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Inverse binomial series and values of Arakawa–Kaneko zeta functions

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ABSTRACT

In this article, we present a variety of evaluations of series of polylogarithmic nature. More precisely, we express the special values at positive integers of two classes of zeta functions of Arakawa–Kaneko-type by means of certain inverse binomial series involving harmonic sums which appeared fifteen years ago in physics in relation to the Feynman diagrams. In some cases, these series may be explicitly evaluated in terms of zeta values and other related numbers. Incidentally, this connection allows us to deduce new identities for the constant $C = \sum_{n\geq 1} \frac{1}{(2n)^3}(1 + \frac{1}{3} + \cdots + \frac{1}{2n-1})$ considered by S. Ramanujan in his notebooks.

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1. Introduction

The function β defined for $\Re(s) > 0$ by the Dirichlet series

$$\beta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^s}$$

has the integral representation

$$\beta(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1+e^{-2t}} t^{s-1}\, dt = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-2t}} \operatorname{Li}_0\left(\frac{1-e^{-2t}}{2}\right) t^{s-1}\, dt$$

where Li_k denotes the classical polylogarithm $\operatorname{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}$. One may also observe that

$$\frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-2t}} \operatorname{Li}_1\big(1-e^{-2t}\big) t^{s-1}\, dt = \big(2-2^{-s}\big) s\zeta(s+1).$$

These preliminary observations lead us to introduce two families of functions α_k and β_k defined by the Mellin transforms

$$\alpha_k(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-2t}} \operatorname{Li}_k\big(1-e^{-2t}\big) t^{s-1}\, dt \quad \text{for } \Re(s) > 0 \text{ and } k \geq 1,$$

$$\beta_k(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-2t}} \operatorname{Li}_k\left(\frac{1-e^{-2t}}{2}\right) t^{s-1}\, dt \quad \text{for } \Re(s) > 0 \text{ and } k \geq 0,$$

so that

$$\alpha_1(s) = 2^{-s}\big(2^{s+1}-1\big) s\zeta(s+1), \quad \text{and} \quad \beta_0(s) = \beta(s).$$

We point out that the function $\alpha_k(s)$ introduced here is (apart from a factor 2^{-s}) nothing less than the special value at $x = 1/2$ of the (generalized) Arakawa–Kaneko zeta function $\xi_k(s,x)$ previously defined in [6], whereas the function $\beta_k(s)$ is a new function of the same type. Let us remind that the original Arakawa–Kaneko zeta function $\xi_k(s) = \xi_k(s,1)$ was introduced by Arakawa and Kaneko in 1999 (cf. [1]) and formed the subject of recent works and further generalizations (cf. [3,6,12]).

In the case where s is a positive integer, the special values $\alpha_k(s)$ and $\beta_k(s)$ can be expressed by means of certain inverse binomial series studied by Kalmykov and Davydychev in relation to the Feynman diagrams (cf. [7]). More precisely, we obtain the following identities:

$$\alpha_k(m+1) = \sum_{n=1}^{\infty} \frac{2^{2n-1}}{\binom{2n}{n} n^{k+1}} P_m\big(O_n^{(1)}, \dots, O_n^{(j)}, \dots, O_n^{(m)}\big) \quad (\text{for } k \geq 1),$$

$$\beta_k(m+1) = \sum_{n=1}^{\infty} \frac{2^{n-1}}{\binom{2n}{n} n^{k+1}} P_m\big(O_n^{(1)}, \dots, O_n^{(j)}, \dots, O_n^{(m)}\big) \quad (\text{for } k \geq 0),$$

where P_m is the modified Bell polynomial of order m and $O_n^{(j)} = \sum_{k=1}^{n} \frac{1}{(2k-1)^j}$ is the "odd" harmonic number of order j. For small values of k and s, these series may be explicitly evaluated in terms of zeta values and other related constants which are real periods in the sense of Kontsevich and Zagier (cf. [8]). For instance, we have the following evaluations:

$$\alpha_1(2) = \sum_{n=1}^{\infty} \frac{2^{2n-1}}{\binom{2n}{n}} \frac{O_n}{n^2} = \frac{7}{2}\zeta(3),$$

$$\beta_1(2) = \sum_{n=1}^{\infty} \frac{2^{n-1}}{\binom{2n}{n}} \frac{O_n}{n^2} = \frac{7}{4}\zeta(3) - \frac{\pi}{2}G,$$

$$\alpha_2(2) = \sum_{n=1}^{\infty} \frac{2^{2n-1}}{\binom{2n}{n}} \frac{O_n}{n^3} = 7\zeta(3)\ln 2 - \frac{\pi^4}{32} - 8G(1),$$

$$\alpha_3(1) = \sum_{n=1}^{\infty} \frac{2^{2n-1}}{\binom{2n}{n}} \frac{1}{n^4} = \frac{\pi^2}{2}(\ln 2)^2 - \frac{7}{2}\zeta(3)\ln 2 + \frac{\pi^4}{96} + 4G(1),$$

where we use the following notation:

$$O_n := O_n^{(1)} = \sum_{j=1}^{n} \frac{1}{2j-1},$$

$$G := \beta(2) \text{ is the Catalan constant,}$$

$$G(1) := \sum_{n=1}^{\infty} \frac{O_n}{(2n)^3} \text{ is the Ramanujan constant (cf. [2, p. 257], [11]).}$$

The evaluation of $\alpha_1(2)$ above was given previously in [5], whereas the following ones are new. In particular, the two last relations provide new interesting formulae for Ramanujan's constant $G(1)$ as explained in detail in Section 6.

2. Bell polynomials and "odd" harmonic numbers

Definition 1. The *modified Bell polynomials* are the polynomials

$$P_m \in \mathbb{Q}[x_1, x_2, \dots, x_m]$$

defined for all natural numbers m by $P_0 = 1$ and the generating function

$$\exp\left(\sum_{k=1}^{\infty} x_k \frac{z^k}{k}\right) = \sum_{m=0}^{\infty} P_m(x_1, \ldots, x_m) z^m.$$

The general explicit expression for P_m is

$$P_m(x_1, \ldots, x_m) = \sum_{k_1+2k_2+\cdots+mk_m=m} \frac{1}{k_1! k_2! \cdots k_m!} \left(\frac{x_1}{1}\right)^{k_1} \left(\frac{x_2}{2}\right)^{k_2} \cdots \left(\frac{x_m}{m}\right)^{k_m}.$$

Example 1. For the first values of m, one has

$$\begin{aligned}
P_0 &= 1,\\
P_1 &= x_1,\\
P_2 &= \frac{1}{2}x_1^2 + \frac{1}{2}x_2,\\
P_3 &= \frac{1}{6}x_1^3 + \frac{1}{2}x_1x_2 + \frac{1}{3}x_3,\\
P_4 &= \frac{1}{24}x_1^4 + \frac{1}{4}x_1^2x_2 + \frac{1}{8}x_2^2 + \frac{1}{3}x_1x_3 + \frac{1}{4}x_4.
\end{aligned}$$

Notation. For $s \in \mathbb{C}$ with $\Re(s) \geq 1$ and an integer $n \geq 1$, let $O_n^{(s)}$ be the "odd" harmonic sum:

$$O_n^{(s)} = \sum_{k=1}^{n} \frac{1}{(2k-1)^s} \quad \text{and} \quad O_n := O_n^{(1)}.$$

In the notation of [6], one has

$$O_n^{(s)} = 2^{-s} h_{n-1}^{(s)}(1/2) \quad \text{with } h_n^{(s)}(x) = \sum_{j=0}^{n} \frac{1}{(j+x)^s}.$$

