\begin{array}{lll} -\xi^2 & (\text{KdV}); & \xi^4 + \alpha \xi^2 & (\text{Kawahara}); & |\xi| & (\text{Benjamin - Ono}); \\ \frac{\xi}{\tanh(\xi H)} - \frac{1}{H} & (\text{Intermediate Long Wave}); & \sqrt{1 + \xi^2} - 1 & (\text{Smith}). \end{array}$$

There exists $s_0 > 0$ such that, if there exists $c \in \mathbb{R}$ such that (I) has a nontrivial solitary wave $U_c \in L^2$ which is linearly unstable, then for any $s \geq s_0$, it is also nonlinearly unstable from H^s to H^s , to L^2 and to L^∞ .

By application of Theorem B.5, we are thus able to show the nonlinear instability from H^s to L^2 or L^∞ by tracking the exponentially growing mode (this question was left open in [48] and also in [49]). In particular, we obtain the L^2 nonlinear instability of the linearly unstable solitary waves for these models.

Proof. All the assumptions for Theorem B.5 for these types of models are verified in section 8.1 in [62], except the existence of the multiplier \mathbb{M}_s .

For (KdV), where $\sigma = 1$, we shall take (for $s \geq 2$ integer)

$$\mathbb{M}_s \equiv (-1)^s \partial_x^{2s} + \frac{1+2s}{3} (-1)^{s-1} \partial_x^{s-1} \{f'(Q) \partial_x^{s-1} \cdot\},$$

as the computations from [62], section 8.1 show. For the Kawahara equation, with $\sigma = 2$, we take (for $s \geq 4$ integer)

$$\mathbb{M}_s \equiv (-1)^s \partial_x^{2s} + \frac{1+2s}{5} (-1)^{s-1} \partial_x^{s-2} \{f'(Q) \partial_x^{s-2} \cdot\}$$

and since the computations are very similar, we omit them. For the Benjamin-Ono equation, we have $\mathcal{M}(\xi) = |\xi|$ and $\sigma = 1/2$, and we will then have to deal with pseudo-differential operator which are Fourier multipliers with homogeneous symbol. For this type of operator, we shall need some commutator estimates. We denote by $\mathcal{F}(w)$ or \hat{w} the Fourier transform of w , and \mathcal{H} the Fourier multiplier with symbol $-i \operatorname{sgn}(\xi)$ (this is the Hilbert transform).

Lemma B.1 (i) Let $h \in L^\infty(\mathbb{R})$ with $\mathcal{F}(\mathcal{M}^{\frac{1}{2}}h) \in L^1(\mathbb{R})$ (for instance, $h \in H^\sigma(\mathbb{R})$ for some $\sigma > 1$). Then, there exists $C > 0$ such that, for any $v \in H^{\frac{1}{2}}(\mathbb{R})$,

$$\|\mathcal{M}^{\frac{1}{2}}(hv) - h\mathcal{M}^{\frac{1}{2}}v\|_{L^2(\mathbb{R})} \leq C\|v\|_{L^2(\mathbb{R})}.$$

(ii) Let $h \in L^\infty(\mathbb{R})$ with $\mathcal{F}(\mathcal{M}^{\frac{3}{2}}h) \in L^1(\mathbb{R})$ (for instance, $h \in H^\sigma(\mathbb{R})$ for some $\sigma > 2$). Then, there exists $C > 0$ such that, for any $v \in H^{\frac{3}{2}}(\mathbb{R})$,

$$\left\| \mathcal{M}^{\frac{3}{2}}\{hv\} - h\mathcal{M}^{\frac{3}{2}}v - \frac{3}{2}[\partial_x h]\mathcal{M}^{\frac{1}{2}}\mathcal{H}v \right\|_{L^2(\mathbb{R})} \leq C\|v\|_{L^2(\mathbb{R})}.$$

(iii) Let $h \in L^\infty(\mathbb{R})$ with $\mathcal{F}(\partial_x \mathcal{M}^{\frac{1}{2}}h) \in L^1(\mathbb{R})$ (for instance, $h \in H^\sigma(\mathbb{R})$ for some $\sigma > 2$). Then, there exists $C > 0$ such that, for any $v \in H^{\frac{3}{2}}(\mathbb{R})$,

$$\left\| \partial_x \mathcal{M}^{\frac{1}{2}}\{hv\} - h\partial_x \mathcal{M}^{\frac{1}{2}}v - \frac{3}{2}[\partial_x h]\mathcal{M}^{\frac{1}{2}}v \right\|_{L^2(\mathbb{R})} \leq C\|v\|_{L^2(\mathbb{R})}.$$

Proof. We have

$$\mathcal{F}\left(\mathcal{M}^{\frac{1}{2}}(hv) - h\mathcal{M}^{\frac{1}{2}}v\right)(\xi) = \int_{\mathbb{R}} |\xi|^{\frac{1}{2}} \hat{h}(\xi - \zeta) \hat{v}(\zeta) d\zeta - \int_{\mathbb{R}} |\zeta|^{\frac{1}{2}} \hat{h}(\xi - \zeta) \hat{v}(\zeta) d\zeta.$$

Using the inequality $\left| |\xi|^{\frac{1}{2}} - |\zeta|^{\frac{1}{2}} \right| \leq C|\xi - \zeta|^{\frac{1}{2}}$, we thus obtain

$$\left| \mathcal{F}\left(\mathcal{M}^{\frac{1}{2}}(hv) - h\mathcal{M}^{\frac{1}{2}}v\right)(\xi) \right| \leq C \int_{\mathbb{R}} |\xi - \zeta|^{\frac{1}{2}} |\hat{h}(\xi - \zeta)| \cdot |\hat{v}(\zeta)| d\zeta = C\{|\mathcal{F}(\mathcal{M}^{\frac{1}{2}}h)| * |\hat{v}|\}(\xi)$$

and we conclude with the classical convolution estimate $L^1 * L^2 \subset L^2$. This argument does not provide the sharpest bound in h , since it involves $\|\mathcal{F}(\mathcal{M}^{\frac{1}{2}}h)\|_{L^1}$, whereas the use of paradifferential calculus will use only $\|h\|_{C^{\frac{1}{2}}}$. However, we shall to use this refinement here.

