Semigroup estimates and stability/instability results for the linearized three waves interaction equations

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Abstract

We consider the three waves interaction system and its linearization (the "pump-wave approximation"). We give some estimates on the semigroup as well as stability or instability results for the linearized problem in suitable norms. We work in the whole space and with periodic boundary condition, and our analysis relies on energy estimates and not on the complete integrability of the system.

Key-words: three waves interaction equations, pump-wave approximation, stability, eigenvalue, semigroup.
MSC (2010): 35Q99, 35F05, 35F20.

1 The three waves interaction equations

In this paper, we consider the three waves interaction system in dimension d

$$\begin{cases} \partial_t A_1 + \mathrm{v}_1 \cdot \nabla A_1 + i\sigma_1 \bar{A}_2 \bar{A}_3 = 0 \\ \partial_t A_2 + \mathrm{v}_2 \cdot \nabla A_2 + i\sigma_2 \bar{A}_3 \bar{A}_1 = 0 \\ \partial_t A_3 + \mathrm{v}_3 \cdot \nabla A_3 + i\sigma_3 \bar{A}_1 \bar{A}_2 = 0. \end{cases} \tag{TWI}$$

The amplitudes A_ℓ are complex-valued, the speeds v_ℓ are fixed vectors in $\mathbb{R}^d$, and the coupling constants σ_ℓ are real-valued. To take into account a domain with periodic condition or in the whole space, the system (TWI) will be posed in

$$\Omega = \Omega_\Lambda \equiv \prod_{j=1}^{d} \mathbb{R}/(2\pi\Lambda_j\mathbb{Z}),$$

with $\Lambda = (\Lambda_j)_{1\leq j\leq d} \in (0, +\infty]^d$ and the convention that $\mathbb{R}/(2\pi\Lambda_j\mathbb{Z}) = \mathbb{R}$ if $\Lambda_j = +\infty$.

The system (TWI) appears in various areas of physics: plasma physics, fluid mechanics, optics, acoustics, mechanical or electrical oscillations... We refer, for instance, to the book [6] and to the paper [8]. One classical way to derive the system (TWI) is as follows. Consider a scalar wave equation with quadratic nonlinearity

$$\partial_t u + \mathcal{Q}u + \mathcal{R}(u^2) = 0 \tag{1}$$

(for example the KdV equation, the KP equation... but this can be extended to systems). The operators $\mathcal{Q}$ and $\mathcal{R}$ are pseudo-differential operators of symbol $iQ = iQ(k) \in i\mathbb{R}$ and $iR = iR(k) \in i\mathbb{R}$ respectively, and the dispersion relation for the linearized equation is $\omega = Q(k)$. If some wave numbers (k_1, k_2, k_3) form a resonant triad, *i.e.*

$$k_1 + k_2 + k_3 = 0, \qquad \omega_1 + \omega_2 + \omega_3 = 0, \qquad \text{with} \qquad \omega_\ell \equiv Q(k_\ell), \qquad \ell = 1, 2, 3,$$

we can search for a solution $u = u(\mathrm{t}, \mathrm{x})$ of the wave equation (1) which is the sum of three wave packets of small amplitude with these resonant wave numbers (k_1, k_2, k_3) and slowly modulated. More precisely, u is assumed to have an expansion of the form

$$u(\mathrm{t}, \mathrm{x}) = \varepsilon \sum_{\ell=1}^{3} A_\ell(t = \varepsilon \mathrm{t}, x = \varepsilon \mathrm{x}) \mathrm{e}^{i(k_\ell \cdot \mathrm{x} - \omega_\ell \mathrm{t})} + c.c. + \mathcal{O}(\varepsilon^2).$$

Notice that the lengthscale of the modulation is of the same order as the typical packet amplitude. Collecting the terms of order ε^2 and cancelling out the secular terms, we infer that, on a formal level, A_1, A_2, A_3 satisfy (on the scales $(t = \varepsilon \mathrm{t}, x = \varepsilon \mathrm{x})$), the system (TWI), with $\mathrm{v}_\ell = \nabla Q(k_\ell)$ (the velocity group for the wave number k_ℓ) and (real-valued) coefficients $\sigma_\ell = R(k_\ell)$ depending on the operator $\mathcal{R}$. In optics, for instance, the time coordinate can be a space coordinate.

For the system (TWI), the so-called Manley-Rowe relations give conserved quantities:

$$\frac{d}{dt} \int_\Omega \sigma_3 |A_2|^2 - \sigma_2 |A_3|^2 \, dx = \frac{d}{dt} \int_\Omega \sigma_1 |A_3|^2 - \sigma_3 |A_1|^2 \, dx = \frac{d}{dt} \int_\Omega \sigma_2 |A_1|^2 - \sigma_1 |A_2|^2 \, dx = 0. \tag{2}$$

Actually, a Lax pair can be found for the system (TWI) (see [13] for $d = 1$, [12] for $d = 2$ and [14] for $d = 3$) thus is a completely integrable system for which inverse scattering can be used. Therefore, there exists an infinite sequence of (formally) conserved quantities (see [1], [7]). Notice however that they do not give H^s bounds uniform in time (as for the KdV equation for instance). Furthermore, explicit soliton solutions can be computed, and blow-up can occur in finite time (under some conditions on the parameters, see [13]). One can also focus on solutions depending only on time or on space (we get a system of ODE's). In order to use inverse scattering, we have to work with data and solutions sufficiently smooth and strongly localized in space. In the physical literature, characteristic functions are of frequent use. The inverse scattering has been well developed when Ω is the whole space $\mathbb{R}^d$, but is always more difficult in the periodic setting. The paper [3] deals rigorously with the inverse scattering problem in the Schwartz space on the line. It is restricted to "generic data", but this fact has been removed in [15]. In general, it is not always clear in which functional spaces the data and the solution are, since the solution obtained by inverse scattering can lie in a space larger than the one used for the initial data. The paper [16] clarifies this point for data in Sobolev spaces with weight $W^{s,2}_m$ defined below. In this paper, we shall work in L^p and Sobolev spaces and shall not use inverse scattering techniques.

1.1 A well-posedness result

We denote $J \equiv \{1 \le j \le d,\ \Lambda_j = +\infty\}$ and, for $x \in \mathbb{R}^d$, $x^J \equiv (x_j)_{j \in J} \in \mathbb{R}^J \subset \mathbb{R}^d$. We define for s, $m \in \mathbb{N}$ and $1 \le p \le \infty$ the Sobolev space with weight

$$W^{s,p}_m(\Omega) \equiv \Big\{ f \in W^{s,p}(\Omega),\ (1 + |x^J|^2)^{m/2} f \in L^p(\Omega) \Big\},$$

equipped with the natural norm

$$\|f\|_{W^{s,p}_m(\Omega)} \equiv \|f\|_{W^{s,p}(\Omega)} + \|(1 + |x^J|^2)^{m/2} f\|_{L^p(\Omega)}.$$

The space $W^{s,p}_0$ is then the standard Sobolev space $W^{s,p}$. For $s > d/p$, $W^{s,p}_m(\Omega)$ is an algebra, and

$$\forall f, g \in W^{s,p}_m(\Omega), \qquad \|fg\|_{W^{s,p}_m(\Omega)} \le C(s, p, \Omega) \|f\|_{W^{s,p}_m(\Omega)} \|g\|_{W^{s,p}_m(\Omega)}. \tag{3}$$

Furthermore, it follows from interpolation theory (see [10]) and Sobolev embedding that if $k \in \mathbb{N}$, $s > k + d/p$ and $f \in W^{s,p}_m(\Omega)$, then $(1 + |x^J|^\mu) \nabla^k f \in \mathcal{C}_b(\Omega)$ for $0 \le \mu < m(1 - d/(p(s-k)))$. As a consequence, one has

$$\bigcap_{s, m \in \mathbb{N}} W^{s,p}_m(\Omega) = \mathscr{S}(\Omega), \tag{4}$$

where the Schwartz space $\mathscr{S}(\Omega)$ is defined as the set of complex valued functions f, smooth and such that for any $\alpha \in \mathbb{N}^d$ and $\mu \in \mathbb{N}$,

$$(1 + |x^J|^\mu) \partial_x^\alpha f \in \mathcal{C}_b(\Omega),$$

endowed with the usual topology. The intersection (4) has to be understood with the induced topology.

The Cauchy problem for (TWI) is locally well-posed in the spaces $W^{s,p}_m(\Omega)$ as well as in the Schwartz space, where inverse scattering can be used.

Proposition 1 *We fix $s \in \mathbb{N}$, $1 \le p \le \infty$ and $m \in \mathbb{N}$, with $s > d/p$. Then, for any initial data $A^{\rm in} = (A_1^{\rm in}, A_2^{\rm in}, A_3^{\rm in}) \in W_m^{s,p}(\Omega)$, there exists a unique maximal solution $A = (A_1, A_2, A_3) \in \mathcal{C}([0,T^*), W_m^{s,p}(\Omega))$ to* (TWI). *Moreover, T^* does not depend on s provided $s > d/p$. Finally, for some constant $c > 0$ depending on p, Ω and $({\rm v}_\ell, \sigma_\ell)_{1\le\ell\le3}$, there holds*

$$T^* \ge \frac{c}{\|A^{\rm in}\|_{W_m^{s,p}(\Omega)}}.$$

Consequently, the Cauchy problem for (TWI) *is locally well-posed in the Schwartz space $\mathscr{S}(\Omega)$.*

Proof. We just write (TWI) under the form

$$A(t,x) = (A_\ell(t,x))_{1\le\ell\le3} = \Big(A_\ell^{\rm in}(x - t{\rm v}_\ell) - i\sigma_\ell \int_0^t \bar{A}_{\ell'}\bar{A}_{\ell''}(\tau, x - (t-\tau){\rm v}_\ell)\, d\tau\Big)_{1\le\ell\le3},$$

where $\{\ell, \ell', \ell''\} = \{1,2,3\}$. Using the fact that $W_m^{s,p}(\Omega)$ is an algebra, the easy estimate

$$\Big\| \int_0^t \bar{A}_{\ell'}\bar{A}_{\ell''}(\tau, x - (t-\tau){\rm v}_\ell)\, d\tau \Big\|_{W_m^{s,p}(\Omega)} \le Ct \sup_{0\le\tau\le t} \|A(\tau)\|^2_{W_m^{s,p}(\Omega)}$$

allows to prove by a standard fixed point argument the local well-posedness of (TWI), with the lower bound for T^*. Since T^* does not depend on $s > d/2$, the result in the Schwartz space follows directly from (4). ▲

If we want to work with data having a certain decay at infinity given by some weight $\varpi : \mathbb{R}^J \to \mathbb{R}_+^*$ (for instance an exponential decay and $\varpi(y) = {\rm e}^{\gamma|y|}$ for $y \in \mathbb{R}^J$ and some fixed $\gamma > 0$), we may define

$$W_\varpi^{s,p}(\Omega) \equiv \Big\{ f \in W^{s,p}(\Omega),\ \varpi(x^J) f \in L^p(\Omega) \Big\},$$

endowed with the natural norm, which is also an algebra (satisfying an estimate like (3)) for $s > d/p$. The above result also holds for $W_\varpi^{s,p}(\Omega)$ (and if $f \in W_\varpi^{s,p}(\Omega)$, $s > k + d/p$, with s, $k \in \mathbb{N}$, then $\varpi^\mu \nabla^k f \in \mathcal{C}_b(\Omega)$ if $0 \le \mu < 1 - d/(p(s-k))$).

1.2 The linearized problem and stability/instability results by inverse scattering

We shall focus in the sequel on the linearized problem when A_1 and A_2 are small compared to A_3, that is

$$\begin{cases} \partial_t A_1 + {\rm v}_1 \cdot \nabla A_1 + i\sigma_1 \bar{A}_2 \bar{A}_3 = 0 \\ \partial_t A_2 + {\rm v}_2 \cdot \nabla A_2 + i\sigma_2 \bar{A}_3 \bar{A}_1 = 0 \\ \partial_t A_3 + {\rm v}_3 \cdot \nabla A_3 = 0. \end{cases}$$

Working in the frame moving with the speed ${\rm v}_3$, replacing A_2 by $\bar{A}_2$ and $i\bar{A}_3$ by A_3, we are led to

$$\begin{cases} \partial_t A_1 + \upsilon_1 \cdot \nabla A_1 + \sigma_1 A_3 A_2 = 0 \\ \partial_t A_2 + \upsilon_2 \cdot \nabla A_2 + \sigma_2 \bar{A}_3 A_1 = 0, \end{cases} \tag{5}$$

where $(\upsilon_1, \upsilon_2) \equiv ({\rm v}_1 - {\rm v}_3, {\rm v}_2 - {\rm v}_3)$ and where $A_3 = A_3(x)$ is a given function independent of time. The datum A_3 is called the pump-wave and (5) the pump-wave approximation. For the system (5), the Manley-Rowe identity (2) becomes

$$\frac{d}{dt} \int_\Omega \sigma_2 |A_1|^2 - \sigma_1 |A_2|^2 \, dx = 0. \tag{6}$$

This suggests that the solution $(A_1, A_2) = (0,0)$ is stable for (5) if $\sigma_1\sigma_2 < 0$, which is what is expected from inverse scattering (see [6], [8]). Note that the inverse scattering problem for the linearized system (5) is

$$\begin{cases} i\lambda\psi_1 + \upsilon_1 \cdot \nabla\psi_1 + \sigma_1 \psi_2 A_3 = 0 \\ i\lambda\psi_2 + \upsilon_2 \cdot \nabla\psi_2 + \sigma_2 \psi_1 \bar{A}_3 = 0, \end{cases} \tag{7}$$

which is exactly the system we obtain when applying Fourier transform in time for (5), with λ the dual variable of t (hence expected real). Here, we consider only bounded eigenvectors. For the inverse scattering point of view, (forward in time) stability for (5) means that (7) has no nontrivial solution with $\mathrm{Im}(\lambda) < 0$ and (forward in time) instability that (7) has at least one nontrivial solution with $\mathrm{Im}(\lambda) < 0$. For $\sigma_1\sigma_2 > 0$, the situation is more delicate. In dimension $d = 1$, with $\Omega = \mathbb{R}$ at least, it is expected (see [8], [6]) that when

$$v_1 v_2 < 0,$$

that is if among the original speeds v_1, v_2, v_3, A_3 is associated to the middle one, then if one takes A_3 as the soliton of (TWI), there is linear instability for (5). This instability leads, for the original system (TWI), to the creation of two solitons, with respective speeds v_1 and v_2 whereas the soliton A_3 disappears (this is called "decay instability"). This result of soliton exchange interaction is classical (see [8], [4]), and is supported by experiments and numerical simulation (see [2]). We would like to point out that the unstable eigenvalues appear in the scattering problem as zeros or poles of some holomorphic functions, and come from numerical studies or asymptotic expansions in some particular regimes. More generally, the decay instability happens each time A_3 is not too small so that the two solitons can appear. The quantities

$$\sqrt{\frac{\sigma_1\sigma_2}{|v_1 v_2|}} \times \Big|\int_{\mathbb{R}} A_3 \, dx\Big| \qquad \text{and} \qquad \sqrt{\frac{\sigma_1\sigma_2}{|v_1 v_2|}} \int_{\mathbb{R}} |A_3| \, dx,$$

called the *area* and the *absolute area*, are proportional to an upper bound for the possible number of solitons contained in the solution of (TWI) with initial data $(A_1 \approx 0, A_2 \approx 0, A_3)$. Thus, a smallness hypothesis

$$\sqrt{\frac{\sigma_1\sigma_2}{|v_1 v_2|}} \int_{\mathbb{R}} |A_3| \, dx < c, \tag{8}$$

for some absolute positive constant c, prevents the formation of solitons. In [4] and [8], one can find various sharp or sufficient conditions (when $\Omega = \mathbb{R}$) on stability in the case $\sigma_1\sigma_2 > 0$ and $v_1 v_2 < 0$ (they depend on the specific regime of interest), for instance $c = \pi/2$ is necessary and sufficient (for a real-valued function A_3, which never vanishes) in the WKB approximation, that is for a slowly varying amplitude A_3, and in the general case $c \simeq 0.903$ is sufficient. In the case

$$v_1 v_2 > 0,$$

this phenomenon can not occur: solitons can not be created in A_1 or A_2. However, the question of stability for (5) if $v_1 v_2 > 0$ is then not very clear: the absence of solitons only means (by inverse scattering) that (5) has no eigenvalue in the half-space $\{\mathrm{Im} < 0\}$. Let us emphasize once again that all these results hold for strongly localized in space data (notice that the condition (8) requires $A_3 \in L^1(\mathbb{R})$ for example).

As already mentioned, the inverse scattering point of view give precise qualitative information, but the absence of clear functional spaces makes the statements of stability/instability results quite delicate. For the same reason, they do not give estimates for the growth of solutions in the unstable cases. Furthermore, the inverse scattering approach is not adapted to perturbation. Our aim in this paper is to determine, in each case, an L^p or Sobolev norm in which stability or instability occurs for (5). For a given equation like (1), where the (TWI) system arises through asymptotic expansions, the resonant set is often compact (*i.e.* the resonant triads are bounded). Therefore, we hope that our results can help for stating and proving stability/instability results (in an adapted L^p type space) for this original problem without inverse scattering. Indeed, since the functions involved in (TWI) will have only low frequencies, a control in any (reasonable) norm (like L^1, L^2 or L^∞ norms) is acceptable.

1.3 Statement of the results

We first note that if s, $m \in \mathbb{N}$ are given, the initial value problem (5) is well-posed in the spaces $W^{s,p}_m(\Omega)$ provided $A_3 \in W^{s,\infty}(\Omega)$. In what follows, $\Sigma_t A^{\mathrm{in}}$ will denote the solution $A = (A_1, A_2)$ at time $t \geq 0$ of (5) with initial datum $A^{\mathrm{in}} = (A_1^{\mathrm{in}}, A_2^{\mathrm{in}})$. The space $L^p(\Omega, \mathbb{C}^2)$ is endowed with the norm $\|F = (F_1, F_2)\|_{L^p} \equiv \max(\|F_1\|_{L^p}, \|F_2\|_{L^p})$.

The results for (5) below depend whether the map A_3 has only Sobolev regularity, that is $A_3 \in W^{s,p}(\Omega)$, which is a first natural context, or A_3 is strongly localized in space ($A_3 \in W^{s,p}_m(\Omega)$ with m sufficiently large), which is natural in order to use inverse scattering. The results below concern different cases which are far

from being disjoint. It will be convenient in the following to distinguish the two cases
(C_{per}) $\Lambda_j < \infty$ for any $1 \leq j \leq d$ such that $(\upsilon_1)_j \neq 0$ or $(\upsilon_2)_j \neq 0$,
(C_{inf}) $\Lambda_j = \infty$ for any $1 \leq j \leq d$ such that $(\upsilon_1)_j \neq 0$ or $(\upsilon_2)_j \neq 0$,
which correspond to cases where we have periodic conditions all or none of the directions of $\mathrm{Span}(\upsilon_1, \upsilon_2)$.

The case $\sigma_1\sigma_2 = 0$ as well as the cases $\upsilon_1 = 0$ and/or $\upsilon_2 = 0$ are degenerate, and will be studied in the end of the section. When $\upsilon_1 = \upsilon_2$, only one speed of propagation is involved and explicit computations can be carried out. Moreover, if $A_3 \equiv 0$, then Σ_t is trivially unitary in all the spaces $W^{s,p}(\Omega)$. The proofs are given in the next section.

Localized data remain localized. First, we state a result when A_3 and the initial data $A^{\mathrm{in}} = (A_1^{\mathrm{in}}, A_2^{\mathrm{in}})$ are sufficiently localized in space, then the solution (A_1, A_2) of (5) remains localized in space for positive times. However, since we work for large times, we have to take into account the transport terms in (5) with speeds υ_1 and υ_2. Therefore, we denote

$$\Gamma \equiv \{\alpha_1 \upsilon_1^J + \alpha_2 \upsilon_2^J, \quad \alpha_1 \geq 0, \quad \alpha_2 \geq 0, \quad \alpha_1 + \alpha_2 \leq 1\}$$

the convex hull of 0, υ_1^J, υ_2^J and in $\mathbb{R}^J$; and set for $t \geq 0$ and $x \in \Omega$

$$D_t(x) \equiv \mathrm{dist}(x^J, t\Gamma).$$

Then, we have

Proposition 2 *We assume $\upsilon_1 \neq 0$, $\upsilon_2 \neq 0$ and that, for some $\nu > 1$, A_3 verifies*

$$(1 + |x^J|^\nu) A_3 \in L^\infty(\Omega).$$

Then, there exists $C > 0$ and $R \geq 1$, depending only on σ_1, σ_2, υ_1, υ_2, ν and $\|(1 + |x^J|^\nu) A_3\|_\infty$, such that for any $\mu \geq 0$ and any initial data $A^{\mathrm{in}} = (A_1^{\mathrm{in}}, A_2^{\mathrm{in}})$ such that

$$(1 + |x^J|^\mu) A^{\mathrm{in}} \in L^\infty(\Omega),$$

there holds, for $t \geq 0$,

$$\|D_t^\mu(x) \Sigma_t A^{\mathrm{in}}\|_{L^\infty(\{D_t \geq R\})} \leq C \|(1 + |x^J|^\mu) A^{\mathrm{in}}\|_{L^\infty(\Omega)}.$$

The stable case $\boldsymbol{\sigma_1\sigma_2 < 0}$. From the conservation laws (6), it follows rather immediately:

Proposition 3 *We assume $\sigma_1\sigma_2 < 0$ and fix $s \in \mathbb{N}$. If $A_3 \in W^{s,\infty}(\Omega)$, then there exists $C > 0$, depending on A_3, s, σ_1, σ_2 (but not on υ_1 and υ_2) such that for $t \geq 0$,*

$$\left\| \Sigma_t \right\|_{\mathscr{L}_c(H^s(\Omega))} \leq C(1 + t^s).$$

For functions depending only on time (hence in the periodic setting and with A_3 constant), the term t^s is useless. However, it can not be removed in general, as shown by the simple example where $\upsilon_1 = \upsilon_2 = 0$ (recall that the constant C does not depend on the speeds υ_1 and υ_2), in which case (5) becomes

$$\left\{ \begin{array}{l} \partial_t A_1 + \sigma_1 A_2 A_3 = 0 \\ \\ \partial_t A_2 + \sigma_2 \bar{A}_3 A_1 = 0, \end{array} \right. \tag{9}$$

thus

$$A_1(t,x) = \alpha(x) \cos(t\sqrt{|\sigma_1\sigma_2|}\, |A_3|(x)) + \beta(x) \sin(t\sqrt{|\sigma_1\sigma_2|}\, |A_3|(x))$$

and the estimate of Proposition 3 is optimal. Nevertheless, the case $\upsilon_1 = \upsilon_2 = 0$ is very degenerate, since all the speeds are equal. However, in dimension $d \geq 3$, we can argue analogously in case (C_{per}) with functions (A_1, A_2) depending only on coordinates orthogonal to $\mathrm{Span}(\upsilon_1, \upsilon_2)$ (and compactly supported) to deduce the optimality of the estimate in Proposition 3. This is also possible if $d = 2$ and the speeds υ_1, υ_2 are collinear (still in case (C_{per})). The secular term t^s means that in general, oscillations in time are expected. We do not know whether the estimate of Proposition 3 is optimal in the other cases for $d \geq 2$. However, in the one dimensional case, one can always improve the above result by showing stability in all the spaces $H^s(\Omega)$, provided the speeds υ_1 and υ_2 are nonzero.

