Meditations on a singular kind of series *

Leonhard Euler

In the correspondence I once used to have with the most illustrious Goldbach, among other speculations of different kind, we considered series contained in this general form

$$1+\frac{1}{2^m}\left(1+\frac{1}{2^n}\right)+\frac{1}{3^m}\left(1+\frac{1}{2^n}+\frac{1}{3^n}\right)+\frac{1}{4^m}\left(1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}\right)+\text{etc.}$$

and tried to find their sums. Even though series of this kind use to occur rarely and they seem to have hardly any use, those investigations to which their consideration had led us nevertheless seem even more worth that they are published, because the methods, we used on this occasion, extend much further and can maybe once be useful for Analysis. Therefore, I decided to present not only the series, even though they, considered separately, by no means seem to be regarded with contempt, but also the various methods that lead to this summations, in this paper; because these are taken from the mentioned correspondence, I want to let the readers know directly that these investigations for the most part are to be attributed to the ingenuity of the most illustrious Goldbach. Mainly three ways lead to series of this kind; because they differ a lot from each other, I will explain each one separately that each one can be understood a lot easier.

*Original title: " Mediationes circa singulare Serierum Genus", first published in „*Novi commentarii academiae scientiarum Petropolitanae 20* (1775), 1776, pp. 140-186", reprinted in „*Opera Omnia*: Series 1, Volume 15, pp. 291 - 349 ", Eneström-Number E477, translated by: Alexander Aycock, for the „Euler-Kreis Mainz"

The First Method to obtain series of this kind

§1 If one has two arbitrary series whose sum is known, say

$$1 + a + b + c + d + e + \text{etc.} = t$$

and

$$1 + \alpha + \beta + \gamma + \delta + \varepsilon + \text{etc.} = u,$$

but then, in addition to them, the sum of the series conflated of them is known, say,

$$1 + a\alpha + b\beta + c\gamma + d\delta + e\varepsilon + \text{etc.} = v,$$

then, by multiplying those series by each other, it follows:

$$\left.\begin{aligned} 1 + a(1+\alpha) + b(1+\alpha+\beta) + c(1+\alpha+\beta+\gamma) + \text{etc.} \\ +1 + \alpha(1+a) + \beta(1+a+b) + \gamma(1+a+b+c) + \text{etc.} \end{aligned}\right\} = tu + v,$$

which is obvious, since the products of the single terms of the first series and the single terms of the second series occur in these two last series; just note that the products of a certain term of the first series and the corresponding term of the second, e.g., $1 \cdot 1$, $a\alpha$, $b\beta$, $c\gamma$, $d\delta$ etc., occur twice; and because they occur only once in the product tu, the series v has to be added.

§2 Now, if we denote the sum of the series

$$1 + \frac{1}{2^m} + \frac{1}{3^m} + \frac{1}{4^m} + \text{etc.}$$

continued to infinity by $\int \frac{1}{z^m}$ that it is

$$1 + \frac{1}{2^m} + \frac{1}{3^m} + \frac{1}{4^m} + \frac{1}{5^m} + \text{etc.} = \int \frac{1}{z^m}$$

and

$$1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\frac{1}{5^n}+\text{etc.}=\int\frac{1}{z^n}$$

and in like manner the series derived from that one by

$$1+\frac{1}{2^{m+n}}+\frac{1}{3^{m+n}}+\frac{1}{4^{m+n}}+\frac{1}{5^{m+n}}+\text{etc.}=\int\frac{1}{z^{m+n}},$$

and from these we form the following series, which will be contained in the propounded form,

$$1+\frac{1}{2^m}\left(1+\frac{1}{2^n}\right)+\frac{1}{3^m}\left(1+\frac{1}{2^n}+\frac{1}{3^n}\right)+\frac{1}{4^m}\left(1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}\right)+\text{etc.}=P,$$

$$1+\frac{1}{2^n}\left(1+\frac{1}{2^m}\right)+\frac{1}{3^n}\left(1+\frac{1}{2^m}+\frac{1}{3^m}\right)+\frac{1}{4^n}\left(1+\frac{1}{2^m}+\frac{1}{3^m}+\frac{1}{4^m}\right)+\text{etc.}=Q,$$

from the principle established above we will have

$$P+Q=\int\frac{1}{z^m}\cdot\int\frac{1}{z^n}+\int\frac{1}{z^{m+n}},$$

whence, if the sum of the one of these two new series would be known from elsewhere, hence also the sum of the other series can be assigned. But we consider the sums of the series

$$1+\frac{1}{2^m}+\frac{1}{3^m}+\frac{1}{4^m}+\text{etc.}\quad\text{or}\quad\int\frac{1}{z^m}$$

to be known here, since, as often as the exponent m is an even number, I was able to determine these sums by means of the circumference of the circle; and for the cases, in which m is an odd number, the sum can easily be found approximately.

§3 Whenever the exponents m and n are assumed to be equal, the two series found are identical and therefore in this case we obtain the following summation

$$1+\frac{1}{2^n}\left(1+\frac{1}{2^n}\right)+\frac{1}{3^n}\left(1+\frac{1}{2^n}+\frac{1}{3^n}\right)+\frac{1}{4^n}\left(1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}\right)+\text{etc.}$$

$$= \frac{1}{2}\left(\int \frac{1}{z^n}\right)^2 + \frac{1}{2}\int \frac{1}{z^{2n}}.$$

Therefore, if we, going to consider particular cases, put

$$\begin{aligned}
1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \text{etc.} &= \Delta = \int \frac{1}{z},\\
1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \text{etc.} &= A = \int \frac{1}{z^2},\\
1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \text{etc.} &= B = \int \frac{1}{z^3},\\
1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \text{etc.} &= C = \int \frac{1}{z^4},\\
1 + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} + \frac{1}{5^5} + \text{etc.} &= D = \int \frac{1}{z^5}\\
&\text{etc.}
\end{aligned}$$

and in like manner further

$$\int \frac{1}{z^6} = E, \quad \int \frac{1}{z^6} = F, \quad \int \frac{1}{z^8} = G, \quad \frac{1}{z^9} = H, \quad \int \frac{1}{z^{10}} = I \quad \text{etc.},$$

we hence obtain the following summations

$$\begin{aligned}
1 + \frac{1}{2^1}\left(1 + \frac{1}{2^1}\right) + \frac{1}{3^1}\left(1 + \frac{1}{2^1} + \frac{1}{3^1}\right) + \text{etc.} &= \frac{1}{2^1}\Delta\Delta + \frac{1}{2}A,\\
1 + \frac{1}{2^2}\left(1 + \frac{1}{2^2}\right) + \frac{1}{3^2}\left(1 + \frac{1}{2^2} + \frac{1}{3^2}\right) + \text{etc.} &= \frac{1}{2^2}AA + \frac{1}{2}C,\\
1 + \frac{1}{2^3}\left(1 + \frac{1}{2^3}\right) + \frac{1}{3^3}\left(1 + \frac{1}{2^3} + \frac{1}{3^3}\right) + \text{etc.} &= \frac{1}{2^3}BB + \frac{1}{2}E,\\
1 + \frac{1}{2^4}\left(1 + \frac{1}{2^4}\right) + \frac{1}{3^4}\left(1 + \frac{1}{2^4} + \frac{1}{3^4}\right) + \text{etc.} &= \frac{1}{2^4}CC + \frac{1}{2}G,\\
1 + \frac{1}{2^5}\left(1 + \frac{1}{2^5}\right) + \frac{1}{3^5}\left(1 + \frac{1}{2^5} + \frac{1}{3^5}\right) + \text{etc.} &= \frac{1}{2^3}DD + \frac{1}{2}I\\
&\text{etc.,}
\end{aligned}$$

where it is convenient to note that the sum Δ of the first series is infinite, all the remaining ones on the other hand are finite.

§4 But if the exponents n and m are not equal, this way series of the form we are contemplating will be obtained, whose sum cannot be determined by means of this method; but nevertheless, the sum of both taken together can be exhibited, as we showed before. To render this more clear and at the same time introduce a shorthand notation, let us denote the sum of this series

$$1+\frac{1}{2^m}\left(1+\frac{1}{2^n}\right)+\frac{1}{3^m}\left(1+\frac{1}{2^n}+\frac{1}{3^n}\right)+\frac{1}{4^m}\left(1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}\right)+\text{etc.}$$

by this sign $\int\frac{1}{z^m}\left(\frac{1}{y^n}\right)$. In this notation the result we found will be expressed this way

$$\int\frac{1}{z^m}\left(\frac{1}{y^n}\right)+\int\frac{1}{z^n}\left(\frac{1}{y^m}\right)=\int\frac{1}{z^m}\cdot\int\frac{1}{z^n}+\int\frac{1}{z^{m+n}}.$$

Therefore, if the sum of the one of these series is known from elsewhere, hence the sum of the other series will also be known; and more cannot be concluded from this first method, whence I proceed to the explanation of the second.

SECOND METHOD TO GET TO SERIES OF THIS KIND

§5 Still using the notation we just introduced it is perspicuous that the quantity

$$\int \frac{1}{z^m} \cdot \int \frac{1}{z^n} = \int \frac{1}{z^{m+n}}$$

is reduced to the following infinite series

$$\begin{array}{ll}
+\frac{1}{1\cdot 2^n}+\frac{1}{2^m\cdot 3^n}+\frac{1}{3^m\cdot 4^n}+\text{etc.} & +\frac{1}{1\cdot 2^m}+\frac{1}{2^n\cdot 3^m}+\frac{1}{3^n\cdot 4^m}+\text{etc.}\\
+\frac{1}{1\cdot 3^n}+\frac{1}{2^m\cdot 4^n}+\frac{1}{3^m\cdot 5^n}+\text{etc.} & +\frac{1}{1\cdot 3^m}+\frac{1}{2^n\cdot 4^m}+\frac{1}{3^n\cdot 5^m}+\text{etc.}\\
+\frac{1}{1\cdot 4^n}+\frac{1}{2^m\cdot 5^n}+\frac{1}{3^m\cdot 6^n}+\text{etc.} & +\frac{1}{1\cdot 4^m}+\frac{1}{2^n\cdot 5^m}+\frac{1}{3^n\cdot 6^m}+\text{etc.}\\
+\frac{1}{1\cdot 5^n}+\frac{1}{2^m\cdot 6^n}+\frac{1}{3^m\cdot 7^n}+\text{etc.} & +\frac{1}{1\cdot 5^m}+\frac{1}{2^n\cdot 6^m}+\frac{1}{3^n\cdot 7^m}+\text{etc.}\\
\text{etc.} & \text{etc.}
\end{array}$$

and the whole task now reduces to this that these single series are summed in a convenient way; this opens a very broad and fruitful field which contains a peculiar kind of series and because of its inner beauty deserves one's complete attention, even though it is not connected directly to our actual task.

§6 But these summations cannot be investigated more conveniently than by resolving the single terms, whose form is

$$\frac{1}{x^m(x+a)^n},$$

into simpler fractions. But by means of the things I treated on this subject in the *Introctio ad Analysin*, it is plain that the fraction is decomposed into the following ones

$$\frac{1}{a^n}\cdot\frac{1}{x^m}-\frac{n}{1a^{n+1}}\cdot\frac{1}{x^{m-1}}+\frac{n(n+1)}{1\cdot 2a^{n+2}}\cdot\frac{1}{x^{m-2}}-\frac{n(n+1)(n+2)}{1\cdot 2\cdot 3a^{n+3}}\cdot\frac{1}{x^{m-3}}+\text{etc.}$$

$$\pm\frac{1}{a^m}\cdot\frac{1}{(x+a)^n}\pm\frac{m}{1a^{m+1}}\cdot\frac{1}{(x+a)^{n-1}}\pm\frac{m(m+1)}{1\cdot 2a^{m+2}}\cdot\frac{1}{(x+a)^{n-2}}$$

$$\pm\frac{m(m+1)(m+2)}{1\cdot 2\cdot 3a^{m+3}}\cdot\frac{1}{(x+a)^{n-3}}\pm\text{etc.},$$

where it is to be noted that in the lower rows the upper sign $+$ holds, if m is an even number, otherwise the lower sign; but then each of both rows must be continued until the terms, where the exponent of the powers of x and $x+a$ was decreased to 1.

§7 Therefore, hence at first the sum of this series

$$\frac{1}{1(a+1)^n}+\frac{1}{2^m(a+2)^n}+\frac{1}{3^m(a+3)^n}+\frac{1}{4^m(a+4)^n}+\text{etc.}=s$$

can be defined, while in the form just exhibited all numbers 1, 2, 3 etc. to infinity are substituted for x and collected into one single sum. For, since all terms resulting from the formula

$$A\frac{1}{x^\lambda}$$

give a series whose sum we expressed by

$$A\int\frac{1}{x^\lambda},$$

but the terms resulting from the formula

$$A\frac{1}{(x+a)^\lambda}$$

give a series, whose sum is

$$A\int\frac{1}{z^\lambda}-A\left(1+\frac{1}{2^\lambda}+\frac{1}{3^\lambda}+\cdots+\frac{1}{a^\lambda}\right),$$

the sum of our series will be

$$s=\frac{1}{a^n}\int\frac{1}{z^m}-\frac{n}{1a^{n+1}}\int\frac{1}{z^{m-1}}+\frac{n(n+1)}{1\cdot 2a^{n+2}}\int\frac{1}{z^{m-2}}-\frac{n(n+1)(n+2)}{1\cdot 2\cdot 3a^{n+3}}\int\frac{1}{z^{m-3}}+\text{etc.}$$

$$\pm\frac{1}{a^m}\int\frac{1}{z^n}\pm\frac{m}{1a^{m+1}}\int\frac{1}{z^{n-1}}\pm\frac{m(m+1)}{1\cdot 2a^{m+2}}\int\frac{1}{z^{n-2}}\pm\frac{m(m+1)(m+2)}{1\cdot 2\cdot 3a^{m+3}}\int\frac{1}{z^{n-3}}\pm\text{etc.}$$

$$\mp\frac{1}{a^m}\left(1+\frac{1}{2^n}+\frac{1}{3^n}+\cdots+\frac{1}{a^n}\right)$$

$$\mp\frac{m}{1a^{m+1}}\left(1+\frac{1}{2^{n-1}}+\frac{1}{3^{n-1}}+\cdots+\frac{1}{a^{n-1}}\right)$$
$$\mp\frac{m(m+1)}{1\cdot 2a^{m+2}}\left(1+\frac{1}{2^{n-2}}+\frac{1}{3^{n-2}}+\cdots+\frac{1}{a^{n-2}}\right)$$
$$\mp\frac{m(m+1)(m+2)}{1\cdot 2\cdot 3a^{m+3}}\left(1+\frac{1}{2^{n-3}}+\frac{1}{3^{n-3}}+\cdots+\frac{1}{a^{n-3}}\right)$$
$$\text{etc.,}$$

where the upper signs hold, if m is an even number, otherwise the lower signs hold. But this expression is always finite, since both structures of terms have to be continued until $\int\frac{1}{z}$.