The starting point for the second inequality is

$$\left| |\xi|^{\frac{3}{2}} - |\zeta|^{\frac{3}{2}} - \frac{3}{2}|\zeta|^{\frac{1}{2}}\text{sgn}(\zeta)(\xi - \zeta) \right| \leq C|\xi - \zeta|^{\frac{3}{2}}.$$

To prove this, note that by homogeneity $\xi = \theta\zeta$, $\theta \in \mathbb{R}$, it suffices to prove

$$\left| |\theta|^{\frac{3}{2}} - 1 - \frac{3}{2}(\theta - 1) \right| \leq C|\theta - 1|^{\frac{3}{2}},$$

which is easy. Therefore,

$$\begin{aligned} & \left| \mathcal{F}\left(\mathcal{M}^{\frac{3}{2}}\{hv\} - h\mathcal{M}^{\frac{3}{2}}v - \frac{3}{2}[\partial_x h]\mathcal{M}^{\frac{1}{2}}\mathcal{H}v\right)(\xi) \right| \\ &= \left| \int_{\mathbb{R}} |\xi|^{\frac{3}{2}} \hat{h}(\xi - \zeta) \hat{v}(\zeta) d\zeta - \int_{\mathbb{R}} |\zeta|^{\frac{3}{2}} \hat{h}(\xi - \zeta) \hat{v}(\zeta) d\zeta - \int_{\mathbb{R}} \frac{3}{2}|\zeta|^{\frac{1}{2}}\text{sgn}(\zeta)(\xi - \zeta) \hat{h}(\xi - \zeta) \hat{v}(\zeta) d\zeta \right| \\ &\leq C \int_{\mathbb{R}} |\xi - \zeta|^{\frac{3}{2}} |\hat{h}(\xi - \zeta)| \cdot |\hat{v}(\zeta)| d\zeta \\ &= C|\mathcal{F}(\mathcal{M}^{\frac{3}{2}}h)| * |\hat{v}|, \end{aligned}$$

and we conclude as before. For the third inequality, we argue in a similar way with the estimate

$$\left| i\xi|\xi|^{\frac{1}{2}} - i\zeta|\zeta|^{\frac{1}{2}} - i\frac{3}{2}|\zeta|^{\frac{1}{2}}(\xi - \zeta) \right| \leq C|\xi - \zeta|^{\frac{3}{2}}.$$

The proof is complete. \square

For the Benjamin-Ono equation, $\hat{\mathcal{M}}(\xi = |\xi|)$, $\sigma = 1/2$ and the index s will be half an integer: $s \in \mathbb{N}/2$. Therefore, we set $s = [s] + \{s\}$, with $[s]$ integer and $\{s\} \in \{0; 1/2\}$. Let us define, for $s \in \mathbb{N}/2$, $s \geq 1$,

$$\mathbb{M}_s \equiv \begin{cases} (-1)^s \partial_x^{2s} + \gamma_s \mathcal{M}^{\frac{1}{2}} \partial_x^{s-1} \{f'(Q) \partial_x^{s-1} \mathcal{M}^{\frac{1}{2}} \cdot\} & \text{if } \{s\} = 0 \\ (-1)^{[s]} \partial_x^{2[s]} \mathcal{M} + \gamma_s \partial_x^{[s]} \{f'(Q) \partial_x^{[s]} \cdot\} & \text{if } \{s\} = \frac{1}{2}, \end{cases}$$

for some real constant γ_s to be determined later. It is clear that \mathbb{M}_s is self-adjoint on L^2 and that there exists $C_s > 0$ such that

$$|(\mathbb{M}_s u, v)_{L^2}| \leq C_s \|u\|_{H^s} \|v\|_{H^s} \quad \text{and} \quad (\mathbb{M}_s u, u)_{L^2} \geq \|u\|_{H^s}^2 - C_s \|u\|_{H^{s-\frac{1}{2}}}^2.$$

In order to verify the assumptions for the multiplier \mathbb{M}_s , it remains to study $\text{Re}(J(L_0 + \nabla^2 F(Q))u, \mathbb{M}_s u)_{L^2}$. When $\{s\} = 0$, *i.e.* $s \in \mathbb{N}$, this quantity is

$$\begin{aligned} & \text{Re}(\partial_x(\mathcal{M} + c + f'(Q))u, \mathbb{M}_s u)_{L^2} \\ &= \text{Re}(\partial_x \mathcal{M} u, (-1)^s \partial_x^{2s} u)_{L^2} + \gamma_s \text{Re}(\partial_x \mathcal{M} u, \mathcal{M}^{\frac{1}{2}} \partial_x^{s-1} \{f'(Q) \partial_x^{s-1} \mathcal{M}^{\frac{1}{2}} u\})_{L^2} \\ &\quad + \text{Re}(\partial_x [f'(Q)u], (-1)^s \partial_x^{2s} u)_{L^2} + \gamma_s \text{Re}(\partial_x [f'(Q)u], \mathcal{M}^{\frac{1}{2}} \partial_x^{s-1} \{f'(Q) \partial_x^{s-1} \mathcal{M}^{\frac{1}{2}} u\})_{L^2} \\ &\quad + c \text{Re}(\partial_x u, \mathbb{M}_s u)_{L^2}. \end{aligned} \tag{B.1}$$

By skew-adjointness, the first and last scalar products are zero. By integration by parts and Leibniz formula, we deduce, since $Q \in H^\infty$,

$$\begin{aligned} \text{Re}(\partial_x [f'(Q)u], (-1)^s \partial_x^{2s} u)_{L^2} &= \text{Re}(\partial_x^{s+1} [f'(Q)u], \partial_x^s u)_{L^2} \\ &\leq \text{Re}(f'(Q) \partial_x^{s+1} u, \partial_x^s u)_{L^2} + (s+1) \text{Re}(\partial_x [f'(Q)] \partial_x^s u, \partial_x^s u)_{L^2} + C_s \|u\|_{H^s} \|u\|_{H^{s-1}} \\ &\leq \left(s + \frac{1}{2}\right) \text{Re}(\partial_x [f'(Q)] \partial_x^s u, \partial_x^s u)_{L^2} + C_s \|u\|_{H^s} \|u\|_{H^{s-1}}. \end{aligned}$$

Similarly, using the easy estimates $\|\mathcal{M}^{\frac{1}{2}}v\|_{L^2} \leq K\|v\|_{H^{\frac{1}{2}}}$ and $\|hv\|_{H^{\frac{1}{2}}} \leq C(h)\|v\|_{H^{\frac{1}{2}}}$ for $h \in L^\infty$ with $\mathcal{F}(\mathcal{M}^{\frac{1}{2}}h) \in L^1$ (this is an immediate consequence of Lemma B.1)

$$\begin{aligned} \gamma_s \operatorname{Re}(\partial_x[f'(Q)u], \mathcal{M}^{\frac{1}{2}}\partial_x^{s-1}\{f'(Q)\partial_x^{s-1}\mathcal{M}^{\frac{1}{2}}u\})_{L^2} &= \gamma_s(-1)^{s-1} \operatorname{Re}(\mathcal{M}^{\frac{1}{2}}\partial_x^s[f'(Q)u], f'(Q)\partial_x^{s-1}\mathcal{M}^{\frac{1}{2}}u)_{L^2} \\ &\leq \gamma_s(-1)^{s-1} \operatorname{Re}(\mathcal{M}^{\frac{1}{2}}[f'(Q)\partial_x^s u], f'(Q)\partial_x^{s-1}\mathcal{M}^{\frac{1}{2}}u)_{L^2} + C\|u\|_{H^{s-\frac{1}{2}}}^2. \end{aligned}$$

