Finding the sum of any series from a given general term*

Leonhard Euler

1. When I had considered more carefully what I explained by the geometrical method in the previous paper[1] on the summation of series and when I had investigated the same method of summation analytically, I saw that what I had extracted geometrically could be deduced from a special method of summation that I had already mentioned three years earlier in a paper on the summation of series[2]. But I had not thought about this more since then. Having examined more deeply the effectiveness of the analytical method, I perceived that not only was the formula discovered geometrically contained in it, but also that by means of it more could be accomplished by adding more terms, so that it would show the true sum absolutely. The geometrical method however seems to find these same terms with the greatest difficulty.

2. In the former paper on the summation of series, if x is the general term of index n of some series, I exhibited in a universal way the following form for the summatory term

$$\int xdn + \frac{x}{2} + \frac{dx}{12dn} - \frac{d^3x}{720dn^3} + \text{etc.},$$

in which the differentials of x over powers of the differential dn, which is assumed constant, are destroyed, because x is taken to be given by n,[3] so that an algebraic sum is obtained if of course xdn admits integration. In the integration of xdn indeed a constant ought to be added such that the whole expression vanishes by putting $n = 0$.

3. Now since I have set out in this paper to describe more accurately this formula and its use, before everything else I shall explain how I discovered

*Presented to the St. Petersburg Academy on October 13, 1735. Originally published as *Inventio summae cuiusque seriei ex dato termino generali*, Commentarii academiae scientiarum Petropolitanae **8** (1741), 9–22. E47 in the Eneström index. Translated from the Latin by Jordan Bell, Department of Mathematics, University of Toronto, Toronto, Ontario, Canada. Email: jordan.bell@gmail.com

[1]Translator: *Methodus universalis serierum convergentium summas quam proxime inveniendi*, E46.

[2]Translator: *Methodus generalis summandi progressiones*, E25, §2.

[3]Translator: My best reading is that since dn is small but fixed, if $d^kx = 0$ for some k then $\frac{d^kx}{dn^k} = 0$ and also for all higher powers.

the formula. I used some singular arguments which offer much to Analysis, partly new and partly already known, which however as far as I recall are not demonstrated clearly enough elsewhere.

4. It follows from the nature of infinitesimal calculus that if y depends in any fixed way on x, if $x + dx$ is put in place of x then y will turn into $y + dy$. Now, if x is then increased by the element dx, that is x is changed to $x + 2dx$, then in place of y we will have $y + 2dy + ddy$. And if x is again increased with dx, then y will transform into $y+3dy+3ddy+d^3y$, where the coefficients are the same as those of the powers of a binomial. From here it follows that if $x+mdx$ is put in place of x, then y will take on this form

$$y + \frac{m}{1}dy + \frac{m(m-1)}{1\cdot 2}ddy + \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}d^3y + \text{etc.}$$

5. Now for our purpose let m be an infinitely large number such that mdx represents a finite quantity; putting $x+mdx$ in place of x, y will have this value

$$y + \frac{mdy}{1} + \frac{m^2d^2y}{1\cdot 2} + \frac{m^3d^3y}{1\cdot 2\cdot 3} + \frac{m^4d^4y}{1\cdot 2\cdot 3\cdot 4} + \text{etc.}$$

Now if we let $mdx = a$ or $m = \frac{a}{dx}$, if $x + a$ is put for x, then y will assume this form

$$y + \frac{ady}{1dx} + \frac{a^2ddy}{1\cdot 2dx^2} + \frac{a^3d^3y}{1\cdot 2\cdot 3dx^3} + \text{etc.},$$

in which all the terms are of finite magnitude.

6. This series, which exhibits the transformed value of y if $x + a$ is put in place of x, was first found by the very insightful Taylor in the *Methodus incrementorum directa et inversa*, and he applied it to many excellent uses. The first result that follows is the raising of a binomial to any power. So if the value of $(x + a)^m$ is sought, I put

$$y = x^m$$

and if $x + a$ is put in place of x, the value of y will be $(x+a)^m$. Since therefore

$$dy = mx^{m-1}dx, \quad d^2y = m(m-1)x^{m-2}dx^2$$

and so on, it will be

$$(x + a)^m = x^m + \frac{max^{m-1}}{1} + \frac{m(m-1)a^2x^{m-2}}{1\cdot 2} + \text{etc.}$$

7. Then by doing the following the Taylor series lets us find approximately a root of this equation. Let us have an equation involving an unknown z, namely $Z = 0$, where Z is a quantity composed in some known way from the unknown z. Then take x as a value nearly equal to z, and let the quantity of Z which

occurs when x is put in place of z be put $= y$, so that if x were the true value of z then $y = 0$.