Proposition 4 *Let $d = 1$, fix $s \in \mathbb{N}$ and assume that $\sigma_1\sigma_2 < 0$, that $A_3 \in W^{s,\infty}(\Omega)$ and that $\upsilon_1 \neq 0$ and $\upsilon_2 \neq 0$. Then, there exists $C > 0$, depending on s, A_3, Ω, υ_1, υ_2, σ_1 and σ_2, such that, for $t \geq 0$,*

$$\left\|\Sigma_t\right\|_{\mathscr{L}_c(H^s(\Omega))} \leq C.$$

In the above Proposition, the constant C depends on A_3 through some eigenvalue problem. It is clear from these results that when $\sigma_1\sigma_2 < 0$, the scattering problem (7) has, as expected, no eigenvalue in $\{\mathrm{Im} < 0\}$ if we require the corresponding eigenvector to belong to $L^2(\Omega)$.

The case of small and localized data. We now consider strongly localized in space data, with a smallness assumption, in the spirit of (8). Then, stability is expected for (5). Note however that the result below is not restricted to the unstable situation $\sigma_1\sigma_2 > 0$. The hypothesis below on A_3 are natural when one writes (5) under the Duhamel form

$$\begin{cases} A_1(t,x) = A_1^{\mathrm{in}}(x - t\upsilon_1) - \sigma_1 \displaystyle\int_0^t A_2(\tau, x - (t-\tau)\upsilon_1) A_3(x - (t-\tau)\upsilon_1)\, d\tau \\ A_2(t,x) = A_2^{\mathrm{in}}(x - t\upsilon_2) - \sigma_2 \displaystyle\int_0^t A_1(\tau, x - (t-\tau)\upsilon_2) \bar{A}_3(x - (t-\tau)\upsilon_2)\, d\tau. \end{cases} \tag{10}$$

Proposition 5 *We assume $\upsilon_1 \neq 0$, $\upsilon_2 \neq 0$ and consider the case* ($\mathrm{C_{inf}}$)*. We fix $s \in \mathbb{N}$ and we assume that $A_3 \in \mathcal{C}_b^s(\Omega)$ is such that*

$$\forall \alpha \in \mathbb{N}^d, \quad |\alpha| \leq s, \qquad \sup_{x\in\Omega} \Big\{ \int_{x+\mathbb{R}\upsilon_1} |\partial_x^\alpha A_3| + \int_{x+\mathbb{R}\upsilon_2} |\partial_x^\alpha A_3| \Big\} < \infty$$

and satisfies the smallness assumption

$$\frac{|\sigma_1\sigma_2|}{|\upsilon_1|\cdot|\upsilon_2|} \Big(\sup_{x\in\Omega} \int_{x+\mathbb{R}\upsilon_1} |A_3| \Big) \Big(\sup_{x\in\Omega} \int_{x+\mathbb{R}\upsilon_2} |A_3| \Big) < 1. \tag{11}$$

Then, there exists some constant C depending only on s, σ_1, σ_2, υ_1, υ_2, A_3 and (11) *such that for $t \geq 0$,*

$$\left\|\Sigma_t\right\|_{\mathscr{L}_c(W^{s,\infty}(\Omega))} \leq C.$$

We may notice that in dimension $d = 1$, with $\sigma_1\sigma_2 > 0$ and $\upsilon_1\upsilon_2 < 0$, our smallness assumption (11) is

$$\sqrt{\frac{\sigma_1\sigma_2}{|\upsilon_1\upsilon_2|}} \int_{\mathbb{R}} |A_3|\, dx < 1,$$

which is an upper bound like (8) with $c = 1$, which is better than the (sufficient) value (in the general case) $c \simeq 0.903$, but worse than the "exact" value $c = \pi/2$ for the WKB approximation. Here again, the above result shows that (7) has no eigenvalue in $\{\mathrm{Im} < 0\}$ when A_3 is both localized and small. In dimension $d = 1$, (11) requires A_3 to be in L^1, and we would like to point out that for (1) (for (KdV) or (KP) equations, for instance), integrability conditions can be imposed on u in order to have finite *mass* $\displaystyle\int_{\mathbb{R}} u\, dx$ (which is often a formally conserved quantity).

The case $\boldsymbol{\sigma_1\sigma_2 > 0}$. In the case $\sigma_1\sigma_2 > 0$, we first give an estimate of exponential growth.

Proposition 6 *We assume $\sigma_1\sigma_2 > 0$, fix $s \in \mathbb{N}$ and let $A_3 \in W^{s,\infty}(\Omega)$, $A_3 \not\equiv 0$. We define*

$$\gamma \equiv \sqrt{\sigma_1\sigma_2}\|A_3\|_{L^\infty(\Omega)} > 0.$$

Then, for any $1 \leq p \leq \infty$, there exists $C > 0$, depending on A_3, p, s, σ_1 and σ_2 (but not on υ_1, υ_2) such that for any $t \geq 0$

$$\left\|\Sigma_t\right\|_{\mathscr{L}_c(W^{s,p}(\Omega))} \leq C(1 + t^s)\mathrm{e}^{\gamma t}.$$

We can notice (as in the analogous case discussed just after Proposition 3) that the polynomial growth in t^s can be achieved with either $\upsilon_1 = \upsilon_2 = 0$, either $d \geq 3$, either $d = 2$ and υ_1, υ_2 collinear.

In the case ($\mathrm{C_{per}}$), the estimate in Proposition 6 is almost optimal, as shown by the following

Proposition 7 *We assume $\sigma_1\sigma_2 > 0$ and consider the case* $(\mathrm{C}_{\mathrm{per}})$, *with $A_3 \in L^\infty(\Omega)$, $A_3 \not\equiv 0$, and let*

$$\gamma \equiv \sqrt{\sigma_1\sigma_2}\|A_3\|_{L^\infty(\Omega)} > 0.$$

(i) If $\Lambda_j < \infty$ for every $1 \le j \le d$ and A_3 is independent of x, then $-i\gamma \in i\mathbb{R}_-^* \subset \{\mathrm{Im} < 0\}$ is an eigenvalue for (7) *with multiplicity one among the functions constant in space.*
(ii) If $d > \dim \mathrm{Span}(\upsilon_1, \upsilon_2)$ and $A_3 \in \mathcal{C}_b(\Omega)$ depends only on the coordinates orthogonal to $\mathrm{Span}(\upsilon_1, \upsilon_2)$, *then for any $0 < \varepsilon < \gamma$, there exist a ball $B_{2r_\varepsilon}(y_\varepsilon)$ and initial data $A^{\mathrm{in}} = (A_1^{\mathrm{in}}, A_2^{\mathrm{in}}) \in \mathcal{C}_c^\infty(B_{2r_\varepsilon}(y_\varepsilon))$ such that $|A^{\mathrm{in}}|_\infty \le 1$ in $B_{2r_\varepsilon}(y_\varepsilon)$, $|A^{\mathrm{in}}|_\infty = 1$ in $B_{r_\varepsilon}(y_\varepsilon)$ and*

$$|\Sigma_t A^{\mathrm{in}}|_\infty \ge \mathsf{e}^{(\gamma-\varepsilon)t} \qquad \text{in} \quad B_{r_\varepsilon}(y_\varepsilon).$$

(iii) If $d > \dim \mathrm{Span}(\upsilon_1, \upsilon_2)$ and A_3 depends only on the coordinates orthogonal to $\mathrm{Span}(\upsilon_1, \upsilon_2)$ *and is such that, for some ball $B_{2r}(y)$, we have $|A_3|(x) = \|A_3\|_{L^\infty(\Omega)}$ for $x \in B_{2r}(y)$, then $-i\gamma \in i\mathbb{R}_-^* \subset \{\mathrm{Im} < 0\}$ is an eigenvalue for* (7) *with infinite multiplicity.*
(iv) If $1 \le p \le \infty$, $d = 1$, A_3 is real-valued and $\upsilon_1 = \upsilon_2 \ne 0$, then there exists some constant $C > 0$, depending on σ_1, σ_2 and $\upsilon_1 = \upsilon_2$ such that if $t \ge 0$, then

$$\|\Sigma_t\|_{\mathscr{L}_c(L^p(\Omega))} \le C \exp\left(t\left|\frac{\sqrt{\sigma_1\sigma_2}}{|\Omega|}\int_\Omega A_3(y)\ dy\right|\right).$$

Moreover, the eigenvalues for (7) *(with multiplicity) are the*

$$\frac{m\upsilon_1}{\Lambda} \pm i\left|\frac{\sqrt{\sigma_1\sigma_2}}{|\Omega|}\int_\Omega A_3(y)\ dy\right|, \qquad \text{for} \quad m \in \mathbb{Z}.$$

In particular, they are simple as soon as $\displaystyle\int_\Omega A_3(y)\ dy \ne 0.$

We notice that in case *(ii)*, there holds, for any $1 \le p \le \infty$ and $t \ge 0$,

$$\left\|\Sigma_t A^{\mathrm{in}}\right\|_{L^p(\Omega)} \ge C_p \left\|A^{\mathrm{in}}\right\|_{L^p(\Omega)} \mathsf{e}^{(\gamma-\varepsilon)t},$$

where the constant C_p depends only on p (and A_3, σ_1, σ_2 and d) and not on $0 < \varepsilon < \gamma$. Consequently,

$$\left\|\Sigma_t\right\|_{\mathscr{L}_c(L^p(\mathbb{R}))} \ge C_p \mathsf{e}^{(\gamma-\varepsilon)t}.$$

Furthermore, for *(ii)* and *(iii)*, if $d = \dim \mathrm{Span}(\upsilon_1, \upsilon_2)$, then we are in case *(i)*.

We now turn to the case $(\mathrm{C}_{\mathrm{inf}})$. By Proposition 5, we now that stability holds in $W^{s,\infty}$ if A_3 is suitably small. In dimension $d = 1$ and without smallness assumption on A_3, we can show that (7) may have some eigenvalues in $\{\mathrm{Im} < 0\}$.

Proposition 8 *We assume $\Omega = \mathbb{R}$ ($d = 1$, $\Lambda_1 = +\infty$), $\sigma_1\sigma_2 > 0$ and $\upsilon_1\upsilon_2 < 0$ (we are then in case* $(\mathrm{C}_{\mathrm{inf}})$*). For $A_3 \in L^\infty(\mathbb{R})$, we denote*

$$\gamma_* \equiv 2\sqrt{\sigma_1\sigma_2}\frac{\sqrt{|\upsilon_1\upsilon_2|}}{|\upsilon_1 - \upsilon_2|}\|A_3\|_{L^\infty(\mathbb{R})} \in \left(0, \sqrt{\sigma_1\sigma_2}\|A_3\|_{L^\infty(\mathbb{R})}\right].$$

(i) Assume $A_3 = \mathcal{A}_3 \mathbf{1}_{[a,b]}$, with $\mathcal{A}_3 \in \mathbb{C}^*$ and $-\infty < a < b < +\infty$. Then, if

$$\frac{\sqrt{\sigma_1\sigma_2}(b-a)|\mathcal{A}_3|}{\sqrt{|\upsilon_1\upsilon_2|}} = \frac{\sqrt{\sigma_1\sigma_2}}{\sqrt{|\upsilon_1\upsilon_2|}}\left|\int_\mathbb{R} A_3(x)\ dx\right| \le \pi/2,$$

(7) *has no eigenvalue in $\{\mathrm{Im} < 0\}$, and if $m \in \mathbb{N}^*$ is such that*

$$\frac{(2m-1)\pi}{2} < \frac{\sqrt{\sigma_1\sigma_2}(b-a)|\mathcal{A}_3|}{\sqrt{|\upsilon_1\upsilon_2|}} = \frac{\sqrt{\sigma_1\sigma_2}}{\sqrt{|\upsilon_1\upsilon_2|}}\left|\int_\mathbb{R} A_3(x)\ dx\right| \le \frac{(2m+1)\pi}{2},$$

then (7) *has exactly* m *eigenvalues in* $\{\text{Im} < 0\}$, *which are simple and lie in* $(0, -i\gamma_*)$.
(ii) We assume that $A_3 \in L^1 \cap L^\infty(\mathbb{R})$ *is real-valued and satisfies*

$$\sqrt{\frac{\sigma_1\sigma_2}{|\upsilon_1\upsilon_2|}} \times \Big| \int_{\mathbb{R}} A_3(x)\ dx \Big| > \frac{\pi}{2}.$$

Then, there exists at least one eigenvalue $\lambda_0 \in i\mathbb{R}^*_- \subset \{\text{Im} < 0\}$ *for* (7), *with corresponding eigenvector decaying exponentially fast at infinity. Hence, for any* $1 \leq p \leq \infty$, *instability holds for* (5) *in* $L^p(\mathbb{R})$.
(iii) If $A_3 \in L^1 \cap L^\infty(\mathbb{R})$, *then there exists a constant* C, *depending on* σ_1, σ_2, υ_1, υ_2 *and* A_3, *such that, for any* $t \geq 0$,

$$\big\|\Sigma_t\big\|_{\mathscr{L}_c(L^\infty(\mathbb{R}))} \leq C\mathrm{e}^{\gamma_* t}.$$

Remark 1 In (i), by an appropriate choice of the area of A_3, we can make the eigenvalues as close as we want to $-i\gamma_*$, so that the estimate in (iii) is (in general) optimal. We emphasize that, comparing with Proposition 7, the optimal rate γ_* in (iii) does depend on the speeds υ_1 and υ_2. In (iii), if $\upsilon_1 + \upsilon_2 = 0$, then $2\dfrac{\sqrt{|\upsilon_1\upsilon_2|}}{|\upsilon_1 - \upsilon_2|} = 1$, thus $\gamma = \gamma_*$ and Proposition 6 gives the same estimate in all $L^p(\mathbb{R})$ spaces without assuming $A_3 \in L^1(\mathbb{R})$. The case (ii) is consistent with the (necessary and) sufficient condition (for real-valued everywhere nonzero functions A_3) in the WKB approximation $c = \pi/2$ (however, on the one hand, we work neither in a WKB approximation, neither with a constraint on the constant sign of A_3; and on the other hand, in the WKB approximation, the condition is also sufficient). In any case, if $d = 1$, $\sigma_1\sigma_2 > 0$ and $\upsilon_1\upsilon_2 < 0$, the soliton is linearly unstable since it is in $L^1(\mathbb{R})$ and has area equal to $\pi > \pi/2$. Let us mention that we have no result corresponding to (ii) when A_3 is complex-valued. Of course, the results of Propositions 5 and 8 leave open, when $d = 1$ and $\sigma_1\sigma_2 > 0 > \upsilon_1\upsilon_2$, the case where A_3 is real-valued with

$$\sqrt{\frac{\sigma_1\sigma_2}{|\upsilon_1\upsilon_2|}} \times \int_{\mathbb{R}} |A_3|(x)\ dx \geq 1 \qquad \text{but} \qquad \sqrt{\frac{\sigma_1\sigma_2}{|\upsilon_1\upsilon_2|}} \times \Big| \int_{\mathbb{R}} A_3(x)\ dx \Big| \leq \frac{\pi}{2}.$$

A numerical example is given in [8] where A_3 is the characteristic function of an interval with corresponding area between 1 and $\pi/2$: for (TWI), there is instability and blow-up in finite time (this suggests instability for (5), but (i) shows that there is no eigenvalue for (7) in $\{\text{Im} < 0\}$). We have not been able to prove any estimate for the solution of (5) better than the exponential growth given in (iii) when restricting ourselves to energy methods. Even though we could locate precisely the eigenvalues in $\{\text{Im} < 0\}$, using this information to estimate the semigroup seems a difficult task.

In the case $\Omega = \mathbb{R}$ and $\upsilon_1\upsilon_2 > 0$, we do not expect eigenvalues in $\{\text{Im} < 0\}$ for (7). However, the stability for (5) is linked to some integrability conditions on A_3.

Proposition 9 *We assume* $\Omega = \mathbb{R}$ *(*$d = 1$, $\Lambda_1 = +\infty$*),* $\sigma_1\sigma_2 > 0$ *and* $\upsilon_1\upsilon_2 > 0$ *(we are then in case* $(\mathrm{C_{inf}})$*).*
(i) We assume $A_3 \in L^1_{loc}(\mathbb{R})$ *real-valued,* $\upsilon_1 = \upsilon_2 \neq 0$, *and fix* $1 \leq p \leq \infty$. *Then stability holds in* $L^p(\mathbb{R})$ *for* (5) *if and only if*

$$\sup_{-\infty < a < b < +\infty} \Big| \int_a^b A_3(y)\ dy \Big| < \infty, \tag{12}$$

in which case there exists C_p, *depending only on* p, σ_1 *and* σ_2, *such that for any* $t \geq 0$,

$$\big\|\Sigma_t\big\|_{\mathscr{L}_c(L^p(\mathbb{R}))} \leq C_p \exp\Big(\frac{\sqrt{\sigma_1\sigma_2}}{|\upsilon_1|} \sup_{-\infty < a < b < +\infty} \Big| \int_a^b A_3(y)\ dy \Big| \Big).$$

If (12) *is not true, then* $\big\|\Sigma_t\big\|_{\mathscr{L}_c(L^p(\mathbb{R}))} \to +\infty$ *as* $t \to +\infty$ *and there exists* $C_p > 1$, *depending only on* p, σ_1 *and* σ_2, *such that for any* $t \geq 0$,

$$\frac{1}{C_p} \exp\Big(\frac{\sqrt{\sigma_1\sigma_2}}{|\upsilon_1|} \sup_{x \in \mathbb{R}} \Big| \int_{x - t|\upsilon_1|}^x A_3(y)\ dy \Big| \Big) \leq \big\|\Sigma_t\big\|_{\mathscr{L}_c(L^p(\mathbb{R}))} \leq C_p \exp\Big(\frac{\sqrt{\sigma_1\sigma_2}}{|\upsilon_1|} \sup_{x \in \mathbb{R}} \Big| \int_{x - t|\upsilon_1|}^x A_3(y)\ dy \Big| \Big).$$

(ii) There exists a constant C, *depending only on* σ_1, σ_2, υ_1 *and* υ_2 *such that, if* $A_3 \in L^1_{loc}(\mathbb{R})$ *and* $t \geq 0$, *we have*

$$\big\|\Sigma_t\big\|_{\mathscr{L}_c(\mathcal{C}_b(\mathbb{R}))} \leq C \exp\Big(\sqrt{\frac{\sigma_1\sigma_2}{\upsilon_1\upsilon_2}} \sup_{x \in \mathbb{R}} \int_{x - t\max(|\upsilon_1|, |\upsilon_2|)}^x |A_3|(y)\ dy \Big).$$

In particular, if $A_3 \in L^1(\mathbb{R})$, then stability holds in $\mathcal{C}_b(\mathbb{R})$ for (5) *and, for $t \geq 0$,*

$$\|\Sigma_t\|_{\mathscr{L}_c(\mathcal{C}_b(\mathbb{R}))} \leq C \exp\Big(\sqrt{\frac{\sigma_1\sigma_2}{\upsilon_1\upsilon_2}}\|A_3\|_{L^1(\mathbb{R})}\Big).$$

(iii) We assume[1] *that $\upsilon_1 > 0$ and $\upsilon_2 > 0$, that $A_3 \in \mathcal{C}(\mathbb{R})$ is real-valued, $A_3 \notin L^1(\mathbb{R})$, and satisfies*[2]

$$A_3 = 0 \quad \text{in } [1, +\infty), \qquad -\sigma_1 A_3 \geq 0 \quad \text{in } \mathbb{R}, \qquad -\sigma_1 A_3 \text{ is nondecreasing in } \mathbb{R}^-. \tag{13}$$

Then, $\|\Sigma_t\|_{\mathscr{L}_c(\mathcal{C}_b(\mathbb{R}))} \to +\infty$ as $t \to +\infty$. More precisely, there exists a constant $c > 1$ depending only on σ_1, σ_2, υ_1 and υ_2 such that, for $t \geq 0$,

$$\|\Sigma_t\|_{\mathscr{L}_c(L^\infty(\mathbb{R}))} \geq c \exp\Big(\frac{\min(\upsilon_1,\upsilon_2)}{\max(\upsilon_1,\upsilon_2)}\sqrt{\frac{\sigma_1\sigma_2}{\upsilon_1\upsilon_2}} \sup_{x\in\mathbb{R}}\Big|\int_{x-t\max(\upsilon_1,\upsilon_2)}^{x} A_3(y)\, dy\Big|\Big).$$

Remark 2 In view of the right-hand side part of the estimate in *(ii)*, problem (7) has no eigenvalue in $\{\mathrm{Im} < 0\}$ (for eigenvectors in $L^\infty(\mathbb{R})$) if $A_3 \in L^2(\mathbb{R})$. However, instability can occur for (5).

In [9] (section 4), the question of the stability for (5) was studied (for $d = 1$) in the case $\upsilon_1 = \upsilon_2$: stability was suggested when $A_3 \in L^1(\mathbb{R})$ and instability was shown by an explicit example for some $A_3 \notin L^1(\mathbb{R})$. This led G. Schneider to the conjecture that stability holds for (5) (if $d = 1$ and $\sigma_1\sigma_2 > 0$) for sufficiently localized functions A_3 (at least in $L^1(\mathbb{R})$) and instability occurs for nonlocalized functions A_3. As we see from the results obtained by inverse scattering (subsection 1.2) and the above results, the situation is more complex.

If $\sigma_1\sigma_2 > 0$ and $\upsilon_1 = \upsilon_2 = 0$, (5) reduces to (9), with solutions that can be written explicitly:

$$\Sigma_t = \exp\Big(-t\begin{pmatrix} 0 & \sigma_1 A_3(x) \\ \sigma_2 \bar{A}_3(x) & 0 \end{pmatrix}\Big),$$

and the matrix Σ_t has eigenvalues $\exp(\pm t\sqrt{\sigma_1\sigma_2}|A_3|(x))$ hence, the estimates for Σ_t are as in Proposition 7. If $A_3 \in \mathcal{C}^2(\Omega)$, notice that in this simple case, $-i\sqrt{\sigma_1\sigma_2}\|A_3\|_{L^\infty(\Omega)} \in \{\mathrm{Im} < 0\}$ is not an eigenvalue for (7) as soon as $\|A_3\|_{L^\infty(\Omega)}$ is a non-degenerate critical value of $|A_3|$. In dimension $d = 1$, it remains to study the degenerate case where one speed υ_1 or υ_2 is zero. Possibly changing x for $-x$ and exchanging the indices 1 and 2, we do not loose generality assuming $\upsilon_1 = 0 < \upsilon_2$.

Proposition 10 *We assume $\Omega = \mathbb{R}$ ($d = 1$, $\Lambda_1 = +\infty$), $\sigma_1\sigma_2 > 0$ and $\upsilon_1 = 0 < \upsilon_2$ (we are then in case* ($\mathrm{C_{inf}}$)*).*
(i) Assume that $A_3 \in L^\infty_{loc}(\mathbb{R})$ is such that $|A_3| \geq \mathcal{A}_3 > 0$ on an interval I of length $\ell > 0$. Then, for any $0 < \varepsilon < \ell$, there exists a constant $c_\varepsilon > 0$, depending on ε and ℓ, such that

$$\|\Sigma_t\|_{\mathscr{L}_c(L^\infty(\mathbb{R}))} \geq c_\varepsilon \exp\Big(\sqrt{\frac{\sigma_1\sigma_2}{\upsilon_2}}\mathcal{A}_3\sqrt{(\ell-\varepsilon)t}\Big).$$

(ii) Assume that $A_3 \in L^\infty(\mathbb{R})$ is compactly supported in an interval I of length $\ell > 0$. Then, there exists a constant $C > 0$, depending on ℓ, σ_1, σ_2 and υ_2, such that for any $t \geq 0$,

$$\|\Sigma_t\|_{\mathscr{L}_c(L^\infty(\mathbb{R}))} \leq C \exp\Big(\sqrt{\frac{\sigma_1\sigma_2}{\upsilon_2}}\|A_3\|_{L^\infty(\mathbb{R})}\sqrt{\ell t}\Big).$$

Let us mention that if $A_3 = \mathcal{A}_3 \mathbf{1}_{[0,\ell]}$, with $\mathcal{A}_3 \in \mathbb{C}^*$, then estimates in *(i)* and *(ii)* give the same upper and lower bounds in the limit $\varepsilon \to 0$.