Using Lemma B.1, we deduce $\|\mathcal{M}^{\frac{1}{2}}[f'(Q)\partial_x^s u] - f'(Q)\mathcal{M}^{\frac{1}{2}}\partial_x^s u\|_{L^2} \leq C(Q)\|u\|_{H^s}$, thus

$$\begin{aligned} \gamma_s \operatorname{Re}(\partial_x[f'(Q)u], \mathcal{M}^{\frac{1}{2}}\partial_x^{s-1}\{f'(Q)\partial_x^{s-1}\mathcal{M}^{\frac{1}{2}}u\})_{L^2} &\leq \gamma_s(-1)^{s-1} \operatorname{Re}(f'(Q)\partial_x^s \mathcal{M}^{\frac{1}{2}}u, f'(Q)\partial_x^{s-1}\mathcal{M}^{\frac{1}{2}}u)_{L^2} + C\|u\|_{H^s}\|u\|_{H^{s-\frac{1}{2}}} \\ &= \frac{\gamma_s}{2}(-1)^s \operatorname{Re}(\partial_x[f'(Q)]\partial_x^{s-1}\mathcal{M}^{\frac{1}{2}}u, f'(Q)\partial_x^{s-1}\mathcal{M}^{\frac{1}{2}}u)_{L^2} + C\|u\|_{H^s}\|u\|_{H^{s-\frac{1}{2}}} \\ &\leq C\|u\|_{H^{s-\frac{1}{2}}}^2 + C\|u\|_{H^s}\|u\|_{H^{s-\frac{1}{2}}} \leq C\|u\|_{H^s}\|u\|_{H^{s-\frac{1}{2}}}. \end{aligned}$$

We now turn to the term

$$\gamma_s \operatorname{Re}(\partial_x \mathcal{M}u, \mathcal{M}^{\frac{1}{2}}\partial_x^{s-1}\{f'(Q)\partial_x^{s-1}\mathcal{M}^{\frac{1}{2}}u\})_{L^2} = \gamma_s(-1)^{s-1} \operatorname{Re}(\partial_x^s u, \mathcal{M}^{\frac{3}{2}}\{f'(Q)\partial_x^{s-1}\mathcal{M}^{\frac{1}{2}}u\})_{L^2}.$$

Using Lemma B.1, we write

$$\begin{aligned} \left\| \mathcal{M}^{\frac{3}{2}}\{f'(Q)\partial_x^{s-1}\mathcal{M}^{\frac{1}{2}}u\} - f'(Q)\partial_x^{s-1}\mathcal{M}^2u - \frac{3}{2}\partial_x[f'(Q)]\mathcal{M}^{\frac{1}{2}}\mathcal{H}\{\partial_x^{s-1}\mathcal{M}^{\frac{1}{2}}u\} \right\|_{L^2} &\leq C(Q)\|\partial_x^{s-1}\mathcal{M}^{\frac{1}{2}}u\|_{L^2} \\ &\leq C(Q)\|u\|_{H^{s-\frac{1}{2}}}, \end{aligned}$$

which implies

$$\begin{aligned} \gamma_s \operatorname{Re}(\partial_x \mathcal{M}u, \mathcal{M}^{\frac{1}{2}}\partial_x^{s-1}\{f'(Q)\partial_x^{s-1}\mathcal{M}^{\frac{1}{2}}u\})_{L^2} &\leq \gamma_s(-1)^{s-1} \operatorname{Re}(\partial_x^s u, f'(Q)\partial_x^{s-1}\mathcal{M}^2u)_{L^2} + \frac{3}{2}\gamma_s(-1)^{s-1} \operatorname{Re}(\partial_x^s u, \partial_x[f'(Q)]\mathcal{M}^{\frac{1}{2}}\mathcal{H}\{\partial_x^{s-1}\mathcal{M}^{\frac{1}{2}}u\})_{L^2} \\ &\quad + C\|u\|_{H^s}\|u\|_{H^{s-\frac{1}{2}}}. \end{aligned}$$

Noticing that $\mathcal{M}^2 = -\partial_x^2$ and $\mathcal{M}^{\frac{1}{2}}\mathcal{H}\partial_x^{s-1}\mathcal{M}^{\frac{1}{2}} = \partial_x^{s-1}\mathcal{M}\mathcal{H} = -\partial_x^s$ (since $\mathcal{M}\mathcal{H}$ has symbol equal to $-i\xi$), we infer

$$\begin{aligned} \gamma_s \operatorname{Re}(\partial_x \mathcal{M}u, \mathcal{M}^{\frac{1}{2}}\partial_x^{s-1}\{f'(Q)\partial_x^{s-1}\mathcal{M}^{\frac{1}{2}}u\})_{L^2} &\leq \gamma_s(-1)^s \operatorname{Re}(\partial_x^s u, f'(Q)\partial_x^{s+1}u)_{L^2} + \frac{3}{2}\gamma_s(-1)^s \operatorname{Re}(\partial_x^s u, \partial_x[f'(Q)]\partial_x^s u)_{L^2} + C\|u\|_{H^s}\|u\|_{H^{s-\frac{1}{2}}} \\ &= \gamma_s(-1)^s \operatorname{Re}(\partial_x^s u, \partial_x[f'(Q)]\partial_x^s u)_{L^2} + C\|u\|_{H^s}\|u\|_{H^{s-\frac{1}{2}}} \end{aligned}$$

by integration by parts.

Reporting these estimates into (B.1), we infer

$$\begin{aligned} \operatorname{Re}(\partial_x(\mathcal{M} + c + f'(Q))u, \mathbb{M}_s u)_{L^2} &\leq \left(s + \frac{1}{2}\right) \operatorname{Re}(\partial_x[f'(Q)]\partial_x^s u, \partial_x^s u)_{L^2} + \gamma_s(-1)^s \operatorname{Re}(\partial_x^s u, \partial_x[f'(Q)]\partial_x^s u)_{L^2} \\ &\quad + C\|u\|_{H^s}\|u\|_{H^{s-\frac{1}{2}}}. \end{aligned}$$

Therefore, the choice

$$\gamma_s \equiv (-1)^{s-1} \left(s + \frac{1}{2}\right)$$

provides the desired control

$$\operatorname{Re}(\partial_x(\mathcal{M} + c + f'(Q))u, \mathbb{M}_s u)_{L^2} \leq C\|u\|_{H^s}\|u\|_{H^{s-\frac{1}{2}}}.$$