Proposition 1. *For all integers $m \geq 0$ and $n \geq 1$,*

$$P_m\big(O_n, \ldots, O_n^{(m)}\big) = \frac{n\binom{2n}{n}}{2^{2n-1}} \int_0^{+\infty} e^{-t}\big(1 - e^{-2t}\big)^{n-1} \frac{t^m}{m!} dt. \tag{1}$$

Proof. Since

$$P_m\big(O_n, \ldots, O_n^{(m)}\big) = 2^{-m} P_m\big(h_{n-1}^{(1)}(1/2), \ldots, h_{n-1}^{(m)}(1/2)\big),$$

formula (1) follows from Lemma 1 of [6] in the special case $x = 1/2$. To be self-contained, we give a direct proof below. We show that

$$\sum_{m=0}^{\infty} P_m\big(O_n,\dots,O_n^{(m)}\big)z^m = \prod_{j=1}^{n}\frac{2j-1}{2j-1-z} = \frac{n\binom{2n}{n}}{2^{2n-1}}\int_0^{+\infty} e^{tz}\big(1-e^{-2t}\big)^{n-1}e^{-t}dt,$$

and then we shall obtain formula (1) by identification of the coefficients of z^m. On one side, one has

$$\begin{aligned}\prod_{j=1}^{n}\frac{2j-1}{2j-1-z} &= \prod_{j=1}^{n}\left(1-\frac{z}{2j-1}\right)^{-1}\\ &= \exp\left(-\sum_{j=1}^{n}\log\left(1-\frac{z}{2j-1}\right)\right)\\ &= \exp\left(\sum_{j=1}^{n}\sum_{k=1}^{+\infty}\frac{z^k}{k(2j-1)^k}\right)\\ &= \exp\left(\sum_{k=1}^{+\infty}\frac{z^k}{k}\sum_{j=1}^{n}\frac{1}{(2j-1)^k}\right),\end{aligned}$$

thus

$$\prod_{j=1}^{n}\frac{2j-1}{2j-1-z} = \exp\left(\sum_{k=1}^{\infty}O_n^{(k)}\frac{z^k}{k}\right) = \sum_{m=0}^{\infty}P_m\big(O_n,\dots,O_n^{(m)}\big)z^m.$$

On the other side, one has

$$\begin{aligned}\prod_{j=1}^{n}\frac{2j-1}{2j-1-z} &= \frac{1}{\sqrt{\pi}}\frac{\Gamma(n+1/2)\Gamma(-z/2+1/2)}{\Gamma(n-z/2+1/2)}\\ &= \frac{1}{\sqrt{\pi}}\frac{\Gamma(n+1/2)}{\Gamma(n)}\frac{\Gamma(n)\Gamma(-z/2+1/2)}{\Gamma(n-z/2+1/2)}\\ &= \frac{n}{2^{2n}}\binom{2n}{n}B(n,-z/2+1/2),\end{aligned}$$

where B is the Euler Beta function. Thus, for $0<|z|<1$, one has

$$\prod_{j=1}^{n}\frac{2j-1}{2j-1-z} = \frac{n\binom{2n}{n}}{2^{2n}}\int_0^{1}u^{n-1}(1-u)^{-z/2-1/2}du,$$

and making the change of variable $u=1-e^{-2t}$, one then obtains:

$$\prod_{j=1}^{n}\frac{2j-1}{2j-1-z} = \frac{n\binom{2n}{n}}{2^{2n-1}}\int_0^{+\infty}e^{tz}\big(1-e^{-2t}\big)^{n-1}e^{-t}dt,$$

and finally

$$\sum_{m=0}^{\infty} P_m\big(O_n,\dots,O_n^{(m)}\big)z^m = \frac{n\binom{2n}{n}}{2^{2n-1}} \int_0^{+\infty} e^{tz}\big(1-e^{-2t}\big)^{n-1}e^{-t}dt. \qquad \square$$

