THREE LONG WAVE ASYMPTOTIC REGIMES FOR THE NONLINEAR-SCHRÖDINGER EQUATION.

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Abstract. We survey some recent results related to three long wave asymptotic regimes for the Nonlinear-Schrödinger Equation: the Euler regime corresponding to the WKB method, the linear wave regime and finally the KdV/KP-I asymptotic dynamics.

1. Introduction

The nonlinear Schrödinger equation

$$\text{(NLS)} \qquad i\frac{\partial \Psi}{\partial \tau} + \frac{1}{2}\Delta\Psi = \Psi f(|\Psi|^2), \qquad \Psi : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{C}$$

appears as a relevant model in condensed matter physics: in nonlinear Optics (see, for instance, the survey [48]); in Bose-Einstein condensation and superfluidity (see [65], [29], [32], [1]). The nonlinearity f may be $f(\varrho) = \varrho$ or $f(\varrho) = \varrho - 1$, in which case (NLS) is termed the Gross-Pitaevskii equation, or $f(\varrho) = \varrho^2$ (see, *e.g.*, [50]) in the context of Bose-Einstein condensates, and more generally a pure power. In nonlinear Optics, quite often in dimensions 1 or 2, the nonlinearity may be more complicated (*cf.* [48]):

$$(1) \quad f(\varrho) = \alpha\varrho^\nu + \beta\varrho^{2\nu}, \qquad f(\varrho) = \alpha\Big(1 - \frac{1}{\big(1+\frac{\varrho}{\varrho_0}\big)^\nu}\Big), \qquad f(\varrho) = \alpha\varrho\Big(1 + \gamma\tanh\big(\frac{\varrho^2-\varrho_0^2}{\sigma^2}\big)\Big) \quad ...$$

where α, β, γ, $\nu > 0$ and $\sigma > 0$ are given constants.

The hydrodynamic form of (NLS) is obtained in a classical way with the *Madelung transform*. Writing (at least when $|\Psi| > 0$, that is away from vortices)

$$\Psi = \sqrt{\varrho}\exp\big(i\Theta\big),$$

inserting this into (NLS), cancelling the phase factor $\exp\big(i\Theta\big)$, separating real and imaginary parts and setting

$$v \equiv \nabla\Theta,$$

we obtain

$$(2) \qquad \begin{cases} \partial_\tau \varrho + \nabla\cdot\big(\varrho v\big) = 0 \\ \partial_\tau v + (v\cdot\nabla)v + \nabla\big(f(\varrho)\big) = \nabla\Big(\dfrac{\Delta\sqrt{\varrho}}{2\sqrt{\varrho}}\Big). \end{cases}$$

The system (2) is a compressible Euler equation with an additional term in the right-hand side called *quantum pressure*. Our purpose is to review some recent results on some asymptotic regimes of (NLS) that can be identified on (2).

• **Euler asymptotic regime.** Consider an highly oscillating WKB[1] initial datum for (NLS) of the form

$$\Psi_{|\tau=0}(x) = \psi_0^\varepsilon(\varepsilon x) = \sqrt{\rho_0^\varepsilon(\varepsilon x)}\exp\left(\frac{i}{\varepsilon}\varphi_0^\varepsilon(\varepsilon x)\right), \tag{3}$$

which corresponds for (2) to initial data

$$\begin{cases} \varrho_{|\tau=0}(x) &= \rho_0^\varepsilon(\varepsilon x) \\ \upsilon_{|\tau=0}(x) &= \big(\nabla\varphi_0^\varepsilon\big)(\varepsilon x) = u_0^\varepsilon(\varepsilon x). \end{cases}$$

Here, $\varepsilon > 0$ is a small parameter, homogeneous to the inverse of a length, hence this is a long-wave regime for (NLS), with wave-length ε^{-1}. For this type of initial data, the suitable scaling for (NLS) is to look for solutions under the form

$$\Psi(\tau, x) = \psi^\varepsilon\big(t, \varepsilon x\big) = \sqrt{\rho^\varepsilon(t,\varepsilon x)}\exp\left(\frac{i}{\varepsilon}\varphi^\varepsilon(t,\varepsilon x)\right), \qquad t = \varepsilon\tau.$$

This is actually the usual semiclassical scaling for (NLS), since ψ^ε then solves

$$i\varepsilon\frac{\partial\psi^\varepsilon}{\partial t} + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon = \psi^\varepsilon f(|\psi^\varepsilon|^2). \tag{4}$$

In this scaling and with

$$\varrho(\tau, x) \equiv \rho^\varepsilon(t, \varepsilon x) \qquad \upsilon(\tau, x) \equiv u^\varepsilon(t, \varepsilon x),$$

the system (2) writes

$$\begin{cases} \partial_t\rho^\varepsilon + \nabla\cdot\big(\rho^\varepsilon u^\varepsilon\big) = 0 \\ \partial_t u^\varepsilon + (u^\varepsilon\cdot\nabla)u^\varepsilon + \nabla\big(f(\rho^\varepsilon)\big) = \varepsilon^2\nabla\left(\dfrac{\Delta\sqrt{\rho^\varepsilon}}{2\sqrt{\rho^\varepsilon}}\right), \end{cases} \tag{5}$$

that is the quantum pressure becomes small. The formal limit of (5) as $\varepsilon \to 0$ is then expected to be Euler Eq.

$$\begin{cases} \partial_t\rho + \nabla\cdot\big(\rho u\big) = 0 \\ \partial_t u + (u\cdot\nabla)u + \nabla\big(f(\rho)\big) = 0, \end{cases} \tag{6}$$

provided the initial data converge suitably. The convergence is expected to hold for times $t = \varepsilon\tau$ of order one. It has to be noticed that even though (NLS) has, in the defocusing case, global solutions in H^1, the smooth solutions may not be global, as well as the smooth solutions to Euler system (6). The time T^* at which the solution to (6) ceases to be smooth is called the *breaking time.*

For the two other regimes we are interested in, we assume $f(\varrho_0) = 0$ for some $\varrho_0 > 0$, and by scaling, we may take $\varrho_0 = 1$, that is

$$f(1) = 0,$$

so that $\Psi = 1$ is a particular solution of (NLS). We will now focus on solutions Ψ of (NLS) such that $|\Psi| \simeq 1$, and in the defocusing case

$$f'(1) > 0.$$

• **Linear wave asymptotic regime.** We consider initial of the type

$$\Psi_{|\tau=0}(x) = \psi_0^\varepsilon(\varepsilon x) = \sqrt{1 + \varepsilon a_0^\varepsilon(\varepsilon x)}\exp\Big(i\varphi_0^\varepsilon(\varepsilon x)\Big),$$

[1] after G. Wentzel, H. Kramers and L. Brillouin

which corresponds to

$$\begin{cases} \varrho_{|\tau=0}(x) &= 1 + \varepsilon a_0^\varepsilon(\varepsilon x) \\ \upsilon_{|\tau=0}(x) &= \varepsilon\big(\nabla\varphi_0^\varepsilon\big)(\varepsilon x) = \varepsilon u_0^\varepsilon(\varepsilon x). \end{cases}$$

The density ϱ is then a perturbation of order ε of the constant state $\varrho = 1$, and u_0^ε is smaller than before by a factor ε. Denoting

$$\Psi(\tau,x) = \psi^\varepsilon(t,\varepsilon x) \qquad \varrho(\tau,x) \equiv \rho^\varepsilon(t,\varepsilon x) = 1+\varepsilon a^\varepsilon(t,\varepsilon x) \qquad \upsilon(\tau,x) \equiv \varepsilon u^\varepsilon(t,\varepsilon x), \qquad t = \varepsilon\tau,$$

we may rewrite (NLS) as (4) and (2) as

$$\begin{cases} \partial_t a^\varepsilon + \nabla\cdot u^\varepsilon = -\varepsilon\nabla\cdot\big(a^\varepsilon u^\varepsilon\big) \\ \partial_t u^\varepsilon + \dfrac{1}{\varepsilon}\nabla\Big(f\big(1+\varepsilon a^\varepsilon(t,x)\big)\Big) = -\varepsilon(u^\varepsilon\cdot\nabla)u^\varepsilon + \varepsilon\nabla\left(\dfrac{\Delta\sqrt{\rho^\varepsilon}}{2\sqrt{\rho^\varepsilon}}\right). \end{cases} \tag{7}$$

Using that $f(1+r) \simeq c^2 r$ as $r \to 0$, where $c^2 = f'(1) > 0$, we infer that the formal limit of (7) will be the linear wave Eq.

$$\begin{cases} \partial_t a + \nabla\cdot u &= 0 \\ \partial_t u + c^2\nabla a &= 0, \end{cases} \qquad c^2 = f'(1) > 0. \tag{8}$$

However, a variant consists in keeping only linear terms in (7), hence we may take into account the linear part of the quantum pressure (see for example [1]). This yields an additional ε-dependent dispersive term, and changes (8) for

$$\begin{cases} \partial_t \mathbf{a} + \nabla\cdot\mathbf{u} &= 0 \\ \partial_t \mathbf{u} + c^2\nabla\mathbf{a} &= \dfrac{\varepsilon}{4}\nabla\Delta\mathbf{a}. \end{cases} \tag{9}$$

The convergence to the free wave regime is expected to hold for times $t = \varepsilon\tau \ll \varepsilon^{-1}$ for (8), and $t = \varepsilon\tau$ larger but $\ll \varepsilon^{-2}$ for (9).