When $\{s\} = 1/2$, the computations are similar: (B.1) becomes now

$$\begin{aligned}
& \operatorname{Re}(\partial_x(\mathcal{M} + c + f'(Q))u, \mathbb{M}_s u)_{L^2} \\
&= \operatorname{Re}(\partial_x \mathcal{M} u, (-1)^{[s]} \partial_x^{2[s]} \mathcal{M} u)_{L^2} + \gamma_s \operatorname{Re}(\partial_x \mathcal{M} u, \partial_x^{[s]} \{f'(Q) \partial_x^{[s]} u\})_{L^2} \\
&\quad + \operatorname{Re}(\partial_x [f'(Q)u], (-1)^{[s]} \partial_x^{2[s]} \mathcal{M} u)_{L^2} + \gamma_s \operatorname{Re}(\partial_x [f'(Q)u], \partial_x^{[s]} \{f'(Q) \partial_x^{[s]} u\})_{L^2} \\
&\quad + c \operatorname{Re}(\partial_x u, \mathbb{M}_s u)_{L^2}, \tag{B.2}
\end{aligned}$$

and the first and last scalar product still vanish. Moreover, by integration by parts and Leibniz formula, we deduce, since $Q \in H^\infty$,

$$\begin{aligned}
& \gamma_s \operatorname{Re}(\partial_x [f'(Q)u], \partial_x^{[s]} \{f'(Q) \partial_x^{[s]} u\})_{L^2} = \gamma_s (-1)^{[s]} \operatorname{Re}(\partial_x^{[s]+1} [f'(Q)u], f'(Q) \partial_x^{[s]} u)_{L^2} \\
&\leq \gamma_s (-1)^{[s]} \operatorname{Re}(f'(Q) \partial_x^{[s]+1} u, f'(Q) \partial_x^{[s]} u)_{L^2} + C \|u\|_{H^{[s]}}^2 \\
&\leq \gamma_s (-1)^{[s]-1} \operatorname{Re}(\partial_x [f'(Q)] \partial_x^{[s]} u, f'(Q) \partial_x^{[s]} u)_{L^2} + C \|u\|_{H^{[s]}}^2 \\
&\leq C \|u\|_{H^{[s]}}^2 = C \|u\|_{H^{s-\frac{1}{2}}}^2.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
& \operatorname{Re}(\partial_x [f'(Q)u], (-1)^{[s]} \partial_x^{2[s]} \mathcal{M} u)_{L^2} = \operatorname{Re}(\mathcal{M}^{\frac{1}{2}} \partial_x^{[s]+1} [f'(Q)u], \partial_x^{[s]} \mathcal{M}^{\frac{1}{2}} u)_{L^2} \\
&\leq \operatorname{Re}(\partial_x \mathcal{M}^{\frac{1}{2}} \{f'(Q) \partial_x^{[s]} u\}, \partial_x^{[s]} \mathcal{M}^{\frac{1}{2}} u)_{L^2} \\
&\quad + [s] \operatorname{Re}(\partial_x \mathcal{M}^{\frac{1}{2}} \{\partial_x [f'(Q)] \partial_x^{[s]-1} u\}, \partial_x^{[s]} \mathcal{M}^{\frac{1}{2}} u)_{L^2} + C \|u\|_{H^{[s]-\frac{1}{2}}} \|u\|_{H^{[s]+\frac{1}{2}}}.
\end{aligned}$$

For the second scalar product, we write, by Lemma B.1,

$$\begin{aligned}
& \operatorname{Re}(\partial_x \mathcal{M}^{\frac{1}{2}} \{\partial_x [f'(Q)] \partial_x^{[s]-1} u\}, \partial_x^{[s]} \mathcal{M}^{\frac{1}{2}} u)_{L^2} \\
&= \operatorname{Re}(\mathcal{M}^{\frac{1}{2}} \{\partial_x [f'(Q)] \partial_x^{[s]-1} u\}, \partial_x^{[s]} \mathcal{M}^{\frac{1}{2}} u)_{L^2} + \operatorname{Re}(\mathcal{M}^{\frac{1}{2}} \{\partial_x [f'(Q)] \partial_x^{[s]} u\}, \partial_x^{[s]} \mathcal{M}^{\frac{1}{2}} u)_{L^2} \\
&\leq C \|u\|_{H^{[s]-\frac{1}{2}}} \|u\|_{H^{[s]+\frac{1}{2}}} + \operatorname{Re}(\partial_x [f'(Q)] \partial_x^{[s]} \mathcal{M}^{\frac{1}{2}} u, \partial_x^{[s]} \mathcal{M}^{\frac{1}{2}} u)_{L^2} + C \|u\|_{H^{[s]}} \|u\|_{H^{[s]+\frac{1}{2}}} \\
&\leq \operatorname{Re}(\partial_x [f'(Q)] \partial_x^{[s]} \mathcal{M}^{\frac{1}{2}} u, \partial_x^{[s]} \mathcal{M}^{\frac{1}{2}} u)_{L^2} + C \|u\|_{H^{s-\frac{1}{2}}} \|u\|_{H^s}.
\end{aligned}$$

For the first scalar product, we use Lemma B.1 once again:

$$\begin{aligned}
& \operatorname{Re}(\partial_x \mathcal{M}^{\frac{1}{2}} \{f'(Q) \partial_x^{[s]} u\}, \partial_x^{[s]} \mathcal{M}^{\frac{1}{2}} u)_{L^2} \\
&\leq \operatorname{Re}(f'(Q) \partial_x \mathcal{M}^{\frac{1}{2}} \partial_x^{[s]} u, \partial_x^{[s]} \mathcal{M}^{\frac{1}{2}} u)_{L^2} + \frac{3}{2} \operatorname{Re}(\partial_x [f'(Q)] \partial_x^{[s]} \mathcal{M}^{\frac{1}{2}} u, \partial_x^{[s]} \mathcal{M}^{\frac{1}{2}} u)_{L^2} + C \|u\|_{H^{s-\frac{1}{2}}} \|u\|_{H^s} \\
&\leq \operatorname{Re}(\partial_x [f'(Q)] \partial_x^{[s]} \mathcal{M}^{\frac{1}{2}} u, \partial_x^{[s]} \mathcal{M}^{\frac{1}{2}} u)_{L^2} + C \|u\|_{H^{s-\frac{1}{2}}} \|u\|_{H^s}.
\end{aligned}$$

As a consequence, since $[s] = s - \frac{1}{2}$,

$$\operatorname{Re}(\partial_x [f'(Q)u], (-1)^{[s]} \partial_x^{2[s]} \mathcal{M} u)_{L^2} \leq \left(s + \frac{1}{2}\right) \operatorname{Re}(\partial_x [f'(Q)] \partial_x^{[s]} \mathcal{M}^{\frac{1}{2}} u, \partial_x^{[s]} \mathcal{M}^{\frac{1}{2}} u)_{L^2} + C \|u\|_{H^{s-\frac{1}{2}}} \|u\|_{H^s}.$$