We now turn (in the case ($\mathrm{C_{inf}}$)), to the higher dimensional situation.

[1] Otherwise, change x for $-x$.

[2] We can choose a smooth function A_3 such that $A_3(x) = \sigma_1(1-x)^\nu$ for $x \leq 0$, with $\nu \in (1/2, 1)$, for instance, and $A_3 \in H^s(\mathbb{R})$ for every $s \in \mathbb{N}$.

Proposition 11 *We assume $\Omega = \mathbb{R}^2$ ($d = 2$, $\Lambda_1 = \Lambda_2 = \infty$), $\sigma_1\sigma_2 > 0$ and that υ_1, υ_2 are not collinear (we are then in case $(\mathrm{C}_{\mathrm{inf}})$). We assume that A_3 satisfies, for some $M \geq 0$ and $\nu > 1$, the decay estimate*

$$\forall x \in \mathbb{R}^2, \qquad |A_3|(x) \leq \frac{M}{1 + |x|_\infty^\nu}. \tag{14}$$

Then, there exists a constant $C(\nu, M)$, depending on M, ν, σ_1, σ_2, υ_1 and υ_2 such that, for any $t \geq 0$,

$$\|\Sigma_t\|_{\mathscr{L}_c(\mathcal{C}_b(\mathbb{R}^2))} \leq C(\nu, M).$$

If υ_1 and υ_2 are collinear, we can reduce to one of the cases of the one-dimensional situation, with the extra-variables as simple parameters. This is also what happens if $d \geq 3$ and υ_1 and υ_2 are not collinear.

The degenerate case $\boldsymbol{\sigma_1\sigma_2 = 0}$. We now focus on the degenerate case $\sigma_1\sigma_2 = 0$. First, if $\sigma_1 = \sigma_2 = 0$, (5) reduces to two decoupled free transport equations with explicit solutions and conservation of all $W^{s,p}(\Omega)$ norms. If $\sigma_2 = 0 \neq \sigma_1$, the function A_2 can be computed explicitly and we obtain from (10)

$$A_2(t,x) = A_2^{\mathrm{in}}(x - t\upsilon_2), \qquad A_1(t,x) = A_1^{\mathrm{in}}(x - t\upsilon_1) - \sigma_1 \int_0^t A_3(x - (t-\tau)\upsilon_1) A_2^{\mathrm{in}}(x - (t-\tau)\upsilon_1 - \tau\upsilon_2)\, d\tau,$$

for which one infers easily the following estimates :
- if $\upsilon_1 \neq 0$ and $\upsilon_1 \neq \upsilon_2$, there exists C, depending on σ_1, υ_1 and υ_2 such that

$$\|\Sigma_t A^{\mathrm{in}}\|_{L^\infty(\Omega)\times L^2(\Omega)} \leq C\Big[\|A^{\mathrm{in}}\|_{L^\infty(\Omega)\times L^2(\Omega)} + \Big(\sup_{x\in\Omega}\int_{x+\mathbb{R}\upsilon_1} |A_3|^2\Big)^{\frac{1}{2}}\Big(\sup_{x\in\Omega}\int_{x+\mathbb{R}(\upsilon_1-\upsilon_2)} |A_2^{\mathrm{in}}|^2\Big)^{\frac{1}{2}}\Big]$$

and

$$\|\Sigma_t\|_{\mathscr{L}_c(L^\infty(\Omega))} \leq C\Big(1 + \sup_{x\in\Omega}\int_{x+\mathbb{R}\upsilon_1} |A_3|\Big),$$

provided the supremum in the right-hand side is finite, and otherwise

$$\|\Sigma_t\|_{\mathscr{L}_c(L^\infty(\Omega))} \approx |\sigma_1| \sup_{x\in\Omega}\int_{x-[0,t]\upsilon_1} |A_3| \qquad \text{as} \qquad t \to +\infty.$$

Thus, in particular, if $d = 1$ and $A_3 \in L^2(\mathbb{R})$ but $A_3 \notin L^1(\mathbb{R})$, then stability holds for (5) in $L^\infty(\mathbb{R}) \times L^2(\mathbb{R})$ but not in $L^\infty(\mathbb{R})$.
- if $\upsilon_1 = \upsilon_2 = 0$, then for any $1 \leq p \leq \infty$,

$$\|\Sigma_t\|_{\mathscr{L}_c(L^p(\Omega))} \approx t\|A_3\|_{L^\infty(\Omega)} \qquad \text{as} \qquad t \to +\infty;$$

- if $\upsilon_1 = \upsilon_2 \neq 0$, then, there exists C, depending on σ_1 and $\upsilon_1 = \upsilon_2$ such that, for $t \geq 0$,

$$\|\Sigma_t\|_{\mathscr{L}_c(L^p(\Omega))} \leq C\Big(1 + \frac{|\sigma_1|}{|\upsilon_1|}\sup_{x\in\Omega,\tau\geq 0}\Big|\int_{x-[0,\tau]\upsilon_1} A_3\Big|\Big)$$

(if the supremum in the right-hand side is finite), and otherwise

$$\|\Sigma_t\|_{\mathcal{L}_c(L^p(\Omega))} \approx \frac{|\sigma_1|}{|\upsilon_1|}\sup_{x\in\Omega}\Big|\int_{x-[0,t]\upsilon_1} A_3\Big| \qquad \text{as} \qquad t \to +\infty;$$

- if $\upsilon_1 = 0 \neq \upsilon_2$, then

$$A_1(t,x) = A_1^{\mathrm{in}}(x) - \sigma_1 A_3(x)\int_{x-[0,t]\upsilon_2} A_2^{\mathrm{in}},$$

and the situation is very degenerate. We can state, for instance, that if $d = 1$,

$$\|\Sigma_t\|_{\mathscr{L}_c(L^\infty(\Omega)\times L^1(\Omega))} \leq \Big(1 + |\sigma_1|\|A_3\|_{L^\infty(\Omega)}\Big)$$

and for $d \geq 2$, denoting

$$\|f\|_{L^\infty_\perp L^1_{\upsilon_2}(\Omega)} \equiv \sup_{x\in\Omega}\int_{x+\mathbb{R}\upsilon_2} |f|,$$

we have

$$\|\Sigma_t\|_{L^\infty(\Omega)\times L^\infty_\perp L^1_{\upsilon_2}(\Omega)} \leq \Big(1+|\sigma_1|\|A_3\|_{L^\infty(\Omega)}\Big).$$

The paper [9] (see also other papers by the same author) investigates, for a nonlinear wave equation in dimension one in the spirit of (1), the problem of the approximation of a wavetrain by the nonlinear Schrödinger equation. The case studied corresponds to the stable one $\sigma_1\sigma_2 < 0$. In [4], we shall study this problem (in arbitrary dimension, replacing the nonlinear Schrödinger equation by the Davey-Stewartson system if $d \geq 2$) for a general semilinear equation, taking into account all possible cases and using the above semigroup estimates.

2 Proofs of the results

In the sequel, $A = (A_1, A_2)$ always stands for $\Sigma_t(A_1^{\rm in}, A_2^{\rm in}) = \Sigma_t A^{\rm in}$. Unless otherwise stated, the L^p norms are taken in the space variable.

2.1 Proof of Proposition 2

Let $x \in \Omega$ with $D_t(x) \geq R \geq 1$. Then, from (10), it comes

$$\begin{aligned} D_t^\mu(x)|A_1|(t,x) &\leq D_t^\mu(x)|A_1^{\rm in}|(x-t\upsilon_1) + |\sigma_1|\int_0^t D_t^\mu(x)|A_2|(\tau, x-(t-\tau)\upsilon_1)|A_3|(x-(t-\tau)\upsilon_1)\ d\tau \\ &\leq |x-t\upsilon_1|^\mu |A_1^{\rm in}|(x-t\upsilon_1) + |\sigma_1|\int_0^t a_2(\tau)|A_3|(x-(t-\tau)\upsilon_1)\ d\tau, \end{aligned}$$

with

$$a_\ell(\tau) \equiv \|D_\tau^\mu(\cdot)A_\ell(\tau,\cdot)\|_{L^\infty(\{D_\tau\geq R\})}, \qquad \ell = 1,2.$$

Here, we use the fact that D_t is 1-Lipschitz, hence $D_t(x) \leq D_t(x-t\upsilon_1) + D_t(t\upsilon_1) = D_t(x-t\upsilon_1)$, and that $D_\tau(x-(t-\tau)\upsilon_1) \geq D_t(x)$ since, Γ being convex,

$$\begin{aligned} D_\tau(x-(t-\tau)\upsilon_1) &= \inf\{|x-(t-\tau)\upsilon_1 - \tau u|, \quad u \in \Gamma\} \\ &= \inf\{|x - t[(1-\tau/t)\upsilon_1 + (\tau/t)u]|, \quad u\in\Gamma\} \geq \inf\{|x-tU|, \quad U \in \Gamma\} = D_t(x). \end{aligned}$$

Therefore, using the decay of A_3,

$$D_t^\mu(x)|A_1|(t,x) \leq a_1(0) + |\sigma_1|C_0(A_3)\Big(\sup_{t'\in[0,t]} a_2(t')\Big)\int_0^t \frac{d\tau}{1+|x-(t-\tau)\upsilon_1|^\nu}.$$

Similarly, we obtain for A_2

$$D_t^\mu(x)|A_2|(t,x) \leq a_2(0) + |\sigma_2|C_0(A_3)\Big(\sup_{t'\in[0,t]} a_1(t')\Big)\int_0^t \frac{d\tau}{1+|x-(t-\tau)\upsilon_2|^\nu}.$$

It suffices then to prove that if $R \geq 1$ is large enough, then for $t \geq 0$ and $x \in \Omega$ with $D_t(x) \geq R$,

$$\max_{j=1,2}\Big(|\sigma_j|C_0(A_3)\int_0^t \frac{d\tau}{1+|x-(t-\tau)\upsilon_j|^\nu}\Big) \leq \epsilon < 1, \tag{15}$$

with ϵ independent of t and x, since then, one has

$$\max_{j=1,2} a_j(t) \leq \max_{j=1,2} a_j(0) + \epsilon\Big(\max_{j=1,2}\sup_{t'\in[0,t]} a_j(t')\Big)$$

and this will conclude the proof. Letting $s\upsilon_1$, $s \in \mathbb{R}$, be the orthogonal projection of x on $\mathbb{R}\upsilon_1$, we have

$$|x-(t-\tau)\upsilon_1|^2 = |s\upsilon_1 - (t-\tau)\upsilon_1|^2 + |x - s\upsilon_1|^2 \geq |s\upsilon_1-(t-\tau)\upsilon_1|^2 + R^2$$

if $s \in [0, t]$ (since then $D_t(x) = |x - sv_1| \geq R$), and thus

$$\int_0^t \frac{d\tau}{1 + |x - (t - \tau)v_j|^\nu} \leq \int_0^t \frac{d\tau}{1 + ([s - (t - \tau)]^2 |v_1|^2 + R^2)^{\nu/2}} \leq \int_{\mathbb{R}} \frac{d\sigma}{1 + (\sigma^2 |v_1|^2 + R^2)^{\nu/2}}$$

which is as small as we want if R is large enough. When $s \geq t$ (the case $s \leq 0$ is identical), then $D_t(x) = |x - tv_1| \geq R$ and

$$|x - (t - \tau)v_1|^2 = |sv_1 - (t - \tau)v_1|^2 + |x - sv_1|^2 = (s - (t - \tau))^2 |v_1|^2 + |x - sv_1|^2$$

so that

$$\int_0^t \frac{d\tau}{1 + |x - (t - \tau)v_j|^\nu} \leq K_\nu \int_{-\infty}^t \frac{d\sigma}{[1 + |x - sv_1| + (s - \sigma)|v_1|]^\nu} \leq \frac{K_\nu}{|v_1|} \frac{1}{[1 + |x - sv_1| + (s - t)|v_1|]^{\nu - 1}}.$$

Then, $|x - sv_1| + (s - t)|v_1| \geq |x - tv_1| \geq R$ and the last expression can be made as small as we want if R is chosen sufficiently large. The proof is complete.

2.2 Proof of Proposition 3

The proof is rather immediate, and proceeds by induction on s. Using (6), we have

$$\frac{d}{dt}\mathcal{N}(A) = 0, \qquad \text{where} \qquad \mathcal{N}(A) \equiv \frac{1}{2} \int_\Omega |A_1|^2 - \frac{\sigma_1}{\sigma_2} |A_2|^2 \, dx \approx \big\| A = (A_1, A_2) \big\|_{L^2}^2,$$

since $\sigma_1 \sigma_2 < 0$. We then define an H^s functional $\mathcal{N}^s$, $s \in \mathbb{N}$, setting ($\mathcal{N}^0 = \mathcal{N}$)

$$\mathcal{N}^s(A) \equiv \sum_{|\alpha| \leq s} \mathcal{N}(\partial_x^\alpha A) \approx \big\| A \big\|_{H^s}^2,$$

and assume

$$\mathcal{N}^s(A(t)) \leq C_s (1 + t^s)^2 \big\| A^{\rm in} \big\|_{H^s}^2$$

for some $s \geq 0$. Applying ∂_x^α, $|\alpha| = s + 1$, to (5), we infer

$$\left\{ \begin{array}{l} \partial_t \partial_x^\alpha A_1 - v_1 \partial_x \partial_x^\alpha A_1 + \sigma_1 A_3 \partial_x^\alpha A_2 = -\sigma_1 [\partial_x^\alpha, A_3] A_2 \\ \\ \partial_t \partial_x^\alpha A_2 - v_2 \partial_x \partial_x^\alpha A_2 + \sigma_2 \bar{A}_3 \partial_x^\alpha A_1 = -\sigma_2 [\partial_x^\alpha, \bar{A}_3] A_1. \end{array} \right.$$

By the Leibnitz formula, $[\partial_x^\alpha, A_3] A_2$ is a sum of terms of the type $* \partial_x^{\alpha - \beta} A_2 \partial_x^\beta A_3$ for $0 \neq \beta \leq \alpha$ (where the coefficient $*$ depends only on α and β). Hence, by the induction hypothesis, we deduce

$$\big\| [\partial_x^\alpha, \bar{A}_3] A_1 \big\|_{L^2} + \big\| [\partial_x^\alpha, A_3] A_2 \big\|_{L^2} \leq C \big\| A_3 \big\|_{W^{s+1,\infty}} (1 + t^s) \big\| A^{\rm in} \big\|_{H^s},$$

where C depends only on s, σ_1 and σ_2. Therefore,

$$\frac{d}{dt}\mathcal{N}(\partial_x^\alpha A) \leq C \big\| A_3 \big\|_{W^{s+1,\infty}} (1 + t^s) \big\| A^{\rm in} \big\|_{H^s} \sqrt{\mathcal{N}(\partial_x^\alpha A)},$$

and the result follows.

2.3 Proof of Proposition 4

Let us first consider the case $\Omega = \mathbb{R}/(2\pi\Lambda)$ with $0 < \Lambda_1 = \Lambda < +\infty$. We define the linear operator $T : D(T) \equiv H^1(\Omega) \subset L^2(\Omega) \to L^2(\Omega)$ by

$$T \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \equiv \begin{pmatrix} i v_1 \psi_1' + i \sigma_1 A_3 \psi_2 \\ i v_2 \psi_2' + i \sigma_2 \bar{A}_3 \psi_1 \end{pmatrix}.$$

Since $\sigma_1 \sigma_2 < 0$, we can consider the scalar product, for $\psi = (\psi_1, \psi_2)$ and $\phi = (\phi_1, \phi_2)$,

$$\langle \psi, \phi \rangle = \left\langle \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \middle| \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \right\rangle \equiv \int_\Omega \psi_1 \bar{\phi}_1 - \frac{\sigma_1}{\sigma_2} \psi_2 \bar{\phi}_2 \, dx,$$

which is equivalent to the standard scalar product in $L^2(\Omega)$ and for which T is self-adjoint. Furthermore, T has compact resolvent. Indeed, for $\mu \in \mathbb{C}$ with $\mathrm{Im}(\mu) \neq 0$ and $|\mu|$ sufficiently large and $f = (f_1, f_2) \in L^2(\Omega)$, we may write the equation $(T - \mu)\psi = f$ for the unknown $\psi = (\psi_1, \psi_2)$ under the form

$$\hat{\psi}(n) = (\hat{\psi}_1, \hat{\psi}_2)(n) = \Big(\frac{i\hat{A}_3 * \hat{\psi}_2(n) - \hat{f}_1(n)}{\mu + v_1 n/\Lambda_1}, \frac{i\hat{\bar{A}}_3 * \hat{\psi}_1(n) - \hat{f}_2(n)}{\mu + v_2 n/\Lambda_1} \Big)$$

where the $(\hat{\psi}(n))_{n\in\mathbb{Z}}$ are the Fourier coefficients of ψ. This last equation is easily solved by a contraction argument (for $|\mu|$ large enough) in $\ell^2(\mathbb{Z})$, and the easy estimate $\|\psi\|_{H^1(\Omega)} \leq C_\mu \|f\|_{L^2(\Omega)}$ allows to conclude to the compactness of $(T-\mu)^{-1}$ by compact Sobolev embedding. By standard theory, there exists a Hilbert basis (for the scalar product $\langle \cdot | \cdot \rangle$, equivalent to the standard scalar product in $L^2(\Omega)$) of eigenvectors of T, denoted $(e_n)_{n\in\mathbb{N}}$, with corresponding real eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$, with $|\lambda_n| \to +\infty$ as $n \to +\infty$. As a consequence, if $\psi = (\psi_2, \psi_2) \in H^1(\Omega)$ and if we denote $c_n \equiv \langle \psi | e_n \rangle$, $n \in \mathbb{N}$, we have

$$\langle T\psi | T\psi \rangle = \Big\langle \sum_{n\in\mathbb{N}} \lambda_n c_n e_n \Big| \sum_{n\in\mathbb{N}} \lambda_n c_n e_n \Big\rangle = \sum_{n\in\mathbb{N}} \lambda_n^2 |c_n|^2.$$

Since $A_3 \in L^\infty(\Omega)$, we have

$$\|(A_3\psi_1, \bar{A}_3\psi_2)\|^2_{L^2(\Omega)} \leq C\|\psi\|^2_{L^2(\Omega)} \leq C\langle \psi|\psi\rangle = C\sum_{n\in\mathbb{N}} |c_n|^2$$

and thus, since v_1, v_2 are nonzero by hypothesis, we infer

$$\|\psi'\|^2_{L^2(\Omega)} \approx \langle (iv_1\psi_1', iv_2\psi_2') | (iv_1\psi_1', iv_2\psi_2')\rangle = \langle T\psi|T\psi\rangle + \mathcal{O}(\|(A_3\psi_1, \bar{A}_3\psi_2)\|^2_{L^2(\Omega)}) = \sum_{n\in\mathbb{N}} \lambda_n^2|c_n|^2 + \mathcal{O}\Big(\sum_{n\in\mathbb{N}} |c_n|^2\Big)$$

and consequently, we have the equivalence of (squared) norms

$$\|\psi\|^2_{H^1(\Omega)} \approx \sum_{n\in\mathbb{N}} (1+\lambda_n^2)|c_n|^2.$$

More generally, by Leibnitz formula (recall that $A_3 \in W^{s,\infty}(\Omega)$), we deduce analogously that for any $s \in \mathbb{N}$, we have the equivalence of (squared) norms

$$\|\psi\|^2_{H^s(\Omega)} \approx \sum_{n\in\mathbb{N}} (1+\lambda_n^{2s})|c_n|^2. \tag{16}$$

We now turn to the proof of Proposition 4. For any initial data $A^{\mathrm{in}} = (A_1^{\mathrm{in}}, A_2^{\mathrm{in}}) \in H^s(\Omega)$, we may write

$$A^{\mathrm{in}} = \sum_{n\in\mathbb{N}} \langle A^{\mathrm{in}} | e_n \rangle e_n = \sum_{n\in\mathbb{N}} c_n e_n$$

in $H^s(\Omega)$, and the solution $A = (A_1, A_2)$ of (5) is

$$A(t) = \sum_{n\in\mathbb{N}} \mathrm{e}^{it\lambda_n} c_n e_n.$$

Hence, we have, using (16),

$$\|A(t)\|^2_{H^s(\Omega)} \leq C\sum_{n\in\mathbb{N}} (1+\lambda_n^{2s})|\mathrm{e}^{it\lambda_n} c_n|^2 = C\sum_{n\in\mathbb{N}} (1+\lambda_n^{2s})|c_n|^2 \leq C\|A^{\mathrm{in}}\|^2_{H^s(\Omega)}$$

as required.

We now turn to the case $\Omega = \mathbb{R}$, *i.e.* $\Lambda = \Lambda_1 = +\infty$. The point is also to expand on eigenvectors, replacing in the expansion $\psi = \sum_{n=0}^{+\infty} c_n e_n$ the series by an integral, in the same way we obtain Fourier transform from Fourier series. The idea is naturally to let $\Lambda \to +\infty$ in the previous case. When T is a second order, self-adjoint operator, this is the well-known Weyl-Stone-Titchmarsh-Kodaira theory (see, *e.g.* [11], Chap. 5, or [5], Chap. 10). The extension to the operator T is not difficult, and we just give a few details for sake of completeness.