• **KdV/KP-I asymptotic regime.** For this last asymptotic regime, the initial data write

$$\Psi_{|\tau=0}(x) = \psi_0^\varepsilon(\varepsilon x_1, \varepsilon^2 x_\perp) = \big(1+\varepsilon^2 a_0^\varepsilon(\varepsilon x_1,\varepsilon^2 x_\perp)\big)\exp\Big(i\varepsilon\varphi_0^\varepsilon(\varepsilon x_1,\varepsilon^2 x_\perp)\Big),$$

where

$$x = (x_1, x_\perp) \in \mathbb{R}\times\mathbb{R}^{d-1}$$

(if $d=1$, just ignore $x_\perp$), that is

$$\begin{cases} \varrho_{|\tau=0}(x) &= \big(1+\varepsilon^2 a_0^\varepsilon(\varepsilon x_1,\varepsilon^2 x_\perp)\big)^2 = 1 + 2\varepsilon^2 a_0^\varepsilon(\varepsilon x) + \mathcal{O}(\varepsilon^4) \\ \upsilon_{|\tau=0}(x) &= \dfrac{\varepsilon^2}{2c}\Big(\partial_1\varphi_0^\varepsilon, \varepsilon\nabla_\perp\varphi_0^\varepsilon\Big)(\varepsilon x_1,\varepsilon^2 x_\perp) = \dfrac{\varepsilon^2}{2c}u_0^\varepsilon(\varepsilon x_1,\varepsilon^2 x_\perp). \end{cases}$$

The relevant dynamics of (NLS) actually takes place in a moving frame with speed ε^{-3} in the original coordinates for (NLS). Let

$$\varrho(\tau,x) \equiv 1+\varepsilon^2 a_0^\varepsilon\big(t,\varepsilon(x_1-c\tau),\varepsilon^2 x_\perp\big) \qquad \upsilon(\tau,x) \equiv \frac{\varepsilon^2}{2c}u_0^\varepsilon\big(t,\varepsilon(x_1-c\tau),\varepsilon^2 x_\perp\big) \qquad t = c\varepsilon^3\tau,$$

where $c^2 = f'(1)$. This ansatz means that we study a weak amplitude wave propagating to the right, in a long wave regime, which is slowly modulated in the transverse direction. It is commonly expected that the KdV or KP-I equation appear as enveloppe equations in such regimes (see, *e.g.* [6] and the references cited therein in the context of water-waves system).

Then, (2) gives

$$
\left\{
\begin{array}{l}
\partial_t a^\varepsilon - \dfrac{1}{\varepsilon^2}\partial_{x_1} a^\varepsilon + 2u^\varepsilon \cdot \nabla^\varepsilon a^\varepsilon + \dfrac{1}{\varepsilon^2}(1+\varepsilon^2 a^\varepsilon)\nabla^\varepsilon \cdot u^\varepsilon = 0 \\[3ex]
\partial_t u^\varepsilon - \dfrac{1}{\varepsilon^2}\partial_{x_1} u^\varepsilon + 2(u^\varepsilon \cdot \nabla^\varepsilon)u^\varepsilon + \dfrac{1}{\varepsilon^2}\nabla^\varepsilon a^\varepsilon + \dfrac{1}{\varepsilon^4}\,\nabla^\varepsilon\big(g(\varepsilon^2 a^\varepsilon)\big) = \dfrac{1}{4c^2}\nabla^\varepsilon\Big(\dfrac{\Delta^\varepsilon a^\varepsilon}{1+\varepsilon^2 a^\varepsilon}\Big),
\end{array}
\right.
\tag{10}
$$

where $\nabla^\varepsilon \equiv (\partial_{x_1}, \varepsilon\nabla_\perp)$, $\Delta^\varepsilon \equiv \nabla^\varepsilon \cdot \nabla^\varepsilon = \partial^2_{x_1} + \varepsilon^2\Delta_\perp$ and g is defined by

$$
f\big((1+r)^2\big) = c^2\big(2r + g(r)\big) \qquad g(r) = \mathcal{O}(r^2) \quad r \to 0.
$$

As $\varepsilon \to 0$, we infer formally that if $a^\varepsilon \to a$ and $u^\varepsilon \to u$, then $-\partial_{x_1} a + \partial_{x_1} u_1 = 0$, *i.e.*

$$
a = u_1, \tag{11}
$$

which turns out to be a preparedness assumption for this singular PDE limit. In order to derive the limit equation satisfied by a, we can add the two equations in (10) for a^ε and u_1^ε: this cancels out the most singular terms and yields the equation

$$
\begin{aligned}
&\partial_t\big(a^\varepsilon + u_1^\varepsilon\big) \; - \frac{1}{4c^2}\partial_{x_1}\Big(\frac{\partial^2_{x_1} a^\varepsilon}{1+\varepsilon^2 a^\varepsilon}\Big) + \big(1+\varepsilon^2 a^\varepsilon\big)\Delta_\perp \partial_{x_1}^{-1} u_1^\varepsilon + \frac{1}{\varepsilon^4}\,\partial_{x_1}\big(g_3(\varepsilon^2 a^\varepsilon)\big) \\
&\qquad + \Big\{ u_1^\varepsilon \partial_{x_1} a^\varepsilon + \frac{1}{2}\,a^\varepsilon \partial_{x_1} u_1^\varepsilon + \partial_{x_1}\big((u_1^\varepsilon)^2\big) + \big[1 + 2\frac{f''(1)}{c^2}\big]\partial_{x_1}\big((a^\varepsilon)^2\big)\Big\} = 0,
\end{aligned}
\tag{12}
$$

where we have expanded

$$
g(r) = \Big[1 + 2\frac{f''(1)}{c^2}\Big] r^2 + g_3(r), \qquad g_3(r) = \mathcal{O}(r^3) \;\; r \to 0
$$

and used

$$
\varphi^\varepsilon = \partial_{x_1}^{-1}\partial_{x_1}\varphi^\varepsilon = 2c\partial_{x_1}^{-1} u_1^\varepsilon.
$$

The formal limit, as $\varepsilon \to 0$, for this equation and (11) is the system

$$
\left\{
\begin{array}{l}
u_1 = a \\[3ex]
2\partial_t a + \Big[6 + \dfrac{2}{c^2} f''(1)\Big] a\partial_{x_1} a - \dfrac{1}{4c^2}\partial^3_{x_1} a + \Delta_\perp \partial_{x_1}^{-1} a = 0,
\end{array}
\right.
\tag{13}
$$

which is the Korteweg-de Vries (KdV) Eq. in dimension $d = 1$, and the Kadomtsev-Petviashvili I (KP-I) Eq. in higher dimensions $d \geq 2$. For this last asymptotic regime, the convergence will hold for times $t = c\varepsilon^3\tau$ of order one, that is $\tau \ll \varepsilon^{-3}$.

In dimension $d = 1$, the formal derivation of the KdV equation from the (NLS) Eq. in this asymptotic regime may be found, for instance, in [53], [47]. It is relevant in the stability analysis of dark solitons or travelling waves of small energy. In the case of the Gross-Pitaevskii equation, for instance (that is for $f(\varrho) = \varrho - 1$), the travelling waves $\Psi(\tau, x) = U(x - \sigma\tau)$ verify

$$
-i\sigma U' + \frac{1}{2}U'' = U(|U|^2 - 1), \qquad z \in \mathbb{R}
$$

and the condition at infinity $|U| \to 1$. For this nonlinearity, an explicit integration (see, *e.g.* [69]) gives, for $0 < \sigma < 1$ the nontrivial solution

$$
U_\sigma(z) = \sigma - i\sqrt{1-\sigma^2}\tanh\Big(z\sqrt{1-\sigma^2}\Big).
$$

With this normalization for (NLS), the speed of sound at $\varrho = 1$ is 1, and all the travelling waves are subsonic. In the transonic limit $\sigma \simeq 1$, we then set $\sigma^2 = 1 - \varepsilon^2$, $\varepsilon > 0$ small, and we obtain

$$U_\sigma(x) = -i\varepsilon \tanh(\varepsilon x) + \sqrt{1-\varepsilon^2} = \sqrt{1 - \frac{\varepsilon^2}{\cosh^2(\varepsilon x)}} \exp\big(i\varepsilon\varphi^\varepsilon(\varepsilon x)\big),$$

with $\varphi^\varepsilon(\varepsilon x) = -\tanh(\varepsilon x) + \mathcal{O}(\varepsilon^3)$, and we see that this is the ansatz we make. Furthermore, here, $A^\varepsilon = -1/\cosh^2$ is independent of ε and is the KdV soliton (with $c = 1$, $f''(1) = 0$).

In higher dimensions $d = 2, 3$, the convergence of the travelling waves to the Gross-Pitaevskii Eq. (*i.e.* (NLS) with $f(\varrho) = \varrho - 1$) with speed $\simeq 1$ to a soliton of the KP-I equation is formally derived in the paper [42], while in [10], this KP-I asymptotic regime for (NLS) in dimension $d = 3$ is used to investigate the linear instability of the solitary waves of speed $\simeq 1$. On the mathematical level, in dimension $d = 2$, the convergence of the travelling waves of speed $\simeq 1$ for the Gross-Pitaevskii Eq. to a ground state of the KP-I Eq. is proved in [12].

2. The Euler regime for (NLS)

Many recent results concern the convergence of (5) to (6). The first work is in the case of analytic data, by [26], on a time interval $t = \varepsilon\tau \in [0, T]$ for some $0 < T < +\infty$. When $d = 1$ and in the integrable cases for (NLS), this problem has also been studied: for $f(\varrho) = \varrho$ and $f(\varrho) = \varrho - 1$ in [39], and in the focusing case $f(\varrho) = -\varrho$ by [23]. Their results hold for $t = \varepsilon\tau$ in any bounded time interval (even after the breaking time T^*). In contrast with [26] and [23], all the results we present below are valid for defocusing nonlinearities only, and before the breaking time.