3. The operators D and S, and the Euler series transformation

Definition 2. Let a be an analytic function in $P=\{x \mid \Re(x)\geq 1\}$ defined by

$$a(x) = \int_0^{+\infty} e^{-xt}\widehat{a}(t)dt \quad \text{for all } x\in P,$$

where $\widehat{a}\in\mathcal{C}^1([0,+\infty[)$ is such that there exists $\alpha<1$, and $C>0$ with

$$\big|\widehat{a}(t)\big| \leq Ce^{\alpha t} \quad \text{for all } t\in\,]0,+\infty[.$$

For $x\in P$, we define the functions $x\mapsto D(a)(x)$ and $x\mapsto S(a)(x)$ by

$$D(a)(x) = \int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}}\big(1-e^{-t}\big)^x\widehat{a}(t)\,dt,$$

$$S(a)(x) = \int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}}\big(1-e^{-xt}\big)\widehat{a}(t)\,dt.$$

Proposition 2. *For all integers $n\geq 1$, one has*

$$S(a)(n) = \sum_{k=1}^{n} a(k),$$

and for all integers $n\geq 0$,

$$D(a)(n+1) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}a(k+1).$$

Proof. The first relation follows from

$$S(a)(n) = \int_0^{+\infty} \frac{e^{-t}-e^{-(n+1)t}}{1-e^{-t}}\widehat{a}(t)dt = \int_0^{+\infty}\left(\sum_{k=1}^{n} e^{-kt}\right)\widehat{a}(t)dt = \sum_{k=1}^{n}\int_0^{+\infty} e^{-kt}\widehat{a}(t)dt.$$

The second relation results from the binomial expansion of $(1-e^{-t})^n$ since

$$\begin{aligned} D(a)(n+1) &= \int_0^{+\infty} e^{-t}\big(1-e^{-t}\big)^n \widehat{a}(t)dt \\ &= \int_0^{+\infty} e^{-t}\sum_{k=0}^{n}(-1)^k\binom{n}{k}e^{-kt}\widehat{a}(t)dt \\ &= \sum_{k=0}^{n}(-1)^k\binom{n}{k}\int_0^{+\infty} e^{-t}e^{-kt}\widehat{a}(t)dt \\ &= \sum_{k=0}^{n}(-1)^k\binom{n}{k}a(k+1). \qquad \square \end{aligned}$$

Example 2. For s with $\Re(s)\geq 1$ and $x\in P$, let

$$a(x)=\frac{1}{(2x-1)^s}.$$

One has

$$a(x)=\int_0^{+\infty} e^{-(2x-1)t}\frac{t^{s-1}}{\Gamma(s)}dt=\int_0^{+\infty} e^{-xt}\frac{e^{\frac{t}{2}}\big(\frac{t}{2}\big)^{s-1}}{2\Gamma(s)}dt.$$

Thus, for all integers $n\geq 1$,

$$D(a)(n)=\int_0^{+\infty} e^{-\frac{t}{2}}\big(1-e^{-t}\big)^{n-1}\frac{\big(\frac{t}{2}\big)^{s-1}}{2\Gamma(s)}dt=\int_0^{+\infty} e^{-t}\big(1-e^{-2t}\big)^{n-1}\frac{t^{s-1}}{\Gamma(s)}dt. \tag{2}$$

By (1), one has

$$\int_0^{+\infty} e^{-t}\big(1-e^{-2t}\big)^{n-1}\frac{t^m}{m!}dt=\frac{2^{2n-1}}{n\binom{2n}{n}}P_m\big(O_n,\dots,O_n^{(m)}\big).$$

Thus, if s is an integer, $s=m+1$ with $m\geq 0$, then we get for all integers $n\geq 1$ the following formula

$$D\left(\frac{1}{(2x-1)^{m+1}}\right)(n)=\frac{2^{2n-1}}{n\binom{2n}{n}}P_m\big(O_n,\dots,O_n^{(m)}\big). \tag{3}$$