We turn finally to the term

$$\gamma_s \operatorname{Re}(\partial_x \mathcal{M} u, \partial_x^{[s]} \{f'(Q) \partial_x^{[s]} u\})_{L^2} = \gamma_s (-1)^{[s]} \operatorname{Re}(\partial_x^{[s]} \mathcal{M}^{\frac{1}{2}} u, \partial_x \mathcal{M}^{\frac{1}{2}} \{f'(Q) \partial_x^{[s]} u\})_{L^2},$$

and infer, by Lemma B.1,

$$\begin{aligned}
& \gamma_s \operatorname{Re}(\partial_x \mathcal{M} u, \partial_x^{[s]} \{f'(Q) \partial_x^{[s]} u\})_{L^2} \leq \gamma_s (-1)^{[s]} \operatorname{Re}(\partial_x^{[s]} \mathcal{M}^{\frac{1}{2}} u, f'(Q) \partial_x \mathcal{M}^{\frac{1}{2}} \partial_x^{[s]} u)_{L^2} \\
&\quad + \frac{3}{2} \gamma_s (-1)^{[s]} \operatorname{Re}(\partial_x^{[s]} \mathcal{M}^{\frac{1}{2}} u, \partial_x [f'(Q)] \mathcal{M}^{\frac{1}{2}} \partial_x^{[s]} u)_{L^2} + C \|u\|_{H^{s-\frac{1}{2}}} \|u\|_{H^s} \\
&= \gamma_s (-1)^{[s]} \operatorname{Re}(\partial_x^{[s]} \mathcal{M}^{\frac{1}{2}} u, \partial_x [f'(Q)] \mathcal{M}^{\frac{1}{2}} \partial_x^{[s]} u)_{L^2} + C \|u\|_{H^{s-\frac{1}{2}}} \|u\|_{H^s}.
\end{aligned}$$

Therefore,

$$\operatorname{Re}(\partial_x(\mathcal{M} + c + f'(Q))u, \mathbb{M}_s u)_{L^2} \leq \left(s + \frac{1}{2} + \gamma_s(-1)^{[s]}\right) \operatorname{Re}(\partial_x^{[s]} \mathcal{M}^{\frac{1}{2}} u, \partial_x[f'(Q)] \mathcal{M}^{\frac{1}{2}} \partial_x^{[s]} u)_{L^2} + C \|u\|_{H^{s-\frac{1}{2}}} \|u\|_{H^s},$$

hence choosing $\gamma_s \equiv (-1)^{[s]-1}(s + \frac{1}{2})$ gives the result.

It remains to study the cases of the Intermediate Long Wave equation and the Smith equation, for which $\hat{\mathcal{M}}$ is, respectively,

$$\frac{\xi}{\tanh(\xi H)} - \frac{1}{H}; \quad \sqrt{1 + \xi^2} - 1.$$

We denote \mathcal{M}_0 the operator with symbol $|\xi|$ (the one of the Benjamin-Ono equation), and define \mathbb{M}_s as for the Benjamin-Ono case (hence with "M" = \mathcal{M}_0). We observe that in both cases, $\tilde{\mathcal{M}} \equiv \mathcal{M} - \mathcal{M}_0$ is bounded on L^2 . Indeed, its symbol is continuous in \mathbb{R} and, for $\xi \rightarrow \pm\infty$,

$$\hat{\mathcal{M}}(\xi) = \frac{\xi}{\tanh(\xi H)} - \frac{1}{H} = \frac{\xi}{\operatorname{sgn}(\xi) + \mathcal{O}(e^{-2|\xi|H})} - \frac{1}{H} = |\xi| - \frac{1}{H} + \mathcal{O}(|\xi|e^{-2|\xi|H})$$

and

$$\hat{\mathcal{M}}(\xi) = \sqrt{1 + \xi^2} - 1 = |\xi|\sqrt{1 + \xi^{-2}} - 1 = |\xi| - 1 + \mathcal{O}(|\xi|^{-1})$$

respectively. In the quantity $\operatorname{Re}(\partial_x(\mathcal{M} + c + f'(Q))u, \mathbb{M}_s u)_{L^2}$, we then have to bound from above the extra term $\operatorname{Re}(\partial_x(\tilde{\mathcal{M}}u), \mathbb{M}_s u)_{L^2}$, that is (using the skew-adjointness for the higher order derivatives in \mathbb{M}_s),

$$\operatorname{Re}(\partial_x(\tilde{\mathcal{M}}u), \gamma_s \mathcal{M}_0^{\frac{1}{2}} \partial_x^{s-1} \{f'(Q) \partial_x^{s-1} \mathcal{M}_0^{\frac{1}{2}} u\})_{L^2} = \gamma_s (-1)^{s-1} \operatorname{Re}(\partial_x^s \mathcal{M}_0^{\frac{1}{2}}(\tilde{\mathcal{M}}u), f'(Q) \partial_x^{s-1} \mathcal{M}_0^{\frac{1}{2}} u)_{L^2} \quad \text{if } \{s\} = 0;$$

$$\operatorname{Re}(\partial_x(\tilde{\mathcal{M}}u), \gamma_s \partial_x^{[s]} \{f'(Q) \partial_x^{[s]} u\})_{L^2} = \gamma_s (-1)^{[s]} \operatorname{Re}(\partial_x^{[s]+1}(\tilde{\mathcal{M}}u), f'(Q) \partial_x^{[s]} u)_{L^2} \quad \text{if } \{s\} = \frac{1}{2}.$$

We then note that in both cases, one may actually split $\tilde{\mathcal{M}} = \mathcal{M} - \mathcal{M}_0 = \tilde{\mathcal{M}}_c + \tilde{\mathcal{M}}_h$, where $\tilde{\mathcal{M}}_c$ is the multiplication by $-1/H$ (resp. -1) and $\tilde{\mathcal{M}}_h$ has a symbol which is continuous in \mathbb{R} and $\mathcal{O}(|\xi|^{-1})$ at infinity, so that $\tilde{\mathcal{M}}_h$ is bounded from H^σ to $H^{\sigma+1}$ if $\sigma \geq 0$. Therefore, when $\{s\} = 0$, we easily get

$$\begin{aligned} & \operatorname{Re}(\partial_x(\tilde{\mathcal{M}}u), \gamma_s \mathcal{M}_0^{\frac{1}{2}} \partial_x^{s-1} \{f'(Q) \partial_x^{s-1} \mathcal{M}_0^{\frac{1}{2}} u\})_{L^2} \\ &= \gamma_s (-1)^{s-1} \operatorname{Re}(\partial_x^s \mathcal{M}_0^{\frac{1}{2}}(\tilde{\mathcal{M}}_c u), f'(Q) \partial_x^{s-1} \mathcal{M}_0^{\frac{1}{2}} u)_{L^2} + \gamma_s (-1)^{s-1} \operatorname{Re}(\partial_x^s \mathcal{M}_0^{\frac{1}{2}}(\tilde{\mathcal{M}}_h u), f'(Q) \partial_x^{s-1} \mathcal{M}_0^{\frac{1}{2}} u)_{L^2} \\ &\leq \frac{1}{2} \gamma_s (-1)^s \operatorname{Re}(\partial_x^{s-1} \mathcal{M}_0^{\frac{1}{2}} u, \tilde{\mathcal{M}}_c \partial_x [f'(Q)] \partial_x^{s-1} \mathcal{M}_0^{\frac{1}{2}} u)_{L^2} + C \|u\|_{H^{s-\frac{1}{2}}}^2 \leq C \|u\|_{H^{s-\frac{1}{2}}}^2 \end{aligned}$$

and similarly when $\{s\} = 1/2$. Therefore, the estimate

$$\operatorname{Re}(\partial_x(\mathcal{M} + c + f'(Q))u, \mathbb{M}_s u)_{L^2} \leq C \|u\|_{H^{s-\frac{1}{2}}} \|u\|_{H^s}$$

remains true for the Intermediate Long Wave equation and the Smith equation. The proof of Proposition B.1 is thus complete by applying Theorem B.6. \square