2.1. The approach of E. Grenier.

In the framework of Sobolev spaces and a defocusing nonlinearity, $f'(\varrho) > 0$ for $\varrho \geq 0$, E. Grenier in [32] notices that it is more convenient to use the transformation

$$\Psi(\tau, x) = A^\varepsilon(t, \varepsilon x) \exp\big(\frac{i}{\varepsilon}\varphi^\varepsilon(t, \varepsilon x)\big), \qquad t = \varepsilon\tau,$$

to allow A^ε to be complex-valued, and to split (NLS) not in the form (2) but in the form

$$(14) \qquad \begin{cases} \partial_t A^\varepsilon + v^\varepsilon \cdot \nabla A^\varepsilon + \dfrac{A^\varepsilon}{2} \nabla \cdot v^\varepsilon = \dfrac{i\varepsilon}{2} \Delta A^\varepsilon \\ \partial_t v^\varepsilon + \big(v^\varepsilon \cdot \nabla\big) v^\varepsilon + \nabla\big(f(|A^\varepsilon|^2)\big) = 0 \end{cases} \qquad v^\varepsilon = \nabla\varphi^\varepsilon.$$

Indeed, since A^ε may be complex-valued, one can solve (14), then compute φ^ε from v^ε, and finally infer that $A^\varepsilon \exp\big(\frac{i}{\varepsilon}\varphi^\varepsilon\big)$ solves (NLS). The main advantages of (14) compared to (2) are that the quantum pressure now appears as a linear term, and that (14) is a symmetrizable hyperbolic system, provided $f'(\varrho) > 0$ for $\varrho \geq 0$, with the symmetrizers

$$S(A) = \mathrm{Diag}\,\Big(Id_{\mathbb{C}}, \frac{1}{4f'(|A|^2)} Id_{\mathbb{R}^d}\Big) \qquad \text{or} \qquad \tilde{S}(A) = \mathrm{Diag}\Big(4f'(|A|^2) Id_{\mathbb{C}}, Id_{\mathbb{R}^d}\Big).$$

This approach proves that the solution ψ^ε (actually A^ε and φ^ε) of (4) exists and is smooth on some time interval $[0, T]$ independent of $0 < \varepsilon < 1$, and at the same time that the Cauchy problem for the Euler Eq. has a unique solution which remains smooth at least on $[0, T]$. The convergence holds for times $0 \leq t = \varepsilon\tau \leq T$, *i.e.* before the breaking time T^*. We emphasize that if $f'(\varrho_*) < 0$, then the Euler system is no longer hyperbolic and the "wave equation" obtained by linearization around the constant state ($\rho = \varrho_*, u = 0$) becomes elliptic in space-time. The case where $f' \geq 0$ but $f'(0) = 0$, for instance $f(\varrho) = \varrho^2$, was left open (the symmetrizer S is then not defined for

$A = 0$, and $\tilde{S}(A)$ is no longer positive definite). It is finally usual in the WKB method to expand to higher order the initial datum

$$\Psi_{|\tau=0}(x) = \psi^\varepsilon_{|t=0}(\varepsilon x) = \Big(\sum_{j=0}^{m} \varepsilon^j A_0^j(\varepsilon x) + \mathcal{O}(\varepsilon^{m+1}) \Big) \exp\Big(\frac{i}{\varepsilon}\varphi_0^\varepsilon(\varepsilon x)\Big). \tag{15}$$

and [32] proves that the expansion remains valid for $0 \le t = \varepsilon\tau \le T$, and that the equations for the A^j's are obtained by formal cancellation of the powers of ε in (14). The approach of [32] has been extended to the case with smooth potential (including the quadratic case) [18], to the modified (NLS) Eq. [22], to Schrödinger-Poisson Eq. [2] by T. Alazard and R. Carles. More recently, these authors in [4] have extended the results in [32] to the case of pure power nonlinearities $f(\varrho) = \varrho^\sigma$, though with some restrictions on σ and d. Since the Euler system (6) is more convenient in variables (a, φ) instead of $(\rho = a^2, u = \nabla\varphi)$, let us rewrite it as

$$\begin{cases} \partial_t a + \nabla\varphi \cdot \nabla a + \dfrac{a}{2}\,\Delta\varphi &= 0 \\ \partial_t \varphi + \dfrac{1}{2}|\nabla\varphi|^2 + f(a^2) &= 0. \end{cases} \tag{16}$$

By [60], when $f(\varrho) = \varrho^\sigma$ for some $\sigma \in \mathbb{N}^*$, the system (16) with an initial condition in $H^\infty \times H^\infty$ has a smooth maximal solution in H^∞. The main result of [4] for the rescaled version (4) of (NLS) reads

Theorem 1 ([4])**.** *Let $1 \le d \le 3$, $\sigma \in \mathbb{N}^*$ and consider $f(\varrho) = \varrho^\sigma$. We assume the initial data a_0^ε, $\varphi_0^\varepsilon \equiv \varphi_0$ in H^∞ such that, for some function $a_0 \in H^\infty$ and for every $s \ge 0$,*

$$\big\| a_0^\varepsilon - a_0 \big\|_{H^s} = \mathcal{O}(\varepsilon).$$

Let $(a, \varphi) \in \mathcal{C}\big([0, T^*[, H^\infty \times H^\infty\big)$, where $T^* > 0$, be the smooth maximal solution of (16). Then, there exists $T \in (0, T^*)$ independent of $0 < \varepsilon < 1$, such that the solution of (4) with initial datum (3) exists and remains smooth on $[0, T]$ and verifies the estimate

$$\sup_{\varepsilon \in (0,1]} \Big\| \psi^\varepsilon \exp\big(-\frac{i}{\varepsilon}\varphi\big) \Big\|_{L^\infty([0,T], H^s)} < +\infty, \tag{17}$$

where

- *if $\sigma = 1$, then $s \in \mathbb{N}$ is arbitrary,*
- *if $\sigma = 2$ and $d = 1$, then one can take $s = 2$,*
- *if $\sigma = 2$ and $2 \le d \le 3$, then one can take $s = 1$,*
- *if $\sigma \ge 3$ then one can take $s = \sigma$.*

As a consequence,

$$|\psi^\varepsilon|^2 \to a^2 \quad \text{in} \quad \mathcal{C}\big([0,T], L^{\sigma+1}\big) \qquad \text{and} \qquad \varepsilon\langle i\psi^\varepsilon, \nabla\psi^\varepsilon\rangle \to a^2 u \quad \text{in} \quad \mathcal{C}\big([0,T], L^{\sigma+1} + L^1\big).$$

Here, $\langle \cdot, \cdot \rangle$ denotes the real scalar product in $\mathbb{C} \simeq \mathbb{R}^2$. The main ingredient used in [4] is a subtle transformation of (4) into a perturbation of a quasilinear symmetric hyperbolic system with non smooth coefficients when $\sigma \ge 2$. Though the assumption

$$\big\| a_0^\varepsilon - a_0 \big\|_{H^\infty} = \mathcal{O}(\varepsilon)$$

is natural in the context of WKB expansions, it is noticed in [4] that the result becomes false with the hypothesis $\big\| a_0^\varepsilon - a_0 \big\|_{H^\infty} = o(1)$, or even $\mathcal{O}(\varepsilon^\alpha)$ as $\varepsilon \to 0$ for some $0 < \alpha < 1$, because the solution can exhibit oscillations in the phase of size $\varepsilon^{\alpha-1}$.

2.2. **The modulated energy functional.** Another approach to prove the convergence of (NLS) to Euler Eq. came from Y. Brenier[2] [15], following an idea due to P.-L. Lions in [58]. Notice that the Hamiltonian associated to (4) is given by the Ginzburg-Landau energy (when this makes sense)

$$\mathcal{E}(\psi) = \frac{1}{2}\int_{\mathbb{R}^d} |\varepsilon\nabla\psi|^2 + F(|\psi|^2)\, dx, \tag{18}$$

where $F' = 2f$ (F is defined only up to an additive constant, which can be fixed when a condition at infinity, for instance $|\psi|^2(x) \to \varrho_0$ as $|x| \to +\infty$, where $f(\varrho_0) = 0$, in order to have a finite energy). The idea is then to use a *modulated energy functional* in order to compare ψ^ε and the smooth solution (a, φ) of (16), namely

$$\mathcal{H}^\varepsilon(\psi) \equiv \frac{1}{2}\int_{\mathbb{R}^d} |\varepsilon\nabla\psi - iu\,\psi|^2 + \left(|\psi|^2 - a^2\right)^2\, dx,$$

if $f(\varrho) = \varrho - \varrho_0$ for some $\varrho_0 \in \mathbb{R}_+$, and more generally

$$\mathcal{H}^\varepsilon(\psi) = \frac{1}{2}\int_{\mathbb{R}^d} |\varepsilon\nabla\psi - iu\,\psi|^2 + \Big[F(|\psi|^2) - F(a^2) - F'(a^2)\big(|\psi|^2 - a^2\big)\Big]\, dx,$$

for a smooth nonlinearity f such that $f' > 0$ on $[0, +\infty)$ (hence F is strictly convex).

This approach has been followed by F-H. Lin and P. Zhang [56] in the context of an exterior domain (the complement of a compact obstacle ω) in $\mathbb{R}^2$ with smooth boundary, with $f(\varrho) = \varrho - 1$ for simplicity. They work with the conditions at infinity

$$\psi^\varepsilon(t,x) \sim \exp\Big(-i\,t\,\frac{|u^\infty|^2}{2\varepsilon} + i\,\frac{u^\infty\cdot x}{\varepsilon}\Big), \qquad |x| \to +\infty, \tag{19}$$

that we can write in hydrodynamical variables

$$\rho^\varepsilon(t,x) = \big|\psi^\varepsilon(t,x)\big|^2 \to 1, \qquad u^\varepsilon(t,x) \to u^\infty, \qquad |x| \to +\infty,$$

where $u^\infty \in \mathbb{R}^d$ is a constant vector. This condition appears naturally when we study an obstacle moving in the fluid. Indeed, we may start from (4) with the Neumann boundary condition on the obstacle which moves at constant velocity in a fluid at rest at infinity. Then, we use the Galilean invariance of (NLS) to change the problem for the study of (4) in a fixed domain Ω, but with the condition (19) at infinity. The main result of [56] is:

Theorem 2 ([56])**.** *Let $d = 2$ and $\Omega = \mathbb{R}^2 \setminus \omega$ be an exterior domain. Assume that the initial data satisfy*

$$\sqrt{\rho_0^\varepsilon}\exp\big(\frac{i}{\varepsilon}\varphi_0^\varepsilon\big) - \exp\big(\frac{i}{\varepsilon}u^\infty\cdot x\big), \quad \rho_0 - 1, \quad u_0 - u^\infty \in H^3(\Omega), \qquad \rho_0 \geq \frac{1}{2}$$

$$\sup_{0<\varepsilon<1}\Big\{\big\|\nabla\sqrt{\rho_0^\varepsilon}\big\|_{L^2} + \Big\|\exp\big(\frac{i}{\varepsilon}\varphi_0^\varepsilon\big) - \exp\big(\frac{i}{\varepsilon}u^\infty\cdot x\big)\Big\|_{L^2} + \big\|\sqrt{\rho_0^\varepsilon}\nabla\varphi_0^\varepsilon\big\|_{L^2}\Big\} < +\infty$$

$$\rho_0^\varepsilon - \rho_0 \to 0 \qquad \textit{and} \qquad \sqrt{\rho_0^\varepsilon}\big(\nabla\varphi_0^\varepsilon - u_0\big) \to 0 \qquad \textit{in} \quad L^2,$$

as well as some compatibility conditions for (ρ_0, u_0). Then, the Euler system (6) in Ω with initial data (ρ_0, u_0) and the Neumann condition $u\cdot n = 0$ on $\partial\Omega$ has a unique solution $(\rho, u) \in (1, u^\infty) + \mathcal{C}([0,T], H^3)$ and, as $\varepsilon \to 0$,

$$|\psi^\varepsilon|^2 - \rho \to 0 \qquad \textit{in} \quad L^\infty\big([0,T], L^2(\Omega)\big)$$

and

$$\varepsilon\langle i\psi^\varepsilon, \nabla\psi^\varepsilon\rangle - \rho u \to 0 \qquad \textit{in} \quad L^\infty\big([0,T], L^1_{loc}(\Omega)\big).$$