Lemma 1. *The operators D and S are linked by the following relation:*

$$D\left(\frac{1}{x}S(a)\right) = \frac{1}{x}D(a) \quad \textit{for all } x \in P.$$

Proof. By the definition of $S(a)$,

$$\frac{1}{x}S(a)(x) = \int_0^{+\infty} \frac{1-e^{-xt}}{x}\left[\frac{e^{-t}}{1-e^{-t}}\widehat{a}(t)\right] dt,$$

integrating by parts, we get

$$\frac{1}{x}S(a)(x) = \int_0^{+\infty} e^{-xt}\left(\int_t^{\infty} \frac{e^{-u}}{1-e^{-u}}\widehat{a}(u)\,du\right) dt,$$

this gives

$$\widehat{\frac{1}{x}S(a)}(t) = \int_t^{\infty} \frac{e^{-u}}{1-e^{-u}}\widehat{a}(u)\,du.$$

Thus

$$D\left(\frac{1}{x}S(a)\right)(x) = \int_0^{+\infty} e^{-t}\left(1-e^{-t}\right)^{x-1}\left(\int_t^{\infty} \frac{e^{-u}}{1-e^{-u}}\widehat{a}(u)\,du\right) dt,$$

and integrating again by parts, we get

$$D\left(\frac{1}{x}S(a)\right)(x) = \int_0^{+\infty} \frac{1}{x}\left(1-e^{-t}\right)^{x} \frac{e^{-t}}{1-e^{-t}}\widehat{a}(t)\,dt = \frac{1}{x}D(a)(x). \qquad \square$$

Remark 1. The relation between D and S given above is a reformulation of a result that we called the "harmonic property" in an earlier paper (where the operator S is denoted by A): cf. [3, Theorem 6].

Proposition 3. *For all complex numbers z such that $|z| < \frac{1}{2}$, one has*

$$\sum_{n=1}^{+\infty} D(a)(n)z^n = -\sum_{n=1}^{+\infty} a(n)\left(\frac{z}{z-1}\right)^n,$$
$$\sum_{n=1}^{+\infty} D(a)(n)\frac{z^n}{n} = -\sum_{n=1}^{+\infty} \frac{1}{n}S(a)(n)\left(\frac{z}{z-1}\right)^n. \tag{4}$$

Proof. For the first relation we write

$$\sum_{n=0}^{+\infty} D(a)(n+1)z^{n+1} = \int_0^{+\infty} e^{-t} \frac{z}{1-(1-e^{-t})z} \widehat{a}(t)dt$$
$$= -\int_0^{+\infty} \left(\frac{z}{z-1}\right) \frac{e^{-t}}{1-e^{-t}\frac{z}{z-1}} \widehat{a}(t)dt.$$

The expansion

$$\left(\frac{z}{z-1}\right) \frac{e^{-t}}{1-e^{-t}\frac{z}{z-1}} \widehat{a}(t) = \sum_{n=1}^{+\infty} e^{-nt} \left(\frac{z}{z-1}\right)^n \widehat{a}(t)$$

gives

$$\sum_{n=0}^{+\infty} D(a)(n+1)z^{n+1} = -\int_0^{+\infty} \sum_{n=1}^{+\infty} e^{-nt} \left(\frac{z}{z-1}\right)^n \widehat{a}(t)dt$$
$$= -\sum_{n=1}^{+\infty} \left(\frac{z}{z-1}\right)^n \int_0^{+\infty} e^{-nt} \widehat{a}(t)dt,$$

the order of $\int_0^{+\infty}$ and $\sum_{n=1}^{\infty}$ may be interchanged because

$$\int_0^{+\infty} \sum_{n=1}^{+\infty} e^{-nt} \left(\frac{|z|}{1-|z|}\right)^n |\widehat{a}(t)|dt = \left(\frac{|z|}{1-|z|}\right) \int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}\frac{|z|}{1-|z|}} |\widehat{a}(t)| < +\infty.$$