For $\lambda \in \mathbb{R}$, let us denote $(\Phi^1(\cdot,\lambda), \Phi^2(\cdot,\lambda))$ the solutions of

$$T\Phi^1 - \lambda\Phi^1 = T\Phi^2 - \lambda\Phi^2 = 0, \qquad \Phi^1(x=0,\lambda) = (1,0), \quad \Phi^2(x=0,\lambda) = (0,1).$$

By classical results, Φ^1 and Φ^2 are smooth functions of λ for fixed x. For $0 < \Lambda < \infty$, we denote $(e_n^\Lambda)_{n\geq 0}$ an Hilbert basis for $T : H^1(\Omega_\Lambda) \to L^2(\Omega_\Lambda)$, $(\lambda_n^\Lambda)_{n\geq 0}$ the corresponding eigenvalues, and $\langle\cdot|\cdot\rangle_\Lambda$ and $\langle\cdot|\cdot\rangle$ will denote the scalar product in $L^2(\Omega_\Lambda)$ and $L^2(\mathbb{R})$. We expand the eigenvectors e_n^Λ, $n \in \mathbb{N}$, on the basis of solutions $(\Phi^1(\cdot,\lambda_n^\Lambda), \Phi^2(\cdot,\lambda_n^\Lambda))$:

$$e_n^\Lambda = \sum_{j=1}^{2} r_{j,n}^\Lambda \Phi^j(\cdot,\lambda_n^\Lambda) \qquad r_{j,n}^\Lambda \in \mathbb{C}.$$

If $\psi \in \mathcal{C}_c(\mathbb{R})$, we can view ψ as an element of $L^2(\Omega_\Lambda)$ provided Λ is sufficiently large so that $\mathrm{Supp}(\psi) \subset [-\pi\Lambda, +\pi\Lambda]$, say $\Lambda \geq \Lambda_0$. Then, the formula $\psi = \sum_{n=0}^{+\infty} c_n^\Lambda e_n^\Lambda$ in $L^2(\Omega_\Lambda)$, for $\Lambda \geq \Lambda_0$ can be written, in $L^2(\Omega_\Lambda)$,

$$\psi(x) = \sum_{n=0}^{+\infty} c_n^\Lambda e_n^\Lambda(x) = \sum_{n=0}^{+\infty} \sum_{j,k=1}^{2} \langle \psi \big| \Phi^k(\cdot,\lambda_n^\Lambda)\rangle r_{j,n}^\Lambda \bar{r}_{k,n}^\Lambda \Phi^j(x,\lambda_n^\Lambda) = \sum_{j,k=1}^{2} \int_{\mathbb{R}} \tilde{\psi}^k(\lambda)\Phi^j(x,\lambda)\, d\rho_{j,k}^\Lambda(\lambda), \tag{17}$$

in the sense of Stieltjes integrals, where (recall that ψ is compactly supported)

$$\tilde{\psi}^k(\lambda) \equiv \langle \psi \big| \Phi^k(\cdot,\lambda)\rangle \qquad (k=1,2)$$

and $\rho_{j,k}^\Lambda$ is a piecewise constant function, normalized so that $\rho_{j,k}^\Lambda(0+0) = 0$, and with discontinuities at the λ_n^Λ's, where

$$\rho_{j,k}^\Lambda(\lambda_n^\Lambda + 0) - \rho_{j,k}^\Lambda(\lambda_n^\Lambda - 0) = \sum_{m,\lambda_m^\Lambda = \lambda_n^\Lambda} r_{j,m}^\Lambda \bar{r}_{k,m}^\Lambda.$$

We observe that the right-hand side is a nonnegative hermitian matrix. The Parseval identity for ψ becomes

$$\langle\psi|\psi\rangle = \langle\psi|\psi\rangle_\Lambda = \sum_{j,k=1}^{2} \int_{\mathbb{R}} \tilde{\psi}^k(\lambda)\overline{\tilde{\psi}^j(\lambda)}\, d\rho_{j,k}^\Lambda(\lambda). \tag{18}$$

In order to prove local in space compactness for the matrix $\rho^\Lambda = (\rho_{j,k}^\Lambda)_{1\leq j,k\leq 2}$, we shall prove that for any $\Lambda_0 > 0$, there exists some $\eta > 0$ and a constant $C(\Lambda_0)$ such that, if $|\lambda_0| \leq \Lambda_0 \leq \Lambda - 1$, then

$$\sum_{j,k=1}^{2} \int_{\lambda_0-\eta}^{\lambda_0+\eta} |d\rho_{j,k}^\Lambda|(\lambda) \leq C(\Lambda_0). \tag{19}$$

First, from the definition of ρ^Λ, we have

$$2\int_{\lambda_0-\eta}^{\lambda_0+\eta} |d\rho_{1,2}^\Lambda|(\lambda) \leq \int_{\lambda_0-\eta}^{\lambda_0+\eta} d\rho_{1,1}^\Lambda(\lambda) + \int_{\lambda_0-\eta}^{\lambda_0+\eta} d\rho_{2,2}^\Lambda(\lambda), \tag{20}$$

hence, for (19), it suffices to show that

$$\int_{\lambda_0-\eta}^{\lambda_0+\eta} d\rho_{1,1}^\Lambda(\lambda) + \int_{\lambda_0-\eta}^{\lambda_0+\eta} d\rho_{2,2}^\Lambda(\lambda) \leq C(\Lambda_0). \tag{21}$$

For fixed $\Lambda_0 > 0$, $\lambda_0 \in [-\Lambda_0, +\Lambda_0]$ and $\Lambda \geq \Lambda_0 + 1$, we choose $\chi \in \mathcal{C}_c((-1,+1))$ with $\int\chi = 1$ and for ψ the mollifier $\psi_\epsilon(x) = \epsilon^{-1}\chi(x/\epsilon)(1, -\sigma_2/\sigma_1)$. Then, we infer, by continuity of the Φ^k, $k = 1$, 2 that, as $\epsilon \to 0$,

$$\tilde{\psi}_\epsilon^k(\lambda) \to \langle (1,1) | \Phi^k(0,\lambda)\rangle = 1$$

for $k = 1$ and $k = 2$ locally uniformly in λ. As a consequence, there exists $\eta > 0$, independent of Λ, such that, for ϵ small enough,

$$\sum_{j,k=1}^{2} \int_{\mathbb{R}} \tilde{\psi}_\epsilon^k(\lambda)\overline{\tilde{\psi}_\epsilon^j(\lambda)}\, d\rho_{j,k}^\Lambda(\lambda) \geq \frac{1}{2}\int_{\lambda_0-\eta}^{\lambda_0+\eta} d\rho_{1,1}^\Lambda(\lambda) + \frac{1}{2}\int_{\lambda_0-\eta}^{\lambda_0+\eta} d\rho_{2,2}^\Lambda(\lambda) - \frac{1}{4}\int_{\lambda_0-\eta}^{\lambda_0+\eta} |d\rho_{1,2}^\Lambda|(\lambda).$$

Inserting (20) and the Parseval identity, we obtain, for ϵ sufficiently small, $\Lambda \geq \Lambda_0 + 1$ and $|\lambda_0| \leq \Lambda_0$,

$$\langle \psi_\epsilon | \psi_\epsilon \rangle \geq \frac{1}{4} \int_{\lambda_0 - \eta}^{\lambda_0 + \eta} d\rho_{1,1}^{\Lambda}(\lambda) + \frac{1}{4} \int_{\lambda_0 - \eta}^{\lambda_0 + \eta} d\rho_{2,2}^{\Lambda}(\lambda),$$

which yields (21), and thus (19).

Therefore, on every bounded interval $[-\Lambda_0, +\Lambda_0]$, the functions $(\rho_{j,k}^{\Lambda})_{1 \leq j,k \leq 2}$ have bounded variation, with a uniform bound for $\Lambda \geq \Lambda_0 + 1$. By Helly's compactness theorem, there exists a matrix valued mapping ρ and a sequence $\Lambda_q \to +\infty$ such that $(\rho_{j,k}^{\Lambda_q})_{1 \leq j,k \leq 2}$ converges to ρ pointwise and weakly in $BV(I)$ for every bounded interval I as $q \to = \infty$. This allows to pass to the limit in (17):

$$\psi(x) = \sum_{j,k=1}^{2} \int_{\mathbb{R}} \tilde{\psi}^k(\lambda) \Phi^j(x, \lambda) \, d\rho_{j,k}(\lambda) \qquad \text{in } L^2(\mathbb{R})$$

and in (18)

$$\langle \psi | \psi \rangle = \sum_{j,k=1}^{2} \int_{\mathbb{R}} \tilde{\psi}^k(\lambda) \overline{\tilde{\psi}^j(\lambda)} \, d\rho_{j,k}(\lambda).$$

These formulas are extended by density if $\psi \in L^2(\mathbb{R})$. Finally, we have, for $\psi \in H^1(\mathbb{R})$,

$$T\psi(x) = \sum_{j,k=1}^{2} \int_{\mathbb{R}} \lambda \tilde{\psi}^k(\lambda) \Phi^j(x, \lambda) \, d\rho_{j,k}(\lambda) \qquad \text{in } L^2(\mathbb{R})$$

and, as in the periodic case, since v_1 and v_2 are non zero,

$$\|\psi'\|_{L^2(\mathbb{R})}^2 \approx \langle (iv_1 \psi_1' | iv_2 \psi_2' \rangle = \langle T\psi | T\psi \rangle + \mathcal{O}\Big(\|\psi\|_{L^2(\mathbb{R})}^2 \Big)$$

since $A_3 \in L^\infty(\mathbb{R})$. Hence,

$$\|\psi\|_{H^1(\mathbb{R})}^2 \approx \sum_{j,k=1}^{2} \int_{\mathbb{R}} (1 + \lambda^2) \tilde{\psi}^k(\lambda) \overline{\tilde{\psi}^j(\lambda)} \, d\rho_{j,k}(\lambda),$$

and, for $s \in \mathbb{N}$ arbitrary,

$$\|\psi\|_{H^s(\mathbb{R})}^2 \approx \sum_{j,k=1}^{2} \int_{\mathbb{R}} (1 + \lambda^{2s}) \tilde{\psi}^k(\lambda) \overline{\tilde{\psi}^j(\lambda)} \, d\rho_{j,k}(\lambda).$$

The conclusion then follows. It can be actually shown (as in [5]) that ρ is unique up to some constant matrix and that ρ^Λ converges to ρ as $\Lambda \to +\infty$ (locally in space) and not only for some sequence.

2.4 Proof of Proposition 5

From the system (10), we deduce, for $0 \leq t' \leq t$,

$$\begin{aligned} |A_1|(t', x) &\leq \|A_1^{\text{in}}\|_{L^\infty} + |\sigma_1| \int_0^{t'} \|A_2(\tau, \cdot)\|_{L^\infty} |A_3|(x - (t' - \tau) v_1) \, d\tau \\ &\leq \|A_1^{\text{in}}\|_{L^\infty} + \frac{|\sigma_1|}{|v_1|} \Big(\sup_{0 \leq \tau \leq t} \|A_2(\tau, \cdot)\|_{L^\infty} \Big) \Big(\sup_{x \in \Omega} \int_{x + \mathbb{R} v_1} |A_3| \Big) \end{aligned}$$

and similarly

$$|A_2|(t', x) \leq \|A_2^{\text{in}}\|_{L^\infty} + \frac{|\sigma_2|}{|v_2|} \Big(\sup_{0 \leq \tau \leq t} \|A_1(\tau, \cdot)\|_{L^\infty} \Big) \Big(\sup_{x \in \Omega} \int_{x + \mathbb{R} v_2} |A_3| \Big).$$

As a consequence,

$$\|A_1(t', \cdot)\|_{L^\infty} \leq \|A_1^{\text{in}}\|_{L^\infty} + \frac{|\sigma_1|}{|v_1|} \|A_2^{\text{in}}\|_{L^\infty} \Big(\sup_{x \in \Omega} \int_{x + \mathbb{R} v_1} |A_3| \Big) + \delta \sup_{0 \leq \tau \leq t} \|A_1(\tau, \cdot)\|_{L^\infty},$$

with

$$\delta \equiv \frac{|\sigma_1\sigma_2|}{|\upsilon_1|\cdot|\upsilon_2|}\Big(\sup_{x\in\Omega}\int_{x+\mathbb{R}\upsilon_1}|A_3|\Big)\Big(\sup_{x\in\Omega}\int_{x+\mathbb{R}\upsilon_2}|A_3|\Big).$$

From the smallness assumption (11), $\delta<1$ and it follows

$$\sup_{0\le t'\le t}\big\|A_1(\tau,\cdot)\big\|_{L^\infty}\le\frac{1}{1-\delta}\Big(\big\|A_1^{\rm in}\big\|_{L^\infty}+\frac{|\sigma_1|}{|\upsilon_1|}\big\|A_2^{\rm in}\big\|_{L^\infty}\sup_{x\in\Omega}\int_{x+\mathbb{R}\upsilon_1}|A_3|\Big)$$

and analogously for A_2. Therefore, the estimate for $s=0$ is proved: there exists $C>0$ such that, for $t\ge 0$,

$$\big\|\Sigma_t A^{\rm in}\big\|_{L^\infty}\le C\big\|A^{\rm in}\big\|_{L^\infty}.$$

We now argue by induction and assume the result for $s\in\mathbb{N}$. Let $\alpha\in\mathbb{N}^d$ with $|\alpha|=s+1$, and let us apply ∂_x^α to (10) to obtain

$$\begin{cases}\partial_x^\alpha A_1(t,x)=\partial_x^\alpha A_1^{\rm in}(x-t\upsilon_1) & -\sigma_1\displaystyle\int_0^t\partial_x^\alpha A_2(\tau,x-(t-\tau)\upsilon_1)A_3(x-(t-\tau)\upsilon_1)\;d\tau\\ & -\sigma_1\displaystyle\int_0^t[\partial_x^\alpha,A_3(x-(t-\tau)\upsilon_1)]A_2(\tau,x-(t-\tau)\upsilon_1)\;d\tau\\ \partial_x^\alpha A_2(t,x)=\partial_x^\alpha A_2^{\rm in}(x-t\upsilon_2) & -\sigma_2\displaystyle\int_0^t\partial_x^\alpha A_1(\tau,x-(t-\tau)\upsilon_2)\bar{A}_3(x-(t-\tau)\upsilon_2)\;d\tau\\ & -\sigma_2\displaystyle\int_0^t[\partial_x^\alpha,\bar{A}_3(x-(t-\tau)\upsilon_2)]A_1(\tau,x-(t-\tau)\upsilon_2)\;d\tau.\end{cases}$$

Using the induction hypothesis and the Leibnitz formula, we infer, for $t\ge 0$,

$$\begin{aligned}\Big|\int_0^t[\partial_x^\alpha,A_3(x-(t-\tau)\upsilon_1)]A_2(\tau,x-(t-\tau)\upsilon_1)\;d\tau\Big|&\\ &\le C_\alpha\big\|A_2\big\|_{L^\infty(\mathbb{R}_+,W^{s,\infty})}\max_{\beta\le\alpha}\int_0^t|\partial_x^\beta A_3(x-(t-\tau)\upsilon_1)|\;d\tau\\ &\le C_s\big\|A^{\rm in}\big\|_{W^{s,\infty}},\end{aligned}$$

and similarly for the last integral. Then, arguing as for the case $s=0$, we infer for $0\le t'\le t$

$$\big\|\partial_x^\alpha A_1(t',\cdot)\big\|_{L^\infty}\le C_s\big\|A^{\rm in}\big\|_{W^{s+1,\infty}}+\delta\sup_{0\le\tau\le t}\big\|\partial_x^\alpha A_1(\tau,\cdot)\big\|_{L^\infty},$$

and similarly for A_2, which allows to conclude.

2.5 Proof of Proposition 6

We recall that, here, $\sigma_1\sigma_2>0$ and $A_3\not\equiv 0$, hence $\gamma>0$. To prove the growth estimate for Σ_t, we argue as for Proposition 5 by induction on s. The first point is to find a suitable norm. For the ODE part (in time) of (5), namely

$$\frac{dA}{dt}+\begin{pmatrix}0 & \sigma_1A_3\\ \sigma_2\bar{A}_3 & 0\end{pmatrix}A=0,$$

in which the matrix has eigenvalues $\pm\sqrt{\sigma_1\sigma_2}|A_3|$, it turns out that the norm

$$\nu_0(A)\equiv|A_1|+\sqrt{\frac{\sigma_1}{\sigma_2}}\,|A_2|$$

is well adapted, since

$$\nu_0\Big(\begin{pmatrix}0 & \sigma_1A_3\\ \sigma_2\bar{A}_3 & 0\end{pmatrix}A\Big)=\sqrt{\sigma_1\sigma_2}|A_3|\nu_0(A).$$

Therefore, for $1\le p\le\infty$, it natural to work with the norm in $L^p(\Omega,\mathbb{C}^2)$

$$\mathscr{N}_p(A)\equiv\big\|A_1\big\|_{L^p}+\sqrt{\frac{\sigma_1}{\sigma_2}}\,\big\|A_2\big\|_{L^p}$$

Taking the L^p norm in (10) yields

$$\begin{cases} \|A_1(t)\|_{L^p} \leq \|A_1^{\text{in}}\|_{L^p} + |\sigma_1| \cdot \|A_3\|_{L^\infty} \int_0^t \|A_2(\tau)\|_{L^p} \, d\tau \\ \\ \|A_2(t)\|_{L^p} \leq \|A_2^{\text{in}}\|_{L^p} + |\sigma_2| \cdot \|A_3\|_{L^\infty} \int_0^t \|A_1(\tau)\|_{L^p} \, d\tau, \end{cases}$$

hence

$$\mathscr{N}_p(A(t)) \leq \mathscr{N}_p(A^{\text{in}}) + \sqrt{\sigma_1\sigma_2}\|A_3\|_{L^\infty} \int_0^t \mathscr{N}_p(A(\tau)) \, d\tau$$

and the Gronwall lemma gives as wished, for $t \geq 0$,

$$\mathscr{N}_p(A(t)) \leq \mathscr{N}_p(A^{\text{in}})\mathsf{e}^{\gamma t}.$$

We assume now that, for some $s \in \mathbb{N}$, we have for $t \geq 0$

$$\|A(t)\|_{W^{s,p}} \leq K_s(A_3)\|A^{\text{in}}\|_{W^{s,p}}(1+t^s)\mathsf{e}^{\gamma t}.$$

We apply ∂_x^α, $|\alpha| = s+1$, to (10):

$$\begin{cases} \partial_x^\alpha A_1(t,x) = \partial_x^\alpha A_1^{\text{in}}(x - t\upsilon_1) & -\sigma_1 \int_0^t \partial_x^\alpha A_2(\tau, x-(t-\tau)\upsilon_1) A_3(x-(t-\tau)\upsilon_1) \, d\tau \\ & -\sigma_1 \int_0^t [\partial_x^\alpha, A_3(x-(t-\tau)\upsilon_1)] A_2(\tau, x-(t-\tau)\upsilon_1) \, d\tau \\ \\ \partial_x^\alpha A_2(t,x) = \partial_x^\alpha A_2^{\text{in}}(x - t\upsilon_2) & -\sigma_2 \int_0^t \partial_x^\alpha A_1(\tau, x-(t-\tau)\upsilon_2) \bar{A}_3(x-(t-\tau)\upsilon_2) \, d\tau \\ & -\sigma_2 \int_0^t [\partial_x^\alpha, \bar{A}_3(x-(t-\tau)\upsilon_2)] A_1(\tau, x-(t-\tau)\upsilon_2) \, d\tau. \end{cases}$$

Using the commutator estimate

$$\left\| \int_0^t [\partial_x^\alpha, A_3(x-(t-\tau)\upsilon_1)] A_2(\tau, x-(t-\tau)\upsilon_1) \, d\tau \right\|_{L^p} \leq C_\alpha \|A_3\|_{W^{s+1,\infty}} \int_0^t \|A_2(\tau)\|_{W^{s,p}} \, d\tau$$

and the fact that, by the induction hypothesis,

$$\int_0^t \|A_2(\tau)\|_{W^{s,p}} \, d\tau \leq K_s(A_3)\|A^{\text{in}}\|_{W^{s,p}} \int_0^t (1+\tau^s)\mathsf{e}^{\gamma\tau} \, d\tau \leq K\|A^{\text{in}}\|_{W^{s,p}}(1+t^s)\mathsf{e}^{\gamma t}$$

since $\gamma > 0$ (and similarly for the other term), we have

$$\begin{cases} \|\partial_x^\alpha A_1(t)\|_{L^p} \leq \|\partial_x^\alpha A_1^{\text{in}}\|_{L^p} + |\sigma_1| \cdot \|A_3\|_{L^\infty} \int_0^t \|\partial_x^\alpha A_2(\tau)\|_{L^p} \, d\tau + K\|A^{\text{in}}\|_{W^{s,p}}(1+t^s)\mathsf{e}^{\gamma t} \\ \\ \|\partial_x^\alpha A_2(t)\|_{L^p} \leq \|\partial_x^\alpha A_2^{\text{in}}\|_{L^p} + |\sigma_2| \cdot \|A_3\|_{L^\infty} \int_0^t \|\partial_x^\alpha A_1(\tau)\|_{L^p} \, d\tau + K\|A^{\text{in}}\|_{W^{s,p}}(1+t^s)\mathsf{e}^{\gamma t}. \end{cases}$$

As a consequence,

$$\begin{aligned} \mathscr{N}_p(\partial_x^\alpha A(t)) &\leq \mathscr{N}_p(\partial_x^\alpha A^{\text{in}}) + K\|A^{\text{in}}\|_{W^{s,p}}(1+t^s)\mathsf{e}^{\gamma t} + \sqrt{\sigma_1\sigma_2}\|A_3\|_{L^\infty} \int_0^t \mathscr{N}_p(\partial_x^\alpha A(\tau)) \, d\tau \\ &\leq K\|A^{\text{in}}\|_{W^{s+1,p}}(1+t^s)\mathsf{e}^{\gamma t} + \sqrt{\sigma_1\sigma_2}\|A_3\|_{L^\infty} \int_0^t \mathscr{N}_p(\partial_x^\alpha A(\tau)) \, d\tau, \end{aligned}$$

and here again, the Gronwall lemma implies, for $t \geq 0$,

$$\mathscr{N}_p(\partial_x^\alpha A(t)) \leq K\|A^{\text{in}}\|_{W^{s+1,p}}(1+t^s)\mathsf{e}^{\gamma t} + \gamma K\|A^{\text{in}}\|_{W^{s+1,p}}\mathsf{e}^{\gamma t} \int_0^t (1+\tau^s) \, d\tau \leq K\|A^{\text{in}}\|_{W^{s+1,p}}(1+t^{s+1})\mathsf{e}^{\gamma t}$$

as claimed.

2.6 Proof of Proposition 7

The proof of (i) follows by considering functions (A_1, A_2) depending only on time. Hence (5) reduces to (9), with eigenvalues $\pm\sqrt{\sigma_1\sigma_2}|A_3|$ and (i) is proved.

For (ii), we consider functions depending only on the variables x' orthogonal to $\mathrm{Span}(\upsilon_1, \upsilon_2)$ (we have $\dim x' \geq 1$). Then, (5) reduces to (9) once again, and therefore

$$A(t, x') = \exp\Big(-t \begin{pmatrix} 0 & \sigma_1 A_3(x') \\ \sigma_2 \bar{A}_3(x') & 0 \end{pmatrix} \Big) A^{\mathrm{in}}(x').$$

If $\varepsilon \in (0, \gamma)$ is given, there exists some $y'_\varepsilon \in \Omega$ such that $\sqrt{\sigma_1\sigma_2}|A_3|(y'_\varepsilon) \geq \gamma - \varepsilon/2$, and then, by continuity of A_3, $r_\varepsilon > 0$ such that $\sqrt{\sigma_1\sigma_2}|A_3|(x') \geq \gamma - \varepsilon > 0$ if $x' \in B_{2r_\varepsilon}(y'_\varepsilon)$. We then consider the eigenvector (for fixed x') for the eigenvalue $\sqrt{\sigma_1\sigma_2}|A_3|(x')$

$$A^{\mathrm{in}}(x') \equiv c\chi(x') \begin{pmatrix} 1 \\ \mathrm{sgn}(\sigma_1)\sqrt{\dfrac{\sigma_2}{\sigma_1}} \dfrac{\bar{A}_3(x')}{|A_3|(x')} \end{pmatrix} \in \mathcal{C}_c^\infty(B_{2r_\varepsilon}(y'_\varepsilon)),$$

where $\chi \in \mathcal{C}_c^\infty(B_{2r_\varepsilon}(y'_\varepsilon), [0,1])$ is equal to one in the ball $B_{r_\varepsilon}(y'_\varepsilon)$ (in the variables x'), and

$$c \equiv \frac{1}{\max(1, \sqrt{\sigma_2/\sigma_1})} > 0$$

is just a normalization constant. Hence,

$$|A^{\mathrm{in}}|_\infty(x') \leq 1 \quad \text{in} \quad B_{2r_\varepsilon}(y'_\varepsilon) \qquad \text{and} \qquad |A^{\mathrm{in}}|_\infty(x') = 1 \quad \text{in} \quad B_{r_\varepsilon}(y'_\varepsilon)$$

and since we have an eigenvector,

$$A(t, x') = \mathrm{e}^{t\sqrt{\sigma_1\sigma_2}|A_3|(x')} A^{\mathrm{in}}(x')$$

and the conclusion follows.