[2] actually, this was for Vlasov-Poisson Eq. instead of (NLS)

The compatibility conditions on (ρ_0, u_0) are determined by the fact that we look for a sufficiently smooth solution of (6), which implies in particular that, for $0 \leq k \leq 2$,

$$n \cdot \partial_t^k u(0) = 0.$$

The main point of this approach is that the functional $\mathcal{H}^\varepsilon$ satisfies a growth estimate of the form

$$\frac{d}{dt}\mathcal{H}^\varepsilon \leq C(u)\big(\mathcal{H}^\varepsilon + \varepsilon^2\big) \tag{20}$$

for a constant $C(u)$ depending on $\|\nabla u\|_{L^\infty}$ and $\|\nabla(\nabla \cdot u)\|_{L^2}$. It follows then that if $\mathcal{H}^\varepsilon$ is small at $t = 0$, then it remains small on the interval of time $[0, T]$. The main advantage of this approach is that we compare the two solutions in more or less the energy space for (4). As a drawback, this does not say anything on the smoothness of ψ^ε in space. Moreover, this method does not seem to allow to justify higher order expansions for an initial datum as in (15).

In comparison with the method of [32], one needs to solve on the one hand Euler Eq. for smooth data, and on the other hand to solve the Cauchy problem[3] for (4) on a time interval $[0, T]$ independent of ε. In dimension $d = 1$ or $d = 2$ with $f(\varrho) \to +\infty$ polynomialy as $\varrho \to +\infty$, the Cauchy problem for (4) is global, using the Brézis-Gallouët trick ([9]) for $d = 2$. However, for nonlinearities f such that $f' > 0$ on an interval I but $f(\varrho) \to -\infty$ as $\varrho \to +\infty$, the classical result on global weak solutions of [28] is no longer valid, and the H^1 solutions may even blow-up in finite time. Furthermore, even in the defocusing case, in higher dimensions, the nonlinearity may be critical, and the Cauchy problem be then much more difficult to solve: see [21] for the case $d = 3$, $f(\varrho) = \varrho^2$ in the whole space, or [7] for $d = 3$ and the Neumann boundary condition in a nontrapping exterior domain, in order to use the Strichartz estimate of [17] (see also [38]). In the context studied in [56], the solution ψ^ε of (4) is proved to be global, hence the comparison with the Euler Eq. holds on $[0, T]$, for arbitrary $0 \leq t \leq T$ with $T < T^*$, T^* being the maximal existence time for the smooth solution (ρ, u) to (6). Nontheless, the fact, in Theorem 1, that ψ^ε remains uniformly smooth on a time interval independent of ε is interesting in itself. For the problem investigated in [56], the method of [32] would require to work with ε-derivatives.

This modulated energy functional method has been extended to the cases of coupled Schrödinger Eq. in [57] and [55] when taking into account trapping smooth potentials and smooth electromagnetic fields. It is finally an important remark that even though $\varphi_0^\varepsilon = 0$, that is the initial datum for (4) is not oscillating at the initial time, and if (a, φ) is the solution of (16) with $(a, \varphi)_{|t=0} = (a_0, 0)$ and a nonzero, then φ is in general nonzero for positive times, since $(\partial_t \varphi)_{|t=0} = -f(a_0^2) \neq 0$. This means that ψ^ε becomes highly oscillating for positive times. This remark is the key point of the proof of the loss of regularity for (NLS) in [3], using the modulated energy functional approach.

2.3. **Linearizing around an approximate solution.** We will consider nonlinearities f such that:

$$(\mathcal{A}) \quad f \in \mathcal{C}^\infty\big([0, +\infty)\big), \qquad f(0) = 0, \qquad f' > 0 \text{ on } (0, +\infty), \qquad \exists n \in \mathbb{N}^*, \quad f^{(n)}(0) \neq 0.$$

In particular, f' may vanish at the origin. This includes all the homogeneous nonlinearities $f(\varrho) = \varrho^\sigma$ for $\sigma \in \mathbb{N}^*$, but also the nonlinearities (1), at least if we work on an intervall I such that $f' > 0$ on I (for example for the first nonlinearity with $\alpha > 0 > \beta$, we can work for ϱ sufficiently small). Indeed, since we will work with smooth data (in particular, uniformly bounded), we may restrict ourselves to maps with values in this interval I. Our main result generalizes in particular the results of Theorem 1 without restriction on s, d and σ.

[3] for, say H^2 initial data in order to fully justify the integrations by parts leading to (20), or for weak solutions that are obtained as limit of smoother maps

Theorem 3 ([19])**.** *We assume ($\mathcal{A}$), and consider an initial data (3) with φ_0^ε real-valued, a_0^ε, φ_0^ε in H^∞ such that, for some real-valued functions $(\varphi_0, a_0) \in H^\infty$, we have for every s,*

$$\left\| a_0^\varepsilon - a_0 \right\|_{H^s} = \mathcal{O}(\varepsilon) \qquad \text{and} \qquad \left\| \varphi_0^\varepsilon - \varphi_0 \right\|_{H^s} = \mathcal{O}(\varepsilon).$$

Then, there exists $T^* > 0$ such that (16) with initial value (a_0, φ_0) has a unique smooth maximal solution $(a, \varphi) \in \mathcal{C}([0, T^*[, H^\infty \times H^\infty)$. Moreover, there exists $T \in (0, T^*]$ such that for every $\varepsilon \in (0,1)$, the solution ψ^ε to (4) with (3) exists at least on $[0,T]$ and satisfies for every s

$$\sup_{\varepsilon\in(0,1]} \left\| \psi^\varepsilon \exp\big(-\frac{i}{\varepsilon}\varphi\big) \right\|_{L^\infty([0,T],H^s)} < +\infty.$$

More precisely, there exists $\varphi^\varepsilon = \varphi + \mathcal{O}_{H^\infty}(\varepsilon)$ such that, for every s,

$$\left\| \psi^\varepsilon \exp\big(-\frac{i}{\varepsilon}\varphi^\varepsilon\big) - a \right\|_{L^\infty([0,T],H^s)} = \mathcal{O}(\varepsilon). \tag{21}$$

From the uniform bound (21), we may immediately derive some convergences of the quadratic physical quantities, as in Theorems 1 and 2

$$|\psi^\varepsilon|^2 = a + \mathcal{O}_{L^\infty(H^s)}(\varepsilon), \qquad \varepsilon\langle i\psi^\varepsilon, \nabla\psi^\varepsilon\rangle = a^2\nabla\varphi + \mathcal{O}_{L^\infty(H^s)}(\varepsilon).$$

In view of the way we construct our solution ψ^ε, namely an approximate solution plus a small perturbation, we are able to take into account various conditions at infinity such as, in hydrodynamic variables, $|\psi^\varepsilon|^2 \to \varrho_0$ at infinity, when $f(\varrho_0) = 0$, and also (19).

Let us give a few comments on the statement of Theorem 3. First, in (21), the correction of order ε in φ is not a surprise, since this modifies the amplitude at leading order.

Second, Theorem 3 contains a result of local existence of smooth solutions for (16) in the case of non necessarily homogeneous nonlinearities satisfying ($\mathcal{A}$) (*cf.* Theorem 4 in [19] for a precise statement with H^s data). This extends the result on homogeneous nonlinearities studied in [60] to nonlinearities satisfying ($\mathcal{A}$). The main point is that when f is not homogeneous, it does not seem possible to use a nonlinear symmetrization as in [60] in order to reduce the problem to a symmetrizable hyperbolic system with smooth coefficients. Nevertheless, we have been able to derive energy estimates by symmetrizing only the first order part of the new system.

In case where the initial datum has a higher order WKB expansion, then we can prove a stronger result, which is also proved in [4] with the restrictions on s, d, σ already mentionned:

Theorem 4 ([19])**.** *Under the assumptions of Theorem 3, we suppose furthermore the expansions*

$$a_0^\varepsilon(x) = \sum_{j=0}^m \varepsilon^j a_0^j(x) + \mathcal{O}_{H^\infty}(\varepsilon^{m+1}) \qquad \varphi_0^\varepsilon(x) = \sum_{j=0}^m \varepsilon^j \varphi_0^j(x) + \mathcal{O}_{H^\infty}(\varepsilon^{m+1})$$

for some $m \geq 1$. Let then $(a^j, \varphi^j) \in \mathcal{C}([0, T^*[, H^\infty)$, $0 \leq j \leq m$, be the solutions of the WKB hierarchy, and

$$a^\varepsilon(x) \equiv \sum_{j=0}^m \varepsilon^j a^j(x), \qquad \varphi^\varepsilon(x) \equiv \sum_{j=0}^m \varepsilon^j \varphi^j(x).$$

Then, for every $T \in (0, T^*)$, there exists $\varepsilon_0(T) > 0$ such that for every $\varepsilon \in (0, \varepsilon_0(T)]$, the solution of (4) with initial data ψ_0^ε remains smooth on $[0,T]$ and satisfies for every $s \in \mathbb{N}$, the estimate

$$\left\| \psi^\varepsilon \exp\big(-\frac{i}{\varepsilon}\varphi^\varepsilon\big) - a^\varepsilon \right\|_{L^\infty([0,T],H^s)} \leq C_{s,T}\varepsilon^{m+1}.$$

It is worthwile to notice that in the above Theorem 4, we have expanded the initial data to the order $m \geq 1$ whereas Theorem 3 corresponds to the case $m = 0$. Besides the possibility to expand the solution, we emphasize that the above result holds for $0 \leq t \leq T$ and $0 < \varepsilon < \varepsilon_0(T)$, where $0 < T < T^*$ is arbitrarily close to the breaking time T^*, whereas when $m = 0$, we are restricted to a time interval $[0, T]$, for some fixed $0 < T < T^*$. It would be quite interesting to understand what happens for ψ^ε for times close to the time T^* where the solution of the Euler system looses its smoothness.