The second relation (4) is an immediate consequence of the first one by Lemma 1 above. □

Proposition 4. *For all integers $p \geq 1$, one has*

$$\sum_{n=1}^{\infty} \frac{D(a)(n)}{p^n n^k} = \int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}} \operatorname{Li}_k\left(\frac{1-e^{-t}}{p}\right) \widehat{a}(t)\, dt. \tag{5}$$

Proof. Let p be a positive integer, then

$$\sum_{n=1}^{\infty} \frac{D(a)(n)}{p^n n^k} = \sum_{n=1}^{\infty} \frac{1}{p^n n^k} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}} \left(1-e^{-t}\right)^n \widehat{a}(t)\, dt$$

$$= \int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}} \sum_{n=1}^{\infty} \frac{(1-e^{-t})^n}{p^n n^k} \widehat{a}(t)\, dt$$

$$= \int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}} \operatorname{Li}_k\left(\frac{1-e^{-t}}{p}\right) \widehat{a}(t)\, dt,$$

the order of $\int_0^{+\infty}$ and $\sum_{n=1}^{\infty}$ may be interchanged since, by the hypothesis on $\widehat{a}$,

$$\left|\widehat{a}(t)\right| \leq Ce^{\alpha t} \quad \text{for all } t \in]0, +\infty[,$$

which gives

$$\int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}} \sum_{n=1}^{\infty} \frac{(1-e^{-t})^n}{p^n n^k} \left|\widehat{a}(t)\right| dt \leq C \int_0^{+\infty} \frac{e^{-t}}{1-e^{-t}} \operatorname{Li}_k\left(\frac{1-e^{-t}}{p}\right) e^{\alpha t}\, dt < +\infty. \qquad \square$$

4. The functions α_k and β_k

Definition 3. Let k be a positive integer. The functions α_k and β_k are respectively defined for all $s \in \mathbb{C}$ with $\Re(s) > 0$ by

$$\alpha_k(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-2t}} \operatorname{Li}_k\left(1-e^{-2t}\right) t^{s-1}\, dt \quad (\text{for } k \geq 1),$$

$$\beta_k(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-2t}} \operatorname{Li}_k\left(\frac{1-e^{-2t}}{2}\right) t^{s-1}\, dt \quad (\text{for } k \geq 0).$$

Example 3.

$$\alpha_1(s) = \frac{2}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-2t}} t^s\, dt = 2^{-s}\left(2^{s+1}-1\right) s\zeta(s+1),$$

$$\beta_1(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{e^{-t}}{1-e^{-2t}} \left[\ln 2 - \ln\left(1+e^{-2t}\right)\right] t^{s-1} dt.$$

Remark 2 *(Link with the Arakawa–Kaneko zeta function).* In [6], we introduced the function $(s,x) \mapsto \xi_k(s,x)$ defined for $\Re(s) > 0$ and $x > 0$ by:

$$\xi_k(s,x) = \frac{1}{\Gamma(s)} \int_0^{+\infty} e^{-xt} \frac{\operatorname{Li}_k(1-e^{-t})}{1-e^{-t}} t^{s-1}\, dt$$

which is a very natural extension of the original Arakawa–Kaneko zeta function in the same way as the Hurwitz zeta function $\zeta(s,x)$ generalizes the Riemann zeta function. In the simplest case $k=1$, $\xi_1(s,x)$ is nothing else than $s\zeta(s+1,x)$, and moreover we deduce immediately from the previous definition the following relation:

$$\alpha_k(s) = 2^{-s}\xi_k\left(s,\frac{1}{2}\right).$$

Proposition 5. *If s is such that $\Re(s)\geq 1$, then*

$$\alpha_k(s) = \sum_{n=1}^{\infty}\frac{1}{n^k}D\left(\frac{1}{(2x-1)^s}\right)(n) \quad (\textit{for } k\geq 1),$$

$$\beta_k(s) = \sum_{n=1}^{\infty}\frac{1}{2^n n^k}D\left(\frac{1}{(2x-1)^s}\right)(n) \quad (\textit{for } k\geq 0).$$