§8 Now, let us also attribute all values from 1 to infinity to the letter a to comprehend all infinite series on the left-hand side in § 5 of the first structure in one sum; and there sum represented this way will be found to be

$$\int\frac{1}{z^n}\cdot\int\frac{1}{z^m}-\frac{n}{1}\int\frac{1}{z^{n+1}}\cdot\int\frac{1}{z^{m-1}}+\frac{n(n+1)}{1\cdot 2}\int\frac{1}{z^{n+2}}\cdot\int\frac{1}{z^{m-2}}$$
$$-\frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}\int\frac{1}{z^{n+2}}\cdot\int\frac{1}{z^{m-3}}+\text{etc.}$$
$$\pm\int\frac{1}{z^m}\cdot\int\frac{1}{z^n}\mp\int\frac{1}{z^m}\left(\frac{1}{y^n}\right)$$
$$\pm\frac{m}{1}\int\frac{1}{z^{m+1}}\cdot\int\frac{1}{z^{n-1}}\mp\frac{m}{1}\int\frac{1}{z^{m+1}}\left(\frac{1}{y^{n-1}}\right)$$
$$\pm\frac{m(m+1)}{1\cdot 2}\int\frac{1}{z^{m+2}}\cdot\int\frac{1}{z^{n-2}}\mp\frac{m(m+1)}{1\cdot 2}\int\frac{1}{z^{m+2}}\left(\frac{1}{y^{n-2}}\right)$$
$$\pm\frac{m(m+1)(m+2)}{1\cdot 2\cdot 3}\int\frac{1}{z^{m+3}}\cdot\int\frac{1}{z^{n-2}}\mp\frac{m(m+1)(m+2)}{1\cdot 2\cdot 3}\int\frac{1}{z^{m+3}}\left(\frac{1}{y^{n-3}}\right)$$
$$\text{etc}$$

And in like manner by exchanging the exponents m and n the sum of the series of the other kind on the right-hand side in § 5 will result. Therefore, having combined these expressions the quantity

$$\int\frac{1}{z^m}\cdot\int\frac{1}{z^n}-\int\frac{1}{z^{m+n}}$$

is transformed into the following form, which is conveniently exhibited in two parts

FIRST PART

$$
\begin{gathered}
+(1\pm 1)\int\frac{1}{z^m}\cdot\frac{1}{z^n}\mp\int\frac{1}{z^m}\left(\frac{1}{y^n}\right)\\
-\frac{m}{1}(1\mp 1)\int\frac{1}{z^{m+1}}\cdot\frac{1}{z^{n-1}}\mp\frac{m}{1}\int\frac{1}{z^{m-1}}\left(\frac{1}{y^{n-1}}\right)\\
+\frac{m(m+1)}{1\cdot 2}(1\pm 1)\int\frac{1}{z^{m+2}}\cdot\frac{1}{z^{n-2}}\mp\frac{m(m+1)}{1\cdot 2}\int\frac{1}{z^{m-2}}\left(\frac{1}{y^{n-2}}\right)\\
-\frac{m(m+1)(m+2)}{1\cdot 2\cdot 3}(1\mp 1)\int\frac{1}{z^{m+3}}\cdot\frac{1}{z^{n-3}}\mp\frac{m(m+1)(m+2)}{1\cdot 2\cdot 3}\int\frac{1}{z^{m-3}}\left(\frac{1}{y^{n-3}}\right)\\
\text{etc.},
\end{gathered}
$$

where the upper signs hold, if m is an even number, the lower sings on the other hand, if m is an odd number.

SECOND PART

$$
\begin{gathered}
+(1\pm 1)\int\frac{1}{z^n}\cdot\frac{1}{z^m}\mp\int\frac{1}{z^n}\left(\frac{1}{y^m}\right)\\
-\frac{n}{1}(1\mp 1)\int\frac{1}{z^{n+1}}\cdot\frac{1}{z^{m-1}}\mp\frac{n}{1}\int\frac{1}{z^{n-1}}\left(\frac{1}{y^{m-1}}\right)\\
+\frac{n(n+1)}{1\cdot 2}(1\pm 1)\int\frac{1}{z^{n+2}}\cdot\frac{1}{z^{m-2}}\mp\frac{n(n+1)}{1\cdot 2}\int\frac{1}{z^{n-2}}\left(\frac{1}{y^{m-2}}\right)\\
-\frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}(1\mp 1)\int\frac{1}{z^{n+3}}\cdot\frac{1}{z^{m-3}}\mp\frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}\int\frac{1}{z^{m-3}}\left(\frac{1}{y^{m-3}}\right)\\
\text{etc.},
\end{gathered}
$$

where the upper signs hold, if n is an even number, otherwise the lower signs hold.

§9 Depending on whether m and n are even or odd, these expressions are reduced to a lower number of terms; it will of course be

FIRST PART, IF m IS AN EVEN NUMBER

$$
+2\int\frac{1}{z^m}\cdot\int\frac{1}{z^n}-\int\frac{1}{z^m}\left(\frac{1}{y^n}\right)-\frac{m}{1}\int\frac{1}{z^{m+1}}\left(\frac{1}{y^{n-1}}\right)
$$

$$+\frac{2m(m+1)}{1\cdot 2}\int\frac{1}{z^{m+2}}\cdot\int\frac{1}{z^{n-2}}-\frac{m(m+1)}{1\cdot 2}\int\frac{1}{z^{m+2}}\left(\frac{1}{y^{n-2}}\right)$$
$$-\frac{m(m+1)(m+2)}{1\cdot 2\cdot 3}\int\frac{1}{z^{m+3}}\left(\frac{1}{y^{n-3}}\right)$$
etc.

or this way

$$2\int\frac{1}{z^{m}}\cdot\int\frac{1}{z^{n}}+\frac{2m(m+1)}{1\cdot 2}\int\frac{1}{z^{m+2}}\cdot\int\frac{1}{z^{n-2}}$$
$$+\frac{2m(m+1)(m+2)(m+3)}{1\cdot 2\cdot 3\cdot 4}\int\frac{1}{z^{m+4}}\cdot\int\frac{1}{z^{n-4}}+\text{etc.}$$
$$-\int\frac{1}{z^{m}}\left(\frac{1}{y^{n}}\right)-\frac{m}{1}\int\frac{1}{z^{m+1}}\left(\frac{1}{y^{n-1}}\right)-\frac{m(m+1)}{1\cdot 2}\int\frac{1}{z^{m+2}}\left(\frac{1}{y^{n-2}}\right)-\text{etc.}$$

THE FIRST PART, IF m IS AN ODD NUMBER

$$-\frac{2m}{1}\int\frac{1}{z^{m+1}}\cdot\int\frac{1}{z^{n-1}}-\frac{2m(m+1)(m+2)}{1\cdot 2\cdot 3}\int\frac{1}{z^{m+3}}\cdot\int\frac{1}{z^{n-3}}-\text{etc.}$$
$$+\int\frac{1}{z^{m}}\left(\frac{1}{y^{n}}\right)+\frac{m}{1}\int\frac{1}{z^{m+1}}\left(\frac{1}{y^{n-1}}\right)+\frac{m(m+1)}{1\cdot 2}\int\frac{1}{z^{m+2}}\left(\frac{1}{y^{n-2}}\right)+\text{etc.}$$

THE SECOND PART, IF n IS AN EVEN NUMBER

$$2\int\frac{1}{z^{n}}\cdot\int\frac{1}{z^{m}}+\frac{2n(n+1)}{1\cdot 2}\int\frac{1}{z^{n+2}}\cdot\int\frac{1}{z^{m-2}}$$
$$+\frac{2n(n+1)(n+2)(n+3)}{1\cdot 2\cdot 3\cdot 4}\int\frac{1}{z^{n+4}}\cdot\int\frac{1}{z^{m-4}}+\text{etc.}$$
$$-\int\frac{1}{z^{n}}\left(\frac{1}{y^{m}}\right)-\frac{n}{1}\int\frac{1}{z^{n+1}}\left(\frac{1}{y^{m-1}}\right)-\frac{n(n+1)}{1\cdot 2}\int\frac{1}{z^{n+2}}\left(\frac{1}{y^{m-2}}\right)-\text{etc.}$$

THE SECOND PART, IF n IS AN ODD NUMBER

$$-\frac{2n}{1}\int\frac{1}{z^{n+1}}\cdot\int\frac{1}{z^{m-1}}-\frac{2n(n+1)(n+2)}{1\cdot 2\cdot 3}\int\frac{1}{z^{n+3}}\cdot\int\frac{1}{z^{m-3}}-\text{etc.}$$
$$+\int\frac{1}{z^{n}}\left(\frac{1}{y^{m}}\right)+\frac{n}{1}\int\frac{1}{z^{n+1}}\left(\frac{1}{y^{m-1}}\right)+\frac{n(n+1)}{1\cdot 2}\int\frac{1}{z^{n+2}}\left(\frac{1}{y^{m-2}}\right)+\text{etc.}$$

§10 It will helpful to have noted in these formulas that series of the form we consider here,

$$\int \frac{1}{z^\mu}\left(\frac{1}{y^\nu}\right),$$

not only occur, but are also all of such a nature that the sum of exponents $\mu+\nu$ is the same everywhere, namely $= m+n$. Therefore, it will be convenient to subdivide the integrations[1] into classes in such a way that all resolutions, in which the sum of the exponents $m+n$ is the same, belong to the same class, since in them the same series, which I decided to expand here, occur; and if the theorem found by means of the first method, by means of which it is

$$\int \frac{1}{z^m}\left(\frac{1}{y^n}\right)+\int \frac{1}{z^n}\left(\frac{1}{y^m}\right)=\int \frac{1}{z^m}\cdot\int \frac{1}{z^n}+\int \frac{1}{z^{m+n}},$$

is recalled, we will hence be able to define the single series of our form $\int \frac{1}{z^\mu}\left(\frac{1}{y^\nu}\right)$ separately. But because the exponents m and n cannot be smaller than 1, for the first class it will be $m+n=2$, for the second $m+n=3$, for the third $m+n=4$ and so forth; but because $\int \frac{1}{z}$ is infinite, for series, whose sum is finite, this infinity must go out of the calculation.

First Order in which it is $m+n=2$

§11 Therefore, here it can only be $m=1$ and $n=1$; the expression

$$\int \frac{1}{z}\cdot\int \frac{1}{z}-\int \frac{1}{z^2}$$

is resolved into the following series

$$\int \frac{1}{z}\left(\frac{1}{y}\right)+\int \frac{1}{z}\left(\frac{1}{y}\right)=2\int \frac{1}{z}\left(\frac{1}{y}\right).$$

The first method gives

$$2\int \frac{1}{z}\left(\frac{1}{y}\right)=\int \frac{1}{z}\cdot\int \frac{1}{z}+\int \frac{1}{z^2},$$

which seems to violate the present form; but because $\int \frac{1}{z}$ is infinite, with respect to the other part $\int \frac{1}{z^2}$ it is to be considered as vanishing. Therefore, hence nothing for our problem can be concluded.

[1] In this context, Euler uses the term "Integrations" and actually means "Summations".

SECOND ORDER IN WHICH IT IS $m+n=3$

§12 Here, again in one single way it is $m=2$ and $n=1$, since the commutation of these exponents makes no difference. Hence this expression

$$\int \frac{1}{z^2} \cdot \int \frac{1}{z} - \int \frac{1}{z^3}$$

is resolved into this one

$$2\int \frac{1}{z^2} \cdot \int \frac{1}{z} - \int \frac{1}{z^2}\left(\frac{1}{y}\right) - 2\int \frac{1}{z^2} \cdot \int \frac{1}{z} + \int \frac{1}{z}\left(\frac{1}{y^2}\right) + \int \frac{1}{z^2}\left(\frac{1}{y}\right),$$

which is contracted to

$$\int \frac{1}{z}\left(\frac{1}{y^2}\right).$$

But by means of the first method it is

$$\int \frac{1}{z^2}\left(\frac{1}{y}\right) + \int \frac{1}{z}\left(\frac{1}{y^2}\right) = \int \frac{1}{z^2} \cdot \int \frac{1}{z} + \int \frac{1}{z^3},$$

whence it seems to follow

$$\int \frac{1}{z^2}\left(\frac{1}{y}\right) = 2\int \frac{1}{z^3};$$

even though this conclusion is correct, as we will see later, it is nevertheless, because of the infinities, hence not possible to be confident about its validity. Writing out this equation it will be

$$1 + \frac{1}{2^2}\left(1+\frac{1}{2}\right) + \frac{1}{3^2}\left(1+\frac{1}{2}+\frac{1}{3}\right) + \frac{1}{4^2}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right) + \text{etc.} = 2\int \frac{1}{z^3}$$

$$= 2\left(1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\frac{1}{5^3}+\text{etc.}\right),$$

which equality is certainly worth one's complete attention.

Third Order in which it is $m+n=4$

§13 Here, two cases are to be considered, the first of which is

$$m=3 \quad \text{and} \quad n=1,$$

whence the form

$$\int \frac{1}{z^3}\cdot\int\frac{1}{z}-\int\frac{1}{z^4}$$

is resolved into the one

$$\int \frac{1}{z^3}\left(\frac{1}{y}\right)-2\int\frac{1}{z^2}\cdot\int\frac{1}{z^2}+\int\frac{1}{z}\left(\frac{1}{y^3}\right)+\int\frac{1}{z^2}\left(\frac{1}{y^2}\right)+\int\frac{1}{z^3}\left(\frac{1}{y}\right),$$

such that it is

$$2\int \frac{1}{z^3}\left(\frac{1}{y}\right)+\int\frac{1}{z^2}\left(\frac{1}{y^2}\right)+\int\frac{1}{z}\left(\frac{1}{y^3}\right)=2\int\frac{1}{z^2}\cdot\int\frac{1}{z^2}+\int\frac{1}{z^3}\cdot\int\frac{1}{z}-\int\frac{1}{z^4}.$$

But from the first method one has

$$\int \frac{1}{z^3}\left(\frac{1}{y}\right)+\int\frac{1}{z}\left(\frac{1}{y^3}\right)=\int\frac{1}{z^3}\cdot\int\frac{1}{z}+\int\frac{1}{z^4},$$

which equality subtracted from that one leaves

$$\int \frac{1}{z^3}\left(\frac{1}{y}\right)+\int\frac{1}{z^2}\left(\frac{1}{y^2}\right)=2\int\frac{1}{z^2}\cdot\int\frac{1}{z^2}-2\int\frac{1}{z^4}.$$

In the other case it is

$$m=2 \quad \text{and} \quad n=2,$$

whence one concludes

$$\int \frac{1}{z^2}\cdot\int\frac{1}{z^2}-\int\frac{1}{z^4}=2\int\frac{1}{z^2}\cdot\int\frac{1}{z^2}-\int\frac{1}{z^2}\left(\frac{1}{y^2}\right)-2\int\frac{1}{z^3}\left(\frac{1}{y}\right)$$
$$+2\int\frac{1}{z^2}\cdot\int\frac{1}{z^2}-\int\frac{1}{z^2}\left(\frac{1}{y^2}\right)-2\int\frac{1}{z^3}\left(\frac{1}{y}\right)$$

and hence further

$$2\int \frac{1}{z^2}\left(\frac{1}{y^2}\right)+4\int \frac{1}{z^3}\left(\frac{1}{y}\right)=3\int \frac{1}{z^2}\cdot \int \frac{1}{z^2}+\int \frac{1}{z^4}.$$

But the first method gives

$$2\int \frac{1}{z^2}\left(\frac{1}{y^2}\right)=\int \frac{1}{z^2}\cdot \int \frac{1}{z^2}+\int \frac{1}{z^4},$$

whence we conclude that it will be

$$\int \frac{1}{z^2}\left(\frac{1}{y^2}\right)=\frac{1}{2}\int \frac{1}{z^2}\cdot \int \frac{1}{z^2}+\frac{1}{2}\int \frac{1}{z^4}$$

and

$$\int \frac{1}{z^3}\left(\frac{1}{y}\right)=\frac{1}{2}\int \frac{1}{z^2}\cdot \int \frac{1}{z^2}.$$

The above conclusion on the other hand yields

$$\int \frac{1}{z^3}\left(\frac{1}{y}\right)=\frac{3}{2}\int \frac{1}{z^2}\cdot \int \frac{1}{z^2}-\frac{5}{2}\int \frac{1}{z^4},$$

which is also true, because it is

$$\int \frac{1}{z^2}\cdot \int \frac{1}{z^2}=\frac{\pi^4}{36}\quad \text{and}\quad \int \frac{1}{z^4}=\frac{\pi^4}{90}$$

such that also the first case, since here no infinity might obstruct it, gives a true result, since it seems to deviate from the truth only then, whenever the square of the infinite, $\int \frac{1}{z}\cdot \int \frac{1}{z}$, as it happened in the first order, enters the calculation; this confirms the conclusion deduced from the second order.