The way we prove Theorem 3 is in two steps. In a first step, we construct an approximate solution of (4) under the form $\psi^\varepsilon_{app} = \mathsf{a}^\varepsilon \exp(i\varphi^\varepsilon/\varepsilon)$, in such a way that

$$i\varepsilon \frac{\partial \psi^\varepsilon_{app}}{\partial t} + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon_{app} - \psi^\varepsilon_{app} f(|\psi^\varepsilon_{app}|^2) = \exp\left(i\varphi^\varepsilon/\varepsilon\right) \times \mathcal{O}_{H^\infty}(\varepsilon).$$

For the proof of Theorem 3, we just take the smooth solution (a, φ) to (16) plus the $\mathcal{O}_{H^\infty}(\varepsilon)$ correction due to the fact that $(a^\varepsilon_0, \varphi^\varepsilon_0) - (a_0, \varphi_0) = \mathcal{O}_{H^\infty}(\varepsilon)$. In this way, $\psi^\varepsilon_{app} = \psi^\varepsilon_0$ at $t = 0$. For an initial datum as in Theorem 4, we have to include the other terms of the expansion, and the remainder $\mathcal{O}_{H^\infty}(\varepsilon)$ is changed for $\mathcal{O}_{H^\infty}(\varepsilon^{m+1})$. To construct ψ^ε_{app}, it is more convenient to split (4) in the form (14) proposed by E. Grenier. The second step is a stability result: we look for an exact solution of (4) under the form

$$\psi^\varepsilon = \psi^\varepsilon_{app} + w \exp\left(\frac{i}{\varepsilon}\varphi^\varepsilon\right) = (\mathsf{a}^\varepsilon + w) \exp\left(\frac{i}{\varepsilon}\varphi^\varepsilon\right), \tag{22}$$

where w should be small, and we find that w solves the nonlinear Schrödinger equation

$$\begin{aligned} i\varepsilon\Big(\partial_t w + u^\varepsilon \cdot \nabla w + \frac{1}{2}\, w\, \nabla \cdot u^\varepsilon\Big) + \frac{\varepsilon^2}{2}\Delta w - 2\langle w, \mathsf{a}^\varepsilon\rangle f'(|\mathsf{a}^\varepsilon|^2)\mathsf{a}^\varepsilon \\ = w \times \mathcal{O}_{H^\infty}(\varepsilon) + \mathcal{O}_{H^\infty}(\varepsilon^2) + Q^\varepsilon(w), \end{aligned} \tag{23}$$

with $u^\varepsilon \equiv \nabla\varphi^\varepsilon$ and $Q^\varepsilon(w)$ contains the at least quadratic terms in w in the nonlinearity f. Since we expect the correction term w to be small, we may work on the linearized version of (23) with a source term, with the adapted energy

$$\frac{1}{2}\int_{\mathbb{R}^d} \varepsilon^2 |\nabla w|^2 + 4f'(|\mathsf{a}^\varepsilon|^2)\langle w, \mathsf{a}^\varepsilon\rangle^2 \; dx,$$

and more precisely the weighted norm

$$N^\varepsilon(w) \equiv \frac{1}{2}\int_{\mathbb{R}^d} \varepsilon^2 |\nabla w|^2 + 4f'(|\mathsf{a}^\varepsilon|^2)\langle w, \mathsf{a}^\varepsilon\rangle^2 + K\varepsilon^2 |w|^2 \; dx$$

for some constant $K > 0$. The corresponding H^s-type norm turns out to be

$$N^\varepsilon_s(w) \equiv \sum_{|\alpha| \leq s-1} N^\varepsilon(\partial^\alpha w) + K \big\|\mathrm{Re}\; w\big\|^2_{H^{s-2}},$$

the last term being here in order to control some commutators. In [33], for example, similar modulated linearized functionals like N^ε were introduced in the study of asymptotic problems in fluid mechanics.

It should be noticed that we do not solve (4) for ψ^ε in H^s or H^1, but only (23) for w in smooth spaces H^s, $s > 1 + d/2$, which presents the big advantage to avoid the problems on the Cauchy problem above mentionned. The control on the growth of the functionals N^ε_s allow to derive a positive lower bound for the existence time for w in H^s independent of ε.

Finally, our approach allows to treat the case of a domain with boundary, with a Neumann condition. For simplicity, we have considered the case of a half-space

$$\mathbb{R}^d_+ = \mathbb{R}^{d-1} \times (0, +\infty),$$

and we denote $x = (x', x_d) \in \mathbb{R}^{d-1} \times (0, +\infty)$. In this context, it is necessary to take into account boundary layers when working with smooth norms (but not for convergences in L^p spaces in space as in [56]). More precisely, since the solution (a, u) of the Euler system (16) with $u \cdot n = \frac{\partial \varphi}{\partial n} = 0$ on $\partial\mathbb{R}^d_+$ may not match the Neumann boundary condition $\partial_{x_d} a(t, x', 0) = 0$, a boundary layer of weak amplitude ε and of size ε appears. These boundary layers are formally described in [64], for a small speed u^∞ at infinity. The WKB expansions $\psi^\varepsilon = \mathsf{a}^\varepsilon \exp(\frac{i}{\varepsilon}\varphi^\varepsilon)$ are then of the form

$$\mathsf{a}^\varepsilon = a^0 + \sum_{k=1}^{m} \varepsilon^k \Big(\mathsf{a}^k(t, x) + A^k(t, x', \frac{x_d}{\varepsilon}) \Big), \quad \varphi^\varepsilon = \varphi^0 + \sum_{k=1}^{m} \varepsilon^k \Big(\varphi^k(t, x) + \Phi^k(t, x', \frac{x_d}{\varepsilon}) \Big)$$

where the profiles $A^k(t, x', X)$, $\Phi^k(t, x', X)$ have an exponentially fast decay in X and are determined in such a way that the approximate WKB expansion $\psi^\varepsilon_{app} = \mathsf{a}^\varepsilon \exp\left(\frac{i}{\varepsilon}\varphi^\varepsilon\right)$ satisfies the Neumann boundary condition

$$\frac{\partial \psi^\varepsilon}{\partial n} = 0.$$

We were able to prove (Theorem 6 in [19]) that this WKB expansion is nonlinearly stable: there exists a smooth solution ψ^ε for (4) in $\mathbb{R}^d_+$ with the Neumann boundary condition $\frac{\partial \psi^\varepsilon}{\partial n} = 0$ and the condition at infinity (19) on a time interval $[0, T]$ independent of ε, which verifies the estimate

$$\left\| \psi^\varepsilon e^{-\frac{i}{\varepsilon}\varphi^\varepsilon} - \mathsf{a}^\varepsilon \right\|_{W^{1,\infty}(\mathbb{R}^d_+)} \leq C\varepsilon. \tag{24}$$

For this solution, we have to include the first boundary layer εA^1 in order to get (24) since its gradient has amplitude one in L^∞.

The case of the Dirichlet boundary condition $\psi = 0$ on $\partial\mathbb{R}^d_+$ is also physically meaningfull (see [24] and also [64] for a formal asymptotic expansion of the boundary layers in this case), but, as often in boundary layer theory in fluid mechanics, seems more complicated to handle. As a matter of fact, in this case, the first boundary layer involved has amplitude one. However, in the context of an obstacle as in [56], the modelization by a Dirichlet condition may be too crude, and one may also use a potential term (see *e.g.* [35], [36]) which is a regularization of the rough potential $V(x) = +\infty$ if x lies in the obstacle and $V(x) = 0$ outside the obstacle of the type, for instance,

$$V^\varepsilon(x) = V_0\big(1 - \tanh(dist(x, \omega)/\varepsilon)\big).$$

Though these different modelizations may not deeply affect the physical results, we do not know how this may change the mathematical analysis.

2.4. **Beyond the breaking time.** For a defocusing subcritical nonlinearity, (NLS) Eq. is globally well-posed in H^1. On the other hand, the smooth solutions to the Euler system (6) are not expected global, due to the formation of shock waves, or caustics in geometric optics, in finite time. Therefore, it is a fundamental question to understand what happens *after* the breaking time T^*. Let us consider the linear Schrödinger Eq.

$$i\varepsilon \frac{\partial \psi}{\partial t} + \frac{\varepsilon^2}{2}\Delta\psi = V(x)\psi, \tag{25}$$

formally obtained by replacing $f(|\psi|^2)$ by a potential term $V(x)$, for which some results are known. In this case, (16) has to be replaced by

$$\left\{\begin{array}{ll}\partial_t a+\nabla\varphi\cdot\nabla a+\dfrac{a}{2}\,\Delta\varphi &=0\\ \partial_t\varphi+\dfrac{1}{2}|\nabla\varphi|^2+V(x) &=0.\end{array}\right. \tag{26}$$

The second Eq. in the above system is a Hamilton-Jacobi Eq., decoupled from the first Eq., and we may consider the associated bicharacteristics. One way to analyse the semiclassical limit for (25) is to use the Wigner transform ([71]), defined as

$$\mathcal{W}^\varepsilon(t,x,\xi)\equiv(2\pi)^{-d}\int_{\mathbb{R}^d}e^{-iy\cdot\xi}\psi^\varepsilon\big(t,x+\frac{\varepsilon}{2}y\big)\overline{\psi^\varepsilon}\big(t,x-\frac{\varepsilon}{2}y\big)\;dy.$$

We may formally recover the quadratic invariants associated to ψ^ε by computing the first two moments

$$|\psi^\varepsilon|^2(t,x)=\int_{\mathbb{R}^d}\mathcal{W}^\varepsilon(t,x,\xi)\;d\xi\qquad\quad \varepsilon\langle i\psi^\varepsilon,\nabla\psi^\varepsilon\rangle=\int_{\mathbb{R}^d}\xi\mathcal{W}^\varepsilon(t,x,\xi)\;d\xi.$$

Moreover, for a WKB initial datum $\psi^\varepsilon=\sqrt{\rho^\varepsilon}\exp(i\varphi^\varepsilon/\varepsilon)$ with $\rho^\varepsilon\to\rho$ and $\varphi^\varepsilon\to\varphi$ suitably, then

$$\mathcal{W}^\varepsilon(t,x,\xi)\to\rho(t,x)\delta_{\xi=\nabla_x\varphi(t,x)}\qquad\text{as}\quad\varepsilon\to 0.$$

The Wigner transform $\mathcal{W}^\varepsilon$ satisfies the following Vlasov (or Liouville) Eq.