8. Now since x differs from the true value of z by a certain amount, put the true value of z to be $x + a$. It is thus clear that if in y we put $x + a$ in place of x then y will vanish. And indeed if one puts $x + a$ in place of x then y will turn into

$$y + \frac{ady}{1dx} + \frac{a^2 ddy}{1 \cdot 2dx^2} + \frac{a^3 d^3 y}{1 \cdot 2 \cdot 3dx^3} + \text{etc.}$$

From this it follows that

$$0 = y + \frac{ady}{1dx} + \frac{a^2 ddy}{1 \cdot 2dx^2} + \text{etc.}$$

9. Since x is set to be very close to z, a will be a very small quantity, so that beside the first two terms all the following ones will vanish. By doing this it arises that $a = -\frac{ydx}{dy}$ and so $z = x - \frac{ydx}{dy}$, which is a value much nearer to z than x. Thus for the equation

$$z^3 - 3z - 20 = 0$$

it will be

$$y = x^3 - 3x - 20 \quad \text{and} \quad \frac{dy}{dx} = 3x^2 - 3$$

and hence

$$z = x - \frac{x^3 - 3x - 20}{3xx - 3} = \frac{2x^3 + 20}{3xx - 3}.$$

Now by first taking $x = 3$ it will be $z = 3\frac{1}{12}$, and then repeating this for a second time taking this value in place of x will lead to a value even closer to z.

10. Next, if some condition is stipulated on the function y by which it is to have a particular relation to x, then the above formula will turn into an equation which contains the character of y. Thus if y is a function of x that vanishes by putting $x = 0$, I put $a = -x$; for thus it turns out that $x + a = 0$ and it will be

$$0 = y - \frac{xdy}{1dx} + \frac{x^2 ddy}{1 \cdot 2dx^2} - \frac{x^3 d^3 y}{1 \cdot 2 \cdot 3dx^3} + \text{etc.}$$

or

$$y = \frac{xdy}{1dx} - \frac{x^2 ddy}{1 \cdot 2dx^2} + \frac{x^3 d^3 y}{1 \cdot 2 \cdot 3dx^3} - \text{etc.}$$

The nature of all functions of x which vanish by putting $x = 0$ are contained in this equation.

11. If we write $\int zdx$ for y, it will be

$$dy = zdx, \quad ddy = dzdx, \quad d^3 y = d^2 zdx \quad \text{etc.;}$$

substituting these values in we get

$$\int zdx = \frac{xz}{1} - \frac{x^2dz}{1\cdot 2dx} + \frac{x^3ddz}{1\cdot 2\cdot 3dx^2} - \text{etc.},$$

in which equation the integral of zdx is expressed by an infinite series. And this is the general quadrature of curves which the most insightful Johann Bernoulli gave in the *Acta eruditorum* of Leipzig; however, he did not attach the analysis which led to this series.

12. However disregarding this, which pertains less to our purpose, I return to series. Therefore let us have some series

$$A + B + C + D + \cdots + X,$$

in which A denotes the first term, B the second, and X that whose index is x, so that X is the general term of the given series. Let us also put the sum of this progression to be

$$A + B + C + D + \cdots + X = S;$$

S will be the summatory term, and if the series is determined it will be composed from x and fixed as much as X is.

13. Now because S will exhibit the sum of as many terms from the series as there are unities in x, if $x-1$ is written in place of x in S, we will obtain the previous sum with the final term X removed. This substitution therefore turns S into $S-X$. Let us compare this with the above formula; it will be $S=y$ and $a=-1$, from which the transformed value of S, or $S-X$, it will be

$$= S - \frac{dS}{1dx} + \frac{ddS}{1\cdot 2dx^2} - \frac{d^3S}{1\cdot 2\cdot 3dx^3} + \text{etc.},$$

from which this equation arises:

$$X = \frac{dS}{1dx} - \frac{ddS}{1\cdot 2dx^2} + \frac{d^3S}{1\cdot 2\cdot 3dx^3} - \frac{d^4S}{1\cdot 2\cdot 3\cdot 4dx^4} + \text{etc.}$$