Fourth Order in which it is $m+n=5$

§14 First, let it be

$$m=4\quad \text{and}\quad n=1,$$

whence for

$$\int \frac{1}{z^4}\cdot \frac{1}{z}-\frac{1}{z^5}$$

this expression results

$$2\int\frac{1}{z^4}\cdot\int\frac{1}{z}-\int\frac{1}{z^4}\left(\frac{1}{y}\right)$$

$$-2\int\frac{1}{z^2}\cdot\int\frac{1}{z^3}-2\int\frac{1}{z^4}\cdot\int\frac{1}{z}+\int\frac{1}{z}\left(\frac{1}{y^4}\right)+\int\frac{1}{z^2}\left(\frac{1}{y^3}\right)+\int\frac{1}{z^3}\left(\frac{1}{y^2}\right)+\int\frac{1}{z^4}\left(\frac{1}{y}\right),$$

whence we conclude

$$\int\frac{1}{z}\left(\frac{1}{y^4}\right)+\int\frac{1}{z^2}\left(\frac{1}{y^3}\right)+\int\frac{1}{z^3}\left(\frac{1}{y^2}\right)=\int\frac{1}{z^4}\cdot\int\frac{1}{z}+2\int\frac{1}{z^2}\cdot\int\frac{1}{z^3}-\int\frac{1}{z^5}.$$

But the first method gives

$$\int\frac{1}{z}\left(\frac{1}{y^4}\right)+\int\frac{1}{z^4}\left(\frac{1}{y}\right)=\int\frac{1}{z^4}\cdot\int\frac{1}{z}+\int\frac{1}{z^5},$$

having subtracted this equality from that one it remains

$$\int\frac{1}{z^2}\left(\frac{1}{y^3}\right)+\int\frac{1}{z^3}\left(\frac{1}{y^2}\right)-\int\frac{1}{z^4}\left(\frac{1}{y}\right)=2\int\frac{1}{z^2}\cdot\int\frac{1}{z^3}-2\int\frac{1}{z^5}.$$

Secondly, let it be

$$m=3\quad\text{and}\quad n=2$$

and for

$$\int\frac{1}{z^3}\cdot\int\frac{1}{z^2}-\int\frac{1}{z^5}$$

it is found

$$-6\int\frac{1}{z^4}\cdot\int\frac{1}{z}+\int\frac{1}{z^3}\left(\frac{1}{y^2}\right)+3\int\frac{1}{z^4}\left(\frac{1}{y}\right)$$

$$+2\int\frac{1}{z^2}\cdot\int\frac{1}{z^3}+6\int\frac{1}{z^4}\cdot\int\frac{1}{z}-\int\frac{1}{z^2}\left(\frac{1}{y^3}\right)-2\int\frac{1}{z^3}\left(\frac{1}{y^2}\right)-\int\frac{1}{z^4}\left(\frac{1}{y}\right)$$

and hence

$$\int\frac{1}{z^2}\left(\frac{1}{y^3}\right)+\int\frac{1}{z^3}\left(\frac{1}{y^2}\right)=\int\frac{1}{z^2}\cdot\int\frac{1}{z^3}+\int\frac{1}{z^5},$$

completely as the first methods yields it; therefore, it hence is

$$\int \frac{1}{z^4}\left(\frac{1}{y}\right) = 3\int \frac{1}{z^5} - \int \frac{1}{z^2} \cdot \int \frac{1}{z^3}.$$

But hence the sums of the series

$$\int \frac{1}{z^2}\left(\frac{1}{y^3}\right) \quad \text{and} \quad \int \frac{1}{z^3}\left(\frac{1}{y^2}\right)$$

are not defined separately. But below [§ 30] we will show that it is

$$\int \frac{1}{z^3}\left(\frac{1}{y^2}\right) = 3\int \frac{1}{z^2} \cdot \int \frac{1}{z^3} - \frac{9}{2}\int \frac{1}{z^5}$$

and

$$\int \frac{1}{z^2}\left(\frac{1}{y^3}\right) = -2\int \frac{1}{z^2} \cdot \int \frac{1}{z^3} + \frac{11}{2}\int \frac{1}{z^5}.$$

Fifth Order in which it is $m + n = 6$

§15 First, let it be

$$m = 5 \quad \text{and} \quad n = 1$$

and it will be

$$\int \frac{1}{z^5} \cdot \int \frac{1}{z} - \int \frac{1}{z^6} = \int \frac{1}{z^5}\left(\frac{1}{y}\right)$$
$$-2\int \frac{1}{z^2} \cdot \int \frac{1}{z^4} - 2\int \frac{1}{z^4} \cdot \int \frac{1}{z^2} + \int \frac{1}{z}\left(\frac{1}{y^5}\right) + \int \frac{1}{z^2}\left(\frac{1}{y^4}\right) + \int \frac{1}{z^3}\left(\frac{1}{y^3}\right)$$
$$+ \int \frac{1}{z^4}\left(\frac{1}{y^2}\right) + \int \frac{1}{z^5}\left(\frac{1}{y}\right),$$

whence it is

$$\int \frac{1}{z}\left(\frac{1}{y^5}\right) + \int \frac{1}{z^2}\left(\frac{1}{y^4}\right) + \int \frac{1}{z^3}\left(\frac{1}{y^3}\right) + \int \frac{1}{z^4}\left(\frac{1}{y^2}\right) + \int \frac{1}{z^5}\left(\frac{1}{y}\right)$$
$$= \int \frac{1}{z^5} \cdot \int \frac{1}{z} + 4\int \frac{1}{z^4} \cdot \int \frac{1}{z^2} - \int \frac{1}{z^6}.$$

But because the first method gives

$$\int \frac{1}{z}\left(\frac{1}{y^5}\right) + \int \frac{1}{z^5}\left(\frac{1}{y}\right) = \int \frac{1}{z^5} \cdot \int \frac{1}{z} + \int \frac{1}{z^6},$$

by throwing out the infinite terms it hence is

$$\int \frac{1}{z^2}\left(\frac{1}{y^4}\right) + \int \frac{1}{z^3}\left(\frac{1}{y^3}\right) + \int \frac{1}{z^4}\left(\frac{1}{y^2}\right) + \int \frac{1}{z^5}\left(\frac{1}{y}\right) = 4\int \frac{1}{z^4} \cdot \int \frac{1}{z^2} - 2\int \frac{1}{z^6}.$$

Secondly, take

$$m = 4 \quad \text{and} \quad n = 2$$

and it will be

$$\int \frac{1}{z^4} \cdot \int \frac{1}{z^2} - \int \frac{1}{z^6} = 2\int \frac{1}{z^4} \cdot \int \frac{1}{z^2} - \int \frac{1}{z^4}\left(\frac{1}{y^2}\right) - \int \frac{1}{z^5}\left(\frac{1}{y}\right)$$

$$+2\int \frac{1}{z^2} \cdot \int \frac{1}{z^4} + 6\int \frac{1}{z^4} \cdot \int \frac{1}{z^2} - \int \frac{1}{z^4}\left(\frac{1}{y^2}\right) - 2\int \frac{1}{z^3}\left(\frac{1}{y^3}\right) - 3\int \frac{1}{z^4}\left(\frac{1}{y^2}\right) - 4\int \frac{1}{z^5}\left(\frac{1}{y}\right)$$

or

$$\int \frac{1}{z^2}\left(\frac{1}{y^4}\right) + 2\int \frac{1}{z^3}\left(\frac{1}{y^3}\right) + 4\int \frac{1}{z^4}\left(\frac{1}{y^2}\right) + 8\int \frac{1}{z^5}\left(\frac{1}{y}\right) = 9\int \frac{1}{z^4} \cdot \int \frac{1}{z^2} + \int \frac{1}{z^6}.$$

Thirdly, let it be

$$m = 3 and n = 3$$

and since both parts become equal, one will have

$$\int \frac{1}{z^3} \cdot \int \frac{1}{z^3} - \int \frac{1}{z^6} = -12\int \frac{1}{z^4} \cdot \int \frac{1}{z^2} + 2\int \frac{1}{z^3}\left(\frac{1}{y^3}\right) + 6\int \frac{1}{z^4}\left(\frac{1}{y^2}\right) + 12\int \frac{1}{z^5}\left(\frac{1}{y}\right)$$

or

$$2\int \frac{1}{z^3}\left(\frac{1}{y^3}\right) + 6\int \frac{1}{z^4}\left(\frac{1}{y^2}\right) + 12\int \frac{1}{z^5}\left(\frac{1}{y}\right) = 12\int \frac{1}{z^4} \cdot \int \frac{1}{z^2} + \int \frac{1}{z^3} \cdot \int \frac{1}{z^3} - \int \frac{1}{z^6}.$$

Combine those with these two resulting from the first method

$$\int \frac{1}{z^2}\left(\frac{1}{y^4}\right) + \int \frac{1}{z^4}\left(\frac{1}{y^2}\right) = \int \frac{1}{z^4} \cdot \int \frac{1}{z^2} + \int \frac{1}{z^6}$$

and

$$2\int \frac{1}{z^3}\left(\frac{1}{y^3}\right) = \int \frac{1}{z^3} \cdot \frac{1}{z^3} + \int \frac{1}{z^6}$$

and hence the single series of our form are determined this way:

$$\begin{aligned}
\int \frac{1}{z^5}\left(\frac{1}{y}\right) &= 3\int \frac{1}{z^4} \cdot \int \frac{1}{z^2} - \frac{1}{2}\int \frac{1}{z^3} \cdot \int \frac{1}{z^3} - \frac{7}{2}\int \frac{1}{z^6}, \\
\int \frac{1}{z^4}\left(\frac{1}{y^2}\right) &= -\frac{16}{3}\int \frac{1}{z^4} \cdot \int \frac{1}{z^2} + \int \frac{1}{z^3} \cdot \int \frac{1}{z^3} + 9\int \frac{1}{z^6}, \\
\int \frac{1}{z^3}\left(\frac{1}{y^3}\right) &= \frac{1}{2}\int \frac{1}{z^3} \cdot \int \frac{1}{z^3} + \frac{1}{2}\int \frac{1}{z^6}, \\
\int \frac{1}{z^2}\left(\frac{1}{y^4}\right) &= \frac{19}{3}\int \frac{1}{z^4} \cdot \int \frac{1}{z^2} - \int \frac{1}{z^3} \cdot \int \frac{1}{z^3} - 8\int \frac{1}{z^6},
\end{aligned}$$

but then by applying the equation found first one obtains

$$\int \frac{1}{z^6} = \frac{4}{7}\int \frac{1}{z^4} \cdot \int \frac{1}{z^2},$$

which because of

$$\int \frac{1}{z^2} = \frac{\pi\pi}{6}, \quad \int \frac{1}{z^4} = \frac{\pi}{90} \quad \text{and} \quad \int \frac{1}{z^6} = \frac{\pi^6}{945}$$

is true.

SIXTH ORDER IN WHICH IT IS $m + n = 7$

§16 First, let it be

$$m = 6 \quad \text{and} \quad n = 1$$

and it will be

$$\int \frac{1}{z^6} \cdot \int \frac{1}{z} - \int \frac{1}{z^7} = 2\int \frac{1}{z^6} \cdot \int \frac{1}{z} - \int \frac{1}{z^6}\left(\frac{1}{y}\right)$$

$$-2\int\frac{1}{z^2}\cdot\int\frac{1}{z^5}-2\int\frac{1}{z^4}\cdot\int\frac{1}{z^3}-2\int\frac{1}{z^6}\cdot\int\frac{1}{z}$$
$$+\int\frac{1}{z}\left(\frac{1}{y^6}\right)+\int\frac{1}{z^2}\left(\frac{1}{y^5}\right)+\int\frac{1}{z^3}\left(\frac{1}{y^4}\right)+\int\frac{1}{z^4}\left(\frac{1}{y^3}\right)+\int\frac{1}{z^5}\left(\frac{1}{y^2}\right)+\int\frac{1}{z^6}\left(\frac{1}{y}\right),$$

whence we conclude this equation

$$\int\frac{1}{z}\left(\frac{1}{y^6}\right)+\int\frac{1}{z^2}\left(\frac{1}{y^5}\right)+\int\frac{1}{z^3}\left(\frac{1}{y^4}\right)+\int\frac{1}{z^4}\left(\frac{1}{y^3}\right)+\int\frac{1}{z^5}\left(\frac{1}{y^2}\right)$$
$$=\int\frac{1}{z^6}\cdot\int\frac{1}{z}+2\int\frac{1}{z^2}\cdot\int\frac{1}{z^5}+2\int\frac{1}{z^4}\cdot\int\frac{1}{z^3}-\int\frac{1}{z^7}.$$

But because it is

$$\int\frac{1}{z}\left(\frac{1}{y^6}\right)+\int\frac{1}{z^6}\left(\frac{1}{y}\right)=\int\frac{1}{z^6}\cdot\int\frac{1}{z}+\int\frac{1}{z^7},$$

it will be

$$\int\frac{1}{z^2}\left(\frac{1}{y^5}\right)+\int\frac{1}{z^3}\left(\frac{1}{y^4}\right)+\int\frac{1}{z^4}\left(\frac{1}{y^3}\right)+\int\frac{1}{z^5}\left(\frac{1}{y^2}\right)-\int\frac{1}{z^6}\left(\frac{1}{y}\right)$$
$$=2\int\frac{1}{z^2}\cdot\int\frac{1}{z^5}+2\int\frac{1}{z^4}\cdot\int\frac{1}{z^3}-2\int\frac{1}{z^7}.$$

But on the other hand it also is

$$\int\frac{1}{z^2}\left(\frac{1}{y^5}\right)+\int\frac{1}{z^5}\left(\frac{1}{y^2}\right)=\int\frac{1}{z^5}\left(\frac{1}{y^2}\right)\cdot\frac{1}{z^2}+\int\frac{1}{z^7}$$

and

$$\int\frac{1}{z^3}\left(\frac{1}{y^4}\right)+\int\frac{1}{z^4}\left(\frac{1}{y^3}\right)=\int\frac{1}{z^4}\cdot\int\frac{1}{z^3}+\int\frac{1}{z^7},$$

whence one will have

$$\int\frac{1}{z^6}\left(\frac{1}{y}\right)=4\int\frac{1}{z^7}-\int\frac{1}{z^2}\cdot\int\frac{1}{z^5}-\int\frac{1}{z^4}\cdot\int\frac{1}{z^3}.$$

Secondly, let it be

$$m=5\quad\text{and}\quad n=2$$

and it will be

$$\int \frac{1}{z^5} \cdot \int \frac{1}{z^2} = -10 \int \frac{1}{z^6} \cdot \int \frac{1}{z} + \int \frac{1}{z^5} \left(\frac{1}{y^2}\right) + 5 \int \frac{1}{z^6} \left(\frac{1}{y}\right)$$
$$+2 \int \frac{1}{z^2} \cdot \int \frac{1}{z^5} + 6 \int \frac{1}{z^4} \cdot \frac{1}{z^3} + 10 \int \frac{1}{z^6} \cdot \int \frac{1}{z}$$
$$-\int \frac{1}{z^2} \left(\frac{1}{y^5}\right) - 2 \int \frac{1}{z^3} \left(\frac{1}{y^4}\right) - 3 \int \frac{1}{z^4} \left(\frac{1}{y^3}\right) - 4 \int \frac{1}{z^5} \left(\frac{1}{y^2}\right) - 5 \int \frac{1}{z^6} \left(\frac{1}{y}\right),$$

whence this equation is concluded

$$\int \frac{1}{z^2} \left(\frac{1}{y^5}\right) + 2 \int \frac{1}{z^3} \left(\frac{1}{y^4}\right) + 3 \int \frac{1}{z^4} \left(\frac{1}{y^3}\right) + 3 \int \frac{1}{z^5} \left(\frac{1}{y^2}\right)$$
$$= \int \frac{1}{z^2} \cdot \int \frac{1}{z^5} + 6 \int \frac{1}{z^4} \cdot \int \frac{1}{z^3} + \int \frac{1}{z^7},$$

which by means of the series derived before from the first method is reduced to this one

$$\int \frac{1}{z^4} \left(\frac{1}{y^3}\right) + 2 \int \frac{1}{z^5} \left(\frac{1}{y^2}\right) = 4 \int \frac{1}{z^4} \cdot \int \frac{1}{z^3} - 2 \int \frac{1}{z^7}.$$

Thirdly, let it be

$$m = 4 \quad \text{and} \quad n = 3;$$

it will be

$$\int \frac{1}{z^4} \cdot \int \frac{1}{z^3} - \int \frac{1}{z^7} = 2 \int \frac{1}{z^4} \cdot \int \frac{1}{z^3} + 20 \int \frac{1}{z^6} \cdot \int \frac{1}{z} - \int \frac{1}{z^4} \left(\frac{1}{y^3}\right) - 4 \int \frac{1}{z^5} \left(\frac{1}{y^2}\right)$$
$$-10 \int \frac{1}{z^6} \left(\frac{1}{y}\right) - 6 \int \frac{1}{z^4} \cdot \int \frac{1}{z^3} - 20 \int \frac{1}{z^6} \cdot \int \frac{1}{z} + \int \frac{1}{z^3} \left(\frac{1}{y^4}\right) + 3 \int \frac{1}{z^4} \left(\frac{1}{y^3}\right)$$
$$+6 \int \frac{1}{z^5} \left(\frac{1}{y^2}\right) + 10 \int \frac{1}{z^6} \left(\frac{1}{y}\right).$$

Hence it is concluded

$$\int \frac{1}{z^3} \left(\frac{1}{y^4}\right) + 2 \int \frac{1}{z^4} \left(\frac{1}{y^3}\right) + 2 \int \frac{1}{z^5} \left(\frac{1}{y^2}\right) = 5 \int \frac{1}{z^4} \cdot \int \frac{1}{z^3} - \int \frac{1}{z^7}$$

or

$$\int \frac{1}{z^4}\left(\frac{1}{y^3}\right) + 2\int \frac{1}{z^5}\left(\frac{1}{y^2}\right) = 4\int \frac{1}{z^4} \cdot \int \frac{1}{z^3} - 2\int \frac{1}{z^7},$$

which is identical the one found before such that hence nothing can be concluded. Therefore, hence first only the sum of the series

$$\int \frac{1}{z^6}\left(\frac{1}{y}\right)$$

is determined, but then these two together

$$\int \frac{1}{z^4}\left(\frac{1}{y^3}\right) + 2\int \frac{1}{z^5}\left(\frac{1}{y^2}\right);$$

but if both of them would be known separately, then also the two remaining ones

$$\int \frac{1}{z^3}\left(\frac{1}{y^4}\right) \quad \text{and} \quad \int \frac{1}{z^2}\left(\frac{1}{y^5}\right)$$

would become known.