The case (iii) follows the same lines, since the hypothesis allows us to take $\varepsilon = 0$ and $r_\varepsilon = r > 0$. The multiplicity is infinite since the choice of $\chi \in \mathcal{C}_c^\infty(B_{2r}(y'))$ such that $\chi = 1$ on $B_r(y')$ is arbitrary.

The proof of (iv) relies on an explicit computation. Let $\upsilon \equiv \upsilon_1 = \upsilon_2$ and let us recall that A_3 is assumed real-valued. Then, (5) becomes, in $\Omega = \mathbb{R}/(2\pi\Lambda\mathbb{Z})$ (with $0 < \Lambda < \infty$),

$$\begin{cases} \partial_t \tilde{A}_1 + \sigma_1 A_3(x + t\upsilon)\tilde{A}_2 = 0 \\ \partial_t \tilde{A}_2 + \sigma_2 A_3(x + t\upsilon)\tilde{A}_1 = 0, \end{cases} \qquad \text{where} \qquad \tilde{A}_j(t, x) \equiv A_j(t, x + t\upsilon) \qquad (j = 1, 2).$$

For fixed $x \in \Omega$, we can solve this ODE explicitly using the fact that the nonsingular matrix

$$P \equiv \begin{pmatrix} \sqrt{\sigma_1\sigma_2} & -\sqrt{\sigma_1\sigma_2} \\ \sigma_2 & \sigma_2 \end{pmatrix}$$

is independent of t and diagonalizes the system: the vector $B \equiv P^{-1}(\tilde{A}_1, \tilde{A}_2)^t$ solves

$$\partial_t B + \begin{pmatrix} \sqrt{\sigma_1\sigma_2}\, A_3(x + t\upsilon) & 0 \\ 0 & -\sqrt{\sigma_1\sigma_2}\, A_3(x + t\upsilon) \end{pmatrix} B = 0,$$

from which we infer after straightforward computations

$$\begin{cases} A_1(t, x) = & A_1^{\mathrm{in}}(x - t\upsilon) \cosh\Big(\sqrt{\sigma_1\sigma_2}\displaystyle\int_0^t A_3(x - (t - \tau)\upsilon)\, d\tau\Big) \\ & -\mathrm{sgn}(\sigma_1)\sqrt{\dfrac{\sigma_1}{\sigma_2}}\, A_2^{\mathrm{in}}(x - t\upsilon) \sinh\Big(\sqrt{\sigma_1\sigma_2}\displaystyle\int_0^t A_3(x - (t - \tau)\upsilon)\, d\tau\Big) \\ A_2(t, x) = & A_2^{\mathrm{in}}(x - t\upsilon) \cosh\Big(\sqrt{\sigma_1\sigma_2}\displaystyle\int_0^t A_3(x - (t - \tau)\upsilon)\, d\tau\Big) \\ & -\mathrm{sgn}(\sigma_1)\sqrt{\dfrac{\sigma_2}{\sigma_1}}\, A_1^{\mathrm{in}}(x - t\upsilon) \sinh\Big(\sqrt{\sigma_1\sigma_2}\displaystyle\int_0^t A_3(x - (t - \tau)\upsilon)\, d\tau\Big). \end{cases} \tag{22}$$

The first statement in (iv) follows easily from this explicit formula and the fact that

$$\Big|\int_0^t A_3(x-(t-\tau)\upsilon)\ d\tau\Big| \leq t\Big|\frac{1}{|\Omega|}\int_\Omega A_3(y)\ dy\Big| + \mathcal{O}(1)$$

as $t \to +\infty$ uniformly in x. We solve the eigenvalue problem (7) in the same way. For $\lambda = -i\mu \in \mathbb{C}$ with $\operatorname{Im}\lambda = -\operatorname{Re}\mu < 0$, (7) is

$$\begin{cases} \upsilon\psi_1' + \sigma_1\psi_2 A_3 + \mu\psi_1 = 0 \\ \\ \upsilon\psi_2' + \sigma_2\psi_1 A_3 + \mu\psi_2 = 0. \end{cases}$$

Here again, the vector $\Psi \equiv \mathrm{e}^{\mu x/\upsilon}P^{-1}\psi$ solves

$$\upsilon\Psi' + \begin{pmatrix} \sqrt{\sigma_1\sigma_2}\,A_3(x) & 0 \\ 0 & -\sqrt{\sigma_1\sigma_2}\,A_3(x) \end{pmatrix}\Psi = 0,$$

which yields, for some vector $\zeta = (\zeta_1, \zeta_2) \in \mathbb{C}^2$,

$$\begin{cases} \mathrm{e}^{-\mu x/\upsilon}\psi_1(t,x) = & \zeta_1 \cosh\Big(\frac{\sqrt{\sigma_1\sigma_2}}{\upsilon}\int_0^x A_3(y)\ dy\Big) - \zeta_2 \operatorname{sgn}(\sigma_1)\sqrt{\frac{\sigma_1}{\sigma_2}}\sinh\Big(\frac{\sqrt{\sigma_1\sigma_2}}{\upsilon}\int_0^x A_3(y)\ dy\Big) \\ \\ \mathrm{e}^{-\mu x/\upsilon}\psi_2(t,x) = & \zeta_2 \cosh\Big(\frac{\sqrt{\sigma_1\sigma_2}}{\upsilon}\int_0^x A_3(y)\ dy\Big) - \zeta_1 \operatorname{sgn}(\sigma_1)\sqrt{\frac{\sigma_2}{\sigma_1}}\sinh\Big(\frac{\sqrt{\sigma_1\sigma_2}}{\upsilon}\int_0^x A_3(y)\ dy\Big). \end{cases} \tag{23}$$

Now, the periodic boundary condition implies that μ is an eigenvalue for (7) if and only if ζ is a non trivial solution of (we recall that $2\pi\Lambda = |\Omega|$)

$$\mathrm{e}^{-\mu|\Omega|/\upsilon}\zeta = \begin{pmatrix} \cosh\Big(\frac{\sqrt{\sigma_1\sigma_2}}{\upsilon}\int_0^{|\Omega|} A_3(y)\ dy\Big) & -\operatorname{sgn}(\sigma_1)\sqrt{\frac{\sigma_1}{\sigma_2}}\sinh\Big(\frac{\sqrt{\sigma_1\sigma_2}}{\upsilon}\int_0^{|\Omega|} A_3(y)\ dy\Big) \\ -\operatorname{sgn}(\sigma_1)\sqrt{\frac{\sigma_2}{\sigma_1}}\sinh\Big(\frac{\sqrt{\sigma_1\sigma_2}}{\upsilon}\int_0^{|\Omega|} A_3(y)\ dy\Big) & \cosh\Big(\frac{\sqrt{\sigma_1\sigma_2}}{\upsilon}\int_0^{|\Omega|} A_3(y)\ dy\Big) \end{pmatrix}\zeta.$$

Since the matrix on the right-hand side has eigenvalues $\exp\Big(\pm\frac{\sqrt{\sigma_1\sigma_2}}{\upsilon}\int_0^{|\Omega|} A_3(y)\ dy\Big)$, this happens only if, for some $m \in \mathbb{Z}$,

$$\mu = \pm\frac{\sqrt{\sigma_1\sigma_2}}{|\Omega|}\Big|\int_0^{|\Omega|} A_3(y)\ dy\Big| + i\frac{2\pi m\upsilon}{|\Omega|} = \pm\sqrt{\sigma_1\sigma_2}\Big|\frac{1}{|\Omega|}\int_\Omega A_3(y)\ dy\Big| + \frac{im\upsilon}{\Lambda}.$$

2.7 Proof of Proposition 8

We recall that here, $\Omega = \mathbb{R}$, $\sigma_1\sigma_2 > 0$ and $\upsilon_1\upsilon_2 < 0$. Possibly changing the sign of x, we may assume, without loss of generality, that $\upsilon_1 > 0 > \upsilon_2$.

2.7.1 Proof of (i)

We shall first focus on statement (i), where we choose A_3 to be a multiple of the characteristic function of an interval: $A_3 \equiv \Gamma\mathbf{1}_{x\in[a,b]}$. We could have assumed $\Gamma \in \mathbb{R}_+$ by just multiplying A_2 by a constant phase factor. We look for eigenvalues for (7) with $\lambda = -i\mu$, $\operatorname{Re}(\mu) > 0$, and (ψ_1, ψ_2) a non-trivial weak solution of (7), *i.e.*

$$\begin{cases} \upsilon_1\psi_1' + \sigma_1\psi_2 A_3 + \mu\psi_1 = 0 \\ \\ \upsilon_2\psi_2' + \sigma_2\psi_1\bar{A}_3 + \mu\psi_2 = 0. \end{cases} \tag{24}$$

Since A_3 vanishes on $(-\infty, a)$ and in view of the fact that $\upsilon_1 > 0 > \upsilon_2$, the subspace of bounded solutions of (24) on $(-\infty, b)$ is exactly $\mathbb{C}\psi_*$, with

$$\psi_*(x) \equiv \begin{cases} (0, \mathrm{e}^{-\mu(x-a)/\upsilon_2}) & \text{on} \quad (-\infty, a] \\ \\ \exp\Big(-(x-a)M\Big)(0,1) & \text{on} \quad [a,b], \end{cases}$$

and where

$$M \equiv \begin{pmatrix} \dfrac{\mu}{\upsilon_1} & \dfrac{\sigma_1 \mathcal{A}_3}{\upsilon_1} \\ \dfrac{\sigma_2 \bar{\mathcal{A}}_3}{\upsilon_2} & \dfrac{\mu}{\upsilon_2} \end{pmatrix}.$$

Hence μ is an eigenvalue for (7) if and only if

$$\Big[\exp\Big(-(b-a)M\Big)(0,1)\Big]_2 = 0,$$

so that ψ_* can be extended to $[b,+\infty)$ by

$$\psi_*(x) \equiv (\beta \mathsf{e}^{-\mu(x-b)/\upsilon_1}, 0)$$

for some nonzero constant β. Denoting

$$\gamma \equiv \sqrt{\sigma_1\sigma_2}|\mathcal{A}_3| \qquad \text{and} \qquad \hat{\mu} \equiv 2\gamma \frac{\sqrt{|\upsilon_1\upsilon_2|}}{\upsilon_1 - \upsilon_2} > 0,$$

the matrix M has eigenvalues

$$\frac{\mu}{2}\frac{\upsilon_1+\upsilon_2}{\upsilon_1\upsilon_2} \pm \frac{\gamma}{\sqrt{|\upsilon_1\upsilon_2|}}\sqrt{\frac{\mu^2}{\hat{\mu}^2} - 1} = \sigma \pm \gamma\tau,$$

where the last square root stands for one fixed complex-valued square root. Thus, a straightforward computation[3] gives that

$$\Big[\exp\Big(-(b-a)M\Big)(0,1)\Big]_2 = 0$$

if and only if (the fraction has value $b-a$ if $\mu = \hat{\mu}$)

$$\exp\Big(-(b-a)\gamma\tau\Big) - \frac{\sinh((b-a)\gamma\tau)}{\gamma\tau}\Big(\frac{\mu}{\upsilon_2} - \sigma - \gamma\tau\Big) = 0,$$

or, after some algebra,

$$\text{for} \quad \mu \neq \hat{\mu}, \qquad \exp\Big(2\gamma(b-a)\tau\Big) = \frac{\dfrac{\mu}{\hat{\mu}} - \sqrt{\dfrac{\mu^2}{\hat{\mu}^2} - 1}}{\dfrac{\mu}{\hat{\mu}} + \sqrt{\dfrac{\mu^2}{\hat{\mu}^2} - 1}}; \qquad \text{for} \quad \mu = \hat{\mu}, \qquad \frac{(b-a)\gamma}{\sqrt{|\upsilon_1\upsilon_2|}} = -1. \tag{25}$$

Therefore, $\mu = \hat{\mu}$ is not an eigenvalue for (24). We assume now $\mu \neq \hat{\mu}$. Since the mapping $\cosh : \mathbb{C} \to \mathbb{C}$ is onto, for $\mu \in \mathbb{C}$, there exists some $y \in \mathbb{C}$ such that

$$\frac{\mu}{\hat{\mu}} = \cosh(y).$$

Moreover, since cosh is even and $2i\pi$-periodic, we can assume $y = u + iv$ with $u \in \mathbb{R}$, $v \in [0,\pi]$, and since $\mu = \cosh(y) = \cosh(u)\cos(v) + i\sinh(u)\sin(v)$ has positive real part, we have $v \in [0,\pi/2)$. Then, (25) becomes

$$\exp\Big(2\gamma(b-a)\tau\Big) = \frac{\cosh(y) - \sinh(y)}{\cosh(y) + \sinh(y)} = \mathsf{e}^{-2y},$$

that is, for some $m \in \mathbb{Z}$,

$$\gamma(b-a)\tau = \frac{\gamma(b-a)}{\sqrt{|\upsilon_1\upsilon_2|}}\sinh(y) = -y + im\pi.$$

[3] using that if $A \in \mathcal{M}_2(\mathbb{C})$ has two distinct eigenvalues (α and β) or is non diagonalizable (double eigenvalue α), the formula $\mathsf{e}^A = \mathsf{e}^\alpha + \dfrac{\mathsf{e}^\beta - \mathsf{e}^\alpha}{\beta - \alpha}(A - \alpha)$ holds, where the fraction is e^α if $\beta = \alpha$.

Taking real and imaginary parts, we infer

$$\alpha \sinh(u)\cos(v) = -u, \qquad -v + m\pi = \alpha\cosh(u)\sin(v), \tag{26}$$

where α is the absolute area:

$$\alpha \equiv \frac{\sqrt{\sigma_1\sigma_2}}{\sqrt{|\upsilon_1\upsilon_2|}}\Big|\int_{\mathbb{R}} A_3(x)\ dx\Big| = \frac{\gamma(b-a)}{\sqrt{|\upsilon_1\upsilon_2|}} > 0.$$

Since $v \in [0,\pi/2)$, the first equation in (26) implies $u = 0$, so that (26) is now reduced to the single equation

$$-v + m\pi = \alpha\sin(v), \tag{27}$$

with $v \in (0,\pi/2)$ ($v = 0$ is excluded since $\mu \neq \hat{\mu}$). Here again, since $v \in (0,\pi/2)$, it is immediate that (27) has no solution for $m \le 0$. An elementary graphical analysis shows the following: for $m = 1$, (27) has no solution (in $(0,\pi/2)$) for $\alpha \le \pi/2$, and exactly one solution $v \in (0,\pi/2)$ for $\alpha > \pi/2$, which goes from $\pi/2^-$ (for $\alpha = (\pi/2)^+$) to 0^+ (for $\alpha \to +\infty$), that is μ goes from 0^+ to $\hat{\mu}$; for $m = 2$, (27) has exactly one solution for $\alpha > 3\pi/2$, and no solution (in $(0,\pi/2)$) for $\alpha \le 3\pi/2$; for $m \ge 1$, (27) has exactly one solution for $\alpha > (2m-1)\pi/2$ (with corresponding μ going from 0^+ to $\hat{\mu}$), and no solution for $\alpha \le (2m-1)\pi/2$. When α crosses the values $\alpha = k\pi/2$, $k \ge 1$ odd, an eigenvalue 0 crosses the real axis to go to the half-plane $\{\mathrm{Im} < 0\}$ up to $-i\hat{\mu}$. Consequently, the equation (27) has no solution for $\alpha \le \pi/2$, and (for $m \ge 1$) exactly m solutions for $(2m-1)\pi/2 < \alpha \le (2m+1)\pi/2$.

2.7.2 Proof of (ii)

We look for eigenvalues for (7) $\lambda = -i\mu$, with $\mu > 0$, and $\psi = (\psi_1,\psi_2)$ a non-trivial, real-valued, solution of (7), that is

$$\begin{cases} \upsilon_1\psi_1' + \sigma_1\psi_2 A_3 + \mu\psi_1 = 0 \\[2ex] \upsilon_2\psi_2' + \sigma_2\psi_1 A_3 + \mu\psi_2 = 0. \end{cases} \tag{28}$$

Notice that if $\mu > 0$, the behavior of ψ at $\pm\infty$ should be given by the solutions of

$$\upsilon_1\psi_1' + \mu\psi_1 = \upsilon_2\psi_2' + \mu\psi_2 = 0,$$

which means, since $\upsilon_2 < 0 < \upsilon_1$, that one expects the solutions (decaying to zero at infinity) of (28) to be such that, for some real constants α_+, α_-

$$\psi(x) = (\psi_1,\psi_2)(x) \approx \begin{cases} (\alpha_+ \mathsf{e}^{-\mu x/\upsilon_1}, 0) & x \to +\infty \\[2ex] (0, \alpha_- \mathsf{e}^{-\mu x/\upsilon_2}) & x \to -\infty. \end{cases}$$

The first step is to construct particular solutions $\psi_\mu^\pm$ with the above mentioned asymptotic behavior at $\pm\infty$ up to $\mu = 0$. It suffices to treat the case $x \to -\infty$ (the other case being analogous). We use a classical fixed point argument (see, *e.g.*, [5]), and the point is to have some uniformity for $\mu \in [0,\bar{\mu}]$, with

$$c \equiv \frac{|\sigma_2|}{|\upsilon_2|} + \frac{|\sigma_1|}{|\upsilon_1|} > 0, \qquad \delta \equiv \frac{1}{2}\Big(\frac{1}{\upsilon_1} - \frac{1}{\upsilon_2}\Big) > 0 \qquad \text{and} \qquad \bar{\mu} \equiv c\big\|A_3\big\|_{L^\infty}/\delta.$$

We set

$$\varphi(x) \equiv \exp\Big(\mu x \begin{pmatrix} 1/\upsilon_1 & 0 \\ 0 & 1/\upsilon_2 \end{pmatrix}\Big)\psi(x) = \begin{pmatrix} \mathsf{e}^{\mu x/\upsilon_1}\psi_1 \\ \mathsf{e}^{\mu x/\upsilon_2}\psi_2 \end{pmatrix},$$

which transforms (28) into

$$\begin{cases} \varphi_1'(x) + \dfrac{\sigma_1}{\upsilon_1}A_3(x)\mathsf{e}^{2\mu\delta x}\varphi_2(x) = 0 \\[3ex] \varphi_1(x) + \dfrac{\sigma_2}{\upsilon_2}A_3(x)\mathsf{e}^{-2\mu\delta x}\varphi_1(x) = 0. \end{cases}$$

We then consider for $R > 0$ that will be chosen sufficiently large, the fixed point problem in $(-\infty,-R)$

$$\varphi(x) = \Upsilon_\mu[\varphi](x) \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \int_{-\infty}^x \begin{pmatrix} \dfrac{\sigma_1}{\upsilon_1}\varphi_2(y)A_3(y)\mathsf{e}^{2\mu\delta y} \\ \dfrac{\sigma_2}{\upsilon_2}\varphi_1(y)A_3(y)\mathsf{e}^{-2\mu\delta y} \end{pmatrix} dy \tag{29}$$

For $0 \le \mu \le \bar{\mu}$, we shall solve (29) in the Banach space

$$E_\mu \equiv \Big\{ \varphi = (\varphi_1, \varphi_2) : (-\infty, -R) \to \mathbb{R}^2, \quad \mathsf{e}^{2\mu\delta|x|}\varphi_1 \in L^\infty(-\infty,-R), \quad \varphi_2 \in L^\infty(-\infty,-R) \Big\}$$

equipped with the norm

$$\|\varphi\|_{E_\mu} \equiv \|\mathsf{e}^{2\mu\delta|x|}\varphi_1\|_{L^\infty(-\infty,-R)} + \|\varphi_2\|_{L^\infty(-\infty,-R)}.$$

In view of the easy estimates

$$\Big| \int_{-\infty}^{x} \frac{\sigma_2}{\upsilon_2} \varphi_1(y) A_3(y) \mathsf{e}^{-2\mu\delta y} \Big| \le \Big| \frac{\sigma_2}{\upsilon_2} \Big| \Big(\int_{-\infty}^{x} |A_3(y)| \; dy \Big) \|\mathsf{e}^{2\mu\delta|x|}\varphi_1\|_{L^\infty(-\infty,-R)} \tag{30}$$

and

$$\Big| \int_{-\infty}^{x} \frac{\sigma_1}{\upsilon_1} \varphi_2(y) A_3(y) \mathsf{e}^{2\mu\delta y} \Big| \le \mathsf{e}^{-2\mu\delta|x|} \Big| \frac{\sigma_1}{\upsilon_1} \Big| \Big(\int_{-\infty}^{x} |A_3(y)| \; dy \Big) \|\varphi_2\|_{L^\infty(-\infty,-R)}, \tag{31}$$

we see that $\Upsilon_\mu : E_\mu \to E_\mu$ is a well-defined affine mapping. Moreover, if R is chosen sufficiently large so that

$$\Big| \frac{\sigma_1}{\upsilon_1} \Big| \Big(\int_{-\infty}^{-R} |A_3(y)| \; dy \Big) \le \frac{1}{2} \qquad \text{and} \qquad \Big| \frac{\sigma_2}{\upsilon_2} \Big| \Big(\int_{-\infty}^{-R} |A_3(y)| \; dy \Big) \le \frac{1}{2},$$

the above inequalities show that $\Upsilon_\mu : E_\mu \to E_\mu$ is $\frac{1}{2}$-Lipschitz. Therefore, $\Upsilon_\mu : E_\mu \to E_\mu$ has a unique fixed point $\varphi_\mu \in E_\mu$. Furthermore,

$$\|\varphi_\mu\|_{E_\mu} = \|\Upsilon_\mu[\varphi_\mu]\|_{E_\mu} \le \|\Upsilon_\mu[\varphi_\mu] - \Upsilon_\mu[0]\|_{E_\mu} + \|\Upsilon_\mu[0]\|_{E_\mu} \le \frac{1}{2}\|\varphi_\mu\|_{E_\mu} + 1,$$

hence

$$\|\varphi_\mu\|_{E_\mu} = \|\mathsf{e}^{2\mu\delta|x|}(\varphi_\mu)_1\|_{L^\infty(-\infty,-R)} + \|(\varphi_\mu)_2\|_{L^\infty(-\infty,-R)} \le 2. \tag{32}$$

We then set

$$\psi_\mu^-(x) \equiv \exp\Big(-\mu x \begin{pmatrix} 1/\upsilon_1 & 0 \\ 0 & 1/\upsilon_2 \end{pmatrix} \Big) \varphi_\mu(x) = \begin{pmatrix} \mathsf{e}^{-\mu x/\upsilon_1}(\varphi_\mu)_1 \\ \mathsf{e}^{-\mu x/\upsilon_2}(\varphi_\mu)_2 \end{pmatrix}.$$

From (30), (31) and (32) and the fact that $\int_{-\infty}^{x} |A_3| \; dy \to 0$ as $x \to -\infty$, it follows that

$$\psi_\mu^-(x) = \mathsf{e}^{-\mu x/\upsilon_2} \Big[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + o(1) \Big] \qquad \text{as} \quad x \to -\infty, \tag{33}$$

with $o(1)$ uniform with respect to $\mu \in [0, \bar{\mu}]$. Following the same lines for $x \to +\infty$, we construct, for $0 \le \mu \le \bar{\mu}$, a solution ψ_μ^+ of (28) such that

$$\psi_\mu^+(x) = \mathsf{e}^{-\mu x/\upsilon_1} \Big[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + o(1) \Big] \qquad \text{as} \quad x \to +\infty, \tag{34}$$

with here again a uniform $o(1)$ for $\mu \in [0, \bar{\mu}]$.