$$\frac{\partial\mathcal{W}^\varepsilon}{\partial t}+\xi\cdot\nabla_x\mathcal{W}^\varepsilon+\Lambda^\varepsilon\mathcal{W}^\varepsilon=0, \tag{27}$$

where

$$\Lambda^\varepsilon\mathcal{W}^\varepsilon(t,x,\xi)\equiv i(2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{1}{\varepsilon}\Big(V\big(x+\frac{\varepsilon}{2}y\big)-V\big(x-\frac{\varepsilon}{2}y\big)\Big)e^{-i(\xi-\eta)\cdot y}\mathcal{W}^\varepsilon(t,x,\eta)\;d\eta dy.$$

If $\mathcal{W}^\varepsilon\to\mathcal{W}$ as $\varepsilon\to 0$ in a suitable sense, it is then expected that the *Wigner measure* $\mathcal{W}$ solves

$$\frac{\partial\mathcal{W}}{\partial t}+\xi\cdot\nabla_x\mathcal{W}+\Big(\nabla_x V\Big)\cdot\nabla_\xi\mathcal{W}=0. \tag{28}$$

From [40] and [61] and the references cited therein, $\mathcal{W}^\varepsilon\to\mathcal{W}$ in $w*-L^\infty(\mathbb{R}_+,\mathcal{M}_+)$, where $\mathcal{M}_+$ denotes the cone of nonnegative finite measures. We emphasize that the convergence holds actually for arbitrarily large times. Furthermore, in [61] (see also [40]), the Wigner measure $\mathcal{W}$ is shown to have, under some non degeneracy assumption, the following structure, locally and away from the caustics:

$$\mathcal{W}(t,x,\xi)=\sum_{\ell=1}^{N(x,t)}\rho_\ell(t,x)\delta_{\xi=\nabla_x\varphi_\ell(t,x)},$$

where each (ρ_ℓ,φ_ℓ) solves (26). In particular, the quadratic quantities split as (locally) finite sums in this case:

$$\rho^\varepsilon\simeq\sum_{\ell=1}^{N(x,t)}\rho_\ell\qquad\quad\varepsilon\langle i\psi^\varepsilon,\nabla\psi^\varepsilon\rangle\simeq\sum_{\ell=1}^{N(x,t)}\rho_\ell\nabla_x\varphi_\ell.$$

This suggests that ψ^ε may be written, after the breaking time, as a *sum* of WKB approximate solutions, that is

$$\psi^\varepsilon(t,x)\simeq\sum_{\ell=1}^{N}a_\ell\exp\big(\frac{i}{\varepsilon}\varphi_\ell\big),$$

since then, as $\varepsilon\to 0$,

$$\mathcal{W}^\varepsilon_{\psi^\varepsilon}\to\sum_{\ell=1}^{N}|a_\ell|^2(t,x)\delta_{\xi=\nabla_x\varphi_\ell(t,x)}.$$

In view of the linearity of (25), finite sums of oscillatory integrals may actually be exhibited by the stationnary phase method when $V = 0$. This underlines the fact that the viscosity solutions for the Hamilton-Jacobi Eq. are not appropriate: the *multivalued* solutions are the physically relevant ones. The numerical study of the multivalued solutions of (26) (hence even after the formation of shocks), is investigated through various approaches: the Whitham ([70]) averaging method (see [40] for example); the reduction of (28) to a finite moment system (see [40], [30]); and the level set approach (see [41]). On the theoretical level, it turns out that the wave function ψ^ε is indeed approximable by a finite sum

$$\sum_{\ell=1}^{N(t,x)} a_\ell \exp\big(\frac{i}{\varepsilon}\varphi_\ell + i\frac{\pi}{2}m_\ell\big),$$

where the integers m_ℓ are the so-called Keller-Maslov indices (see [44] and [62]). This is acheived with the construction of the canonical operator, which acts on ψ^ε as differentiation along the characteristics for the phase (plus a small remainder term).

In the nonlinear setting (4), to our knowledge, the only result concerning the description of ψ^ε after the breaking time T^* can be found in [39], for the one-dimensional cubic Schrödinger Eq. It relies on the Lax-Levermore theory for integrable systems developed for the dispersionless limit for KdV (see [54]). The Whitham averaging method can also be used for integrable systems.

3. The linear wave regime for (NLS)

The linear wave regime for (NLS) is investigated by F. Béthuel, R. Danchin and D. Smets in [11]. Comparing with the Euler asymptotic regime, this may be thought as the particular WKB development, with $a_0^0 = 1$ (recall $f(1) = 0$) and $\varphi_0^0 = 0$,

$$a_0^\varepsilon = 1 + \varepsilon a_0^1 + ..., \qquad \varphi_0^\varepsilon = 0 + \varepsilon\varphi_0^1 + ... \, .$$

Indeed, inserting this development in (5), we formally derive $a^0 = 1$ and $\varphi^0 = 0$, and then (a^1, φ^1) verifies the linearized Euler Eq. around $(1, 0)$, that is the free wave Eq. However, the previous analysis for the Euler asymptotic regime was specific for times $t = \varepsilon\tau$ of order one and, as we will see, the free wave limiting behaviour holds on much larger times, namely $0 \le t = \varepsilon\tau \ll \varepsilon^{-1}$. The first result obtained by [11] contains a uniform bound in high order Sobolev spaces as well as a comparison estimate with the corresponding solution of the free wave Eq. by treating the right-hand side of (7) as a source term.

Theorem 5 ([11])**.** *Let $s > 1 + d/2$. Then, there exists $C = C(s, d) > 0$ such that for every initial data $(a_0^\varepsilon, u_0^\varepsilon) \in H^{s+1} \times H^s$ for (7) with $C\varepsilon\|(a_0^\varepsilon, u_0^\varepsilon)\|_{H^{s+1}\times H^s} \le 1$, there exists, for some*

$$T^\varepsilon \ge \frac{1}{C\varepsilon\|(a_0^\varepsilon, u_0^\varepsilon)\|_{H^{s+1}\times H^s}},$$

a unique solution $(a^\varepsilon, u^\varepsilon) \in \mathcal{C}\big([0, T^\varepsilon], H^{s+1} \times H^s\big)$ to (7), which verifies, for $0 \le t = \varepsilon\tau \le T^\varepsilon$

$$\big\|(a^\varepsilon, u^\varepsilon)(t)\big\|_{H^{s+1}\times H^s} \le C\big\|(a_0^\varepsilon, u_0^\varepsilon)\big\|_{H^{s+1}\times H^s} \qquad \text{and} \qquad \frac{1}{2} \le 1 + \varepsilon a^\varepsilon \le 2.$$

As a consequence, if $(\mathfrak{a}^\varepsilon, \mathfrak{u}^\varepsilon)$ denotes the solution of the free wave Eq., where $c^2 = f'(1) > 0$,

$$\begin{cases} \partial_t \mathfrak{a}^\varepsilon + \nabla\cdot\mathfrak{u}^\varepsilon &= 0 \\ \partial_t \mathfrak{u}^\varepsilon + c^2\nabla\mathfrak{a}^\varepsilon &= 0, \end{cases}$$

with initial data $(a_0^\varepsilon, u_0^\varepsilon)$, then, if $\varepsilon < 1$ and $0 \le t \le T^\varepsilon$,

$$\big\|(a^\varepsilon, u^\varepsilon) - (\mathfrak{a}^\varepsilon, \mathfrak{u}^\varepsilon)\big\|_{H^{s-2}} \le C\Big[\varepsilon t\big\|(a_0^\varepsilon, u_0^\varepsilon)\big\|^2_{H^{s-1}\times H^s} + \varepsilon^2 t\big\|(a_0^\varepsilon, u_0^\varepsilon)\big\|_{H^{s+1}\times H^s}\Big].$$

Remark 1. In the case where a_0^ε and u_0^ε are of order ε^{-1}, but with $1/2 \le 1+\varepsilon a^\varepsilon \le 2$, the above uniform estimates on the time scale $t \sim 1$, *i.e.* $\tau \sim \varepsilon^{-1}$ give bounds similar to those derived in the previous section for the Euler asymptotic regime.

The last statement of Theorem 5 implies that the convergence to the free wave regime holds for times of order $0 \le t = \varepsilon\tau \ll 1$.

In [11], the uniform H^s bounds have been established starting from the system (7) and using an augmented system as in [8], involving the unknown

$$U^\varepsilon = \begin{pmatrix} a^\varepsilon \\ u^\varepsilon \\ \nabla \ln(1+\varepsilon a^\varepsilon) \end{pmatrix}.$$

Notice that in the linear wave asymptotic regime, we write $\Psi_{\tau=0} = \sqrt{1+\varepsilon a_0^\varepsilon(\varepsilon x)}\exp\big(i\varphi_0^\varepsilon(\varepsilon x)\big)$, which corresponds to the usual Madelung transform, and not the way to write Ψ used in [32]. Following an idea of F. Coquel, [8] works with the variables $\big(a^\varepsilon, u^\varepsilon + i\nabla\ln(1+\varepsilon a^\varepsilon)\big) \in \mathbb{R}\times\mathbb{C}^d$. The resulting system is not symmetrizable, whereas it is with the approach of E. Grenier. However, the principal part is somehow symmetrizable using the weight $(\rho^\varepsilon)^2 = 1+\varepsilon a^\varepsilon$ for the vector field part.

The second main result of [11] is a consequence of the dispersive properties of the operator, depending on ε and acting on $(a,u)^t$,

$$\frac{\partial}{\partial t} + \begin{pmatrix} 0 & \nabla\cdot \\ c^2\nabla - \varepsilon\nabla\Delta & 0 \end{pmatrix}.$$

This operator was alo involved in the scattering analysis of the Gross-Pitaevskii Eq. of [34] (in dimension $d \ge 4$). The use of Strichartz estimates allows then to improve the time T^ε.