Seventh Order in which it is $m + n = 8$

First, let it be

$$m = 7 \quad \text{and} \quad n = 1;$$

it will be

$$\int \frac{1}{z^7} \cdot \frac{1}{z} - \int \frac{1}{z^8} = \int \frac{1}{z^7}\left(\frac{1}{y}\right) - 2\int \frac{1}{z^2} \cdot \int \frac{1}{z^6} - 2\int \frac{1}{z^4} \cdot \int \frac{1}{z^4} - 2\int \frac{1}{z^6} \cdot \int \frac{1}{z^2}$$

$$+ \int \frac{1}{z}\left(\frac{1}{y^7}\right) + \int \frac{1}{z^2}\left(\frac{1}{y^6}\right) + \int \frac{1}{z^3}\left(\frac{1}{y^5}\right) + \int \frac{1}{z^4}\left(\frac{1}{y^4}\right) + \int \frac{1}{z^3}\left(\frac{1}{y^5}\right) + \int \frac{1}{z^6}\left(\frac{1}{y^2}\right) + \int \frac{1}{z^7}\left(\frac{1}{y}\right),$$

which last line goes over into this one

$$\int \frac{1}{z} \cdot \int \frac{1}{z^7} + \int \frac{1}{z^2} \cdot \int \frac{1}{z^6} + \int \frac{1}{z^3} \cdot \int \frac{1}{z^5} + \frac{1}{2}\int \frac{1}{z^4} \cdot \int \frac{1}{z^4} + \frac{7}{2}\int \frac{1}{z^8}$$

and so it will be

$$\int \frac{1}{z^7}\left(\frac{1}{y}\right) = 3\int \frac{1}{z^2}\cdot\frac{1}{z^6} - \int\frac{1}{z^3}\cdot\int\frac{1}{z^5} + \frac{3}{2}\int\frac{1}{z^4}\cdot\int\frac{1}{z^4} - \frac{9}{2}\int\frac{1}{z^8}.$$

Secondly, let it be

$$m = 6 \quad \text{and} \quad n = 2;$$

it will be

$$\int\frac{1}{z^6}\cdot\int\frac{1}{z^2} - \int\frac{1}{z^8} = 2\int\frac{1}{z^6}\cdot\int\frac{1}{z^2} - \int\frac{1}{z^6}\left(\frac{1}{y^2}\right) - 6\int\frac{1}{z^7}\left(\frac{1}{y}\right)$$

$$+2\int\frac{1}{z^2}\cdot\int\frac{1}{z^6} + 6\int\frac{1}{z^4}\cdot\int\frac{1}{z^4} + 10\int\frac{1}{z^6}\cdot\int\frac{1}{z^2}$$

$$-\int\frac{1}{z^2}\left(\frac{1}{y^6}\right) - 2\int\frac{1}{z^3}\left(\frac{1}{y^5}\right) - 3\int\frac{1}{z^4}\left(\frac{1}{y^4}\right) - 4\int\frac{1}{z^5}\left(\frac{1}{y^3}\right) - 5\int\frac{1}{z^6}\left(\frac{1}{y^2}\right) - 6\int\frac{1}{z^7}\left(\frac{1}{y}\right)$$

or

$$\int\frac{1}{z^2}\left(\frac{1}{y^6}\right) + 2\int\frac{1}{z^3}\left(\frac{1}{y^5}\right) + 3\int\frac{1}{z^4}\left(\frac{1}{y^4}\right) + 4\int\frac{1}{z^5}\left(\frac{1}{y^3}\right) + 6\int\frac{1}{z^6}\left(\frac{1}{y^2}\right)$$

$$+12\int\frac{1}{z^7}\left(\frac{1}{y}\right) = 12\int\frac{1}{z^6}\cdot\int\frac{1}{z^2} + \int\frac{1}{z^8} + 6\int\frac{1}{z^4}\cdot\int\frac{1}{z^4},$$

which is reduced to this one

$$2\int\frac{1}{z^5}\left(\frac{1}{y^3}\right) + 5\int\frac{1}{z^6}\left(\frac{1}{y^2}\right) + 12\int\frac{1}{z^7}\left(\frac{1}{y}\right)$$

$$= 12\int\frac{1}{z^6}\cdot\int\frac{1}{z^2} - 2\int\frac{1}{z^3}\cdot\int\frac{1}{z^5} + \frac{9}{2}\int\frac{1}{z^4}\cdot\int\frac{1}{z^4} - \frac{7}{2}\int\frac{7}{2}\int\frac{1}{z^8}$$

Thirdly, let it be

$$m = 5 \quad \text{and} \quad n = 3;$$

it will be

$$\int\frac{1}{z^5}\cdot\int\frac{1}{z^3} - \int\frac{1}{z^8} = -10\int\frac{1}{z^6}\cdot\int\frac{1}{z^2} + \int\frac{1}{z^5}\left(\frac{1}{y^3}\right) + 5\int\frac{1}{z^6}\left(\frac{1}{y^2}\right) + 15\int\frac{1}{z^7}\left(\frac{1}{y}\right)$$

$$-6\int\frac{1}{z^4}\cdot\int\frac{1}{z^4}-20\int\frac{1}{z^6}\cdot\int\frac{1}{z^2}$$
$$+\int\frac{1}{z^3}\left(\frac{1}{y^5}\right)+3\int\frac{1}{z^4}\left(\frac{1}{y^4}\right)+6\int\frac{1}{z^5}\left(\frac{1}{y^3}\right)+10\int\frac{1}{z^6}\left(\frac{1}{y^2}\right)+15\int\frac{1}{z^7}\left(\frac{1}{y}\right),$$

whence it is

$$\int\frac{1}{z^3}\left(\frac{1}{y^5}\right)+3\int\frac{1}{z^4}\left(\frac{1}{y^4}\right)+7\int\frac{1}{z^5}\left(\frac{1}{y^3}\right)+15\int\frac{1}{z^6}\left(\frac{1}{y^2}\right)+30\int\frac{1}{z^7}\left(\frac{1}{y}\right)$$
$$=30\int\frac{1}{z^6}\cdot\int\frac{1}{z^2}+\int\frac{1}{z^5}\cdot\frac{1}{z^3}+6\int\frac{1}{z^4}\cdot\int\frac{1}{z^4}-\int\frac{1}{z^8}$$

or

$$6\int\int\frac{1}{z^5}\left(\frac{1}{y^3}\right)+15\int\frac{1}{z^6}\left(\frac{1}{y^2}\right)+30\int\frac{1}{z^7}\left(\frac{1}{y}\right)$$
$$=30\int\frac{1}{z^6}\cdot\int\frac{1}{z^2}+\frac{9}{2}\int\frac{1}{z^4}\cdot\int\frac{1}{z^4}-\frac{7}{2}\int\frac{1}{z^8}.$$

Furthermore, let it be

$$m=4\quad\text{and}\quad n=4$$

and it will be

$$\frac{1}{2}\int\frac{1}{z^4}\cdot\int\frac{1}{z^4}-\frac{1}{2}\int\frac{1}{z^8}=2\int\frac{1}{z^4}\cdot\int\frac{1}{z^4}+20\int\frac{1}{z^6}\cdot\int\frac{1}{z^2}-\int\frac{1}{z^4}\left(\frac{1}{y^4}\right)-4\int\frac{1}{z^5}\left(\frac{1}{y^3}\right)$$
$$-10\int\frac{1}{z^6}\left(\frac{1}{y^2}\right)-20\int\frac{1}{z^7}\left(\frac{1}{y}\right)$$

and hence

$$4\int\frac{1}{z^5}\left(\frac{1}{y^3}\right)+10\int\frac{1}{z^6}\left(\frac{1}{y^2}\right)+20\int\frac{1}{z^7}\left(\frac{1}{y}\right)=20\int\frac{1}{z^6}\cdot\int\frac{1}{z^2}+\int\frac{1}{z^4}\cdot\int\frac{1}{z^4},$$

which equation together with the preceding contains the same determination and is reduced to this property

$$6\int\frac{1}{z^4}\cdot\int\frac{1}{z^4}=7\int\frac{1}{z^8},$$

which is in extraordinary agreement with the values I found a long time before,

$$\int \frac{1}{z^4} = \frac{\pi^4}{90} \quad \text{and} \quad \int \frac{1}{z^8} = \frac{\pi^8}{9450}.$$

But if this last equation is compared to the second case, one hence concludes

$$4\int \frac{1}{z^7}\left(\frac{1}{y}\right) = 4\int \frac{1}{z^2}\cdot\int \frac{1}{z^6} - 4\int \frac{1}{z^3}\cdot\int \frac{1}{z^5} + 8\int \frac{1}{z^8}\cdot\int \frac{1}{z^4} - 7\int \frac{1}{z^8}$$

or

$$\int \frac{1}{z^7}\left(\frac{1}{y}\right) = \int \frac{1}{z^2}\cdot\int \frac{1}{z^6} - \int \frac{1}{z^3}\cdot\int \frac{1}{z^5} + \frac{1}{2}\int \frac{1}{z^4}\cdot\int \frac{1}{z^4},$$

which value in combination with the first case yields

$$2\int \frac{1}{z^2}\cdot\int \frac{1}{z^6} + \int \frac{1}{z^4}\cdot\int \frac{1}{z^4} = \frac{9}{2}\int \frac{1}{z^8},$$

which equation is also true. Hence except for the series

$$\int \frac{1}{z^7}\left(\frac{1}{y}\right)$$

and the determinations of the first method we obtain only this one new determination

$$2\int \frac{1}{z^5}\left(\frac{1}{y^3}\right) + 5\int \frac{1}{z^6}\left(\frac{1}{y^2}\right) = 10\int \frac{1}{z^3}\cdot\int \frac{1}{z^5} - \frac{9}{2}\int \frac{1}{z^4}\cdot\int \frac{1}{z^4}$$

and therefore the sum of these series

$$\int \frac{1}{z^5}\left(\frac{1}{y^3}\right) \quad \text{and} \quad \int \frac{1}{z^6}\left(\frac{1}{y^2}\right)$$

can be defined separately.

Eighth Order in which it is $m + n = 9$

§18 For this order the first method gives these equations

$$\int \frac{1}{z}\left(\frac{1}{y^8}\right) + \int \frac{1}{z^8}\left(\frac{1}{y}\right) = \int \frac{1}{z} \int \frac{1}{z^8} + \int \frac{1}{z^9},$$
$$\int \frac{1}{z^2}\left(\frac{1}{y^7}\right) + \int \frac{1}{z^7}\left(\frac{1}{y^2}\right) = \int \frac{1}{z^2} \int \frac{1}{z^7} + \int \frac{1}{z^9},$$
$$\int \frac{1}{z^3}\left(\frac{1}{y^6}\right) + \int \frac{1}{z^6}\left(\frac{1}{y^3}\right) = \int \frac{1}{z^3} \int \frac{1}{z^6} + \int \frac{1}{z^9},$$
$$\int \frac{1}{z^4}\left(\frac{1}{y^5}\right) + \int \frac{1}{z^5}\left(\frac{1}{y^4}\right) = \int \frac{1}{z^4} \int \frac{1}{z^5} + \int \frac{1}{z^9}$$

But the second method additionally yields these determinations

$$\int \frac{1}{z^8}\left(\frac{1}{y}\right) = 5\int \frac{1}{z^9} - \int \frac{1}{z^2}\cdot\int \frac{1}{z^7} - \int \frac{1}{z^3}\cdot\int \frac{1}{z^6} - \int \frac{1}{z^4}\cdot\frac{1}{z^5},$$
$$2\int \frac{1}{z^5}\left(\frac{1}{y^4}\right) + 5\int \frac{1}{z^6}\left(\frac{1}{y^3}\right) + 5\int \frac{1}{z^7}\left(\frac{1}{y^2}\right) = 10\int \frac{1}{z^3}\cdot\int \frac{1}{z^6},$$
$$\int \frac{1}{z^6}\left(\frac{1}{y^3}\right) + 3\int \frac{1}{z^7}\left(\frac{1}{y^2}\right) = 6\int \frac{1}{z^4}\cdot\int \frac{1}{z^5} + 6\int \frac{1}{z^3}\cdot\int \frac{1}{z^6} - 10\int \frac{1}{z^9}.$$

Therefore, because in this order 8 series of the form occur, which we contemplate here, these seven equations do not suffice for the definition of all of them; but if except for the series $\int \frac{1}{z^8}\left(\frac{1}{y}\right)$ one of the remaining series could be summed from elsewhere, hence completely all sums would become known.

Ninth Order in which it is $m + n = 10$

§19 From the first method we obtain these equations for this order

$$\int \frac{1}{z}\left(\frac{1}{y^9}\right) + \int \frac{1}{z^9}\left(\frac{1}{y}\right) = \int \frac{1}{z} \cdot \int \frac{1}{z^9} + \int \frac{1}{z^{10}}$$
$$\int \frac{1}{z^2}\left(\frac{1}{y^8}\right) + \int \frac{1}{z^8}\left(\frac{1}{y^2}\right) = \int \frac{1}{z^2} \cdot \int \frac{1}{z^8} + \int \frac{1}{z^{10}}$$
$$\int \frac{1}{z^3}\left(\frac{1}{y^7}\right) + \int \frac{1}{z^7}\left(\frac{1}{y^3}\right) = \int \frac{1}{z^3} \cdot \int \frac{1}{z^7} + \int \frac{1}{z^{10}}$$
$$\int \frac{1}{z^4}\left(\frac{1}{y^6}\right) + \int \frac{1}{z^6}\left(\frac{1}{y^4}\right) = \int \frac{1}{z^4} \cdot \int \frac{1}{z^6} + \int \frac{1}{z^{10}}$$

$$\int \frac{1}{z^5}\left(\frac{1}{y^5}\right) = \frac{1}{2}\int \frac{1}{z^5} \cdot \int \frac{1}{z^5} + \frac{1}{2}\int \frac{1}{z^{10}};$$

therefore, since here 9 series occur, for their summation the second method first gives

$$\int \frac{1}{z^9}\left(\frac{1}{y}\right) = 3\int \frac{1}{z^2}\cdot\frac{1}{z^8} - \int \frac{1}{z^3}\cdot\int\frac{1}{z^7} + 3\int\frac{1}{z^4}\cdot\int\frac{1}{z^6} - \frac{1}{2}\int\frac{1}{z^5}\cdot\int\frac{1}{z^5} - \frac{11}{2}\int\frac{1}{z^{10}},$$

but of the four remaining equations, which are deduced from this, two define nothing except the known relation, according to which it is

$$\int \frac{1}{z^{10}} = \frac{10}{11}\int\frac{1}{z^4}\cdot\int\frac{1}{z^6},$$

but the remaining two yield

$$\int \frac{1}{z^6}\left(\frac{1}{y^4}\right) + \int\frac{1}{z^7}\left(\frac{1}{y^3}\right) = 6\int\frac{1}{z^4}\cdot\int\frac{1}{z^6} - \int\frac{1}{z^5}\cdot\int\frac{1}{z^5} - \frac{7}{2}\int\frac{1}{z^{10}},$$

$$\int \frac{1}{z^7}\left(\frac{1}{y^3}\right) + 7\int\frac{1}{z^8}\left(\frac{1}{y^2}\right) = 14\int\frac{1}{z^3}\cdot\int\frac{1}{z^7} - 45\int\frac{1}{z^4}\cdot\int\frac{1}{z^4}\cdot\int\frac{1}{z^6}$$
$$+8\int\frac{1}{z^5}\cdot\int\frac{1}{z^5} + 33\int\frac{1}{z^{10}},$$

such that still one equation is missing to sum all series of this order.