In a second step, we use polar coordinates $\psi = (\psi_1, \psi_2) = \rho(\cos\theta, \sin\theta)$, and rewrite the system (28) under the form

$$\begin{cases} \rho' = -\mu\rho\Big(\dfrac{\cos^2(\theta)}{\upsilon_1} + \dfrac{\sin^2(\theta)}{\upsilon_2}\Big) - \Big(\dfrac{\sigma_1}{\upsilon_1} + \dfrac{\sigma_2}{\upsilon_2}\Big)\rho A_3 \\ \\ \theta' = \mu\Big(\dfrac{1}{\upsilon_1} - \dfrac{1}{\upsilon_2}\Big)\cos\theta\sin\theta + A_3\Big(\dfrac{\sigma_1}{\upsilon_1}\sin^2\theta - \dfrac{\sigma_2}{\upsilon_2}\cos^2\theta\Big). \end{cases}$$

Note that the equation for θ does not involve ρ. We shall denote $\psi_\mu^\pm = \rho_\mu^\pm(\cos\theta_\mu^\pm, \sin\theta_\mu^\pm)$, and from , we see that we can choose θ_μ^- (resp. θ_μ^+) such that $\theta_\mu^-(-\infty) = \pi/2$ (resp. $\theta_\mu^+(+\infty) = 0$), and that the mapping $[0,\bar{\mu}] \times (-\infty, -R] \ni (\mu, x) \mapsto \theta_\mu^-(x) \in \mathbb{R}$ (resp. $[0,\bar{\mu}] \times ([+R, +\infty) \ni (\mu, x) \mapsto \theta_\mu^+(x) \in \mathbb{R}$) is uniformly continuous. We focus then on the second equation of the above system, namely

$$\theta' = \mu\Big(\frac{1}{\upsilon_1} - \frac{1}{\upsilon_2}\Big)\cos\theta\sin\theta + A_3\Big(\frac{\sigma_1}{\upsilon_1}\sin^2\theta - \frac{\sigma_2}{\upsilon_2}\cos^2\theta\Big). \tag{35}$$

The point is to show that for some $\mu > 0$, there exists $m \in \mathbb{Z}$ such that $\theta_\mu^- = \theta_\mu^+ + m\pi$ on $\mathbb{R}$. Since θ_μ^- and θ_μ^+ solve (35), it is sufficient to show that $\theta_\mu^-(x) = \theta_\mu^+(x) \bmod \pi$ for some $x \in \mathbb{R}$, which will be done by using the intermediate value theorem in $[0, \bar{\mu}]$.

We consider first the case $\mu = 0$. Then, we have for $x \in \mathbb{R}$,

$$\int_{-\infty}^{x} A_3(y)\, dy = \int_{-\infty}^{x} \frac{(\theta_0^-)'}{\frac{\sigma_2}{\upsilon_2}\cos^2\theta_0^- - \frac{\sigma_1}{\upsilon_1}\sin^2\theta_0^-}\, dy = \int_{\pi/2}^{\theta_0^-(x)} \frac{d\Theta}{\frac{\sigma_2}{\upsilon_2}\cos^2\Theta - \frac{\sigma_1}{\upsilon_1}\sin^2\Theta}.$$

The integrand in the right-hand side is π-periodic, even, of constant sign (since $\sigma_1\sigma_2 > 0 > \upsilon_1\upsilon_2$) equal to $\epsilon \equiv \mathrm{sgn}(\sigma_2/\upsilon_2) = \mathrm{sgn}(\sigma_1/\upsilon_1) = -\mathrm{sgn}(\sigma_2/\upsilon_2) = \pm 1$, and has integral over one period equal to (using the change of variable $t = \tan\Theta$)

$$\forall \alpha \in \mathbb{R}, \qquad \int_{\alpha}^{\alpha+\pi} \frac{d\Theta}{\frac{\sigma_2}{\upsilon_2}\cos^2\Theta - \frac{\sigma_1}{\upsilon_1}\sin^2\Theta} = \int_{-\pi/2}^{+\pi/2} \frac{d\Theta}{\frac{\sigma_2}{\upsilon_2}\cos^2\Theta - \frac{\sigma_1}{\upsilon_1}\sin^2\Theta} = \int_{-\infty}^{+\infty} \frac{dt}{\frac{\sigma_2}{\upsilon_2} - \frac{\sigma_1}{\upsilon_1}t^2} = \epsilon\pi\sqrt{\frac{|\upsilon_1\upsilon_2|}{\sigma_1\sigma_2}}.$$

Since, by assumption,

$$\Big|\int_{\mathbb{R}} A_3(y)\, dy\Big| > \frac{\pi}{2}\sqrt{\frac{|\upsilon_1\upsilon_2|}{\sigma_1\sigma_2}},$$

it follows that θ_0^- has a finite limit at $+\infty$ which is such that

$$\eta \equiv \Big|\theta_0^-(+\infty) - \theta_0^-(-\infty)\Big| - \frac{\pi}{2} = \Big|\theta_0^-(+\infty) - \frac{\pi}{2}\Big| - \frac{\pi}{2} > 0.$$

Since θ_μ^- and θ_μ^+ tend to $\pi/2$ and 0 at $-\infty$ and $+\infty$ uniformly with respect to $[0, \bar{\mu}]$, we can choose some $r \gg 1$ such that, for $0 \le \mu \le \bar{\mu}$,

$$\Big|\theta_\mu^-(-r) - \frac{\pi}{2}\Big| \le \frac{\pi}{4}, \qquad \Big|\theta_\mu^+(+r)\Big| \le \frac{1}{10}\min(\eta, \pi) \qquad \text{and} \qquad \Big|\theta_0^-(+r) - \theta_0^-(+\infty)\Big| \le \frac{\eta}{10}.$$

It then follows that

$$\Big|\theta_0^-(+r) - \theta_0^+(+r) - \frac{\pi}{2}\Big| = \Big|\Big(\theta_0^-(+r) - \theta_0^-(+\infty)\Big) + \Big(\theta_0^-(+\infty) - \frac{\pi}{2}\Big) - \theta_0^+(+r)\Big| \ge \frac{\pi}{2} + \frac{4\eta}{5},$$

thus $\theta_0^-(+r) - \theta_0^+(+r) \notin [0, \pi]$.

Now, we focus on the case where $\mu = \bar{\mu}$. We have

$$-c\|A_3\|_{L^\infty} \le \frac{d\theta_{\bar{\mu}}^-}{dx} - 2\delta\bar{\mu}\cos\theta_{\bar{\mu}}^-\sin\theta_{\bar{\mu}}^- \le c\|A_3\|_{L^\infty}$$

Denoting $\underline{\theta}$ and $\bar{\theta}$ the solutions of

$$\begin{cases} \dfrac{d\underline{\theta}}{dx} = \underline{F}(\underline{\theta}) \equiv 2\delta\bar{\mu}\cos\underline{\theta}\sin\underline{\theta} - c\|A_3\|_{L^\infty} = \delta\bar{\mu}(-1 + \sin(2\underline{\theta})) & \text{with} \quad \underline{\theta}(-r) = \theta_\mu^-(-r) \\ \dfrac{d\bar{\theta}}{dx} = \bar{F}(\bar{\theta}) \equiv 2\delta\bar{\mu}\cos\bar{\theta}\sin\bar{\theta} + c\|A_3\|_{L^\infty} = \delta\bar{\mu}(1 + \sin(2\bar{\theta})) & \text{with} \quad \bar{\theta}(-r) = \theta_\mu^-(-r). \end{cases}$$

(for which we have global existence) we then have

$$\underline{\theta} \le \theta_{\bar{\mu}}^- \le \bar{\theta} \qquad \text{in } [-r, +r].$$

It is immediate that $\bar{F}$ is positive on the interval $(\pi/4, 3\pi/4)$ and vanishes at its right end; similarly, $\underline{F}$ is negative on the interval $(\pi/4, 3\pi/4)$ and vanishes at its left end. We then infer, since $\theta_{\bar{\mu}}^-(-r) \in (\pi/4, 3\pi/4)$, that on $[-r, +r]$,

$$\frac{\pi}{4} \le \theta^- \le \theta_{\bar{\mu}}^- \le \theta^+ \le \frac{3\pi}{4}.$$

Consequently,

$$\Big|\theta_{\bar{\mu}}^-(+r) - \theta_{\bar{\mu}}^+(+r) - \frac{\pi}{2}\Big| \le \Big|\theta_{\bar{\mu}}^-(+r) - \frac{\pi}{2}\Big| + \Big|\theta_{\bar{\mu}}^+(+r)\Big| \le \frac{\pi}{4} + \frac{\pi}{10} < \frac{\pi}{2}$$

and $\theta_{\bar{\mu}}^-(+r) - \theta_{\bar{\mu}}^+(+r) \in (0, \pi)$. Since the mapping $[0, \bar{\mu}] \ni \mu \mapsto \theta_{\bar{\mu}}^-(+r) - \theta_{\bar{\mu}}^+(+r) \in \mathbb{R}$ is continuous (up to $\mu = 0$), it follows from the intermediate value theorem that for some $0 < \mu < \bar{\mu}$, $\theta_{\bar{\mu}}^-(+r) - \theta_{\bar{\mu}}^+(+r)$ is an integer multiple of π. For this value of μ, ψ_μ^+ is collinear to ψ_μ^-, and we have then obtained a solution $\psi = \psi_\mu^- = K\psi_\mu^+$ (for some $K \in \mathbb{R}^*$) to (28) which tends to 0 at $\pm\infty$ exponentially fast.

2.7.3 Proof of (iii)

We first recall that if $\upsilon_1+\upsilon_2=0$, then $\gamma=\gamma_*$ and the estimates (iii) and (iv) come from Proposition 6. We thus assume $\upsilon_1+\upsilon_2\neq 0$. We introduce the variables (close to the characteristic variables)

$$\hat{x}\equiv\Big(\frac{1}{\upsilon_1}-\frac{1}{\upsilon_2}\Big)x \qquad \text{and} \qquad \hat{t}\equiv 2t-\Big(\frac{1}{\upsilon_1}+\frac{1}{\upsilon_2}\Big)x,$$

and denote

$$\hat{A}_j(\hat{t},\hat{x})=A_j(t,x) \qquad j=1,2, \qquad \hat{A}_3(\hat{x})\equiv A_3(x).$$

Writing (5) in terms of $\hat{A}=(\hat{A}_1,\hat{A}_2)$, we infer the system in $(\hat{t},\hat{x})$

$$\begin{cases} \partial_{\hat{t}}\hat{A}_1+\partial_{\hat{x}}\hat{A}_1+\sigma_1\dfrac{\upsilon_2}{\upsilon_2-\upsilon_1}\hat{A}_3\hat{A}_2=0 \\ \\ \partial_{\hat{t}}\hat{A}_2-\partial_{\hat{x}}\hat{A}_2+\sigma_2\dfrac{\upsilon_1}{\upsilon_1-\upsilon_2}\hat{A}_3\hat{A}_1=0. \end{cases}$$

This system in variables $(\hat{t},\hat{x})$ is of the type (5) with (σ_1,σ_2) replaced by $(\hat{\sigma}_1,\hat{\sigma}_2)\equiv\Big(\sigma_1\dfrac{\upsilon_2}{\upsilon_2-\upsilon_1},\sigma_2\dfrac{\upsilon_1}{\upsilon_1-\upsilon_2}\Big)$, for which $\hat{\sigma}_1\hat{\sigma}_2>0$. Thus, we may apply the result of Proposition 6 (forward and backward in time) to get

$$\big\|\hat{A}(\hat{t})\big\|_{L^\infty}\leq C\big\|\hat{A}(\hat{t}=0)\big\|_{L^\infty}\exp\Big(|\hat{t}|\sqrt{\sigma_1\sigma_2}\frac{\sqrt{|\upsilon_1\upsilon_2|}}{|\upsilon_1-\upsilon_2|}\big\|\hat{A}_3\big\|_{L^\infty}\Big). \tag{36}$$

We recall that we have assumed $\upsilon_1+\upsilon_2\neq 0$, thus we can set

$$\hat{\upsilon}\equiv\frac{2\upsilon_1\upsilon_2}{\upsilon_1+\upsilon_2}\in\mathbb{R}^*,$$

so that the line $\{\hat{t}=0\}$ is the line $\{t=x/\hat{\upsilon}\}$. In order to be able to use (36), we need to estimate

$$\big\|\hat{A}(\hat{t}=0)\big\|_{L^\infty}=\big\|\theta\mapsto A(\theta,\theta\hat{\upsilon})\big\|_{L^\infty}.$$

Since $\upsilon_1>0>\upsilon_2$,

$$\frac{1}{\hat{\upsilon}}=\frac{1}{\upsilon_1}+\frac{1}{\upsilon_2}\in\Big(\frac{1}{\upsilon_2},\frac{1}{\upsilon_1}\Big),$$

and then $\hat{\upsilon}<\upsilon_2$ (if $\upsilon_1>-\upsilon_2$) or $\hat{\upsilon}>\upsilon_1$ (if $\upsilon_1<-\upsilon_2$). We first treat the case $\hat{\upsilon}>\upsilon_1$. By assumption, $A_3\in L^1$, thus there exists $R>0$ such that

$$\sqrt{\frac{\sigma_1\sigma_2}{|\upsilon_1\upsilon_2|}}\int_{|y|>R}|A_3|(y)\ dy\leq\frac{1}{2}. \tag{37}$$

We shall follow the first lines of the proof of Proposition 5 to obtain a large time estimate for

$$\mathrm{N}(t)\equiv\big\|A_1(t)\big\|_{L^\infty(\{x\geq R+t\upsilon_1\})}+\sqrt{\frac{\sigma_1}{\sigma_2}\Big|\frac{\upsilon_2}{\upsilon_1}\Big|}\,\big\|A_2(t)\big\|_{L^\infty(\{x\geq R+t\upsilon_1\})}.$$

Let us fix $0\leq t'\leq t$ and $x\geq R+t'\upsilon_1$. Note that for $0\leq\tau\leq t'$, we have

$$x-(t'-\tau)\upsilon_1\geq(R+t'\upsilon_1)-(t'-\tau)\upsilon_1=R+\tau\upsilon_1 \qquad \text{and} \qquad x-(t'-\tau)\upsilon_2\geq x\geq R+t'\upsilon_1\geq R+\tau\upsilon_1.$$

Hence, from (10), we infer

$$\begin{aligned}|A_1|(t',x)&\leq|A_1^{\rm in}(x-t'\upsilon_1)|+|\sigma_1|\int_0^{t'}\big\|A_2(\tau,\cdot)\big\|_{L^\infty(\{y\geq R+\tau\upsilon_1\})}|A_3|(x-(t'-\tau)\upsilon_1)\ d\tau\\ &\leq\big\|A_1^{\rm in}\big\|_{L^\infty}+\frac{|\sigma_1|}{|\upsilon_1|}\Big(\sup_{0\leq\tau\leq t}\big\|A_2(\tau,\cdot)\big\|_{L^\infty(\{y\geq R+\tau\upsilon_1\})}\Big)\Big(\int_R^{+\infty}|A_3|(y)\ dy\Big)\end{aligned}$$

and similarly

$$|A_2|(t',x)\leq\big\|A_2^{\rm in}\big\|_{L^\infty}+\frac{|\sigma_2|}{|\upsilon_2|}\Big(\sup_{0\leq\tau\leq t}\big\|A_1(\tau,\cdot)\big\|_{L^\infty(\{y\geq R+\tau\upsilon_1\})}\Big)\Big(\int_R^{+\infty}|A_3|(y)\ dy\Big).$$

Consequently, using (37), we deduce that for $0 \le t' \le t$,

$$\mathrm{N}(t') \le C\|A^{\mathrm{in}}\|_{L^\infty} + \frac{1}{2}\sup_{0\le\tau\le t}\mathrm{N}(\tau),$$

from which it comes, for any $t \ge 0$,

$$\mathrm{N}(t) \le \sup_{0\le\tau\le t}\mathrm{N}(\tau) \le 2C\|A^{\mathrm{in}}\|_{L^\infty}. \tag{38}$$

Since $\hat{\upsilon} > \upsilon_1$, for $\theta \ge R/(\hat{\upsilon}-\upsilon_1)$, we have $\theta\hat{\upsilon} \ge R + \theta\upsilon_1$. Therefore, the above estimate yields in particular

$$\|\theta \mapsto A(\theta, \theta\hat{\upsilon})\|_{L^\infty(\{\theta\ge R/(\hat{\upsilon}-\upsilon_1)\})} \le C\|A^{\mathrm{in}}\|_{L^\infty}.$$

In the case $\hat{\upsilon} < \upsilon_2 < 0$, we deduce by similar arguments (computing the L^∞ norms in $\{y \le -R + t\upsilon_2\}$) that

$$\|\theta \mapsto A(\theta, \theta\hat{\upsilon})\|_{L^\infty(\{\theta\ge R/(\upsilon_2-\hat{\upsilon})\})} \le C\|A^{\mathrm{in}}\|_{L^\infty}.$$

The estimates for $\theta \ll -1$ are derived in the same way. Hence, there exists some $\Theta > 0$, depending only on R, υ_1 and υ_2, such that

$$\|\theta \mapsto A(\theta, \theta\hat{\upsilon})\|_{L^\infty(\{|\theta|\ge\Theta\})} \le C\|A^{\mathrm{in}}\|_{L^\infty}.$$

Since $A_3 \in L^\infty$, the Cauchy problem (5) is locally well-posed in L^∞ and it follows that

$$\|\theta \mapsto A(\theta, \theta\hat{\upsilon})\|_{L^\infty(\{|\theta|\le\Theta\})} \le C\|A^{\mathrm{in}}\|_{L^\infty}$$

for some constant C depending on A_3, σ_1, σ_2, υ_1 and υ_2. Collecting the above inequalities, it comes

$$\|\hat{A}(\hat{t}=0)\|_{L^\infty} = \|\theta \mapsto A(\theta, \theta\hat{\upsilon})\|_{L^\infty} \le C\|A^{\mathrm{in}}\|_{L^\infty}.$$

Inserting this into (36) yields for $t \ge 0$ and $x \in \mathbb{R}$

$$|A(t,x)|_\infty = |\hat{A}(\hat{t},\hat{x})|_\infty \le C\|A^{\mathrm{in}}\|_{L^\infty} \exp\left(\left|2t - \frac{x}{\hat{\upsilon}}\right|\frac{\gamma_*}{2}\right). \tag{39}$$

This estimate is sufficient to show that eigenvalues for (7) (if they exist) lie in $\{\mathrm{Im} \ge -\gamma_*\}$. Indeed, assume that $\lambda \in \mathbb{C}$ is an eigenvalue for (7) with corresponding eigenvector ψ, so that $A(t,x) = \mathsf{e}^{i\lambda t}\psi(x)$ solves (5). Then, (39) yields, for any $x \in \mathbb{R}$,

$$|A(t,x)|_\infty = \mathsf{e}^{-t\mathrm{Im}\lambda}|\psi(x)|_\infty \le C\|\psi\|_{L^\infty} \exp\left(\left|2t - \frac{x}{\hat{\upsilon}}\right|\frac{\gamma_*}{2}\right).$$

Choosing $x = x_0 \in \mathbb{R}$ such that $2|\psi(x_0)|_\infty \ge \|\psi\|_{L^\infty}$, this gives

$$\mathsf{e}^{-t\mathrm{Im}\lambda} \le 2C \exp\left(\left|2t - \frac{x_0}{\hat{\upsilon}}\right|\frac{\gamma_*}{2}\right),$$

which yields the result letting $t \to +\infty$.

To prove (iii), we use (39) only for $x \in [-R, +R]$:

$$\|A(t)\|_{L^\infty(-R,+R)} \le C_R\|A^{\mathrm{in}}\|_{L^\infty}\mathsf{e}^{t\gamma_*}. \tag{40}$$

We now derive an estimate for

$$N(t) \equiv \|A_1\|_{L^\infty((0,t)\times(R,+\infty))} + \sqrt{\frac{\sigma_1}{\sigma_2}\left|\frac{\upsilon_2}{\upsilon_1}\right|}\|A_2\|_{L^\infty((0,t)\times(R,+\infty))}.$$

For $x \ge R$ and $0 \le t' \le t$, we have, by (10) and since $\upsilon_2 < 0$,

$$|A_2|(t',x) \le \|A_2^{\mathrm{in}}\|_{L^\infty} + \frac{|\sigma_2|}{|\upsilon_2|}\|A_1\|_{L^\infty((0,t)\times(R,+\infty))}\int_R^{+\infty}|A_3|(y)\,dy. \tag{41}$$

Furthermore, (38) gives in particular

$$\|A_1\|_{L^\infty(\{0\le t'\le t,\ x\ge R+t'\upsilon_1\})} \le C\|A^{\mathrm{in}}\|_{L^\infty}. \tag{42}$$

Let now $0 \leq t' \leq t$ and $R \leq x \leq R + t' v_1$. By the method of characteristics, we obtain

$$A_1(t', x) = A_1\Big(t' - \frac{x-R}{v_1}, R\Big) - \sigma_1 \int_{t' - \frac{x-R}{v_1}}^{t'} A_2(\tau, x - (t'-\tau)v_1) A_3(x - (t'-\tau)v_1)\ d\tau.$$

We use (40) to bound the first term ($t' - (x-R)/v_1 \leq t' \leq t$), and since $x - (t'-\tau)v_1 \geq R$ in the integral, we obtain, for $R \leq x \leq R + t' v_1$,

$$\big\| A_1 \big\|_{L^\infty(\{0 \leq t' \leq t,\ R \leq x \leq R + t' v_1\})} \leq C_R \big\| A^{\text{in}} \big\|_{L^\infty} \mathsf{e}^{t\gamma_*} + \frac{|\sigma_1|}{v_1} \big\| A_2 \big\|_{L^\infty((0,t)\times(R,+\infty))} \int_R^{+\infty} |A_3|(y)\ dy. \tag{43}$$

the combination of (41), (42) and (43) yields

$$N(t) \leq C_R \big\| A^{\text{in}} \big\|_{L^\infty} \mathsf{e}^{t\gamma_*} + N(t) \Big(\sqrt{\frac{\sigma_1 \sigma_2}{|v_1 v_2|}} \int_R^{+\infty} |A_3|(y)\ dy \Big)^2.$$

Possibly taking R larger if necessary, we can assume the term in parenthesis $\leq 1/\sqrt{2}$, so that

$$N(t) \leq 2 C_R \big\| A^{\text{in}} \big\|_{L^\infty} \mathsf{e}^{t\gamma_*}.$$

The estimate for $x \leq -R$ follows the same lines, and combining with (40), we infer as claimed

$$\big\| A(t) \big\|_{L^\infty} \leq C \big\| A^{\text{in}} \big\|_{L^\infty} \mathsf{e}^{t\gamma_*}.$$

2.8 Proof of Proposition 9

We recall that $\sigma_1 \sigma_2 > 0$. Possibly changing x for $-x$, we may, without loss of generality, assume that $v_1 > 0$ and $v_2 > 0$. Moreover, if necessary, we may exchange the indices 1 and 2 so that $v_2 \geq v_1 > 0$.