B.3 Proofs

B.3.1 Proof of Theorem B.3

We shall prove the resolvent estimate required in Corollary B.1. Let us consider $\lambda = \gamma + i\tau \in \mathbb{C}$ with $\gamma \neq 0$ and the resolvent equation $(\mathcal{J}\mathcal{L} - \lambda)v = \Sigma$, or

$$(\gamma + i\tau)v = \mathcal{J}\mathcal{L}(v) - \Sigma. \tag{B.3}$$

By hypothesis, the essential spectrum of $\mathcal{J}\mathcal{L}$ is of the form $i[\mathbb{R} \setminus (-\vartheta_0, +\vartheta_0)]$. Moreover, we have seen that $\mathcal{J}\mathcal{L}$ has a finite number of eigenvalues in the half-space $\{\text{Re} > 0\}$, hence, for $|\tau| \geq \tau_0$ sufficiently large, we know that there exists a unique solution v to (B.3). By taking the scalar product with $\mathcal{L}(v)$, we deduce the conservation law

$$\gamma(v, \mathcal{L}(v))_{\mathcal{X}} = -\text{Re}(\Sigma, \mathcal{L}(v))_{\mathcal{X}}. \quad (\text{B.4})$$

By our assumption, there exists a finite (possibly empty) number of eigenvalues in $(-\infty, 0]$, $(-\mu_1, \dots, -\mu_q)$, each one of finite multiplicity. For any $1 \leq k \leq q$, we fix an orthonormal basis $(\chi_{k,\ell})_{1 \leq \ell \leq n_k}$ of the eigenspace $\text{Ker}(\mathcal{L} + \mu_k)$. By assumption (A), any eigenvector $\chi_{k,\ell}$ is smooth in the sense that $\chi_{k,\ell} \in D(\mathcal{J})$ and $\mathcal{J}\chi_{k,\ell} \in D(\mathcal{L})$.

We then make a spectral orthogonal decomposition

$$v = \sum_{k=1}^q \sum_{\ell=1}^{n_k} \alpha_{k,\ell} \chi_{k,\ell} + v_+,$$

where $\mathcal{L}(\chi_{k,\ell}) = \mu_k \chi_{k,\ell}$ and $(v_+, \mathcal{L}(v_+))_{\mathcal{X}} \geq \delta \|v_+\|_{\mathcal{X}}^2$ for some positive δ . In the double sum, we have a finite number (independent of v) of terms. Inserting this into (B.4) yields

$$|\gamma| \delta \|v_+\|_{\mathcal{X}}^2 \leq |\gamma| \delta (v_+, \mathcal{L}(v_+))_{\mathcal{X}} \leq \delta \left[|\text{Re}(\Sigma, \mathcal{L}(v))_{\mathcal{X}}| + \sum_{k,\ell} \mu_k |\alpha_{k,\ell}|^2 \right] \leq K \|\Sigma\|_{\mathcal{X}} \|v\|_{\mathcal{X}} + K \sum_{k,\ell} |\alpha_{k,\ell}|^2$$

Using the inequality $ab \leq \varepsilon a^2 + b^2/(4\varepsilon)$ with $a = \|v\|_{\mathcal{X}}$, $b = K \|\Sigma\|_{\mathcal{X}}$ and $\varepsilon = |\gamma| \delta / 2$, the equality $\|v\|_{\mathcal{X}}^2 = \|v_+\|_{\mathcal{X}}^2 + \sum_{k,\ell} |\alpha_{k,\ell}|^2$ and incorporating the term $|\gamma| \delta \|v_+\|_{\mathcal{X}}^2 / 2$ in the left-hand side, we infer

$$\frac{|\gamma| \delta}{2} \|v_+\|_{\mathcal{X}}^2 \leq K' \sum_{k,\ell} |\alpha_{k,\ell}|^2 + K'' \|\Sigma\|_{\mathcal{X}}^2. \quad (\text{B.5})$$

On the other hand, since $\chi_{k,\ell} \in D(\mathcal{J})$ and $\mathcal{J}\chi_{k,\ell} \in D(\mathcal{L})$ by assumption (A), taking the scalar product of (B.1) with $\chi_{k,\ell}$ provides

$$(\gamma + i\tau)\alpha_{k,\ell} = -(v, \mathcal{L}\mathcal{J}\chi_{k,\ell})_{\mathcal{X}} - (\Sigma, \chi_{k,\ell})_{\mathcal{X}}.$$

Consequently,

$$(|\gamma| + |\tau|)|\alpha_{k,\ell}| \leq K_{k,\ell} \|v\|_{\mathcal{X}} + K \|\Sigma\|_{\mathcal{X}},$$

thus

$$(|\gamma| + |\tau|)^2 \sum_{k,\ell} |\alpha_{k,\ell}|^2 \leq K_0 \|v\|_{\mathcal{X}}^2 + K \|\Sigma\|_{\mathcal{X}}^2 = K_0 \sum_{k,\ell} |\alpha_{k,\ell}|^2 + K \|v_+\|_{\mathcal{X}}^2 + K \|\Sigma\|_{\mathcal{X}}^2,$$

which implies, if $|\tau| \geq 1 + \sqrt{K_0} - |\gamma|$,

$$\sum_{k,\ell} |\alpha_{k,\ell}|^2 \leq K \frac{\|v_+\|_{\mathcal{X}}^2 + \|\Sigma\|_{\mathcal{X}}^2}{(|\gamma| + |\tau|)^2 - K_0}.$$

Reporting this into (B.5) gives

$$\frac{|\gamma| \delta}{2} \|v_+\|_{\mathcal{X}}^2 \leq K' K \frac{\|v_+\|_{\mathcal{X}}^2 + \|\Sigma\|_{\mathcal{X}}^2}{(|\gamma| + |\tau|)^2 - K_0} + K'' \|\Sigma\|_{\mathcal{X}}^2.$$

If $|\tau| \geq 1 + \sqrt{K_0 + 4K'K'/|\gamma|\delta} - |\gamma|$, we deduce

$$\frac{|\gamma| \delta}{4} \|v_+\|_{\mathcal{X}}^2 \leq \left(K'' + \frac{K'K}{(|\gamma| + |\tau|)^2 - K_0} \right) \|\Sigma\|_{\mathcal{X}}^2 \leq K_1 \|\Sigma\|_{\mathcal{X}}^2,$$

and it follows that

$$\|v\|_{\mathcal{X}}^2 = \|v_+\|_{\mathcal{X}}^2 + \sum_{k,\ell} |\alpha_{k,\ell}|^2 \leq K_2 \|\Sigma\|_{\mathcal{X}}^2,$$

where K_2 does not depend on $|\tau|$ (large enough) as wished.