§20 The following things are to be observed on the determinations, which this second method yields:

First, only in the first, second, third and fifth order all series of our form are defined, in all remaining ones one equation is missing that all series of that order can be summed such that, if such a determination would be known from elsewhere, the whole task could be completed.

Further, also for these orders, in which $m+n$ is an even number, it deserves to be noted that this method yields the same relation among the sums of the even powers

$$\int\frac{1}{z^2},\quad \int\frac{1}{z^4},\quad \int\frac{1}{z^6}\quad \text{etc.},$$

which I had once found from completely different principles, although here the quadrature of the circle, on which these sums depend, was not mentioned at all. Hence it could also have been expected that for the orders in which $m+n$ is an odd number a similar relation among the sums of odd powers should have been resulted; but quite the opposite happened, since the equations we found for these orders, are identical such that absolutely nothing can be concluded from this. Because this happened against all expectations, this defect of a complete determination is worth one's complete attention.

§21 Thirdly, it is to be observed that in all orders always one series of our form is determined perfectly, of course the one which is denoted by the formula

$$\int \frac{1}{z^{m+n-1}}\left(\frac{1}{y}\right);$$

but because its determination, depending on whether $m+n$ was an even or odd number, follows another law, we want to list it for the respective orders here separately.

For the Orders in which $m+n$ is an even Number

$$\int \frac{1}{z^3}\left(\frac{1}{y}\right) = \frac{3}{2}\int \frac{1}{z^2}\cdot\int \frac{1}{z^2} - \frac{5}{2}\int \frac{1}{z^4},$$

$$\int \frac{1}{z^5}\left(\frac{1}{y}\right) = 3\int \frac{1}{z^2}\cdot\int \frac{1}{z^4} - \frac{1}{2}\int \frac{1}{z^3}\cdot\int \frac{1}{z^3} - \frac{7}{2}\int \frac{1}{z^6},$$

$$\int \frac{1}{z^7}\left(\frac{1}{y}\right) = 3\int \frac{1}{z^2}\cdot\int \frac{1}{z^6} - \int \frac{1}{z^3}\cdot\int \frac{1}{z^7} + 3\int \frac{1}{z^4}\cdot\int \frac{1}{z^6} - \frac{1}{2}\int \frac{1}{z^5}\cdot\int \frac{1}{z^5} - \frac{11}{2}\int \frac{1}{z^{10}}$$

$$\int \frac{1}{z^{11}}\left(\frac{1}{y}\right) = 3\int \frac{1}{z^2}\cdot\int \frac{1}{z^{10}} - \int \frac{1}{z^3}\cdot\int \frac{1}{z^9} + 3\int \frac{1}{z^4}\cdot\int \frac{1}{z^8} - \int \frac{1}{z^5}\cdot\int \frac{1}{z^7} + \frac{3}{2}\int \frac{1}{z^6}\cdot\int \frac{1}{z^6} - \frac{13}{2}\int \frac{1}{z^{13}}$$

etc.,

which expressions are restricted to the parity of the number m and n that they cannot be transferred to the odd numbers by means of interpolation.

$$\int \frac{1}{z^2}\left(\frac{1}{y}\right) = 2\int \frac{1}{z^3},$$
$$\int \frac{1}{z^4}\left(\frac{1}{y}\right) = 3\int \frac{1}{z^5} - \int \frac{1}{z^2}\cdot\int \frac{1}{z^3},$$
$$\int \frac{1}{z^6}\left(\frac{1}{y}\right) = 4\int \frac{1}{z^7} - \int \frac{1}{z^2}\cdot\int \frac{1}{z^5} - \int \frac{1}{z^3}\cdot\int \frac{1}{z^4},$$
$$\int \frac{1}{z^8}\left(\frac{1}{y}\right) = 5\int \frac{1}{z^9} - \int \frac{1}{z^2}\cdot\int \frac{1}{z^7} - \int \frac{1}{z^3}\cdot\int \frac{1}{z^6} - \int \frac{1}{z^4}\cdot\int \frac{1}{z^5},$$
$$\int \frac{1}{z^{10}}\left(\frac{1}{y}\right) = 6\int \frac{1}{z^{11}} - \int \frac{1}{z^2}\cdot\int \frac{1}{z^9} - \int \frac{1}{z^3}\cdot\int \frac{1}{z^8} - \int \frac{1}{z^4}\cdot\int \frac{1}{z^7} - \int \frac{1}{z^5}\cdot\int \frac{1}{z^6}$$

etc.

But here there is no obstruction that these expression are also transferred to even numbers.

§22 Having done the interpolation correctly these summations for all orders will be as follows:

$$2\int \frac{1}{z^2}\left(\frac{1}{y}\right) = 4\int \frac{1}{z^3},$$
$$2\int \frac{1}{z^3}\left(\frac{1}{y}\right) = 5\int \frac{1}{z^4} - \int \frac{1}{z^2}\cdot\int \frac{1}{z^2},$$
$$2\int \frac{1}{z^4}\left(\frac{1}{y}\right) = 6\int \frac{1}{z^5} - 2\int \frac{1}{z^2}\cdot\int \frac{1}{z^3},$$
$$2\int \frac{1}{z^5}\left(\frac{1}{y}\right) = 7\int \frac{1}{z^6} - 2\int \frac{1}{z^2}\cdot\int \frac{1}{z^4} - \int \frac{1}{z^3}\cdot\int \frac{1}{z^3},$$
$$2\int \frac{1}{z^6}\left(\frac{1}{y}\right) = 8\int \frac{1}{z^7} - 2\int \frac{1}{z^2}\cdot\int \frac{1}{z^5} - 2\int \frac{1}{z^3}\cdot\int \frac{1}{z^4},$$
$$2\int \frac{1}{z^7}\left(\frac{1}{y}\right) = 9\int \frac{1}{z^8} - 2\int \frac{1}{z^2}\cdot\int \frac{1}{z^6} - 2\int \frac{1}{z^3}\cdot\int \frac{1}{z^5} - \int \frac{1}{z^4}\cdot\int \frac{1}{z^4},$$

$$2\int\frac{1}{z^8}\left(\frac{1}{y}\right)=10\int\frac{1}{z^9}\ -2\int\frac{1}{z^2}\cdot\int\frac{1}{z^7}-2\int\frac{1}{z^3}\cdot\int\frac{1}{z^6}-2\int\frac{1}{z^4}\cdot\int\frac{1}{z^5},$$
$$2\int\frac{1}{z^9}\left(\frac{1}{y}\right)=11\int\frac{1}{z^{10}}-2\int\frac{1}{z^2}\cdot\int\frac{1}{z^8}-2\int\frac{1}{z^3}\cdot\int\frac{1}{z^7}-2\int\frac{1}{z^4}\cdot\int\frac{1}{z^6}\ -\int\frac{1}{z^5}\cdot\int\frac{1}{z^5}$$
etc.,

whence in general, if one puts $m+n=\lambda$, it will be

$$2\int\frac{1}{z^{\lambda-1}}\left(\frac{1}{y}\right)=(\lambda+1)\int\frac{1}{z^\lambda}-\int\frac{1}{z^2}\cdot\int\frac{1}{z^{\lambda-2}}-\int\frac{1}{z^3}\cdot\int\frac{1}{z^{\lambda-3}}$$
$$-\int\frac{1}{z^\lambda}\cdot\int\frac{1}{z^{\lambda-4}}-\cdots-\int\frac{1}{z^{\lambda-2}}\cdot\int\frac{1}{z^2}.$$

§23 But in order to confirm these interpolations, compare these expressions for the even orders to the ones exhibited before and hence the following relations will be obtained:

$$5\int\frac{1}{z^4}\ =2\int\frac{1}{z^2}\cdot\int\frac{1}{z^2}$$
$$7\int\frac{1}{z^6}\ =4\int\frac{1}{z^2}\cdot\int\frac{1}{z^4}$$
$$9\int\frac{1}{z^8}\ =4\int\frac{1}{z^2}\cdot\int\frac{1}{z^6}\ +2\int\frac{1}{z^4}\cdot\int\frac{1}{z^4},$$
$$11\int\frac{1}{z^{10}}=4\int\frac{1}{z^2}\cdot\int\frac{1}{z^8}\ +4\int\frac{1}{z^4}\cdot\int\frac{1}{z^6},$$
$$13\int\frac{1}{z^{12}}=4\int\frac{1}{z^2}\cdot\int\frac{1}{z^{10}}+4\int\frac{1}{z^4}\cdot\int\frac{1}{z^8}\ +2\int\frac{1}{z^6}\cdot\int\frac{1}{z^6}$$
etc.,

which are in perfect agreement with those which I found once. For, if we put

$$\int\frac{1}{z^2}=\alpha\pi^2,\quad \int\frac{1}{z^4}=\beta\pi^4,\quad \int\frac{1}{z^6}=\gamma\pi^6,\quad \int\frac{1}{z^8}=\delta\pi^8,\quad \int\frac{1}{z^{10}}=\varepsilon\pi^{10}\quad \text{etc.}$$

it will, as I demonstrated, be

$$
\begin{aligned}
5\beta &= 2\alpha\alpha, \\
7\gamma &= 4\alpha\beta, \\
9\delta &= 4\alpha\gamma + 2\beta\beta, \\
11\varepsilon &= 4\alpha\delta + 4\beta\gamma, \\
13\zeta &= 4\alpha\varepsilon + 4\beta\delta + 2\gamma\gamma, \\
15\eta &= 4\alpha\zeta + 4\beta\varepsilon + 4\gamma\delta
\end{aligned}
$$

etc.

§24 The first method only for the even orders yielded the sum of one series contained in our general form, which series having put $m + n = 2\mu$ will be as follows

$$\int \frac{1}{z^\mu}\left(\frac{1}{y^\mu}\right) = \frac{1}{2}\int \frac{1}{z^\mu} \cdot \int \frac{1}{z^\mu} + \frac{1}{2}\int \frac{1}{z^{2\mu}}.$$

But now by means of the second method we are furthermore able to sum one series of our form of each order and hence it was possible to sum all these series of the order $m + n = 6$. From this one could conjecture that this summation also succeeds for all orders, even though the second method does not solve the task completely; but in most cases the task must be considered to be completed, since, if one series of a certain order except the two mentioned ones could be summed from elsewhere, hence immediately the sums of all remaining series are obtained. The situation indeed always is as this in the orders expanded here; but if we proceed further, a lot more determinations are detected to be missing.

§25 In order to see the nature of the equations, which both the first and the second method yield for any arbitrary order, more clearly, let us represent our formulas even shorter that for any order $m + n = \lambda$ instead of

$$\int \frac{1}{z^\mu} \cdot \int \frac{1}{z^\nu}$$

either p^μ or p^ν is written, which two formulas because of $\mu + \nu = \lambda$ are to be considered to be equivalent. And in like manner write p^λ for $\int \frac{1}{z^\lambda}$; but then write q^μ instead of the formulas

$$\int \frac{1}{z^\mu}\left(\frac{1}{y^\nu}\right) \quad \text{and} \quad \int \frac{1}{z^\mu}\left(\frac{1}{y^{\lambda-\nu}}\right);$$

and hence the equations of the single orders will become more perspicuous.

For the order $m + n = 3$

$$q + q^2 = p + p^3,$$

and

$$\begin{array}{l} q + q^2 \;=\; +2p^2 + p - p^3 \quad \text{or} \quad q = p - p^3. \\ \;\;-1 \qquad\quad -2 \end{array}$$

For the order $m + n = 4$

$$\begin{aligned} q + q^3 &= p + p^4, \\ 2q^2 &= p^2 + p^4, \end{aligned}$$

and

$$\begin{array}{l} q + q^2 + q^3 = 2p^2 + p - p^4, \\ \;\;+1 \end{array}$$

and

$$\begin{array}{l} \qquad q^2 + 2q^3 \;=\; +2p^2 - p^2 + p^4. \\ +1 \end{array}$$

For the order $m + n = 5$

$$\begin{aligned} q \;\; + q^4 &= p \;\; + p^5, \\ q^2 + q^3 &= p^2 + p^5, \end{aligned}$$

and

$$\begin{array}{l} q + q^2 + q^3 + \;q^4 = 2p^3 + 2p^4 + p - p^6, \\ \qquad\qquad\quad -1 \qquad\qquad\;\; -2 \end{array}$$

and

$$\begin{array}{l} q^2 + 2q^3 + 3q^4 = 2p^2 + 6p^4 - p^2 + p^5. \\ \quad -1 \quad -3 \qquad\qquad -6 \end{array}$$

For the Order $m + n = 6$

$$\begin{aligned} q \ + q^5 &= p \ + p^6, \\ q^2 + q^4 &= p^2 + p^6, \\ 2q^2 \ &= p^3 + p^6 \end{aligned}$$

and

$$\begin{array}{l} q + q^2 + q^3 + q^4 + \quad q^5 = 2p^2 + 2p^4 + p - p^6 \\ \qquad\qquad\qquad\quad +1 \end{array}$$

furthermore

$$\begin{array}{l} q^2 + 2q^3 + 3q^4 + 4q^5 = 2p^2 + 6p^4 - p^2 + p^6, \\ \qquad\quad +1 \quad +4 \qquad\qquad +2 \end{array}$$

and finally

$$\begin{array}{l} \qquad q^3 + 3q^4 + 6q^5 \ = \quad 6p^4 + p^3 - p^6. \\ +1 \quad +3 \quad +6 \qquad\quad +6 \end{array}$$

For the Order $m + n = 7$

$$\begin{aligned} q \ + q^6 &= p \ + p^7, \\ q^2 + q^5 &= p^2 + p^7, \\ q^3 + q^4 &= p^3 + p^7, \end{aligned}$$

and

$$\begin{array}{l} q + q^2 + q^3 + q^4 + q^5 + \ q^6 = 2p^2 + 2p^4 + 2p^6 + p - p^7, \\ \qquad\qquad\qquad\qquad\quad -1 \qquad\qquad\qquad\qquad -2 \end{array}$$

further

$$\begin{array}{l} q^2 + 2q^3 + 3q^4 + 4q^5 + 5q^6 = 2p^2 + 6p^4 + 10p^6 - p^2 + p^7, \\ \qquad\qquad\quad -1 \quad -5 \qquad\qquad\qquad\qquad -10 \end{array}$$

and finally

$$\begin{array}{rcl} q^3 + 3q^4 + 6q^5 + 10q^6 & = & 6p^4 + 20p^6 + p^3 - p^7. \\ -1 \quad +4 \quad -10 & & +2 \quad -20 \end{array}$$

For the Order in which it is $m + n = 8$

$$\begin{aligned} q \ + q^7 &= p \ + p^8, \\ q^2 + q^6 &= p^2 + p^8, \\ q^3 + q^5 &= p^3 + p^8, \\ 2q^4 \quad &= p^4 + p^8 \end{aligned}$$

and

$$\begin{array}{l} q + q^2 + q^3 + q^4 + q^5 + q^6 + \quad q^7 = 2p^2 + 2p^4 + 2p^6 + p - p^8 \\ \qquad\qquad\qquad\qquad\qquad\quad +1 \end{array}$$

further

$$\begin{array}{l} q^2 + 2q^3 + 3q^4 + 4q^5 + 5q^6 + 6q^7 = 2p^2 + 6p^4 + 10p^6 - p^2 + p^8, \\ \qquad\qquad\qquad\quad +1 \quad +6 \qquad\qquad\qquad\quad +2 \end{array}$$

also

$$\begin{array}{l} q^3 + 3q^4 + 6q^5 + 10q^6 + 15q^7 = 6p^4 + 20p^6 + p^3 - p^8, \\ \qquad\qquad +1 \quad +5 \quad +15 \qquad\qquad +10 \end{array}$$

finally

$$\begin{array}{rcl} q^4 + 4q^5 + 10q^6 + 20q^7 & = & 2p^4 + 20p^6 - p^4 + p^8. \\ +1 \quad +4 \quad +10 \quad +20 & & +2 \quad +20 \end{array}$$

Therefore, this way these equations are easily continued arbitrarily far.