14. Therefore by means of this equation, the general term of any series is found from the given summatory term. However, since this is already very easy, it would be superfluous to use this method for finding the general term from the summatory term. Rather this equation is most useful if all the terms are expanded, and it can thus be applied to all uses. For by a known method the series

$$X = \frac{dS}{1dx} - \frac{ddS}{1\cdot 2dx^2} + \frac{d^3S}{1\cdot 2\cdot 3dx^3} - \text{etc.}$$

can be inverted, so that from the general term X the summatory term S can be determined, which is desired most.

15. Let us therefore put

$$\frac{dS}{dx} = \alpha X + \frac{\beta dX}{dx} + \frac{\gamma ddX}{dx^2} + \frac{\delta d^3X}{dx^3} + \frac{\epsilon d^4X}{dx^4} + \text{etc.},$$

so it will be

$$S = \alpha \int X dx + \beta X + \frac{\gamma dX}{dx} + \frac{\delta ddX}{dx^2} + \text{etc.}$$

Next, it will be

$$\frac{ddS}{dx^2} = \frac{\alpha dX}{dx} + \frac{\beta ddX}{dx^2} + \frac{\gamma d^3X}{dx^3} + \frac{\delta d^4X}{dx^4} + \text{etc.}$$

and

$$\frac{d^3S}{dx^3} = \frac{\alpha ddX}{dx^2} + \frac{\beta d^3X}{dx^3} + \frac{\gamma d^4X}{dx^4} + \text{etc.}$$

and

$$\frac{d^4S}{dx^4} = \frac{\alpha d^3X}{dx^3} + \frac{\beta d^4X}{dx^4} + \text{etc.}$$

and then

$$\frac{d^5S}{dx^5} = \frac{\alpha d^4X}{dx^4} + \text{etc.}$$

16. Let us substitute the series in place of each of the terms of the above series, and put the similar terms among these equal to 0. By doing this, the coefficients α, β, γ etc. will be determined as follows

$$\begin{aligned}
\alpha &= 1, \\
\beta &= \frac{\alpha}{2}, \\
\gamma &= \frac{\beta}{2} - \frac{\alpha}{6}, \\
\delta &= \frac{\gamma}{2} - \frac{\beta}{6} + \frac{\alpha}{24}, \\
\epsilon &= \frac{\delta}{2} - \frac{\gamma}{6} + \frac{\beta}{24} - \frac{\alpha}{120}, \\
\zeta &= \frac{\epsilon}{2} - \frac{\delta}{6} + \frac{\gamma}{24} - \frac{\beta}{120} + \frac{\alpha}{720} \\
&\quad \text{etc.}
\end{aligned}$$

17. Thus the coefficients $\alpha, \beta, \gamma, \delta$ etc. constitute a series of such a nature that each term is determined by all the preceding terms, with the first term being $= 1$. Also, the numbers which all the final terms need to be divided by constitute the progression called *hypergeometric* by Wallis

$$2, \quad 6, \quad 24, \quad 120, \quad 720, \quad 5040 \quad \text{etc.}$$

However, this series of coefficients α, β, γ etc. is thus constituted that I could hardly believe that each could be exhibited by some general term.

18. Therefore for our purposes we should be contented with the series of coefficients being continued as far as we want, which can easily be done perfectly from the law of the progression. I have worked out this series as follows,

$$+1,\ +\frac{1}{1\cdot 2},\ +\frac{1}{1\cdot 2\cdot 3\cdot 2},\ +0,\ -\frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6},\ -0,\ +\frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 6},\ +0,$$
$$-\frac{3}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10},\ -0,\ +\frac{5}{1\cdots 11\cdot 6},\ +0,\ -\frac{691}{1\cdots 13\cdot 210},\ -0,$$
$$+\frac{35}{1\cdots 15\cdot 2},+0,-\frac{3617}{1\cdots 17\cdot 30}\quad \text{etc.}$$

It is notable that in this series all the even terms besides the second vanish.