2.8.1 Proof of case (i)

In case (i), $v_1 = v_2 > 0$ will be denoted v. We can notice that the explicit computation yielding (22) is still valid in this case, thus

$$\begin{cases} A_1(t,x) = & A_1^{\text{in}}(x - tv) \cosh\Big(\frac{\sqrt{\sigma_1\sigma_2}}{v} \int_{x-tv}^{x} A_3(y)\ dy \Big) \\ & -\text{sgn}(\sigma_1) \sqrt{\frac{\sigma_1}{\sigma_2}}\, A_2^{\text{in}}(x - tv) \sinh\Big(\frac{\sqrt{\sigma_1\sigma_2}}{v} \int_{x-tv}^{x} A_3(y)\ dy \Big) \\ A_2(t,x) = & A_2^{\text{in}}(x - tv) \cosh\Big(\frac{\sqrt{\sigma_1\sigma_2}}{v} \int_{x-tv}^{x} A_3(y)\ dy \Big) \\ & -\text{sgn}(\sigma_1) \sqrt{\frac{\sigma_2}{\sigma_1}}\, A_1^{\text{in}}(x - tv) \sinh\Big(\frac{\sqrt{\sigma_1\sigma_2}}{v} \int_{x-tv}^{x} A_3(y)\ dy \Big) \end{cases}$$

and (i) follows easily.

2.8.2 Proof of case (ii)

Let $(T, X) \in \mathbb{R}^+ \times \mathbb{R}$, and assume that A solves (5). We recall that $A^{\text{in}} \in \mathcal{C}_b(\mathbb{R})$, hence $A \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}, \mathbb{C})$. Due to the finite speeds of propagation $v_2 \geq v_1 > 0$, the value of $A(T, X)$ will depend only on A_3 and A^{in} in $[X - Tv_2, X]$. We wish to prove, for some constant $C > 0$ depending only on σ_1, σ_2, v_1 and v_2, the pointwise estimate

$$|A(T,X)|_\infty \leq C \big\| A^{\text{in}} \big\|_{\mathcal{C}([X - Tv_2, X])} \exp\Big(\sqrt{\frac{\sigma_1\sigma_2}{v_1 v_2}} \int_{X - Tv_2}^{X} |A_3|(y)\ dy \Big). \tag{44}$$

For some given $r \in \mathbb{N}^*$ (that will tend to $+\infty$), we divide the interval $[X - Tv_2, X]$ into r subintervals $[x^j, x^{j+1}]$, $0 \leq j < r$, with $x^j \equiv X - Tv_2(1 - j/r)$, $j \in \mathbb{Z}$ (we omit the dependency on r in the notations). Since $A_3 \in L^1_{loc}(\mathbb{R})$, there exists $r_0 \in \mathbb{N}^*$ (depending on T and X) such that, if $r \geq r_0$,

$$\forall\, 0 \leq j < r, \qquad \alpha^j \equiv \sqrt{\frac{\sigma_1\sigma_2}{v_1 v_2}} \int_{x^j}^{x^{j+1}} |A_3|(y)\ dy < 1. \tag{45}$$

On each subinterval (x^j, x^{j+1}) $(0 \le j < r)$, A solves, in the space-time trapezoid

$$\mathcal{D}^j \equiv \{(t,x) \in \mathbb{R}^+ \times (x^j, x^{j+1}),\ x - t\upsilon_2 \ge X - T\upsilon_2\},$$

the boundary value problem

$$\begin{cases} \partial_t A_1 + \upsilon_1 \partial_x A_1 + \sigma_1 A_3 A_2 = 0 \\ \partial_t A_2 + \upsilon_2 \partial_x A_2 + \sigma_2 \bar{A}_3 A_1 = 0, \end{cases} \quad \text{for} \quad (t,x) \in \mathcal{D}^j \tag{46}$$

with the boundary conditions (since $\upsilon_2 \ge \upsilon_1 > 0$, they are naturally on the left-hand side of the domain and at the initial time)

$$\begin{cases} A(t, x = x^j) = A(t, x = x^j), & 0 \le t \le T, \\ A(t = 0, x) = A^{\text{in}}(x), & x^j \le x \le x^{j+1}. \end{cases} \tag{47}$$

Using the method of characteristics, we find for $(t,x) \in \mathcal{D}^j$:

$$\begin{aligned} A_1(t,x) = &\mathbf{1}_{x \ge x^j + t\upsilon_1} \Big\{ A_1^{\text{in}}(x - t\upsilon_1) - \sigma_1 \int_0^t A_3(x - (t-\tau)\upsilon_1) A_2(\tau, x - (t-\tau)\upsilon_1)\ d\tau \Big\} \\ &+ \mathbf{1}_{x \le x^j + t\upsilon_1} \Big\{ A_1\Big(t - \frac{x - x^j}{\upsilon_1}, x^j\Big) - \sigma_1 \int_{t - \frac{x-x^j}{\upsilon_1}}^t A_3(x - (t-\tau)\upsilon_1) A_2(\tau, x - (t-\tau)\upsilon_1)\ d\tau \Big\} \\ A_2(t,x) = &\mathbf{1}_{x \ge x^j + t\upsilon_2} \Big\{ A_2^{\text{in}}(x - t\upsilon_2) - \sigma_2 \int_0^t \bar{A}_3(x - (t-\tau)\upsilon_2) A_1(\tau, x - (t-\tau)\upsilon_2)\ d\tau \Big\} \\ &+ \mathbf{1}_{x \le x^j + t\upsilon_2} \Big\{ A_2\Big(t - \frac{x - x^j}{\upsilon_2}, x^j\Big) - \sigma_2 \int_{t - \frac{x-x^j}{\upsilon_2}}^t \bar{A}_3(x - (t-\tau)\upsilon_2) A_1(\tau, x - (t-\tau)\upsilon_2)\ d\tau \Big\}. \end{aligned}$$

Denoting $t^j \equiv T - (X - x^j)/\upsilon_2$ for $0 \le j \le r$, this yields the inequalities (since $\upsilon_2 \ge \upsilon_1 > 0$)

$$\begin{aligned} \|A_1\|_{\mathcal{C}(\mathcal{D}^j)} &\le \max \Big\{ \|A_1^{\text{in}}\|_{\mathcal{C}([x^j, x^{j+1}])} + \frac{|\sigma_1|}{\upsilon_1} \|A_2\|_{\mathcal{C}(\mathcal{D}^j)} \int_{x^j}^{x^{j+1}} |A_3|(y)\ dy; \\ &\qquad \|A_1(\cdot, x^j)\|_{\mathcal{C}([0,t^j])} + \frac{|\sigma_1|}{\upsilon_1} \|A_2\|_{\mathcal{C}(\mathcal{D}^j)} \int_{x^j}^{x^{j+1}} |A_3|(y)\ dy \Big\} \\ &= \max \Big\{ \|A_1^{\text{in}}\|_{\mathcal{C}([x^j, x^{j+1}])}; \|A_1(\cdot, x^j)\|_{\mathcal{C}([0,t^j])} \Big\} + \frac{|\sigma_1|}{\upsilon_1} \|A_2\|_{\mathcal{C}(\mathcal{D}^j)} \int_{x^j}^{x^{j+1}} |A_3|(y)\ dy \end{aligned} \tag{48}$$

$$\begin{aligned} \|A_2\|_{\mathcal{C}(\mathcal{D}^j)} &\le \max \Big\{ \|A_2^{\text{in}}\|_{\mathcal{C}([x^j, x^{j+1}])} + \frac{|\sigma_2|}{\upsilon_2} \|A_1\|_{\mathcal{C}(\mathcal{D}^j)} \int_{x^j}^{x^{j+1}} |A_3|(y)\ dy; \\ &\qquad \|A_2(\cdot, x^j)\|_{\mathcal{C}([0,t^j])} + \frac{|\sigma_2|}{\upsilon_2} \|A_1\|_{\mathcal{C}(\mathcal{D}^j)} \int_{x^j}^{x^{j+1}} |A_3|(y)\ dy \Big\} \\ &= \max \Big\{ \|A_2^{\text{in}}\|_{\mathcal{C}([x^j, x^{j+1}])}; \|A_2(\cdot, x^j)\|_{\mathcal{C}([0,t^j])} \Big\} + \frac{|\sigma_2|}{\upsilon_2} \|A_1\|_{\mathcal{C}(\mathcal{D}^j)} \int_{x^j}^{x^{j+1}} |A_3|(y)\ dy. \end{aligned} \tag{49}$$

Setting, for $0 \le j \le r$,

$$\begin{cases} \mathfrak{a}_1^j \equiv \max \Big\{ \|A_1^{\text{in}}\|_{\mathcal{C}([X - T\upsilon_2, X])}; \|A_1\|_{\mathcal{C}(\mathcal{D}^j)} \Big\}, \\ \mathfrak{a}_2^j \equiv \sqrt{\dfrac{\sigma_1 \upsilon_2}{\sigma_2 \upsilon_1}} \max \Big\{ \|A_2^{\text{in}}\|_{\mathcal{C}([X - T\upsilon_2, X])}; \|A_2\|_{\mathcal{C}(\mathcal{D}^j)} \Big\}, \end{cases}$$

we infer from (48) and (49) that for $0 \le j \le r$,

$$\begin{cases} \mathfrak{a}_1^j \le \mathfrak{a}_1^{j-1} + \alpha^j \mathfrak{a}_2^j, \\ \mathfrak{a}_2^j \le \mathfrak{a}_2^{j-1} + \alpha^j \mathfrak{a}_1^j. \end{cases}$$

As a consequence, the quantity

$$\mathfrak{a}^j \equiv \mathfrak{a}_1^j + \mathfrak{a}_2^j$$

satisfies (in view of (45))

$$\mathfrak{a}^j \leq \frac{\mathfrak{a}^{j-1}}{1-\alpha^j}.$$

Therefore, for some constant $C > 0$ depending only on σ_1, σ_2, υ_1 and υ_2,

$$\frac{1}{C}|A(T,X)|_\infty \leq \mathfrak{a}^r \leq \mathfrak{a}^0 \times \Big(\prod_{j=1}^r (1-\alpha^j)\Big)^{-1}.$$

Letting $r \to +\infty$, since $\alpha^j \to 0$ uniformly for $0 \leq j \leq r$, we have

$$\Big(\prod_{j=1}^r (1-\alpha^j)\Big)^{-1} = \exp\Big(-\sum_{j=1}^r \ln(1-\alpha^j)\Big) \to \exp\Big(\sqrt{\frac{\sigma_1\sigma_2}{\upsilon_1\upsilon_2}} \int_{X-T\upsilon_2}^X |A_3|(y)\ dy\Big).$$

Moreover, as $r \to +\infty$,

$$\mathfrak{a}^0 \to \|A_1^{\text{in}}\|_{\mathcal{C}([X-T\upsilon_2,X])} + \sqrt{\frac{\sigma_1\upsilon_2}{\sigma_2\upsilon_1}}\|A_2^{\text{in}}\|_{\mathcal{C}([X-T\upsilon_2,X])},$$

thus (44) follows, and hence statement (ii).

2.8.3 Proof of case (iii)

We consider here A_3 satisfying hypothesis (13). It is sufficient to show that if $(T,X) \in \mathbb{R}^+ \times \mathbb{R}^-$ and $A^{\text{in}} \in \mathcal{C}_c(\mathbb{R},\mathbb{R}^2)$ is such that

$$0 \leq A_1^{\text{in}} \leq 1, \quad 0 \leq A_2^{\text{in}} \leq \sqrt{\frac{\sigma_1\upsilon_2}{\sigma_2\upsilon_1}} \qquad\qquad A_1^{\text{in}} = 1, \quad A_2^{\text{in}} = \sqrt{\frac{\sigma_1\upsilon_2}{\sigma_2\upsilon_1}} \quad \text{in} \quad [X-T\upsilon_2, X]$$

and A_1^{in} and A_2^{in} are nondecreasing in $\mathbb{R}^-$, then

$$|A(T,X)|_\infty \geq c\exp\Big(\frac{\upsilon_1}{\upsilon_2}\sqrt{\frac{\sigma_1\sigma_2}{\upsilon_1\upsilon_2}} \int_{X-T\upsilon_2}^X |A_3|(y)\ dy\Big) \tag{50}$$

for some constant $c > 0$ depending only on σ_1, σ_2, υ_1 and υ_2. Indeed, (iii) will then follow from the fact that A_3 vanishes in $[1,+\infty)$, hence

$$\sup_{x\leq 0} \int_{x-T\upsilon_1}^x |A_3|(y)\ dy \geq \sup_{x\in\mathbb{R}} \int_{x-T\upsilon_1}^x |A_3|(y)\ dy - \int_0^1 |A_3|(y)\ dy.$$

We recall that (10) is a fixed point problem $A = \Upsilon[A]$, where $\Upsilon = (\Upsilon_1, \Upsilon_2)$,

$$\begin{cases} \Upsilon_1[A](t,x) \equiv A_1^{\text{in}}(x-t\upsilon_1) - \sigma_1 \displaystyle\int_0^t A_2(\tau, x-(t-\tau)\upsilon_1)A_3(x-(t-\tau)\upsilon_1)\ d\tau \\ \Upsilon_2[A](t,x) \equiv A_2^{\text{in}}(x-t\upsilon_2) - \sigma_2 \displaystyle\int_0^t A_1(\tau, x-(t-\tau)\upsilon_2)A_3(x-(t-\tau)\upsilon_2)\ d\tau. \end{cases}$$

The solution A is real-valued since A^{in} and A_3 are. From (13) (recall that $\sigma_1\sigma_2 > 0$), $-\sigma_1 A_3$ and $-\sigma_2 A_3$ are nonnegative and nondecreasing in $\mathbb{R}^-$. Moreover, A_1^{in} and A_2^{in} are nonnegative and nondecreasing in $\mathbb{R}^-$. As a consequence, if $A_1(t)$ and $A_2(t)$ are nonnegative and nondecreasing in $\mathbb{R}^-$ for every $t \geq 0$, then $\Upsilon_1[A](t)$ and $\Upsilon_2[A](t)$ are also nonnegative and nondecreasing in $\mathbb{R}^-$ for every $t \geq 0$ (since $0 \leq \upsilon_1 < \upsilon_2$, thus for $0 \leq \tau \leq t$, $x-(t-\tau)\upsilon_1 \leq x-(t-\tau)\upsilon_2 \leq x \leq 0$). It follows that if $A = \Sigma_t A^{\text{in}}$, then $A_1(t)$ and $A_2(t)$ are nonnegative and nondecreasing in $\mathbb{R}^-$ for every $t \geq 0$.

As for the case (ii), for $r \in \mathbb{N}^*$, we split $[X-T\upsilon_2, X] \subset \mathbb{R}^-$ into r subintervals $[x^j, x^{j+1}]$, $0 \leq j < r$, with $x^j \equiv X - T\upsilon_2(1-j/r)$ for $j \in \mathbb{Z}$ and set $t^j \equiv T - (X-x^j)/\upsilon_2 = T/r$ for $0 \leq j \leq r$. We fix j, $k \in \mathbb{N}$ with $0 \leq j \leq k < r$, $k \in \mathbb{N}$ and work in the (space-time) triangle $\mathcal{D}^{j,k}$ with vertices (x^{j-1}, t^k), (x^j, t^k) and (x^j, t^{k+1}). We show a bound from below for A_1 and A_2 at (x^j, t^k) depending on their values at (x^{j-1}, t^k).

First, since $A_1(t)$ and $A_2(t)$ are nonnegative and nondecreasing in $\mathbb{R}^-$ for every $t \geq 0$, we infer as a first step that for $(t, x) \in \mathcal{D}^{j,k}$,

$$A_1(t,x) \geq A_1(t^k, x-(t-t^k)\upsilon_1) \geq A_1(t^k, x^{j-1}) \qquad \text{and} \qquad A_2(t,x) \geq A_2(t^k, x-(t-t^k)\upsilon_2) \geq A_2(t^k, x^{j-1}).$$

Consequently,

$$\begin{aligned} A_1(t^{k+1}, x^j) &\geq A_1(t^k, x^j - (t^{k+1} - t^k)\upsilon_1) - \sigma_1 \int_{t^k}^{t^{k+1}} A_2(\tau, x^j - (t^{k+1} - \tau)\upsilon_1) A_3(x^j - (t^{k+1} - \tau)\upsilon_1) \, d\tau \\ &\geq A_1(t^k, x^{j-1}) - \frac{\sigma_1}{\upsilon_1} A_2(t^k, x^{j-1}) \int_{x^j - (t^{k+1} - t^k)\upsilon_1}^{x^j} A_3(y) \, dy \end{aligned}$$

(since $0 \leq \upsilon_1 \leq \upsilon_2$, we have $(\tau, x^j - (t^{k+1} - \tau)\upsilon_1) \in \mathcal{D}^{j,k}$), and similarly

$$\begin{aligned} A_2(t^{k+1}, x^j) &\geq A_2(t^k, x^j - (t^{k+1} - t^k)\upsilon_2) - \sigma_2 \int_{t^k}^{t^{k+1}} A_1(\tau, x^j - (t^{k+1} - \tau)\upsilon_2) A_3(x^j - (t^{k+1} - \tau)\upsilon_2) \, d\tau \\ &\geq A_2(t^k, x^{j-1}) - \frac{\sigma_2}{\upsilon_2} A_1(t^k, x^{j-1}) \int_{x^j - (t^{k+1} - t^k)\upsilon_2}^{x^j} A_3(y) \, dy. \end{aligned}$$

From the fact that $0 \leq \upsilon_1 \leq \upsilon_2$, we deduce that the quantity

$$\mathfrak{a}^{j,k} \equiv \min \Big\{ A_1(t^k, x^j); \sqrt{\frac{\sigma_1 \upsilon_2}{\sigma_2 \upsilon_1}} A_2(t^k, x^j) \Big\}$$

verifies

$$\mathfrak{a}^{j,k+1} \geq \mathfrak{a}^{j-1,k} \Big(1 + \sqrt{\frac{\sigma_1 \sigma_2}{\upsilon_1 \upsilon_2}} \int_{x^j - (t^{k+1} - t^k)\upsilon_1}^{x^j} |A_3|(y) \, dy \Big).$$

Therefore,

$$\begin{aligned} \mathfrak{a}^{r,r} &\geq \mathfrak{a}^{0,0} \prod_{j=1}^{r} \Big(1 + \sqrt{\frac{\sigma_1 \sigma_2}{\upsilon_1 \upsilon_2}} \int_{x^j - T\upsilon_1/r}^{x^j} |A_3|(y) \, dy \Big) \\ &\geq \exp \Big(\sum_{j=1}^{r} \sqrt{\frac{\sigma_1 \sigma_2}{\upsilon_1 \upsilon_2}} \int_{x^j - T\upsilon_1/r}^{x^j} |A_3|(y) \, dy + \frac{C(X,T)}{r} \Big). \end{aligned}$$

since $\mathfrak{a}^{0,0} = 1$ and $\int_{x^j - T\upsilon_1/r}^{x^j} |A_3|(y) \, dy \leq C_0(X,T)/r$. Now, since $|A_3|$ is nondecreasing in $\mathbb{R}^-$,

$$\sum_{j=1}^{r} \int_{x^j - T\upsilon_1/r}^{x^j} |A_3|(y) \, dy \geq \frac{T\upsilon_1}{r} \sum_{j=1}^{r} |A_3|(x^j) \to \frac{\upsilon_1}{\upsilon_2} \int_{X - T\upsilon_2}^{X} |A_3|(y) \, dy$$

as $r \to +\infty$. Hence, passing to the limit as $r \to +\infty$ in the previous inequality yields

$$\min \Big\{ A_1(T,X); \sqrt{\frac{\sigma_1 \upsilon_2}{\sigma_2 \upsilon_1}} A_2(T,X) \Big\} \geq \exp \Big(\frac{\upsilon_1}{\upsilon_2} \sqrt{\frac{\sigma_1 \sigma_2}{\upsilon_1 \upsilon_2}} \int_{X - T\upsilon_2}^{X} |A_3|(y) \, dy \Big).$$

This establishes (50) for some constant $c > 0$ depending only on σ_1, σ_2, υ_1 and υ_2.

2.9 Proof of Proposition 10

Here, we recall that $\upsilon_1 = 0 \neq \upsilon_2$. Let us first recall that the telegraph equation

$$\partial^2_{XY} U + K^2 U = 0,$$

for $U = U(X,Y) : \mathbb{R}^2 \to \mathbb{R}$ and $K > 0$ is a constant, has a well-known particular explicit solution given by

$$U(X,Y) \equiv J_0(2K\sqrt{XY}),$$

where, J_0 is the Bessel function defined by

$$J_0(z) = \frac{1}{\pi}\int_0^{\pi} \cos\Big(z\sin(u)\Big)\,du = \sum_{j=0}^{+\infty} \frac{(-1)^j z^{2j}}{(j!)^2}.$$

Indeed,

$$\partial^2_{XY} U = -K^2\Big(J_0''(2K\sqrt{XY}) + \frac{J_0'(2K\sqrt{XY})}{2K\sqrt{XY}}\Big) \qquad \text{and} \qquad J_0''(z) + \frac{J_0'(z)}{z} + J_0(z) = 0.$$

Similarly, the above computation yields that

$$A_2(t,x) \equiv J_0\Big(2|\mathcal{A}_3|\sqrt{\frac{\sigma_1\sigma_2}{\upsilon_2}}\sqrt{x\Big(\frac{x}{\upsilon_2} - t\Big)}\Big)$$

solves, in $\mathbb{R}_+ \times (0,\ell)$,

$$\partial_t\Big(\partial_t + \upsilon_2\partial_x\Big)A_2 = \sigma_1\sigma_2|\mathcal{A}_3|^2 A_2.$$

One could then define A_1 such that (A_1, A_2) solves (5) in $\mathbb{R}_+ \times (0,\ell)$, with appropriate boundary and initial data. Moreover, the Laplace method yields the asymptotic behavior of J_0 on the imaginary axis:

$$J_0(i\xi) \sim \frac{\mathrm{e}^{\xi}}{\sqrt{2\pi\xi}} \qquad \text{for } \xi \to +\infty \quad (\xi \in \mathbb{R}). \tag{51}$$

Hence, for fixed $x \in (0,\ell)$,

$$A_2(t,x) \sim \frac{c_0}{|\mathcal{A}_3|\sqrt{t}} \exp\Big(2\sqrt{\frac{\sigma_1\sigma_2}{\upsilon_2}}|\mathcal{A}_3|\sqrt{xt}\Big),$$

where the positive constant c_0 depends only on σ_1, σ_2 and υ_2. Due to the fact that the boundary datum involved in A_1 is unbounded, the growth rate in the exponential has an extra factor 2.