Theorem 6 ([11])**.** *Under the assumption of Theorem 5 with if $s > 2 + d/2$ and $0 \le \varepsilon < 1$, then*

$$\begin{aligned}
&\textit{if } d \ge 4, && && T^\varepsilon \ge \frac{c}{\varepsilon^2\|(a_0^\varepsilon,u_0^\varepsilon)\|^2_{H^{s+1}\times H^s}};\\
&\textit{if } d = 3, && \forall\,\alpha\in(0,1), && T^\varepsilon \ge \min\Big(\frac{c_\alpha}{\varepsilon^{1+\alpha}\|(a_0^\varepsilon,u_0^\varepsilon)\|^{1+\alpha}_{H^{s+1}\times H^s}}, \frac{1}{\varepsilon^3\|(a_0^\varepsilon,u_0^\varepsilon)\|^2_{H^{s+1}\times H^s}}\Big);\\
&\textit{if } d = 2, && \forall\,q\in\big(\frac{2}{s-2},2\big), && T^\varepsilon \ge \min\Big(\frac{1}{\varepsilon^{4/3}\|(a_0^\varepsilon,u_0^\varepsilon)\|^{4/3}_{H^{s+1}\times H^s}}, \frac{c_q}{\varepsilon^{1+q}\|(a_0^\varepsilon,u_0^\varepsilon)\|^q_{H^{s+1}\times H^s}}\Big).
\end{aligned}$$

Denoting $(\mathfrak{a}^\varepsilon,\mathfrak{u}^\varepsilon)$ the solution of the linear equation

$$\frac{\partial}{\partial t}\begin{pmatrix} \mathfrak{a}^\varepsilon \\ \mathfrak{u}^\varepsilon \end{pmatrix} + \begin{pmatrix} 0 & \nabla\cdot \\ c^2\nabla - \varepsilon\nabla\Delta & 0 \end{pmatrix}\begin{pmatrix} \mathfrak{a}^\varepsilon \\ \mathfrak{u}^\varepsilon \end{pmatrix} = 0$$

with initial data $(a_0^\varepsilon,u_0^\varepsilon)$, then, if $\varepsilon < 1$, $\alpha\in(0,\frac12)$ and $0 \le t \le T^\varepsilon$,

$$\big\|(a^\varepsilon,u^\varepsilon) - (\mathfrak{a}^\varepsilon,\mathfrak{u}^\varepsilon)\big\|_{H^{s-1}} \le \begin{cases} C\varepsilon\sqrt{t}\big\|(a_0^\varepsilon,u_0^\varepsilon)\big\|^2_{H^{s+1}\times H^s} & d \ge 4 \\ C_\alpha\big(t^{1-\alpha}\varepsilon + \varepsilon^{3/2}\sqrt{t}\big)\big\|(a_0^\varepsilon,u_0^\varepsilon)\big\|^2_{H^{s+1}\times H^s} & d = 3 \\ C_\alpha\big(t^{3/4}\varepsilon + \varepsilon^{2-\alpha}t^{1-\alpha}\big)\big\|(a_0^\varepsilon,u_0^\varepsilon)\big\|^2_{H^{s+1}\times H^s} \quad \textit{for } \alpha > 2 - s/2 & d = 2. \end{cases}$$

Remark 2. In [11] the last statement is actually more precise, since it is shown that the low frequency part of $u^\varepsilon - \mathfrak{u}^\varepsilon$, corresponding to $|\xi| \le \varepsilon^{-1}$, is even smaller by a factor ε.

For this particular asymptotic regime, the convergence is proved for times $t = \varepsilon\tau$ always much smaller than ε^{-2}.

4. The KdV/KP-I regime for (NLS)

We recall that for the KdV/KP-I regime for (NLS), the data are such that $|\Psi| \simeq 1$ and $|\Psi| \to 1$ at infinity. In this context, it is natural to choose for the Ginzburg-Landau energy (18)

$$\mathcal{E}(\Psi) \equiv \frac{1}{2} \int_{\mathbb{R}^n} |\nabla \Psi|^2 + F\big(|\Psi|^2\big)\ dx,$$

where $F(R) \equiv 2 \int_1^R f(r)\ dr$.

4.1. **The KdV regime for (NLS).** In this subsection, we consider the case $d = 1$. To begin with, we may work first only in the energy space for (NLS) and the H^1 energy space for KdV. Notice that the condition at infinity $|\Psi| \to 1$ makes the Cauchy problem for (NLS) nonusual. Several recent works are devoted to this question: [73], [25] and the survey [27]. We shall use[4] the following:

Theorem 7 ([73]). *There exists $\mathcal{E}_0 > 0$ such that, for every $\Psi_0 \in H^1_{loc}(\mathbb{R})$ verifying $\mathcal{E}(\Psi_0) \le \mathcal{E}_0$ and $|\Psi_0|(x) \to 1$ as $|x| \to +\infty$, there exists a unique solution Ψ to (NLS) such that $\Psi - \Psi_0 \in \mathcal{C}\big(\mathbb{R}_+, H^1(\mathbb{R})\big)$. Moreover, $\mathcal{E}\big(\Psi(t)\big) = \mathcal{E}(\Psi_0)$ for $t \ge 0$.*

The Cauchy problem for the KdV equation[5] is also known to be well-posed in the H^1 energy space by [45].

Theorem 8 ([45]). *We consider the Cauchy problem for the KdV equation*

$$2\partial_t w + k\, w \partial_x w - \frac{1}{4c^2}\, \partial_{xxx} w = 0, \qquad w_{|t=0} = w_0.$$

If $w_0 \in H^1(\mathbb{R})$, then there exists a unique solution of the KdV equation satisfying $w \in \mathcal{C}\big(\mathbb{R}_+, H^1(\mathbb{R})\big)$ and $\partial_x w \in L^4_{loc}\big(\mathbb{R}_+, L^\infty(\mathbb{R})\big)$. Furthermore, $\|w(t)\|_{L^2(\mathbb{R})}$ does not depend on $t \in \mathbb{R}_+$.

The well-posedness of KdV has been shown in spaces of much lower regularity (see [46] and [74]), but we do not use these results. We finally use the natural scaling for the KdV/KP-I regime:

$$\Psi(\tau, x) = \psi^\varepsilon(t, \varepsilon(x_1 - c\tau), \varepsilon^2 x_\perp), \qquad t = c\varepsilon^3 \tau$$

where $c = \sqrt{f'(1)} > 0$, so that (NLS) reads now

$$ic\varepsilon^3 \frac{\partial \psi^\varepsilon}{\partial t} - ic\varepsilon \partial_{x_1} \psi^\varepsilon + \frac{\varepsilon^2}{2} \partial^2_{x_1} \psi^\varepsilon + \frac{\varepsilon^4}{2} \Delta_\perp \psi^\varepsilon = \psi^\varepsilon f(|\psi^\varepsilon|^2). \tag{29}$$

Theorem 9 ([20]). *Assume that $(a_0^\varepsilon)_{0<\varepsilon<1} \in H^1$ and $(\varphi_0^\varepsilon)_{0<\varepsilon<1} \in \dot{H}^1$ are uniformly bounded and well-prepared in the sense that*

$$M \equiv \sup_{0<\varepsilon<1} \left\{ \big\| a_0^\varepsilon \big\|_{H^1} + \frac{1}{\varepsilon} \big\| \partial_x \varphi_0^\varepsilon - 2c a_0^\varepsilon \big\|_{L^2} \right\} < +\infty$$

and assume that

$$a_0^\varepsilon \to a_0 \quad in \quad L^2 \quad as \quad \varepsilon \to 0.$$

Consider the initial datum

$$\psi_0^\varepsilon = \big(1 + \varepsilon^2 a_0^\varepsilon\big) \exp\big(i\varepsilon\varphi_0^\varepsilon\big)$$

for (29), and let $\psi^\varepsilon \in \psi_0^\varepsilon + \mathcal{C}(\mathbb{R}_+, H^1)$ be the associated solution (given by Theorem 7).

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[4] In order to use Theorem III.3.1 in [73], we notice that if $\mathcal{E}(\Psi)$ is small, with $\Psi = \rho e^{i\phi}$, then $\|\partial_x \rho\|_{L^2} + \|\rho - 1\|_{L^\infty} + \|\partial_x \phi\|_{L^2}$ is small. For the middle term, this follows by Sobolev embedding, since $F(\rho) \simeq f'(1)(\rho - 1)^2$ as $\rho \to 1$.

[5] Notice that it might happen that $k = 0$, in which case the KdV equation reduces to the so-called (linear) Airy equation $2\partial_t w - \frac{1}{4c^2}\partial_x^3 w = 0$, and the Cauchy problem is then trivial to solve.

Then, there exists $\varepsilon_0 > 0$, depending only on M, such that, for $0 < \varepsilon \leq \varepsilon_0$, there exist two real-valued functions φ^ε, $a^\varepsilon \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$ such that $(a^\varepsilon, \varphi^\varepsilon)_{|t=0} = (a_0^\varepsilon, \varphi_0^\varepsilon)$, and

$$\psi^\varepsilon = \left(1 + \varepsilon^2 a^\varepsilon\right) \exp\left(i\varepsilon\varphi^\varepsilon\right)$$

with $1 + \varepsilon^2 a^\varepsilon \geq \frac{1}{2}$. Furthermore, as $\varepsilon \to 0$, we have the convergences

$$a^\varepsilon \to a \quad \text{in} \quad \mathcal{C}([0,T], H^s), \quad \partial_x \varphi^\varepsilon \to 2ca, \quad \text{in} \quad \mathcal{C}([0,T], L^2)$$

for every $s < 1$ and every $T > 0$, where a is the solution of KdV with initial value a_0.

Let us emphasize that the initial data are well-prepared (compare with (11), where, we recall, $\partial_x \varphi^\varepsilon = 2cu^\varepsilon$) in the sense that

$$\left\|\partial_x \varphi_0^\varepsilon - 2ca_0^\varepsilon\right\|_{L^2} = \mathcal{O}(\varepsilon).$$

Under a stronger assumption on the preparedness of the initial data, namely

$$\left\|\partial_x \varphi_0^\varepsilon - 2ca_0^\varepsilon\right\|_{L^2} = o(\varepsilon),$$

we can obtain

Corollary 1 ([20]). *Under the assumptions of Theorem 9, if we assume moreover*

$$a_0^\varepsilon \to a_0 \qquad \text{in} \quad H^1 \quad \text{as} \quad \varepsilon \to 0$$

and

$$\left\|\partial_x \varphi_0^\varepsilon - 2cA_0^\varepsilon\right\|_{L^2} = o(\varepsilon),$$

then, for every $T > 0$

$$A^\varepsilon \to A \qquad \text{in} \quad \mathcal{C}\big([0,T], H^1(\mathbb{R})\big) \qquad \text{and} \qquad \left\|\partial_x \varphi^\varepsilon - 2cA^\varepsilon\right\|_{L^\infty([0,T],L^2)} = o(\varepsilon).$$

In this case, the convergence to the KdV asymptotic regime takes place for times $t = \mathcal{O}(1)$, that is $\tau = \mathcal{O}(\varepsilon^{-3})$.