19. Therefore if these terms are substituted in place of α, β, γ etc., we will obtain the following summatory term

$$\begin{aligned} S \;=\; & \int X dx + \frac{X}{1\cdot 2} + \frac{dX}{1\cdot 2\cdot 3\cdot 2dx} - \frac{d^3X}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6dx^3} + \frac{d^5X}{1\cdots 7\cdot 6dx^5} \\ & - \frac{3d^7X}{1\cdots 9\cdot 10dx^7} + \frac{5d^9X}{1\cdots 11\cdot 6dx^9} - \frac{691d^{11}X}{1\cdots 13\cdot 210dx^{11}} \\ & + \frac{35d^{13}X}{1\cdots 15\cdot 2dx^{13}} - \frac{3617d^{15}X}{1\cdots 17\cdot 30dx^{15}} + \text{etc.} \end{aligned}$$

20. This series has an important use in finding the sums of algebraic progressions, in which x does not appear in the denominator of the general term. For this reason x will have positive exponents everywhere, and hence some differential of it will vanish and thus the series will stop, and therefore the summatory term will be represented by a finite number of terms. Immediately we see that all the terms which do not contain x can be ignored, since already some constant needs to be added in $\int X dx$, to make $S = 0$ when we put $x = 0$.

21. To clearly see the use of this formula, it is worthwhile to offer some examples. Thus let $X = x$, that is, let the series to be summed be

$$1 + 2 + 3 + \cdots + x;$$

since

$$\int X dx = \frac{x^2}{2}$$

the sum will be

$$S = \frac{x^2 + x}{2};$$

for $\frac{dX}{dx}$ is constant and is therefore ignored, and the following differentials spontaneously vanish.

Next let $X = x^2$, or let this be the series to be summed

$$1 + 4 + 9 + \cdots + x^2;$$

it will be

$$\int X dx = \frac{x^3}{3} \quad \text{and} \quad \frac{dX}{dx} = 2x$$

and hence the sum of the series is

$$S = \frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{6}.$$

22. Now let the general series of powers of the natural numbers be given

$$1 + 2^n + 3^n + 4^n + 5^n + \text{etc.},$$

whose general term is x^n. One will therefore have $X = x^n$ and

$$\int X dx = \frac{x^{n+1}}{n+1}.$$

Furthermore the differentials will be thus obtained,

$$\begin{aligned}
\frac{dX}{dx} &= nx^{n-1}, \\
\frac{d^3X}{dx^3} &= n(n-1)(n-2)x^{n-3}, \\
\frac{d^5X}{dx^5} &= n(n-1)(n-2)(n-3)(n-4)x^{n-5} \\
& \text{etc.}
\end{aligned}$$

Therefore with these values substituted the summatory term of the given series will be

$$\begin{aligned}
S = {} & \frac{x^{n+1}}{n+1} + \frac{x^n}{2} + \frac{nx^{n-1}}{2 \cdot 6} - \frac{n(n-1)(n-2)x^{n-3}}{2 \cdot 3 \cdot 4 \cdot 30} + \frac{n(n-1)(n-2)(n-3)(n-4)x^{n-5}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 42} \\
& - \frac{n(n-1)\cdots(n-6)x^{n-7}}{2 \cdot 3 \cdots 8 \cdot 30} + \frac{n(n-1)\cdots(n-8)5x^{n-9}}{2 \cdot 3 \cdots 10 \cdot 66} - \frac{n(n-1)\cdots(n-10)691x^{n-11}}{2 \cdot 3 \cdots 12 \cdot 2730} \\
& + \frac{n(n-1)\cdots(n-12)7x^{n-13}}{2 \cdot 3 \cdots 14 \cdot 6} - \frac{n(n-1)\cdots(n-14)3617x^{n-15}}{2 \cdot 3 \cdots 16 \cdot 510} + \text{etc.}
\end{aligned}$$

The above series α, β, γ etc. should be continued as far necessary for this series, which is worth continuing.