2.9.1 Proof of case (i)

Without loss of generality, we may assume that the interval I is $(0,\ell)$. Then, (10) becomes

$$\begin{cases} A_1(t,x) = A_1^{\mathrm{in}}(x) - \sigma_1 A_3(x)\displaystyle\int_0^t A_2(\tau,x)\,d\tau \\ A_2(t,x) = A_2^{\mathrm{in}}(x - t\upsilon_2) - \sigma_2\displaystyle\int_0^t A_1(\tau, x-(t-\tau)\upsilon_2)\bar{A}_3(x-(t-\tau)\upsilon_2)\,d\tau. \end{cases}$$

Choosing $A_1^{\mathrm{in}} \equiv 0$ and $A_2^{\mathrm{in}} \equiv 1$, we infer

$$A_2(t,x) = 1 + \frac{\sigma_1\sigma_2}{\upsilon_2}\int_{x-t\upsilon_2}^{x} |A_3|^2(y)\int_0^{t-(x-y)/\upsilon_2} A_2(\theta,y)\,d\theta dy.$$

It is then clear that A_1 and A_2 remain non negative for $t > 0$. Moreover, for $0 \le x \le \ell$ and $t \ge x/\upsilon_2$,

$$A_2(t,x) \ge 1 + \frac{\sigma_1\sigma_2}{\upsilon_2}|\mathcal{A}_3|^2\int_0^x\int_{y/\upsilon_2}^{t-(x-y)/\upsilon_2} A_2(\theta,y)\,d\theta dy. \tag{52}$$

This implies, as a first step, for $0 \le x \le \ell$ and $t \ge x/\upsilon_2$,

$$A_2(t,x) \ge 1,$$

Inserting this lower bound into (52) yields as a second step

$$A_2(t,x) \ge 1 + \frac{\sigma_1\sigma_2}{\upsilon_2}|\mathcal{A}_3|^2\int_0^x\int_{y/\upsilon_2}^{t-(x-y)/\upsilon_2} 1\,d\theta dy = 1 + \frac{\sigma_1\sigma_2}{\upsilon_2}|\mathcal{A}_3|^2 x\Big(t - \frac{x}{\upsilon_2}\Big).$$

Arguing by induction and using the fact that, for $n \in \mathbb{N}$,

$$\int_0^x \int_{y/\upsilon_2}^{t-(x-y)/\upsilon_2} y^n \Big(\theta - \frac{y}{\upsilon_2}\Big)^n \, d\theta dy = \frac{1}{(n+1)^2} \, x^{n+1} \Big(t - \frac{x}{\upsilon_2}\Big)^{n+1},$$

it follows that, for any $n \in \mathbb{N}$,

$$A_2(t,x) \geq \sum_{j=0}^{n} \frac{1}{(j!)^2} \Big[\frac{\sigma_1\sigma_2}{\upsilon_2} |\mathcal{A}_3|^2\Big]^j x^j \Big(t - \frac{x}{\upsilon_2}\Big)^j.$$

Letting $n \to +\infty$ and in view of the power series expansion of J_0, it comes

$$A_2(t,x) \geq J_0\Big(i\sqrt{\frac{\sigma_1\sigma_2}{\upsilon_2}} |\mathcal{A}_3| \sqrt{x\Big(t - \frac{x}{\upsilon_2}\Big)}\Big).$$

Hence, it follows from the asymptotic behavior (51) of J_0 on the imaginary axis that for any $0 < \varepsilon < \ell$, there exists $c_\varepsilon > 0$ such that

$$A_2(t, x = \ell) \geq c_\varepsilon \exp\Big(\sqrt{\frac{\sigma_1\sigma_2}{\upsilon_2}} |\mathcal{A}_3| \sqrt{(\ell - \varepsilon)t}\Big)$$

provided t is sufficiently large.

2.9.2 Proof of case (ii)

We consider $A_3 \in L^\infty(\mathbb{R})$ and compactly supported, say in $(0,\ell)$ without loosing generality. In the regions $\{x < 0\}$ and $\{x > \ell\}$, A_3 vanishes and A_2 is explicitable:

$$A_2(t,x) = A_2^{\text{in}}(x - t\upsilon_2) \qquad \text{or} \qquad A_2(t,x) = A_2\Big(t - \frac{x-\ell}{\upsilon_2}, \ell\Big)$$

if ($x \leq 0$ or $x \geq \ell + t\upsilon_2$) or $\ell \leq x \leq \ell + t\upsilon_2$, thus it suffices to prove the estimate in $[0,\ell]$. As in the previous case, (10) yields

$$A_2(t,x) = A_2^{\text{in}}(x - t\upsilon_2) - \frac{\sigma_2}{\upsilon_2} \int_{x-t\upsilon_2}^x A_1^{\text{in}}(y) \bar{A}_3(y) \, dy + \frac{\sigma_1\sigma_2}{\upsilon_2} \int_{x-t\upsilon_2}^x |A_3|^2(y) \int_0^{t-(x-y)/\upsilon_2} A_2(\theta, y) \, d\theta dy,$$

thus, for $0 \leq x \leq \ell$,

$$|A_2|(t,x) \leq \big\|A_2^{\text{in}}\big\|_{L^\infty} + \frac{|\sigma_2|}{\upsilon_2} \ell \big\|A_3\big\|_{L^\infty} \big\|A_1^{\text{in}}\big\|_{L^\infty} + \frac{\sigma_1\sigma_2}{\upsilon_2} \big\|A_3\big\|_{L^\infty}^2 \int_0^x \int_0^t |A_2|(\theta, y) \, d\theta dy. \tag{53}$$

The idea is to construct a super solution to this inequality, using a power series expansion as in the previous case. Let

$$\Phi(t,x) \equiv \sum_{j=0}^{+\infty} \Big[\frac{\sigma_1\sigma_2}{\upsilon_2} \big\|A_3\big\|_{L^\infty}^2\Big]^j \frac{x^j t^j}{(j!)^2} = J_0\Big(i\sqrt{\frac{\sigma_1\sigma_2}{\upsilon_2}} \big\|A_3\big\|_{L^\infty} \sqrt{xt}\Big) > 0.$$

Then, an immediate computation yields

$$1 + \frac{\sigma_1\sigma_2}{\upsilon_2} \big\|A_3\big\|_{L^\infty}^2 \int_0^x \int_0^t \Phi(\theta, y) \, d\theta dy = \Phi(t,x). \tag{54}$$

Therefore, for some constant K_0 depending only on σ_1, σ_2, υ_2 and $\ell\|A_3\|_{L^\infty}$, we have, for $0 \leq x \leq \ell$,

$$\begin{aligned}\Big\{|A_2| - K_0 \big\|A^{\text{in}}\big\|_{L^\infty} \Phi\Big\}_+(t,x) &\leq \frac{\sigma_1\sigma_2}{\upsilon_2} \big\|A_3\big\|_{L^\infty}^2 \int_0^x \int_0^t \Big\{|A_2| - K_0 \big\|A^{\text{in}}\big\|_{L^\infty} \Phi\Big\}(\theta, y) \, d\theta dy \\ &\leq \frac{\sigma_1\sigma_2}{\upsilon_2} \big\|A_3\big\|_{L^\infty}^2 \ell \int_0^t \Big\|\Big\{|A_2| - K_0 \big\|A^{\text{in}}\big\|_{L^\infty} \Phi\Big\}_+(\theta)\Big\|_{L^\infty(0,\ell)} \, d\theta.\end{aligned}$$

Hence, by Gronwall inequality, for any $t \geq 0$ and $x \in [0,\ell]$,

$$\Big\{|A_2| - K_0 \big\|A^{\text{in}}\big\|_{L^\infty} \Phi\Big\}(t,x) \leq 0.$$

Consequently, from the asymptotics (51),

$$|A_2| \le K_0 \|A^{\rm in}\|_{L^\infty} \Phi(t,x) \le \frac{C}{\sqrt{t}} \|A^{\rm in}\|_{L^\infty} \exp\Big(\sqrt{\frac{\sigma_1\sigma_2}{\upsilon_2}} \|A_3\|_{L^\infty} \sqrt{\ell t}\Big),$$

for some constant C depending on σ_1, σ_2, υ_2, ℓ and $\|A_3\|_{L^\infty}$. Inserting this upper bound in the first line of (10) yields

$$\begin{aligned} |A_1|(t,x) &\le \|A_1^{\rm in}\|_{L^\infty} + |\sigma_1| \|A_3\|_{L^\infty} \|A^{\rm in}\|_{L^\infty} \int_0^t \frac{C}{\sqrt{\tau}} \exp\Big(\sqrt{\frac{\sigma_1\sigma_2}{\upsilon_2}} \|A_3\|_{L^\infty} \sqrt{\ell\tau}\Big)\, d\tau \\ &\le C' \|A^{\rm in}\|_{L^\infty} \exp\Big(\sqrt{\frac{\sigma_1\sigma_2}{\upsilon_2}} \|A_3\|_{L^\infty} \sqrt{\ell t}\Big). \end{aligned}$$

The proof is complete.

2.10 Proof of Proposition 11

We denote $(\tilde{x}_1, \tilde{x}_2)$ the coordinates in the basis $(\upsilon_1; \upsilon_2)$ of $\mathbb{R}^2$. We follow closely the arguments used in the proof of Proposition 9. We fix $(T, X) \in \mathbb{R}_+ \times \mathbb{R}^2$. For $r \in \mathbb{N}^*$ and $j = (j_1, j_2) \in \mathbb{Z}^2$, we denote

$$V^j \equiv \tilde{X}_1 - T(1 - j_1/r)\upsilon_1 + \tilde{X}_2 - T(1 - j_1/r)\upsilon_2 \in \mathbb{R}^2.$$

We define $\mathcal{R}$ as the parallelogram with vertices 0, $\tilde{X}_2\upsilon_2 = V^{(0,r)}$, $\tilde{X}_1\upsilon_1 = V^{(r,0)}$ and $X = V^{(r,r)}$, and $\mathcal{R}^j$ ($j \in \mathbb{Z}^2$) as the parallelogram with vertices $V^{(j_1,j_2)}$, $V^{(j_1+1,j_2)}$, $V^{(j_1,j_2+1)}$ and $V^{(j_1+1,j_2+1)}$. We then decompose the parallelogram $\mathcal{R}$ into the r^2 subparallelograms $\mathcal{R}^j$, $j \in \mathbb{Z}^2$, $0 \le j_1, j_2 < r$.

On each parallelogram $\mathcal{R}^j$, A solves, in the space-time domain

$$\mathcal{D}^j \equiv \{(t,x) \in \mathbb{R}^+ \times \mathcal{R}^j,\ \tilde{x}_1 + \tilde{x}_2 - t \ge \tilde{X}_1 + \tilde{X}_2 - T\},$$

the boundary value problem

$$\begin{cases} \partial_t A_1 + \partial_{\tilde{x}_1} A_1 + \sigma_1 A_3 A_2 = 0 \\ \partial_t A_2 + \partial_{\tilde{x}_2} A_2 + \sigma_2 \bar{A}_3 A_1 = 0, \end{cases} \qquad \text{for} \quad (t,x) \in \mathcal{D}^j \tag{55}$$

with the boundary conditions on the faces $\mathcal{F}_1^j \equiv \mathcal{D}^j \cap \{\tilde{x}_1 = V_1^j\}$ and $\mathcal{F}_2^j \equiv \mathcal{D}^j \cap \{\tilde{x}_2 = V_2^j\}$:

$$\begin{aligned} A_1(t, \tilde{x}_1 = \tilde{V}_1^j, \tilde{x}_2) &= A_1(t, \tilde{x}_1 = \tilde{V}_1^j, \tilde{x}_2) && (t, \tilde{x}_1 = \tilde{V}_1^j, \tilde{x}_2) \in \mathcal{F}_1^j, \\ A_2(t, \tilde{x}_1, \tilde{x}_2 = \tilde{V}_2^j) &= A_2(t, \tilde{x}_1, \tilde{x}_2 = \tilde{V}_2^j) && (t, \tilde{x}_1, \tilde{x}_2 = \tilde{V}_2^j) \in \mathcal{F}_2^j, \\ A(t = 0, \tilde{x}) &= A^{\rm in}(\tilde{x}), && \tilde{x} \in \mathcal{R}^j. \end{aligned}$$

By the method of characteristics, we find for $(t,x) \in \mathcal{D}^j$:

$$\begin{aligned} |A_1|(t,\tilde{x}) \le\, & \mathbf{1}_{\tilde{x}_1 \ge \tilde{x}_1^j + t} \Big\{ \|A_1^{\rm in}\|_{\mathcal{C}(\mathcal{R}^j)} + |\sigma_1| \int_0^t \|A_3\|_{L^\infty(\mathcal{R}^j)} \|A_2\|_{\mathcal{C}(\mathcal{D}^j)}\, d\tau \Big\} \\ & + \mathbf{1}_{\tilde{x}_1 \le \tilde{V}_1^j + t} \Big\{ \|A_{1|\tilde{x}_1 = \tilde{V}_1^j}\|_{\mathcal{C}(\mathcal{F}_1^j)} + |\sigma_1| \int_{t - \frac{\tilde{x}_1 - \tilde{x}_1^j}{|\upsilon_1|}}^t \|A_3\|_{L^\infty(\mathcal{R}^j)} \|A_2\|_{\mathcal{C}(\mathcal{D}^j)}\, d\tau \Big\} \\ \le\, & \max \Big\{ \|A_1^{\rm in}\|_{\mathcal{C}(\mathcal{R})}; \|A_{1|\tilde{x}_1 = \tilde{V}_1^j}\|_{\mathcal{C}(\mathcal{F}_1^j)} \Big\} + |\sigma_1| \frac{T}{r} \|A_3\|_{L^\infty(\mathcal{R}^j)} \|A_2\|_{\mathcal{C}(\mathcal{D}^j)} \end{aligned}$$

and similarly (here, the two quantities A_1 and A_2 play symmetric roles)

$$|A_2|(t,\tilde{x}) \le \max \Big\{ \|A_2^{\rm in}\|_{\mathcal{C}(\mathcal{R})}; \|A_{2|\tilde{x}_2 = \tilde{x}_2^j}\|_{\mathcal{C}(\mathcal{F}_2^j)} \Big\} + |\sigma_2| \frac{T}{r} \|A_3\|_{L^\infty(\mathcal{R}^j)} \|A_1\|_{\mathcal{C}(\mathcal{D}^j)}.$$

For $j \in \mathbb{Z}^2$, with $0 \le j_1, j_2 < r$, we set

$$\alpha^j \equiv \frac{T}{r} \sqrt{\sigma_1\sigma_2} \|A_3\|_{L^\infty(\mathcal{R}^j)}, \qquad \begin{cases} \mathfrak{a}_1^j \equiv \max \Big\{ \|A_1^{\rm in}\|_{\mathcal{C}(\mathcal{R})}; \|A_1\|_{\mathcal{C}(\mathcal{D}^j)} \Big\}, \\ \mathfrak{a}_2^j \equiv \sqrt{\dfrac{\sigma_1\upsilon_2}{\sigma_2\upsilon_1}} \max \Big\{ \|A_2^{\rm in}\|_{\mathcal{C}(\mathcal{R})}; \|A_2\|_{\mathcal{C}(\mathcal{D}^j)} \Big\}. \end{cases}$$

Then, we have

$$\begin{cases} \mathfrak{a}_1^{(j_1,j_2)} \leq \mathfrak{a}_1^{(j_1-1,j_2)} + \alpha^{(j_1,j_2)}\mathfrak{a}_2^{(j_1,j_2)}, \\ \mathfrak{a}_2^{(j_1,j_2)} \leq \mathfrak{a}_2^{(j_1,j_2-1)} + \alpha^{(j_1,j_2)}\mathfrak{a}_1^{(j_1,j_2)}, \end{cases}$$

from which it follows

$$\begin{cases} \mathfrak{a}_1^{(j_1,j_2)} \leq \dfrac{\mathfrak{a}_1^{(j_1-1,j_2)} + \alpha^{(j_1,j_2)}\mathfrak{a}_2^{(j_1,j_2-1)}}{1-(\alpha^{(j_1,j_2)})^2}, \\ \mathfrak{a}_2^{(j_1,j_2)} \leq \dfrac{\mathfrak{a}_1^{(j_1,j_2-1)} + \alpha^{(j_1,j_2)}\mathfrak{a}_2^{(j_1-1,j_2)}}{1-(\alpha^{(j_1,j_2)})^2}, \end{cases}$$

as soon as $\alpha^{(j_1,j_2)} < 1$, which is the case if $r \geq 1 + T\sqrt{\sigma_1\sigma_2}\big\|A_3\big\|_{L^\infty(\mathbb{R}^2)}$. Let us set

$$K^0 \equiv \max\Big\{\mathfrak{a}_\ell^{(i,0)}; \ell = 1,2 \ 0 \leq i \leq r\Big\}. \tag{56}$$

By an immediate induction, we infer

$$\mathfrak{a}_1^{(i,1)} \leq K^0\delta_i, \tag{57}$$

where the δ_i's are defined by $\delta_0 \equiv 1$ and

$$\delta_i \equiv \frac{1+\delta_{i-1}\alpha^{(i,0)}}{1-(\alpha^{(i,0)})^2}.$$

It can be easily checked by induction on i that

$$\delta_i = \Big\{\prod_{h=0}^{i}(1-(\alpha^{(h,0)})^2)\Big\}^{-1}\Big[(1-(\alpha^{(0,0)})^2)\prod_{h=0}^{i}\alpha^{(h,0)} + \sum_{\ell=0}^{i-1}\prod_{h=\ell+2}^{i}\alpha^{(h,0)} \times \prod_{h=0}^{\ell}(1-(\alpha^{(h,0)})^2)\Big].$$

Therefore, using that

$$\prod_{h=0}^{\ell}(1-(\alpha^{(h,0)})^2) \leq 1$$

and noticing that if $C_0 \equiv T\sqrt{\sigma_1\sigma_2}\|A_3\|_{L^\infty(\mathbb{R}^2)}$, then, for $0 \leq j_1, j_2 < r$,

$$\alpha^{(j_1,j_2)} \leq \frac{C_0}{r},$$

we infer, for $0 \leq i < r$,

$$\begin{aligned} \delta_i &\leq \exp\Big(-\sum_{h=0}^{i}\ln(1-(\alpha^{(h,0)})^2)\Big)\Big[1+\alpha^{(i,0)} + \frac{C_0^2}{r^2} + \sum_{\ell=0}^{i-3}\Big(\frac{C_0^2}{r^2}\Big)^{i-\ell-1}\Big] \\ &\leq \exp\Big(\alpha^{(i,0)} - \sum_{h=0}^{i}\ln(1-(\alpha^{(h,0)})^2) + \frac{C_0'}{r^2}\Big). \end{aligned}$$

Reporting this into (57) yields

$$\mathfrak{a}_1^{(i,1)} \leq K^0\exp\Big(\max_{0\leq h\leq r}\alpha^{(h,0)} - \sum_{h=0}^{r}\ln(1-(\alpha^{(h,0)})^2) + \frac{C_0'}{r^2}\Big),$$

and, by similar arguments,

$$\mathfrak{a}_1^{(1,i)} \leq K^0\exp\Big(\max_{0\leq h\leq r}\alpha^{(0,h)} - \sum_{h=0}^{r}\ln(1-(\alpha^{(0,h)})^2) + \frac{C_0'}{r^2}\Big).$$

Consequently, denoting $\hat{\alpha}^{(j_1,j_2)} \equiv \max\Big\{\alpha^{(j_1,j_2)}; \alpha^{(j_2,j_1)}\Big\}$, it comes

$$\max\Big\{\mathfrak{a}_\ell^{(i,1)}; \ell = 1,2 \ 1 \leq i \leq r\Big\} \leq K^1 \equiv K^0\exp\Big(\max_{0\leq h\leq r}\hat{\alpha}^{(h,0)} - \sum_{h=0}^{r}\ln(1-(\hat{\alpha}^{(h,0)})^2) + \frac{C_0'}{r^2}\Big),$$

which is an estimate analoguous to (56). Hence, arguing by induction on the lines $j_1 = Cte$ and $j_2 = Cte$, we deduce

$$\max\Big\{\mathfrak{a}_\ell^{(j_1,j_2)};\ell=1,2\ 1\le j_1,j_2\le r\Big\}\le K^0\exp\Big(\sum_{i=0}^{r}\max_{i\le h\le r}\hat\alpha^{(h,i)}-\sum_{0\le i\le h\le r}\ln(1-(\hat\alpha^{(h,i)})^2)+\frac{C_0'}{r}\Big).$$

We now let $r\to+\infty$ in the previous estimate (C_0' depends on T). We bound K^0 using that (for $\ell = 1, 2$)

$$\mathfrak{a}_\ell^{(i,0)}\to\|A_\ell^{\rm in}\|_{L^\infty(\mathcal{R})}$$

uniformly for $0\le i\le r$ (since in the L^∞ norm, the time interval shrinks to $\{0\}$). Moreover, in view of (14), there holds

$$\sum_{i=0}^{r}\max_{i\le h\le r}\hat\alpha^{(h,i)}-\sum_{0\le i\le h\le r}\ln(1-(\hat\alpha^{(h,i)})^2)\le C(\nu,M),$$

where $C(\nu,M)$ depends on ν and M (and υ_1, υ_2), but not on X, T or r. Therefore, for some constant C depending only on σ_1 and σ_2, we have as wished

$$\frac{1}{C}|A(T,X)|_\infty\le\max\Big\{\mathfrak{a}_1^{(r-1,r-1)};\mathfrak{a}_2^{(r-1,r-1)}\Big\}\le\|A^{\rm in}\|_{L^\infty(\mathcal{R})}\mathrm{e}^{C(\nu,M)}.$$

References

[1] M. Ablowitz, D. Kaup, A. Newell and H. Segur, *The inverse scattering transform-Fourier analysis for nonlinear problems.* Studies in Appl. Math. **53**, no. 4 (1974), 249-315.

[2] F. Baronio, M. Conforti, M. Andreana, V. Couderc, C. De Angelis, S. Wabnitz, A. Barthélémy and A. Degasperis, *Frequency Generation and Solitonic Decay in Three-Wave Interactions.* Opt. Express **17** (2009), 13889-13894.

[3] R. Beals and R. Coifman, *Inverse scattering and evolution equations.* Comm. Pure Appl. Math. **38**, no. 1 (1985), 29-42.

[4] D. Chiron, *On the Schrödinger approximation for wavetrains in semilinear equations.* In preparation.

[5] E. Coddington and N. Levinson, *Theory of ordinary differential equations.* McGraw-Hill Book Company, Inc., New York-Toronto-London, (1955).

[6] A. Craik, *Wave interactions and fluid flows.* Cambridge Monographs on Mechanics and Applied Mathematics. Cambridge University Press, Cambridge, (1985).

[7] D. Kaup, *The inverse scattering solution for the full three-dimensional three-wave resonant interaction.* Phys. D **1**, no. 1 (1980), 45-67.

[8] D. Kaup, A. Rieman and A. Bers, *Space-time evolution of nonlinear three-wave interactions. I. Interaction in a homogeneous medium.* Rev. Mod. Phys. **51**, no. 2 (1979), 275-309.

[9] G. Schneider, *Justification and failure of the nonlinear Schrödinger equation in case of non-trivial quadratic resonances.* J. Differential Equations **216**, no. 2 (2005), 354-386.

[10] H. Triebel, *Interpolation theory, function spaces, differential operators.* Johann Ambrosius Barth, Heidelberg, (1995) (second edition).

[11] K. Yosida, *Lectures on differential and integral equations.* Pure and Applied Mathematics, Vol. X Interscience Publishers, New York-London (1960).

[12] V. Zakharov, *Exact solutions to the problem of the parametric interaction of three-dimensional wave packets.* Dokl. Akad. Nauk. S.S.S.R. **228** (1976), 1314-1316. [Translated in Soviet Phys. Dokl. **21** (1976), 322-323].

[13] V. Zakharov and S. Manakov, *Resonant interaction of wave packets in nonlinear media.* Zh. Eksp. Tero. Fiz. Pis'ma Red. **18**, (1973), 413-417 . [Translated in JETP Lett. **18**, no. 7 (1973), 243-245].

[14] V. Zakharov and S. Manakov, *The theory of resonance interaction of wave packets in nonlinear media.* Zh. Eksper. Teoret. Fiz. **69**, no. 5 (1975), 1654-1673. [Translated in Soviet Physics JETP **42**, no. 5 (1975), 842-850].

[15] X. Zhou, *Direct and inverse scattering transforms with arbitrary spectral singularties.* Comm. Pure Appl. Math. **42** (1989), 895-938.

[16] X. Zhou, L^2*-Sobolev space bijectivity of the scattering and inverse scattering transforms.* Comm. Pure Appl. Math. **51** (1998), 697-731.