The proof of the first semigroup estimate follows then easily, see, for instance, Proposition 2 in [59].

Proof of the semigroup estimate when $\gamma_0 > 0$. Here, we assume $\gamma_0 > 0$. As a consequence, the spectrum of $[\mathcal{J}\mathcal{L}]_{\mathbb{C}}$ is of the form $\sigma_s \cup \sigma_u$, where $\sigma_{\text{ess}}([\mathcal{J}\mathcal{L}]_{\mathbb{C}}) \subset \sigma_s \subset \{\text{Re} \leq 0\}$ and $\emptyset \neq \sigma_u \subset \{\text{Re} > 0\}$ consists in a finite number of eigenvalues of finite algebraic multiplicities. Therefore, we may define (see, *e.g.* [41], [39]) the spectral Riesz projection

$$\mathbb{P} \equiv \frac{1}{2i\pi} \int_{\Gamma} ([\mathcal{J}\mathcal{L}]_{\mathbb{C}} - z)^{-1} dz,$$

where Γ is any simple (positively oriented) closed curve enclosing σ_u . As a consequence, \mathbb{P} is bounded, commutes with $[\mathcal{J}\mathcal{L}]_{\mathbb{C}}$ on $D([\mathcal{J}\mathcal{L}]_{\mathbb{C}})$ and verifies $\sigma([\mathcal{J}\mathcal{L}]_{\mathbb{C}}\mathbb{P}) = \sigma_u$, $\sigma([\mathcal{J}\mathcal{L}]_{\mathbb{C}}(\text{Id} - \mathbb{P})) = \sigma_s$. Moreover, $[\mathcal{J}\mathcal{L}]_{\mathbb{C}}\mathbb{P}$ is bounded, hence generates a continuous semigroup, $e^{t[\mathcal{J}\mathcal{L}]_{\mathbb{C}}\mathbb{P}}$, given by the exponential series

$$e^{t[\mathcal{J}\mathcal{L}]_{\mathbb{C}}\mathbb{P}} = \sum_{n=0}^{+\infty} \frac{t^n ([\mathcal{J}\mathcal{L}]_{\mathbb{C}}\mathbb{P})^n}{n!}.$$

In addition, $[\mathcal{J}\mathcal{L}]_{\mathbb{C}}(\text{Id} - \mathbb{P}) = [\mathcal{J}\mathcal{L}]_{\mathbb{C}} - [\mathcal{J}\mathcal{L}]_{\mathbb{C}}\mathbb{P}$ also generates a continuous semigroup and that we have $e^{t[\mathcal{J}\mathcal{L}]_{\mathbb{C}}} = e^{t[\mathcal{J}\mathcal{L}]_{\mathbb{C}}\mathbb{P}} e^{t[\mathcal{J}\mathcal{L}]_{\mathbb{C}}(\text{Id} - \mathbb{P})}$.

The semigroup generated by the bounded operator $[\mathcal{J}\mathcal{L}]_{\mathbb{C}}\mathbb{P}$ is easily analyzed. We shall now apply the spectral mapping theorem of J. Prüss (Theorem B.2) to $[\mathcal{J}\mathcal{L}]_{\mathbb{C}}(\text{Id} - \mathbb{P})$ in order to control the growth of its norm. By Corollary B.1, it suffices to estimate its resolvent $[[\mathcal{J}\mathcal{L}]_{\mathbb{C}}(\text{Id} - \mathbb{P}) - (\gamma + i\tau)]^{-1}$ for large $|\tau|$ (note that $\sigma([\mathcal{J}\mathcal{L}]_{\mathbb{C}}(\text{Id} - \mathbb{P})) = \sigma_s \subset \{\text{Re} \leq 0\}$). If $\Sigma \in \mathcal{X}_{\mathbb{C}}$ and $|\tau|$ is large, it is clear that the solution $u \in \mathcal{X}_{\mathbb{C}}$ to $[[\mathcal{J}\mathcal{L}]_{\mathbb{C}}(\text{Id} - \mathbb{P}) - (\gamma + i\tau)]u = \Sigma$ is given by

$$u = [[\mathcal{J}\mathcal{L}]_{\mathbb{C}} - (\gamma + i\tau)]^{-1}(\text{Id} - \mathbb{P})\Sigma - \frac{1}{\gamma + i\tau}\mathbb{P}\Sigma,$$

thus, for $|\tau|$ large,

$$\left\| [[\mathcal{J}\mathcal{L}]_{\mathbb{C}}(\text{Id} - \mathbb{P}) - (\gamma + i\tau)]^{-1} \right\|_{\mathcal{L}_c(\mathcal{X}_{\mathbb{C}})} \leq \left\| [[\mathcal{J}\mathcal{L}]_{\mathbb{C}} - (\gamma + i\tau)]^{-1} \right\|_{\mathcal{L}_c(\mathcal{X}_{\mathbb{C}})} \|\text{Id} - \mathbb{P}\|_{\mathcal{L}_c(\mathcal{X})} + \frac{1}{|\gamma + i\tau|} \|\mathbb{P}\|_{\mathcal{L}_c(\mathcal{X}_{\mathbb{C}})}$$

is bounded. Consequently, by Theorem B.2 and since $\sigma_s \subset \{\text{Re} \leq 0\}$, $\sigma(e^{t[\mathcal{J}\mathcal{L}]_{\mathbb{C}}(\text{Id} - \mathbb{P})}) = e^{t\sigma([\mathcal{J}\mathcal{L}]_{\mathbb{C}}(\text{Id} - \mathbb{P}))} = e^{t\sigma_s} \subset \bar{D}(0, 1)$. It follows that for any $\epsilon > 0$, there exists $K_{\epsilon} > 0$ such that

$$\forall t \geq 0, \quad \left\| e^{t[\mathcal{J}\mathcal{L}]_{\mathbb{C}}(\text{Id} - \mathbb{P})} \right\|_{\mathcal{L}_c(\mathcal{X}_{\mathbb{C}})} \leq K_{\epsilon} e^{\epsilon t}.$$

Since $e^{t[\mathcal{J}\mathcal{L}]_{\mathbb{C}}\mathbb{P}}$ is given by the exponential series, we also have the optimal estimate

$$\forall t \geq 0, \quad \left\| e^{t[\mathcal{J}\mathcal{L}]_{\mathbb{C}}\mathbb{P}} \right\|_{\mathcal{L}_c(\mathcal{X}_{\mathbb{C}})} \leq K_0(1+t)^{m-1} e^{\gamma_0 t}$$

by definition of m . We conclude by taking $\epsilon = \gamma_0/2$ for instance.