23. Thus from this general summation of the series whose general term is

x^n, sums of series of particular powers can be constructed, as follows,

$$\begin{aligned}
\int x^1 &= \frac{x^2}{2} + \frac{x}{2}, \\
\int x^2 &= \frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{6}, \\
\int x^3 &= \frac{x^4}{4} + \frac{x^3}{2} + \frac{x^2}{4}, \\
\int x^4 &= \frac{x^5}{5} + \frac{x^4}{2} + \frac{x^3}{3} - \frac{x}{30}, \\
\int x^5 &= \frac{x^6}{6} + \frac{x^5}{2} + \frac{5x^4}{12} - \frac{x^2}{12}, \\
\int x^6 &= \frac{x^7}{7} + \frac{x^6}{2} + \frac{x^5}{2} - \frac{x^3}{6} + \frac{x}{42}, \\
\int x^7 &= \frac{x^8}{8} + \frac{x^7}{7} + \frac{7x^6}{12} - \frac{7x^4}{24} + \frac{x^2}{12}, \\
\int x^8 &= \frac{x^9}{9} + \frac{x^8}{2} + \frac{2x^7}{3} - \frac{7x^5}{15} + \frac{2x^3}{9} - \frac{x}{30}, \\
\int x^9 &= \frac{x^{10}}{10} + \frac{x^9}{2} + \frac{3x^8}{4} - \frac{7x^6}{10} + \frac{x^4}{2} - \frac{3x^2}{20}, \\
\int x^{10} &= \frac{x^{11}}{11} + \frac{x^{10}}{2} + \frac{5x^9}{6} - x^7 + x^5 - \frac{x^3}{2} + \frac{5x}{66}, \\
\int x^{11} &= \frac{x^{12}}{12} + \frac{x^{11}}{2} + \frac{11x^{10}}{12} - \frac{11x^8}{8} + \frac{11x^6}{6} - \frac{11x^4}{8} + \frac{5x^2}{12}, \\
\int x^{12} &= \frac{x^{13}}{13} + \frac{x^{12}}{2} + x^{11} - \frac{11x^9}{6} + \frac{22x^7}{7} - \frac{33x^5}{10} + \frac{5x^3}{3} - \frac{691x}{2730}, \\
\int x^{13} &= \frac{x^{14}}{14} + \frac{x^{13}}{2} + \frac{13x^{12}}{12} - \frac{143x^{10}}{60} + \frac{143x^8}{28} - \frac{143x^6}{20} + \frac{65x^4}{12} - \frac{691x^2}{420}, \\
\int x^{14} &= \frac{x^{15}}{15} + \frac{x^{14}}{2} + \frac{7x^{13}}{6} - \frac{91x^{11}}{30} + \frac{143x^9}{18} - \frac{143x^7}{10} + \frac{91x^5}{6} - \frac{691x^3}{90} + \frac{7x}{6}, \\
\int x^{15} &= \frac{x^{16}}{16} + \frac{x^{15}}{2} + \frac{5x^{14}}{4} - \frac{91x^{12}}{24} + \frac{143x^{10}}{12} - \frac{429x^8}{16} + \frac{455x^6}{12} - \frac{691x^4}{24} + \frac{35x^2}{4}, \\
\int x^{16} &= \frac{x^{17}}{17} + \frac{x^{16}}{2} + \frac{4x^{15}}{3} - \frac{14x^{13}}{3} + \frac{52x^{11}}{3} - \frac{143x^9}{3} + \frac{260x^7}{3} - \frac{1382x^5}{15} \\
&\quad + \frac{140x^3}{3} - \frac{3617x}{510}.
\end{aligned}$$

24. But if on the other hand x does not always have positive exponents in the general term of a series, then too the expression will come out to be a sum of infinitely many terms, because series of this kind do not admit general summation, but rather involve quadratures. Still though, I have observed that

by means of this formula this type of series can easily be summed very closely. This has a great utility in series which converge slowly and others that are difficult to sum. I will explain through examples how this is done.

25. Thus before any others I will first consider the harmonic series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\text{etc.},$$

whose general term is $\frac{1}{x}$, and let S be the summatory term which is sought. So it is $X=\frac{1}{x}$ and

$$\int X dx = \text{Const.} + lx.$$

And then

$$\frac{dX}{dx}=\frac{-1}{x^2},\quad \frac{d^3X}{dx^3}=\frac{-1\cdot 2\cdot 3}{x^4},\quad \frac{d^5X}{dx^5}=\frac{-1\cdot 2\cdot 3\cdot 4\cdot 5}{x^6}\quad \text{etc.}$$

Substituting these yields

$$\begin{aligned} S \quad = \quad & \text{Const.} + lx + \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \frac{1}{240x^8} - \frac{1}{132x^{10}} \\ & + \frac{691}{32760x^{12}} - \frac{1}{12x^{14}} + \text{etc.}, \end{aligned}$$

where the constant that is added needs to be such that by putting $x=0$ it makes $S=0$. Certainly however the constant cannot be determined from this, because all the terms are infinitely large.