1 Third Method to get to Series of this Kind

§26 This method is almost equal to the preceding one; for, I consider the series

$$\int \frac{1}{z^n}\left(1 + \frac{1}{2^m} + \frac{1}{3^m} + \cdots + \frac{1}{)z-1)^m}\right),$$

whose value expressed in the above way is

$$= \int \frac{1}{z^n}\left(\frac{1}{y^m}\right) - \int \frac{1}{z^{m+n}},$$

but in the way of the preceding paragraph the same value is $= q^n - p^{m+n}$; note that by means of the first method it is

$$q^m + q^n = p^m + p^{m+n} = p^m + p^{m+n}$$

because of $p^m = p^n$. Now, any arbitrary term of this form

$$\frac{1}{z^n}\left(1 + \frac{1}{2^m} + \frac{1}{3^m} + \cdots + \frac{1}{(z-1)^m}\right)$$

is contained in this form

$$\frac{1}{(x+a)^n x^m},$$

which, as we saw in § 6, is resolved into these parts

$$\frac{1}{a^n}\cdot\frac{1}{x^m} - \frac{n}{1a^{n+1}}\cdot\frac{1}{x^{m-1}} + \frac{n(n+1)}{1\cdot 2a^{n+2}}\cdot\frac{1}{x^{m-2}} - \frac{n(n+1)(n+2)}{1\cdot 2\cdot 3a^{n+3}}\cdot\frac{1}{x^{m-3}} + \text{etc.}$$

and

$$\pm\frac{1}{a^m}\cdot\frac{1}{(x+a)^n}\pm\frac{m}{1a^{m+1}}\cdot\frac{1}{(x+a)^{n-1}}\pm\frac{m(m+1)}{1\cdot 2a^{m+2}}\cdot\frac{1}{(x+a)^{n-2}}\pm\text{etc.},$$

where the upper of he ambiguous signs hold, if m is an even number, the lower ones, if m is an odd number; but then both progressions have to be continued until in the upper the exponent of the factor $\frac{1}{x}$, in the lower the exponent of the factor $\frac{1}{x+a}$ becomes 1.

§27 Therefore, to obtain the sum of the propounded series, in the single terms of the expanded formula one has to write all numbers in natural order from 1 to infinity so instead of a as instead of x and has to collect all terms to result from this into one single sum. But then for the terms of the expanded above part it will be

$$\int\frac{1}{a^n}\cdot\frac{1}{x^m}=\int\frac{1}{z^n}\cdot\int\frac{1}{z^m}=p^n=p^m,$$

$$\int\frac{1}{a^{n+1}}\cdot\frac{1}{x^{m-1}}=\int\frac{1}{z^{n+1}}\cdot\int\frac{1}{z^{m-1}}=p^{n+1}=p^{m-1},$$

$$\int\frac{1}{a^{n+2}}\cdot\frac{1}{x^{m-2}}=\int\frac{1}{z^{n+2}}\cdot\int\frac{1}{z^{m-2}}=p^{n+2}=p^{m-2}$$

$$\text{etc.},$$

but for the terms of the expanded lower part

$$\int\frac{1}{a^m}\cdot\frac{1}{(x+a)^n}=\int\frac{1}{z^m}\cdot\int\frac{1}{z^n}-\int\frac{1}{z^m}\left(\frac{1}{y^n}\right)=p^m-q^m=q^n-p^{m+n},$$

$$\int\frac{1}{a^{m+1}}\cdot\frac{1}{(x+a)^{n-1}}=\int\frac{1}{z^{m+1}}\cdot\int\frac{1}{z^{n-1}}-\int\frac{1}{z^{m+1}}\left(\frac{1}{y^{n-1}}\right)=p^{m+1}-q^{m+1}=q^{n-1}-p^{m+n},$$

$$\int\frac{1}{a^{m+2}}\cdot\frac{1}{(x+a)^{n-2}}=\int\frac{1}{z^{m+2}}\cdot\int\frac{1}{z^{n-2}}-\int\frac{1}{z^{m+2}}\left(\frac{1}{y^{n-2}}\right)=p^{m+2}-q^{m+2}=q^{n-2}-p^{m+n}$$

$$\text{etc.}$$

Therefore, having substituted these expressions the value of our series, which is $=q^n-p^{m+n}$, is expanded into the following expression

$$p^m - \frac{n}{1}p^{m-1} + \frac{n(n+1)}{1\cdot 2}p^{m-2} - \frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}p^{m-3} + \text{etc.}$$
$$\pm q^n \pm \frac{m}{1}q^{n-1} \pm \frac{m(m+1)}{1\cdot 2}q^{n-2} \pm \frac{m(m+1)(m+2)}{1\cdot 2\cdot 3}q^{n-3} \pm \text{etc.}$$
$$\mp p^{m+n} \mp \frac{m}{1}p^{m+n} \mp \frac{m(m+1)}{1\cdot 2}p^{m+n} \mp \frac{m(m+1)(m+2)}{1\cdot 2\cdot 3}p^{m+n} \mp \text{etc.}$$

But because it is

$$p^\mu = q^\mu + q^{m+n-\mu} - p^{m+n},$$

we will have

$$\begin{aligned}
0 = q^m \quad & - \frac{n}{1}(q^{m-1} + q^{n+1}) + \frac{n(n+1)}{1\cdot 2}(q^{m-2} + q^{n+2}) - \frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}(q^{m-3} + q^{n+3}) + \text{etc.} \\
& - \frac{n}{1}p^{m+n} \qquad\quad - \frac{n(n+1)}{1\cdot 2}p^{m+n} \qquad\quad - \frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}p^{m+n} \qquad - \text{etc.} \\
\pm q^n \quad & \pm \frac{m}{1}q^{n-1} \qquad\quad \pm \frac{m(m+1)}{1\cdot 2}q^{n-2} \qquad\quad \pm \frac{m(m+1)(m+2)}{1\cdot 2\cdot 3}q^{n-3} \qquad \pm \text{etc.} \\
\mp p^{m+n} & \mp \frac{m}{1}p^{m+n} \qquad\quad \mp \frac{m(m+1)}{1\cdot 2}p^{m+n} \qquad\quad \mp \frac{m(m+1)(m+2)}{1\cdot 2\cdot 3}p^{m+n} \qquad \mp \text{etc.}
\end{aligned}$$

§28 Before we descend to the orders considered above, let us expand some other cases in general.

I. Therefore, let it be

$$m = 1$$

and the found equation will obtain this form

$$0 = q - q^n - q^{n-1} - q^{n-2} - \cdots - q + np^{n+1}$$

or

$$q^2 + q^3 + q^4 + \cdots + q^n = np^{n+1}$$

while the exponent of the order is $n+1$ such that it is

$$q^\mu + q^{m+1-\mu} = p^\mu + p^{n+1}.$$

II. Let it be

$$m = 2$$

and the exponent of the order $n+2$ that it is

$$q^\mu + q^{n+2-\mu} = p^\mu + p^{n+2},$$

and our equation will be

$$\begin{aligned} 0 = q^2 - n(q + q^{n+1}) + q^n + 2q^{n-1} + 3q^{n-2} + 4q^{n-3} + \cdots + nq \\ + np^{n+2} \qquad\qquad + - \frac{n(n+1)}{1 \cdot 2} p^{n+2} \end{aligned}$$

or

$$q^n + 2q^{n-1} + 3q^{n-2} + \cdots + (n-1)q^2 + q^2 - nq^{n+1} = \frac{n(n-1)}{1 \cdot 2} p^{n+2}.$$

III. Let it be

$$m = 3$$

and the exponent of the order $n+3$ that it is

$$q^\mu + q^{n+3-\mu} = p^\mu + p^{n+3},$$

and our equation will be

$$\begin{aligned} 0 = q^3 - n(q^2 + q^{n+1}) + \frac{n(n+1)}{1 \cdot 2}(q + q^{n+2}) - \frac{n(n-1)}{1 \cdot 2} p^{n+3} \\ -q^n - 3q^{n-1} - 6q^{n-2} - 10q^{n-3} - \cdots - \frac{n(n+1)}{1 \cdot 2} q + \frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3} p^{n+3} \end{aligned}$$

or

$$\begin{aligned} q^n + 3q^{n-1} + 6q^{n-2} + \cdots + \frac{n(n-1)}{1 \cdot 2} q^3 + \frac{n(n-1)}{1 \cdot 2} q^2 + nq^{n+1} \\ = \frac{n(nn+5)}{6} p^{n+3} \qquad -1 \qquad +n \qquad - \frac{n(n+1)}{1 \cdot 2} q^{n+2} \end{aligned}$$

or this way more distinctly

$$q^n + 3q^{n-1} + 6q^{n-2} + 10q^{n-1} + \cdots + \frac{n(n-1)}{1\cdot 2}q^2$$
$$-q^3 + n(q^2 + q^{n+1}) - \frac{n(n+1)}{1\cdot 2}q^{n+2} = \frac{n(nn+5)}{6}p^{n+3}$$

IV. Let it be

$$m = 4$$

and the exponent of the order $n+4$ and

$$q^\mu + q^{n+4-\mu} = p^\mu + p^{n+4},$$

and our equation will be

$$0 = q^4 - n(q^3 + q^{n+1}) + \frac{n(n+1)}{1\cdot 2}(q^2 + q^{n+2}) - \frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}(q + q^{n+3})$$
$$+\frac{n(nn+5)}{1\cdot 2\cdot 3}p^{n+4}$$
$$+q^n + 4q^{n-1} + 10q^{n-2} + 20q^{n-3} + \cdots + \frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}q$$
$$-\frac{n(n+1)(n+2)(n+3)}{1\cdot 2\cdot 3\cdot 4}p^{n+4}$$

or

$$q^n + 4q^{n-1} + 10q^{n-2} + \cdots + \frac{(n+1)n(n-1)}{1\cdot 2\cdot 3}q^2 = \frac{n(n-1)(nn+3n+14)}{24}p^{n+4}$$
$$+q^4 - n(q^3 + q^{n+1}) + \frac{n(n+1)}{1\cdot 2}(q^2 + q^{n+2}) - \frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}q^{n+3}.$$

§29 Let us in like manner also expand certain cases for the exponent n.

I. First, let it be

$$n = 1$$

and the exponent of the order $m+1$ and our equation will be

$$0 = q^m - q^{m-1} + q^{m-2} - q^{m-3} + \cdots \mp q \left.\begin{matrix} +1 \\ +0 \end{matrix}\right\} p^{m+1} \pm q \mp p^{m+1}$$

$$-q^2 \quad +q^3 \quad -q^4 \qquad \mp q^m,$$

whence it is plain, if m was an even number, in which case the upper signs hold, that the whole equation becomes the identical one; but if m is an odd number, one will have

$$q^2 - q^3 + q^4 - q^5 + \cdots - q^m = \frac{1}{2}p^{m+1}.$$

II. Let it be

$$n = 2$$

and the exponent of the order $m+2$ and our equation will be

$$0 = q^m - 2q^{m-1} + 3q^{m-2} - 4q^{m-3} + \cdots\cdots\cdots\cdots \mp mq \left.\begin{matrix} +\frac{1}{2}(m+2) \\ -\frac{1}{2}(m-1) \end{matrix}\right\} p^{m+2}$$

$$-2q^3 \quad +3q^4 \quad -4q^5 \quad + \cdots\cdots\cdots\cdots \mp q^{m+1}$$

$$\pm q^2 \quad \pm mq \quad \mp (m+1)p^{m+2},$$

where the above signs hold for the even values of m, the lower for the even values. Now for the various values of m we will have:

First, for the even values

$$\begin{aligned} m = 2: &\quad q^2 - q^3 = \frac{1}{2}p^4, \\ m = 4: &\quad q^2 - q^3 + q^4 - q^5 = \frac{1}{2}p^6, \\ m = 6: &\quad q^2 - q^3 + q^4 - q^5 + q^6 - q^7 = \frac{1}{2}p^8, \\ m = 8: &\quad q^2 - q^3 + q^4 - q^5 + q^6 - q^7 + q^8 - q^9 = \frac{1}{2}p^{10} \end{aligned}$$

etc.

further, for the even values

$$
\begin{aligned}
m = 1: &\quad q^2 = 2p^3,\\
m = 3: &\quad 3q^2 + q^3 - 3q^4 = 3p^5,\\
m = 5: &\quad 5q^2 - q^3 - q^4 + 3q^5 - 5q^6 = 4p^7,\\
m = 7: &\quad 7q^2 - 3q^3 + q^4 + q^5 - 3q^6 + 5q^7 - 7q^8 = 5p^9,\\
m = 9: &\quad 9q^2 - 5q^3 + 3q^4 - q^5 - q^6 + 3q^7 - 5q^8 + 7q^9 - 9q^{10} = 6p^{11},\\
&\quad \text{etc.}
\end{aligned}
$$

III. If we put

$$n = 3,$$

the single cases are considered more conveniently and they are

first, for an odd number m

$$
\begin{aligned}
m = 1: &\quad q^2 + q^3 = 3p^4,\\
m = 3: &\quad 6q^2 + 0q^3 + 3q^4 - 6q^5 = 7p^6,\\
m = 5: &\quad 15q^2 - 5q^3 + 6q^4 - 7q^5 + 10q^6 - 15q^7 = 13p^8,\\
m = 7: &\quad 28q^2 - 14q^3 + 13q^4 - 12q^5 + 13q^6 - 16q^7 + 21q^8 - 28q^9 = 21p^{10}\\
&\quad \text{etc.,}
\end{aligned}
$$

on the other hand for the odd orders

$$
\begin{aligned}
m = 2: &\quad 3q^2 + q^3 - 3q^4 = 3p^5,\\
m = 4: &\quad 10q^2 - 2q^3 - 2q^4 + 6q^5 - 10q^6 = 8p^7,\\
m = 6: &\quad 21q^2 - 9q^3 + 3q^4 + 3q^5 - 9q^6 + 15q^7 - 21q^8 = 15p^9,\\
m = 8: &\quad 36q^2 - 20q^3 + 12q^4 - 4q^5 - 4q^6 + 12q^7 - 20q^8 + 28q^9 - 36q^{10} = 24p^{11}\\
&\quad \text{etc.;}
\end{aligned}
$$

here, for both orders the form of the equation expressed in general will be as follows:

If m is an even number,

$$(m+1)q^2 - (m-3)q^3 + (m-5)q^4 - (m-7)q^5 + \cdots - (m+1)q^{m+2}$$
$$= \frac{m+4}{2}p^{m+3};$$

If m is an odd number,

$$m(m+1)q^2 - (mm-3m)q^3 + (mm-5m+12)q^4 - (mm-7m+24)q^5$$
$$+(mm-9m+40)q^6 - (mm-11m+60)q^7 + \cdots - m(m+1)q^{m+2}$$
$$= \frac{mm+4m+7}{2}p^{m+3}.$$

§30 Now, let us go through the single orders and reduce the equations found by means of the second method using the formula

$$p^{\mu} = q^{\mu} + q^{m+n-\mu} - p^{m+n}$$

to similar forms, of which kind we them obtained here.