B.3.3 Proof of Theorem B.4

Since \mathcal{A} generates a continuous semigroup, v is a solution to $\partial_t v = \mathcal{A}v + \Phi(v)$ if and only if it is a mild solution:

$$v(t) = e^{t\mathcal{A}}v^{\text{in}} + \int_0^t e^{(t-\tau)\mathcal{A}}\Phi(v(\tau)) d\tau.$$

There exists $r_0 > 0$ such that $\|\Phi(v)\|_X \leq M\|v\|_X^2$ if $\|v\|_X \leq r_0$. We choose $v^{\text{in}} = \delta \text{Re } w$, where $\|\text{Re } w\|_X = 1$ and w is an eigenvector for the eigenvalue λ , and write the solution under the form $v = e^{t\mathcal{A}}v^{\text{in}} + \tilde{v} = \text{Re}(e^{t\lambda}w) + \tilde{v}$. If $\lambda \in \mathbb{R}$, we can choose $w \in D(\mathcal{A}) \subset D(\mathcal{A}_{\mathbb{C}})$. Then,

$$\tilde{v}(t) = \int_0^t e^{(t-\tau)\mathcal{A}} \Phi(\delta \text{Re}(e^{t\lambda}w) + \tilde{v}(\tau)) d\tau.$$

Let us denote $r_1 \equiv \min\left(r_0, \frac{\gamma_0 - \beta}{2MM_0}\right)$ and let $T > 0$ be the maximal time such that $T < \ln(r/(2\delta))/\gamma_0$ and $\|\tilde{u}(\tau)\|_X < r_1/2$ in $[0, T]$, where $0 < r < r_1$ will be determined later. We shall work for $0 \leq t < T$, so that $\|\delta \text{Re}(e^{t\lambda}w) + \tilde{v}(\tau)\|_X < \delta e^{t\gamma_0} + r_1/2 \leq r_1 \leq r_0$. Then,

$$\begin{aligned} \|\tilde{v}(t)\| &\leq \int_0^t \left\| e^{(t-\tau)\mathcal{A}} \right\|_{\mathcal{L}_c(X)} M \|\delta \text{Re}(e^{\tau\lambda}w) + \tilde{v}(\tau)\|^2 d\tau \\ &\leq 2M_0M \int_0^t e^{(\gamma_0+\beta)(t-\tau)} \left(\delta^2 e^{2\tau\gamma_0} + \|\tilde{v}(\tau)\|^2 \right) d\tau \\ &\leq \frac{2M_0M}{\gamma_0 - \beta} \delta^2 e^{2t\gamma_0} + r_1 M_0 M \int_0^t e^{(\gamma_0+\beta)(t-\tau)} \|\tilde{v}(\tau)\|_X d\tau, \end{aligned}$$

since $\beta < \gamma_0$. Applying now the Gronwall inequality to $e^{-(\gamma_0+\beta)t}\|\tilde{v}(t)\|_X$ then gives, since $M_0Mr_1 < \gamma_0 - \beta$,

$$\|\tilde{u}(t)\|_X \leq \left[\frac{2M_0M}{\gamma_0 - \beta} + \frac{2r_1M_0^2M^2}{(\gamma_0 - \beta)(\gamma_0 - \beta - r_1M_0M)} \right] \delta^2 e^{2t\gamma_0} = K\delta^2 e^{2t\gamma_0}.$$

We now choose $r \equiv \sqrt{r_1/K}$, so that the right-hand side is $\leq Kr^2/4 < r_1/2$, and this implies that u exists at least on $[0, \ln(r/(2\delta))/\gamma_0]$. In addition, for $0 \leq t < T$,

$$\|u(t)\|_X \geq \delta e^{t\gamma_0} - \|\tilde{u}(t)\|_X \geq \delta e^{t\gamma_0} - K\delta^2 e^{2t\gamma_0},$$

as desired. We conclude choosing $\varepsilon_0 > 0$ so small that $2\varepsilon_0 - K\varepsilon_0^2 \geq \varepsilon_0$.

B.3.3 Proof of Corollary B.2

We pick some $0 < \beta < \gamma_0$ (for instance $\beta = \gamma_0/2$) in order to have the semigroup estimate required in Theorem B.4. The solution $u(t) = T(\omega_*t)(\phi_{\omega_*} + v(t))$ verifies, for $0 \leq t \leq \gamma_0^{-1} \ln(2\varepsilon_0/\delta)$,

$$\|v(t)\|_X = \|T(-\omega_*t)u(t) - (\phi_{\omega_*} + \delta \text{Re}(e^{t\lambda}w))\|_X \leq K\delta^2 e^{2t\gamma_0}.$$

Hence, $T(-\omega_*t)u(t)$ remains at distance $\leq K\varepsilon_0$ from $\phi_{\omega_*} \in \mathfrak{M}$ and therefore

$$\text{dist}_{\mathcal{X}}(u(t), \mathfrak{M}) \geq \text{dist}_{\mathcal{X}}(\delta \text{Re}(e^{t\lambda}w), \mathfrak{M} - \phi) - K\delta^2 e^{2t\gamma_0}.$$

Assume $\lambda \in \mathbb{R}$. Then, we observe that the straight line $\mathbb{R} \ni \theta \mapsto \theta w$ is transverse to the tangent space $T_{\phi}\mathfrak{M}$ of the manifold \mathfrak{M} , since w is an eigenvector of $\mathcal{J}\mathcal{L}$ for $\lambda \neq 0$, hence does not belong to the kernel of \mathcal{L} . Therefore, $\text{dist}_{\mathcal{X}}(\theta w, \mathfrak{M} - \phi) \geq |\theta|/K_1$ for small $|\theta|$. Thus,

$$\text{dist}_{\mathcal{X}}(u(t), \mathfrak{M}) \geq \frac{1}{K_1} \delta e^{t\lambda} - K\delta^2 e^{2t\gamma_0}.$$

If $\lambda \in \mathbb{C} \setminus \mathbb{R}$, the equation $[\mathcal{J}\mathcal{L}]_{\mathbb{C}}(w) = \lambda w$ splits as $\mathcal{J}\mathcal{L}(\text{Re } w) = \text{Re}(\lambda)\text{Re } w - \text{Im}(\lambda)\text{Im } w$ and $\mathcal{J}\mathcal{L}(\text{Im } w) = \text{Im}(\lambda)\text{Re } w + \text{Re}(\lambda)\text{Im } w$. Therefore, $\text{Re } w$ and $\text{Im } w$ do not belong to $\ker(\mathcal{L})$. Consequently, the surface $\mathbb{C} \ni \theta \mapsto \text{Re}(\theta w)$ is transverse to the tangent space $T_{\phi}\mathfrak{M}$ of the manifold \mathfrak{M} , and we conclude as before that

$$\text{dist}_{\mathcal{X}}(u(t), \mathfrak{M}) \geq \frac{1}{K_1} \delta e^{t\gamma_0} - K\delta^2 e^{2t\gamma_0}.$$

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