26. Indeed to determine the constant another case should be considered, in which the sum of the series is known; this can be obtained if a certain number of terms are gathered into a single sum. Therefore let us add the first 10 terms

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{10}$$

whose sum turns out to be

$$= 2,9289682539682539;$$

this should be equal to the sum of the terms from the formula, namely

$$\text{Const.} + l10 + \frac{1}{20} - \frac{1}{1200} + \frac{1}{1200000} - \frac{1}{252000000} + \frac{1}{24000000000} - \frac{1}{1320000000000} + \text{etc.}$$

With this done, because one finds that

$$l10 = 2,302585092994045684$$

the added constant will be

$$= 0,5772156649015329$$

and with this determined once, any sum of terms of this series can be found.

27. I have investigated the sum of 100, 1000, 10000 etc. terms of the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} +$ etc. by this rule, and I have found the following,

$$\begin{aligned}
\int 10 &= 2,9289682539682539, \\
\int 100 &= 5,1873775176396203, \\
\int 1000 &= 7,4854708605503449, \\
\int 10000 &= 9,7876060360443823, \\
\int 100000 &= 12,0901461298634280, \\
\int 1000000 &= 14,3927267228657236.
\end{aligned}$$

28. If the first term of the series, 1, is taken, it will be $S = 1$ and $x = 1$, and hence $lx = 0$. From the equation we therefore get

$$0,4227843350984670 = \frac{1}{2} - \frac{1}{12} + \frac{1}{120} - \frac{1}{252} + \frac{1}{240} - \frac{1}{132} + \frac{691}{32760} - \frac{1}{12} + \text{etc.}$$

This series is very irregular and not even convergent, and the sum is found only approximately. However the sum of the series continued to infinity will be

$$= l\infty + 0,5772156649015329,$$

which happens by putting $x = \infty$.

29. Now let us proceed to considering this series

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \text{etc.}$$

in which $X = \frac{1}{2x-1}$ and

$$\int X dx = \text{Const.} + \frac{1}{2} l(2x-1)$$

and also

$$\frac{dX}{dx} = \frac{-2}{(2x-1)^2}, \quad \frac{d^3X}{dx^3} = \frac{-2 \cdot 4 \cdot 6}{(2x-1)^4}, \quad \frac{d^5X}{dx^5} = \frac{-2 \cdot 4 \cdot 6 \cdot 8 \cdot 10}{(2x-1)^6} \quad \text{etc.}$$

With these found, the sum of the proposed series will be

$$\begin{aligned}
S = {} & \text{Const.} + \frac{1}{2} l(2x-1) + \frac{1}{2(2x-1)} - \frac{1}{6(2x-1)^2} + \frac{1}{15(2x-1)^4} - \frac{8}{63(2x-1)^6} \\
& + \frac{8}{15(2x-1)^8} - \frac{128}{32(2x-1)^{10}} + \frac{256 \cdot 691}{4095(2x-1)^{12}} - \frac{2048}{3(2x-1)^{14}} + \frac{1024 \cdot 3617}{255(2x-1)^{16}} - \text{etc.}
\end{aligned}$$

30. The constant quantity in this case cannot be determined as easily as that in the previous case by actual addition of several terms. Indeed in this case a great help is that this constant can be determined from the preceding one. Namely the sum of the series

$$1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\text{etc.}$$

continued to infinity is $=$ Const. $+\frac{1}{2}l\infty$. Let us subtract the harmonic series from twice this series; we will have

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\text{etc.},$$

whose sum it turns out is $l2$. Therefore it will be

$$l2=2\text{const.}+l\infty-l\infty-0,577215\,\text{etc.}$$

and hence the constant that is sought is

$$=0,6351814227307392.$$

31. I proceed now to more complicated series, and I consider

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\text{etc.}$$

the reciprocals of the squares, whose general term is $\frac{1}{x^2}=X$. Therefore it will be

$$\int X\,dx=\text{Const.}-\frac{1}{x}$$

and

$$\frac{dX}{dx}=\frac{-2}{x^3},\quad \frac{d^3X}{dx^3}=\frac{-2\cdot3\cdot4}{x^5},\quad \frac{d^5X}{dx^5}=\frac{-2\cdot3\cdot4\cdot5\cdot6}{x^7}\quad\text{etc.}$$