Order $m+n=3$

Method I	Method II	Method III
$q+q^2 = p+p^3,$	$q^2 = 2p^3,$	$q^2 = 2p^3.$

Order $m+n=4$

$q+q^3 = p+p^4,$	$3q^2 - q^3 = 4p^4,$	$q^2 + q^3 = 3p^4,$
$2q^2 = p^2+p^4,$	$4q^2 - 4q^3 = 2p^4,$	$2q^2 - 2q^3 = p^4;$

therefore

$$q^2 = \frac{7}{4}p^4 \quad \text{and} \quad q^3 = \frac{5}{4}p^4, \quad \text{because of} \quad p^2 = \frac{5}{2}p^4 \quad \text{therefore} \quad q^3 = \frac{1}{2}p^2.$$

Order $m+n=5$

$$q \; + q^4 = \; p \; + p^5, \qquad q^2+q^3+q^4=4p^5, \qquad q^2+q^3 \; + q^4 = \; 4p^5,$$

$$q^2+q^3 = \; p^2+p^5, \qquad 0=0, \qquad 3q^2+q^3+3q^4 = \; 3p^5,$$

$$q^4 = 3p^5 - p^2, \quad q^3 = -\frac{9}{2}p^5 + 3p^2, \quad q^2 = \frac{11}{2}p^5 - 2p^2.$$

Order $m+n=6$

$$q \; + \; q^5 = \; p \; + p^6, \qquad 3q^2 - \; q^3 + 3q^4 - \; q^5 = \; 6p^6, \qquad q^2 + \; q^3 + q^4 + \; q^5 = 5p^6,$$

$$q^2 + \; q^4 = p^2 + p^6, \qquad 8q^2 - 2q^3 + 5q^4 - \; 8q^5 = \; 8p^6, \qquad 4q^2 + 2q^3 + q^4 - 4q^5 = 6p^6,$$

$$2q^3 = p^3 + p^6, \qquad 12q^2 \; + 6q^4 - 12q^5 = 14p^6, \qquad 6q^2 + 3q^4 \; - 6q^5 = 7p^6,$$

Therefore,

$$q^5 = p^2 - \frac{1}{2}p^3, \quad q^4 = p^3 - \frac{1}{3}p^6, \quad q^3 = \frac{1}{2}p^3 + \frac{1}{2}p^6, \quad q^2 = p^2 - p^3 + \frac{4}{3}p^6,$$

$$q^6 = \frac{7}{2}p^6 - p^2 - \frac{1}{2}p^3.$$

Because of $p^2 = \frac{7}{4}p^6$ it will be

$$q^5 = -\frac{7}{2}p^6 + 3p^2 - \frac{1}{2}p^3,$$

$$q^5 = \frac{7}{4}p^6 - \frac{1}{2}p^3, \quad q^4 = -\frac{1}{3}p^6 + p^3, \quad q^3 = \frac{1}{2}p^6 + \frac{1}{2}p^3, \quad q^2 = \frac{37}{12}p^6 - p^3.$$

Order $m+n=7$

$$q^2 + \; q^3 + \; q^4 + \; q^5 + \; q^6 = \; 6p^7,$$

$$q \; + q^6 = \; p + p^7, \quad q^2+q^3+q^4+q^5+q^6 = 6p^7, \qquad 5q^2 + 3q^3 + 2q^4 + \; q^5 - \; 5q^6 = 10p^7,$$

$$q^2 + q^5 = p^2 + p^7, \qquad 4q^3 + 3q^4 - 2q^5 = 6p^7, \qquad 10q^2 + 2q^3 + \; q^4 + 4q^5 - 10q^6 = 14p^7,$$

$$q^3 + q^4 = p^3 + p^7, \qquad 4q^3 + 3q^4 - 2q^5 = 6p^7, \qquad 10q^2 - 2q^3 - 2q^4 + 6q^5 - 10q^6 = \; 8p^7,$$

$$5q^2 - \; q^3 - \; q^4 + 3q^5 - \; 5q^6 = \; 4p^7;$$

hence it is concluded

$$
\begin{aligned}
q^6 &= +\ 4p^7 - p^2 - p^3,\\
q^5 &= -10p^7 - 5p^2 + 2p^3,\\
q^4 &= +18p^7 - 10p^2,\\
q^3 &= -17p^7 + 10p^2 + p^3,\\
q^2 &= +11p^7 - 4p^2 - 2p^3.
\end{aligned}
$$

Order $m + n = 8$

Method I | Method II

$$
\begin{aligned}
q + q^7 &= p + p^8, & 3q^2 - q^3 + 3q^4 - q^5 + 3q^6 - q^7 &= 8p^8,\\
q^2 + q^6 &= p^2 + p^8, & 12q^2 - 2q^3 + 9q^4 - 4q^5 + 7q^6 - 12q^7 &= 18p^8,\\
q^3 + q^5 &= p^3 + p^8, & 30q^2 \quad * + 9q^4 - 6q^5 + 15q^6 - 30q^7 &= 38p^8,\\
2q^4 &= p^4 + p^8, & 40q^2 \quad * + 4q^4 - 8q^5 + 20q^6 - 40q^7 &= 42p^8,
\end{aligned}
$$

Method III

$$
\begin{aligned}
q^2 + q^3 + q^4 + q^5 + q^6 + q^7 &= 7p^8,\\
6q^2 + 4q^3 + 3q^4 + 2q^5 + q^6 - 6q^7 &= 15p^8,\\
15q^2 + 5q^3 + 3q^4 + q^5 + 5q^6 - 15q^7 &= 25p^8,\\
20q^2 \quad * + 2q^4 - 4q^5 + 10q^6 - 20q^7 &= 21p^8,\\
15q^2 - 5q^3 + 6q^4 - 7q^5 + 10q^6 - 15q^7 &= 13p^8,\\
q^2 - q^3 + q^4 - q^5 + q^6 - q^7 &= \tfrac{1}{2}p^8;
\end{aligned}
$$

hence one finds

$$q^7 = \frac{9}{2}p^8 - p^2 - p^3 - \frac{1}{2}p^4,$$
$$q^7 = -\frac{9}{2}p^8 + 3p^2 - p^3 + \frac{3}{2}p^4,$$
$$q^7 = p^2 - p^3 + \frac{1}{2}p^4;$$

all remaining equation coalesce into this single one

$$4q^5 + 10q^6 = 2pp^3 - 9p^4 \quad \text{because of} \quad 7p^8 = 6p^4$$

and therefore this third method does not yield a complete determination, although it would have been possible for the cases $m + n = 5$ and $m + n = 7$.

Order $m + n = 9$

Here, by means of the third method all are determined and this in an unique way as follows

$$\begin{aligned}
q^8 &= + \ \ 5p^9 - \ \ p^2 - \ \ p^3 - \ \ p^4,\\
q^7 &= - \ \tfrac{35}{2}p^9 + \ \ 7p^2 + 2p^3 + 4p^4,\\
q^6 &= + \ \tfrac{85}{2}p^9 - 21p^2 \quad\ \ * - 6p^4,\\
q^5 &= - \ \tfrac{125}{2}p^9 + 35p^2 \quad\ \ * + 5p^4,\\
q^4 &= + \ \tfrac{127}{2}p^9 - 35p^2 \quad\ \ * - 4p^4,\\
q^3 &= - \ \tfrac{83}{2}p^9 + 21p^2 + \ \ p^3 + 6p^4,\\
q^2 &= - \ \tfrac{37}{2}p^9 - \ \ 6p^2 - 2p^3 - 4p^4.
\end{aligned}$$

Having left out the tenth order I observe that also the eleventh can be determined perfectly; for, having done the calculation one finds

$$
\begin{aligned}
q^{10} &= \quad\; 6p^{11} - \quad p^2 - p^3 - \quad p^4 - p^5, \\
q^9 &= -\; 27p^{11} + \quad 9p^2 + 2p^3 + \quad 6p^4 + 4p^5, \\
q^8 &= +\; 83p^{11} - \; 36p^2 \quad * - \; 15p^4 - 6p^5, \\
q^7 &= -\; \tfrac{329}{2}p^{11} + \; 84p^2 \quad * + 21p^4, + 4p^5, \\
q^6 &= +\; \tfrac{463}{2}p^{11} - 126p^2 \quad * - 21p^4, \\
q^5 &= -\; \tfrac{461}{2}p^{11} + 126p^2 \quad * + \; 21p^4 + \; p^5, \\
q^4 &= +\; \tfrac{331}{2}p^{11} - \; 84p^2 \quad * - \; 20p^4 - 4p^5, \\
q^3 &= -\; 82p^{11} + \; 36p^2 + \; p^3 + \; 15p^4 + 6p^5, \\
q^2 &= +\; 28p^{11} - \quad 8p^2 - 2p^3 - \quad 6p^4 - 4p^5.
\end{aligned}
$$

§31 If we contemplate these equations with more attention, in the coefficients of the terms p^{11}, p^9, p^7 and p^5 we will without any difficulty detect the following structure:

$$m+n=11 \qquad 6=\frac{11+1}{2}$$

$$\begin{aligned}
2\cdot 27 &= 10\cdot 6 - 6\\
3\cdot 83 &= 9\cdot 27 + 6\\
4\cdot \tfrac{329}{2} &= 8\cdot 83 - 6\\
5\cdot \tfrac{463}{2} &= 7\cdot \tfrac{329}{2} + 6\\
6\cdot \tfrac{461}{2} &= 6\cdot \tfrac{463}{2} - 6\\
7\cdot \tfrac{331}{2} &= 5\cdot \tfrac{461}{2} + 6\\
8\cdot 82 &= 4\cdot \tfrac{331}{2} - 6\\
9\cdot 28 &= 3\cdot 82 + 6\\
10\cdot 5 &= 2\cdot 28 - 6
\end{aligned}$$

for, it is

$$q=-5p^{11}+p+p^2+p^3+p^4+p^5$$

$$m+n=9 \qquad \frac{9+1}{2}$$

$$\begin{aligned}
2\cdot \tfrac{35}{2} &= 8\cdot 5 - 5\\
3\cdot \tfrac{85}{2} &= 7\cdot \tfrac{35}{2} + 5\\
4\cdot \tfrac{125}{2} &= 6\cdot \tfrac{85}{2} - 5\\
5\cdot \tfrac{127}{2} &= 5\cdot \tfrac{125}{2} + 5\\
6\cdot \tfrac{83}{2} &= 4\cdot \tfrac{127}{2} - 5\\
7\cdot \tfrac{37}{2} &= 3\cdot \tfrac{83}{2} + 5\\
8\cdot 4 &= 2\cdot \tfrac{37}{2} - 5
\end{aligned}$$

for, it is

$$q=-4p^9+p+p^2+p^3+p^4$$

$$m+n=7 \qquad \frac{7+1}{2}$$

$$\begin{aligned}
2\cdot 10 &= 6\cdot 4 - 4\\
3\cdot 18 &= 5\cdot 10 + 4\\
4\cdot 17 &= 4\cdot 18 - 4\\
5\cdot 11 &= 3\cdot 17 + 4\\
6\cdot 3 &= 2\cdot 11 - 4
\end{aligned}$$

for, it is

$$q=-3p^7+p+p^2+p^3$$

$$m+n=5 \qquad 3=\frac{3+1}{2}$$

$$\begin{aligned}
2\cdot \tfrac{9}{2} &= 4\cdot 3 - 3\\
3\cdot \tfrac{11}{2} &= 3\cdot \tfrac{9}{2} + 3\\
4\cdot 2 &= 4\cdot \tfrac{11}{2} - 3
\end{aligned}$$

for, it is

$$q=-2p^5+p+p^2$$

§32 Let us try to derive the equations for the order $m+n=13$, since at the same time the law of the progression is rendered perspicuous for the higher odd orders.

$$
\begin{aligned}
q^{12} &= +\ Ap^{13} - p^2 - p^3 - p^4 - p^5 - p^6,\\
q^{11} &= +\ Bp^{13} - 11p^2 + 2p^3 + 8p^4 + p^5 + \beta p^6,\\
q^{10} &= +\ Cp^{13} - 55p^2 \ * - 28p^4 - 6p^5 - \gamma p^6,\\
q^{9} &= -\ Dp^{13} + 165p^2 \ * + (56+1)p^4 + 4p^5 - \delta p^6,\\
q^{8} &= +\ Ep^{13} - 330p^2 \ * - (70+8)p^4 \ * - \varepsilon p^6,\\
q^{7} &= -\ Fp^{13} + 462p^2 \ * + (56+28)p^4 \ * + \zeta p^6,\\
q^{6} &= +\ Gp^{13} - 462p^2 \ * - (28+56)p^4 \ * - \eta p^6,\\
q^{5} &= -\ Hp^{13} + 330p^2 \ * - (8+70)p^4 + p^5 - \vartheta p^6,\\
q^{4} &= +\ Ip^{13} - 165p^2 \ * - 56p^4 - 4p^5 + \iota p^6,\\
q^{3} &= -\ Kp^{13} + 55p^2 + p^3 + 28p^4 + 6p^5 + \varkappa p^6,\\
q^{2} &= +\ Lp^{13} - (11-1)p^2 - 2p^3 - 8p^4 - 4p^5 - \lambda p^6.
\end{aligned}
$$

For the unknowns it is

$$
\begin{array}{rll}
A = \frac{13+1}{2}, & A = 7, & \lambda = \beta, \\
2B = 12A - 7, & B = \frac{77}{2}, & \varkappa = \gamma, \\
3C = 11B + 7, & C = \frac{287}{2}, & \iota = \delta, \\
4D = 10C - 7, & D = 357, & \theta = \varepsilon, \\
5E = 9D + 7, & E = 644, & \eta = \zeta - 1, \\
6F = 8E - 7, & F = \frac{1715}{2}, & \text{and it seems to be} \\
7G = 7F + 7, & G = \frac{1717}{2}, & \beta = 6, \lambda = 6, \\
8H = 6G - 7, & H = 643, & \gamma = 15, \varkappa = 15, \\
9I = 5H + 7, & I = 358, & \delta = 20, \ \iota = 20, \\
10K = 4I - 7, & K = \frac{285}{2}, & \varepsilon = 15, \theta = 15, \\
11L = 3K + 7, & L = \frac{79}{2}; & \zeta = 6 + 1, \quad \eta = 6.
\end{array}
$$

§33 That the structure of these equation is seen more clearly and the anomalies occurring here vanish, let us represent these equations according to the single odd orders this way.

Order $m + n = 3$

$$
\begin{array}{l|l}
q^2 = Ap^3 & A = \frac{3+1}{2} = 2 \\
q \ = -Bp^3 + p & 2B = 2A - 2
\end{array}
$$

Order $m+n=5$

$q^4 = +Ap^5 - p^2$	$A = \frac{5+1}{2} = 3$
$q^3 = -Bp^5 + 3p^2$	$2B = 4A - 3$
$q^2 = +Cp^5 - 3p^2 + p^2$	$3C = 3B + 3$
$q = -Dp^5 + p^2 + p$	$4D = 2C - 3$

Order $m+n=7$

$q^6 = +Ap^7 - p^2 - p^3$	$A = \frac{7+1}{2} = 4$
$q^5 = -Bp^7 + 5p^2 + 2p^3$	$2B = 6A - 4$
$q^4 = +Cp^7 - 10p^2 \quad *$	$3C = 5B + 4$
$q^3 = -Dp^7 + 10p^2 \quad * \quad + \quad p^3$	$4D = 4C - 4$
$q^2 = +Ep^7 - 5p^2 - 2p^3 + +p^2$	$5E = 3D + 4$
$q = -Fp^7 + p^2 + p^3 \quad + \quad p$	$6F = 2E - 4$

Order $m+n=9$

$q^8 = +Ap^9 - p^2 - p^3 - p^4$	$A = \frac{9+1}{2} = 5$
$q^7 = -Bp^9 + 7p^2 + 2p^3 + 4p^4$	$2B = 8A - 5$
$q^6 = +Cp^9 - 21p^2 \quad * \quad - 6p^4$	$3C = 7B + 5$
$q^5 = -Dp^9 + 35p^2 \quad * \quad - (4+1)p^4$	$4D = 6C - 5$
$q^4 = +Ep^9 - 35p^2 \quad * \quad - (1+4)p^4 + p^4$	$5E = 5D + 5$
$q^3 = -Fp^9 + 21p^2 \quad * \quad + 6p^4 + p^3$	$6F = 4E - 5$
$q^2 = +Gp^9 - 7p^2 - 2p^3 - 4p^4 + p^2$	$7G = 3F + 5$
$q = -Hp^9 + p^2 + p^3 + p^4 + p$	$8H = 2G - 5$

Order $m + n = 11$

$$
\begin{array}{lcl|rcl}
q^{10} &=& + Ap^{11} - p^2 - p^3 - p^4 - p^5 & A &=& \frac{11+1}{2} = 6 \\
q^9 &=& - Bp^{11} + 9p^2 + 2p^3 + 6p^4 + 4p^5 & 2B &=& 10A - 6 \\
q^8 &=& + Cp^{11} - 36p^2 \quad * \quad - 15p^4 - 6p^5 & 3C &=& 9B + 6 \\
q^7 &=& - Dp^{11} + 84p^2 \quad * \quad + p(20+1)^4 + 4p^5 & 4D &=& 8C - 6 \\
q^6 &=& + Ep^{11} - 126p^2 \quad * \quad - (15+6)p^4 \quad * & 5E &=& 7D + 6 \\
q^5 &=& - Fp^{11} + 126p^2 \quad * \quad - (6+15)p^4 \quad * + p^5 & 6F &=& 6E - 6 \\
q^4 &=& + Gp^{11} - 84p^2 \quad * \quad - (1+20)p^4 - 4p^5 + p^4 & 7G &=& 5F + 6 \\
q^3 &=& - Hp^{11} + 36p^2 \quad * \quad + 15p^4 + 6p^5 + p^3 & 8H &=& 4G - 6 \\
q^2 &=& + Ip^{11} - 9p^2 - 2p^3 - 6p^4 - 4p^5 + p^2 & 9I &=& 3H + 6 \\
q &=& - Kp^{11} + p^2 + p^3 + p^4 + p^5 + p & 10K &=& 2I - 6
\end{array}
$$