The proof of Theorem 9 is based on a compactness argument. From the conservation of energy and momentum, one infers first the uniform bounds

$$\sup_{0<\varepsilon<1} \left\|a^\varepsilon\right\|_{L^\infty(\mathbb{R}_+, H^1)} + \frac{1}{\varepsilon}\left\|\partial_x \varphi^\varepsilon - 2ca^\varepsilon\right\|_{L^\infty(\mathbb{R}_+, L^2)} < +\infty.$$

Then, from the fact that, in (10), the singular terms come from a transport Eq. with high speed ε^{-2}, we deduce local compactness in space-time for $\partial_x \varphi^\varepsilon$ and a^ε in $L^2_{loc}(\mathbb{R}_+ \times \mathbb{R})$, which allows to pass to the limit in (12). We finally recover global strong convergences thanks to the conservation of the energy $\mathcal{E}(\Psi)$ of Ψ and the L^2 norm of the solution to the KdV Eq..

Using different techniques, namely the integrable character of the GP equation (that is (NLS) with $f(\varrho) = \varrho - 1$), F. Béthuel, P. Gravejat, J-C. Saut and D. Smets in [13] proved the following comparison result between A^ε and the corresponding solution to KdV:

Theorem 10 ([13]). *Let ψ^ε be the solution to (29) with $f(\varrho) = \varrho - 1$ for the initial datum*

$$\psi_0^\varepsilon = \left(1 + \varepsilon^2 a_0^\varepsilon\right) \exp\left(i\varepsilon\varphi_0^\varepsilon\right),$$

with

$$\left\|a^\varepsilon\right\|_{H^3} + \left\|u^\varepsilon\right\|_{H^3} \leq M,$$

then, for $0 < \varepsilon < \varepsilon_0(M)$, there holds

$$\left\|a^\varepsilon - w^\varepsilon\right\|_{L^2} \leq C_M \Big(\left\|a_0^\varepsilon - u_0^\varepsilon\right\|_{H^3} + \varepsilon\Big) e^{C_M t},$$

where w^ε is the solution of KdV with

$$w_0^\varepsilon = a_0^\varepsilon.$$

We may observe that the above result allows to compare the functions a^ε and w^ε for times of order $t \ll |\ln \varepsilon|$, that is $\tau \ll \varepsilon^{-3}|\ln \varepsilon|$, provided that $\|a_0^\varepsilon - u_0^\varepsilon\|_{H^3} = \mathcal{O}(\varepsilon^\alpha)$ for some positive α.

In dimension $d = 1$, the solution of the wave equation (7) consists, by Duhamel's formula, in two travelling bumps propagating to the left and to the right with speed $c = \sqrt{f'(1)}$. In the above mentionned results, the focus is on a single bump. It is of high interest to consider the two sliding bumps and to understand the interaction between them. For some results in this direction, see [14].

4.2. **The KdV/KP-I regime for (NLS) in smooth norms.** In this subsection, the dimension d is arbitrary, and we work with H^s norms for a sufficiently large s. Note that for an initial datum w in H^s with $s > 1 + d/2$, the Cauchy problem for the KdV/KP-I equation

$$2\partial_t w + \Big[6 + \frac{2}{c^2} f''(1)\Big] w \partial_{x_1} w - \frac{1}{4c^2}\partial_{x_1}^3 w + \Delta_\perp \partial_{x_1}^{-1} w = 0$$

is well-posed: there exists a unique local in time H^s solution. Note that it is actually known to be well-posed in spaces of much lower regularity [37], [63] in dimension $d = 2$. In dimension $d = 3$, the solution of KP-I may blow-up (in H^1) in finite time (see [59]).

Theorem 11 ([20])**.** *Let $d \geq 1$ and let s such that $s > 1 + \frac{d}{2}$. Assume that*

$$M_s \equiv \sup_{0<\varepsilon<1} \big\| \big(a_0^\varepsilon, \partial_{x_1}\varphi_0^\varepsilon, \varepsilon\nabla_\perp \varphi_0^\varepsilon\big)\big\|_{H^{s+1}(\mathbb{R}^d)} < +\infty$$

and consider the initial datum for (29)

$$\psi_0^\varepsilon = \big(1 + \varepsilon^2 a_0^\varepsilon\big) \exp\big(i\varepsilon\varphi_0^\varepsilon\big).$$

Then, there exist $T > 0$ and $0 < \varepsilon_0 < 1$, depending on M_s, such that, for $0 < \varepsilon \leq \varepsilon_0$, there exist two real-valued functions $a^\varepsilon \in \mathcal{C}\big([0,T], H^{s+1}(\mathbb{R}^d)\big)$ and $\varphi^\varepsilon \in \mathcal{C}\big([0,T], \dot H^{s+1}(\mathbb{R}^d)\big) \cap \mathcal{C}\big([0,T]\times\mathbb{R}^d\big)$ such that $(a^\varepsilon, \varphi^\varepsilon)_{|t=0} = (A_0^\varepsilon, \varphi_0^\varepsilon)$ and, for $0 \leq t \leq T$,

$$\psi^\varepsilon = \big(1 + \varepsilon^2 a^\varepsilon\big) \exp\big(i\varepsilon\varphi^\varepsilon\big), \qquad\qquad 1 + \varepsilon^2 a^\varepsilon \geq 1/2$$

and

$$\sup_{0<\varepsilon<\varepsilon_0,\ t\in[0,T]} \Big\{ \|a^\varepsilon\|_{H^{s+1}(\mathbb{R}^d)} + \big\|\big(\partial_{x_1}\varphi^\varepsilon, \varepsilon\nabla_\perp\varphi^\varepsilon\big)\big\|_{H^s(\mathbb{R}^d)} \Big\} < +\infty.$$

We assume moreover that for some functions $(a_0, \partial_{x_1}\varphi_0) \in H^{s+1}(\mathbb{R}^d)$, there holds

$$(a_0^\varepsilon, \partial_{x_1}\varphi_0^\varepsilon, \varepsilon\nabla_\perp\varphi_0^\varepsilon) \to (a_0, \partial_{x_1}\varphi_0, 0) \qquad in \quad L^2(\mathbb{R}^d)$$

and let a be the solution of the KdV/KP-I equation with initial value

$$a_{|t=0} = \frac{1}{2}\big(a_0 + \frac{1}{2c}\partial_{x_1}\varphi_0\big) = \frac{1}{2}\big(a_0 + (u_0^\varepsilon)_1\big) \in H^{s+1}(\mathbb{R}^d).$$

Then, as $\varepsilon \to 0$, we have the strong convergence

$$\frac{1}{2}\Big(a^\varepsilon + \frac{1}{2c}\partial_{x_1}\varphi^\varepsilon\Big) \to a \qquad in \qquad L^2\big([0,T], H^\sigma(\mathbb{R}^d)\big) \quad \forall\, \sigma < s$$

and the weak convergences

$$a^\varepsilon \rightharpoonup a \qquad \partial_x\varphi^\varepsilon \rightharpoonup 2ca \qquad weakly\ in \quad L^2\big([0,T]\times\mathbb{R}^d\big).$$

Remark 3. In dimension $d = 1$, the above uniform estimate in H^s are similar to those obtained in [11]. In higher dimension, this is no longer the case. From the proof of the above theorem, one may infer the lower bound $T \geq C/M_s$. One can not expect to use some dispersive properties to improve very much the time T as in Theorem 6, at least in dimension three, in view of the blow-up result in [59]. We do not know whether the techniques of [37] could allow to improve the time T in dimension 2.

In [20], the uniform bounds have been established using the trick of E. Grenier, that is to write $\psi^\varepsilon = A^\varepsilon \exp(i\varepsilon\phi)$ with A^ε complex-valued.

One may improve the convergences in the case of well-prepared data:

Theorem 12 ([20]). *Under the same assumptions of Theorem 11, if moreover we have*

$$(30)\qquad \big\|\nabla_\perp \varphi_0^\varepsilon\big\|_{L^2(\mathbb{R}^d)} = \mathcal{O}(1) \quad \textit{and} \quad \begin{cases} (d=1) & \big\|\partial_x\varphi_0^\varepsilon - 2ca_0^\varepsilon\big\|_{L^2(\mathbb{R}^d)} \to 0 \\ (d\geq 2) & \big\|\partial_x\varphi_0^\varepsilon - 2ca_0^\varepsilon\big\|_{L^2(\mathbb{R}^d)} = \mathcal{O}(\varepsilon) \end{cases} \quad \textit{as} \quad \varepsilon\to 0,$$

then, we have the strong convergences

$$a^\varepsilon \to a \quad \textit{strongly in } \mathcal{C}\big([0,T], H^{\sigma+1}(\mathbb{R}^d)\big), \qquad \partial_{x_1}\varphi^\varepsilon \to 2ca \quad \textit{strongly in } \mathcal{C}([0,T], H^{\sigma}(\mathbb{R}^d))$$

for every $\sigma < s$. Furthermore, if $d \geq 2$, there exists $K > 0$ such that, for $0 \leq t \leq T$, $0 < \varepsilon < \varepsilon_0$,

$$(31)\qquad \int_{\mathbb{R}^d} |\nabla_\perp \varphi^\varepsilon|^2 \, dX \leq K.$$

We emphasize that the hypothesis (30) is stronger in dimensions $d \geq 2$ than in dimension $d = 1$, in order to ensure the bound (31). Moreover, in dimension $d = 1$, (30) is weaker than the hypothesis in Theorem 9.

It is well-known, in the physics literature, that the KdV soliton is unstable in the 2-dimensional KP-I Eq. with respect to periodic transverse perturbations (see [5] and [43]). The nonlinear instability can be rigorously settled using the complete integrability of both the KdV and the 2-dimensional KP-I Eqs. Recently, [66] proved this result by using a PDE approach which does not involve integrability arguments.

It would be interesting to tackle the problem for the Gross-Pitaevkii Eq. in the KdV/KP-I asymptotic regime. The paper [52] shows the transverse instability of the one dimensional travelling waves of the Gross-Pitaevskii Eq.. From the formal computations of [51], it is expected that the 2-dimensional soliton of KP-I becomes unstable in dimension 3 for long wavelength periodic transverse nonaxisymetric perturbations, and should create vortices (see [9]).

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