With these substituted it will be

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots+\frac{1}{x^2}=S$$

$$=\text{Const.}-\frac{1}{x}+\frac{1}{2x^2}-\frac{1}{6x^3}+\frac{1}{30x^5}-\frac{1}{42x^7}+\frac{1}{30x^9}-\frac{5}{66x^{11}}+\frac{691}{2730x^{13}}-\frac{7}{6x^{15}}+\text{etc.}$$

where the constant quantity should be determined from a special case.

32. Thus I actually added the first ten terms of this series and I found that their sum is

$$1,549767731166540.$$

Since in this case $x=10$, if this is added to

$$\frac{1}{10}-\frac{1}{200}+\frac{1}{6000}-\frac{1}{3000000}+\frac{1}{420000000}-\frac{1}{30000000000}+\frac{1}{1320000000000}$$
$$-\frac{691}{27300000000000000}+\frac{7}{6000000000000000}-\text{etc.},$$

one gets an added constant $= 1,64493406684822643647$. And this constant is equal to the sum of the series continued to infinity; for by putting $x = \infty$ it will be $S =$ Const., with all the terms vanishing.

33. In a similar way for the reciprocals of the cubes

$$1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \text{etc.}$$

if the first ten terms are added, this sum is obtained

$$1,197531985674193.$$

Whence one finds that the constant which should be added in the summation of this series is

$$= 1,202056903159594.$$

And this number is equal to the sum of the series

$$1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64}$$

continued to infinity.

And for the biquadrates

$$1 + \frac{1}{16} + \frac{1}{81} + \text{etc.}$$

the sum is

$$= 1,0823232337110824.$$

34. Let us now consider by this method the series by which the area of the circle whose diameter is 1 is exhibited, namely

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \text{etc.}$$

or

$$\frac{2}{1\cdot 3} + \frac{2}{5\cdot 7} + \frac{2}{9\cdot 11} + \frac{2}{13\cdot 15} + \text{etc.},$$

whose general term is

$$\frac{2}{(4x-3)(4x-1)}$$

or resolving into factors

$$\frac{1}{4x-3} - \frac{1}{4x-1}.$$

For finding the approximate sum of this series,

$$X = \frac{1}{4x-3} - \frac{1}{4x-1}$$

and

$$\int X dx = \text{Const.} - \frac{1}{4} l \frac{4x-1}{4x-3}$$

and then

$$\frac{dX}{dx} = \frac{-4}{(4x-3)^2} + \frac{4}{(4x-1)^2}, \quad \frac{d^3X}{dx^3} = \frac{-4\cdot 8\cdot 12}{(4x-3)^4} + \frac{4\cdot 8\cdot 12}{(4x-1)^4} \quad \text{etc.}$$

From this the sum of the series

$$\frac{2}{1\cdot 3} + \frac{2}{5\cdot 7} + \ldots + \frac{2}{(4x-3)(4x-1)} + \text{etc.}$$

will be

$$\begin{aligned} S \;=\; & \text{Const.} - \frac{1}{4} l \frac{4x-1}{4x-3} + \frac{1}{2}\left(\frac{1}{4x-3} - \frac{1}{4x-1}\right) - \frac{1}{3}\left(\frac{1}{(4x-3)^2} - \frac{1}{(4x-1)^2}\right) \\ & + \frac{8}{15}\left(\frac{1}{(4x-3)^4} - \frac{1}{(4x-1)^4}\right) - \frac{256}{63}\left(\frac{1}{(4x-3)^6} - \frac{1}{(4x-1)^6}\right) \\ & + \frac{1024}{15}\left(\frac{1}{(4x-3)^8} - \frac{1}{(4x-1)^8}\right) - \frac{4^8}{33}\left(\frac{1}{(4x-3)^{10}} - \frac{1}{(4x-1)^{10}}\right) + \text{etc.} \end{aligned}$$

Truly even if ten terms of this series are added it will not converge enough so that a proper constant could be exhibited. But four times the constant is equal to the periphery of a circular whose diameter is $= 1$.