Order $m + n = 13$

$$
\begin{array}{rcllllllll|rcl}
q^{12} &=& +\,Ap^{13} & - & p^2 - p^3 & - & p^4 - & p^5 - & p^6 & & A &=& \frac{13+1}{2} = 7 \\
q^{11} &=& -\,Bp^{13} & + & 11p^2 + 2p^3 & -+ & 8p^4 + & 4p^5 + & 6p^6 & & 2B &=& 12A - 7 \\
q^{10} &=& +\,Cp^{13} & - & 55p^2 \quad * & - & 28p^4 - & 6p^5 - & 15p^6 & & 3C &=& 11B + 7 \\
q^{9} &=& -\,Dp^{13} & + & 165p^2 \quad * & + & (56+1)p^4 + & 4p^5 + & 20p^6 & & 4D &=& 10C - 7 \\
q^{8} &=& +\,Ep^{13} & - & 330p^2 \quad * & - & (70+8)p^4 & * - & 15p^6 & & 5E &=& 9D + 7 \\
q^{7} &=& -\,Fp^{13} & + & 462p^2 \quad * & - & (28+56)p^4 & * + & (6+1)p^6 & & 6F &=& 8E - 7 \\
q^{6} &=& +\,Gp^{13} & - & 462p^2 \quad * & - & (28+56)p^4 - & * + & (1+6)p^6 & - p^6 & 7G &=& 7F + 7 \\
q^{5} &=& -\,Hp^{13} & + & 330p^2 \quad * & + & (8+70)p^4 & * + & 15p^6 & + p^5 & 8H &=& 6G - 7 \\
q^{4} &=& +\,Ip^{13} & - & 165p^2 \quad * & - & (1+56)p^4 - & 4p^5 - & 20p^6 & + p^4 & 9I &=& 5H + 7 \\
q^{3} &=& -\,Kp^{13} & + & 55p^2 \quad * & + & 28p^4 + & 6p^5 + & 15p^6 & + p^3 & 10K &=& 4I - 7 \\
q^{2} &=& +\,Lp^{13} & - & 11p^2 - 2p^3 & - & 8p^4 - & 4p^5 - & 6p^6 & + p^2 & 11L &=& 3K + 7 \\
q &=& -\,Mp^{13} & + & p^2 + p^3 & + & p^4 + & p^5 + & p^6 & + p & 12M &=& 2L - 7
\end{array}
$$

§34 Here, the coefficients of p^2 seem to recede from the law of the following even orders, since in the order $m + n = 13$ they are formed from the coefficients of the binomial raised to the power 11, while the following are formed from the powers 8, 6. But the same can be represented this way in that coefficient, that they are connected to the law of the following

$$-1, \quad +(10+1), \quad -(45+10), \quad +(120+45), \quad -(210+120) \quad \text{etc.};$$

therefore, I will exhibit the equation of the order $m + n = 15$ this way.

Order $m + n = 15$

$$
\begin{aligned}
q^{14} &= + Ap^{15} - p^2 - p^3 - p^4 - p^5 - p^6 - p^7,\\
q^{13} &= - Bp^{15} + (12+1)p^2 + 2p^3 + 10p^4 + 4p^5 + 8p^6 + 6p^7,\\
q^{12} &= + Cp^{15} - (66+12)p^2 \quad * \quad - 45p^4 - 6p^5 - 28p^6 - 15p^7,\\
q^{11} &= - Dp^{15} + (220+66)p^2 \quad * \quad + (120+1)p^4 + 4p^5 + 56p^6 + 20p^7,\\
q^{10} &= + Ep^{15} - (495+220)p^2 \quad * \quad - (210+10)p^4 \quad * - 70p^6 - 15p^7,\\
q^{9} &= - Fp^{15} - (792+495)p^2 \quad * \quad - (252+45)p^4 \quad * + (56+1)p^6 + 6p^7,\\
q^{8} &= + Gp^{15} - (924+792)p^2 \quad * \quad - (210+10)p^4 \quad * - (28+8)p^6 \quad *,\\
q^{7} &= - Hp^{15} + (792+924)p^2 \quad * \quad + (120+210)p^4 \quad * + (8+28)p^6 \quad * + p^7,\\
q^{6} &= + Ip^{15} - (495+792)p^2 \quad * \quad - (45+252)p^4 \quad * - (1+56)p^6 - 6p^7 + p^6,\\
q^{5} &= - Kp^{15} + (220+495)p^2 \quad * \quad + (10+210)p^4 \quad * + 70p^6 + 15p^7 + p^5,\\
q^{4} &= + Lp^{15} - (66+220)p^2 \quad * \quad - (1+120)p^4 - 4p^5 - 56p^6 - 20p^7 + p^4,\\
q^{3} &= - Mp^{15} + (12+66)p^2 \quad * \quad + 45p^4 + 6p^5 + 28p^6 + 15p^7 + p^3,\\
q^{2} &= + Np^{15} - (1+12)p^2 - 2p^3 - 10p^4 - 4p^5 - 8p^6 - 6p^7 - p^2,\\
q &= - Op^{15} + p^2 + p^3 + p^4 + p^5 + p^6 + p^7 + p.
\end{aligned}
$$

And now the law of progression is not too complex and can easily accommodated to higher orders.

§35 But since the law, based on induction alone, could seem to be incorrect, all doubts will be removed, if instead of the even powers of p the odd ones are introduced, since for the eleventh order the odd powers p^9 and p^7 are equivalent to p^2 and p^4. Further, even the coefficients A, B, C, D etc. can be exhibited by a simpler law, which immediately follows from the coefficients of the binomial raised to the same powers, of which order these equations in question are, this way:

Order $m + n = 11$

p^{11}	p^1	p^3	p^5	p^7	p^9	
$q^{10} = \frac{1}{2}(1 + 11)$	-1	-1	-1	$-\ 1$	$-\ 1$	$+ p \ = p^{10}$
$q^9 \ = \frac{1}{2}(1 - 55)$	$*$	$+2$	$+4$	$+\ 6$	$+\ 8 + \ 1$	
$q^8 \ = \frac{1}{2}(1 + 165)$	$*$	-1	-6	$- 15$	$- 28 - \ 8$	$+ p^3 = \ p^8$
$q^7 \ = \frac{1}{2}(1 - 330)$	$*$	$*$	$+4$	$+ 20 + \ 1$	$+ 56 + 28$	
$q^6 \ = \frac{1}{2}(1 + 462)$	$*$	$*$	-1	$- 15 - \ 6$	$- 70 - 56$	$+ p^5 = \ p^6$
$q^5 \ = \frac{1}{2}(1 - 462)$	$*$	$*$	$+1$	$+\ 6 + 15$	$+ 56 + 70$	
$q^4 \ = \frac{1}{2}(1 + 330)$	$*$	$*$	-4	$- 20 - 20$	$- 28 - 56$	$+ p^7 = \ p^4$
$q^3 \ = \frac{1}{2}(1 - 165)$	$*$	$+1$	$+6$	$+ 15$	$+\ 8 + 28$	
$q^2 \ = \frac{1}{2}(1 + 55)$	$*$	-2	-4	$-\ 6$	$-\ 1 - \ 8$	$+ p^9 = \ p^2$
$q^1 \ = \frac{1}{2}(1 - 11)$	$+1$	$+1$	$+1$	$+\ 1$	$+\ 1$	

In any arbitrary vertical column the coefficients of the binomial raised to a power smaller by 1 are written both from top to bottom and from the bottom to the top and where two occur they are collected into one sum.

§36 Hence it will now be possible to define it in general for all odd orders; but since here the binomial coefficients of the respective power occur, for the sake of brevity let us write

$$\frac{n(n-1)n-2)\cdots(n-\nu+1)}{1\cdot 2\cdot 3\cdots \nu} = n(\nu),$$

that it is

$$n(0) = 1, \quad n(1) = n, \quad n(2) = \frac{n(n-1)}{1\cdot 2}, \quad n(3) = \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} \quad \text{etc.},$$

where it will be helpful to have observed, if ν was larger than n, that it is always $n(\nu) = 0$, but if $n = \nu$, that it will be $n(n) = 1$ and in general

$n(\nu) = n(n-\nu)$. Therefore, using this notation the general equations will be as follows

Order $m + n = \lambda$

$$q^{\lambda-1} = p^{\lambda-1} + \tfrac{1}{2}(1+\lambda(1))p^{\lambda} \left.\begin{matrix} +\,0(0) \\ +\,0(\lambda-2) \end{matrix}\right\} p \left.\begin{matrix} +\,2(0) \\ +\,2(\lambda-2) \end{matrix}\right\} p^3 \left.\begin{matrix} +\,4(0) \\ +\,4(\lambda-2) \end{matrix}\right\} p^5 \left.\begin{matrix} +\,6(0) \\ +\,6(\lambda-2) \end{matrix}\right\} p^7 + \text{etc.}$$

$$q^{\lambda-2} = \ * \ + \tfrac{1}{2}(1-\lambda(2))p^{\lambda} \left.\begin{matrix} -\,0(1) \\ -\,0(\lambda-3) \end{matrix}\right\} p \left.\begin{matrix} -\,2(1) \\ -\,2(\lambda-3) \end{matrix}\right\} p^3 \left.\begin{matrix} -\,4(1) \\ -\,4(\lambda-3) \end{matrix}\right\} p^5 \left.\begin{matrix} -\,6(1) \\ -\,6(\lambda-3) \end{matrix}\right\} p^7 + \text{etc.}$$

$$q^{\lambda-3} = p^{\lambda-3} + \tfrac{1}{2}(1+\lambda(3))p^{\lambda} \left.\begin{matrix} +\,0(2) \\ +\,0(\lambda-4) \end{matrix}\right\} p \left.\begin{matrix} +\,2(2) \\ +\,2(\lambda-4) \end{matrix}\right\} p^3 \left.\begin{matrix} +\,4(2) \\ +\,4(\lambda-4) \end{matrix}\right\} p^5 \left.\begin{matrix} +\,6(2) \\ +\,6(\lambda-4) \end{matrix}\right\} p^7 + \text{etc.}$$

$$q^{\lambda-4} = \ * \ + \tfrac{1}{2}(1-\lambda(4))p^{\lambda} \left.\begin{matrix} -\,0(3) \\ -\,0(\lambda-5) \end{matrix}\right\} p \left.\begin{matrix} -\,2(3) \\ -\,2(\lambda-5) \end{matrix}\right\} p^3 \left.\begin{matrix} -\,4(3) \\ -\,4(\lambda-5) \end{matrix}\right\} p^5 \left.\begin{matrix} -\,6(3) \\ -\,6(\lambda-5) \end{matrix}\right\} p^7 + \text{etc.}$$

$$q^{\lambda-5} = p^{\lambda-5} + \tfrac{1}{2}(1+\lambda(5))p^{\lambda} \left.\begin{matrix} +\,0(4) \\ +\,0(\lambda-6) \end{matrix}\right\} p \left.\begin{matrix} +\,2(4) \\ +\,2(\lambda-6) \end{matrix}\right\} p^3 \left.\begin{matrix} +\,4(4) \\ +\,4(\lambda-6) \end{matrix}\right\} p^5 \left.\begin{matrix} +\,6(4) \\ +\,6(\lambda-6) \end{matrix}\right\} p^7 + \text{etc.}$$

etc.,

whence we conclude that it will be in general:

I. If ν was an odd number,

$$q^{\lambda-\nu} = p^{\lambda-\nu} + \tfrac{1}{2}(1+\lambda(1))p^{\lambda} \left.\begin{matrix} +\,0(\nu-1) \\ +\,0(\lambda-\nu-1) \end{matrix}\right\} p \left.\begin{matrix} +\,2(\nu-1) \\ +\,2(\lambda-\nu-1) \end{matrix}\right\} p^3 \left.\begin{matrix} +\,4(\nu-1) \\ +\,4(\lambda-\nu-1) \end{matrix}\right\} p^5 \left.\begin{matrix} +\,6(\nu-1) \\ +\,6(\lambda-\nu-1) \end{matrix}\right\} p^7 + \text{etc.}$$

II: If ν is an even number,

$$q^{\lambda-\nu} = * + \tfrac{1}{2}(1+\lambda(1))p^{\lambda} \left.\begin{matrix} -0(\nu-1) \\ -0(\lambda-\nu-1) \end{matrix}\right\} p \left.\begin{matrix} -2(\nu-1) \\ -2(\lambda-\nu-1) \end{matrix}\right\} p^3 \left.\begin{matrix} -4(\nu-1) \\ -4(\lambda-\nu-1) \end{matrix}\right\} p^5$$
$$\left.\begin{matrix} -6(\nu-1) \\ -6(\lambda-\nu-1) \end{matrix}\right\} p^7 + \text{etc.}$$

But the terms of these equations do not have to be continued further than the formula $p^{\lambda-2}$, which yields the last term.

§37 But these summations only hold, if the exponent of the order $m+n=\lambda$ was an odd number; and hence by means of these equations the sums of all series contained in this form

$$q^m = 1 + \frac{1}{2^m}\left(1+\frac{1}{2^n}\right) + \frac{1}{3^m}\left(1+\frac{1}{2^n}+\frac{1}{3^n}\right) + \frac{1}{4^m}\left(1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}\right) + \text{etc.}$$

can be exhibited, if only $m+n=\lambda$ was an odd number. But these sums are defined by the sums of the powers of the reciprocals, which by means of the letter p I represent in the following manner, that it is

$$p^\lambda = 1 + \frac{1}{2^\lambda} + \frac{1}{3^\lambda} + \frac{1}{4^\lambda} + \frac{1}{4^\lambda} + \text{etc.}$$

and

$$p^m = p^n = \left(1+\frac{1}{2^m}+\frac{1}{3^m}+\frac{1}{4^m}+\text{etc.}\right)\left(1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\text{etc.}\right).$$

But here two cases have to be distinguished, depending on whether m was an even or an odd number; for, one will have:

in the I. case in which m is an even number,

$$q^m = p^m + \tfrac{1}{2}(1+\lambda(n))p^\lambda \left.\begin{matrix} +0(n-1) \\ +0(m-1) \end{matrix}\right\} p \left.\begin{matrix} +2(n-1) \\ +2(m-1) \end{matrix}\right\} p^3 \left.\begin{matrix} +4(n-1) \\ +4(n-1) \end{matrix}\right\} p^5$$

$$\left.\begin{matrix} + 6(n-1) \\ + 6(m-1) \end{matrix}\right\} p^7 \cdots \left.\begin{matrix} + (\lambda-1)(n-1) \\ + (\lambda-1)(m-1) \end{matrix}\right\} p^{\lambda-2}.$$

in the II. case in which m is an odd number,

$$q^m = * + \tfrac{1}{2}(1+\lambda(n))p^\lambda \quad \left.\begin{matrix} -0(n-1) \\ -0(m-1) \end{matrix}\right\} p \quad \left.\begin{matrix} -2(n-1) \\ -2(m-1) \end{matrix}\right\} p^3 \quad \left.\begin{matrix} -4(n-1) \\ -4(n-1) \end{matrix}\right\} p^5$$

$$\left.\begin{matrix} -6(n-1) \\ -6(m-1) \end{matrix}\right\} p^7 \cdots \left. -(\lambda-1)(n-1) - (\lambda-1)(m-1) \right\} p^{\lambda-2}.$$

According to the law the term following the last would be

$$\left.\begin{matrix} (\lambda-1)(n-1) \\ (\lambda-1)(m-1) \end{matrix}\right\} p^\lambda,$$

where it should be noted that it is

$$(\lambda-1)(n-1) + (\lambda-1)(m-1) = \lambda(n) = \lambda(m).$$

§38 But if the exponent of the order $m+n=\lambda$ was an even number, these formulas can not hold by any means, since in the case of imparity the forms p, p^3, p^5, p^7 etc. because of $p^m = p^n$ also contain these even ones p^2, p^4, p^6 etc., what does not happen in the cases in which $m+n$ is an even number. But the three methods used here do not suffice to define the sums of even orders, since even the third for the eighth order does not yield all determinations. But even though for the fourth and sixth order the sums were assigned above, in them nevertheless no law is seen, whence one could make a conjecture for the following orders. The reason for this difference obviously is that for the even orders each two of these formulas p^λ, p^2 p^4, p^6 etc. can be compared to each other and these comparisons are indicated by means of our methods; therefore, the determinations we are looking for, are to be considered to be missing. Therefore, it is even more remarkable, that in the odd orders no assignable relation among the formulas p^λ, p^2, p^3, p^4 exists at all. Nevertheless, there is no doubt that other methods are given, by which the series of odd orders can be summed, even though the three explained here do not suffice at all.