Proof. This is an immediate consequence of formula (5) applied to the function $a(x)=\frac{1}{(2x-1)^s}$ (for $p=1,2$) since $\widehat{a}(t)=e^{\frac{t}{2}}\frac{(\frac{t}{2})^{s-1}}{2\Gamma(s)}$ as already seen in Example 2. □

Corollary 1. *For all integers $m\geq 0$, then*

$$\alpha_k(m+1) = \sum_{n=1}^{\infty}\frac{2^{2n-1}}{\binom{2n}{n}n^{k+1}}P_m\big(O_n^{(1)},\dots,O_n^{(m)}\big) \quad (\textit{for } k\geq 1), \tag{6}$$

$$\beta_k(m+1) = \sum_{n=1}^{\infty}\frac{2^{n-1}}{\binom{2n}{n}n^{k+1}}P_m\big(O_n^{(1)},\dots,O_n^{(m)}\big) \quad (\textit{for } k\geq 0). \tag{7}$$

Proof. This is an immediate consequence of Proposition 5 by formula (3). □

Example 4. Since $\alpha_1(s)=2^{-s}(2^{s+1}-1)s\zeta(s+1)$, then for all integers $m\geq 1$,

$$\big(2-2^{-m}\big)m\zeta(m+1) = \sum_{n=1}^{\infty}\frac{2^{2n-1}}{\binom{2n}{n}n^2}P_{m-1}\big(O_n^{(1)},\dots,O_n^{(m-1)}\big). \tag{8}$$

In particular, for $m=2$, a nice formula for Apéry's constant (cf. [5, p. 81]) is regained:

$$\sum_{n=1}^{\infty}\frac{2^{2n}}{\binom{2n}{n}}\frac{O_n}{n^2} = 7\zeta(3). \tag{9}$$

Example 5. Since $\beta_0(s)=\beta(s)$, one has for all integers $m\geq 1$,

$$\beta(2m) = \sum_{n=1}^{\infty}\frac{2^{n-1}}{\binom{2n}{n}n}P_{2m-1}\big(O_n^{(1)},\dots,O_n^{(2m-1)}\big). \tag{10}$$

In particular, for $m = 1$, a nice formula for Catalan's constant (cf. [2, p. 293, Entry 34]) is regained:

$$\sum_{n=1}^{\infty} \frac{2^n}{\binom{2n}{n}} \frac{O_n}{n} = 2G, \tag{11}$$

and for $m = 2$, formula (10) is translated into

$$\sum_{n=1}^{\infty} \frac{2^n}{\binom{2n}{n}} \frac{(O_n)^3}{n} + 3\sum_{n=1}^{\infty} \frac{2^n}{\binom{2n}{n}} \frac{O_n O_n^{(2)}}{n} + 2\sum_{n=1}^{\infty} \frac{2^n}{\binom{2n}{n}} \frac{O_n^{(3)}}{n} = 12\beta(4). \tag{12}$$

Remark 3. In a similar way (cf. [3, §5.5]), one can prove for $\xi_k(s) := \xi_k(s,1)$ the following identity:

$$\xi_k(s) = \sum_{n=1}^{\infty} \frac{1}{n^k} D\left(\frac{1}{x^s}\right)(n) \quad \big(\text{for } \Re(s) \geq 1 \text{ and } k \geq 1\big),$$

and, furthermore, one has (cf. [3, §3])

$$D\left(\frac{1}{x^{m+1}}\right)(n) = \frac{P_m(H_n^{(1)}, \ldots, H_n^{(m)})}{n} \quad \text{with } H_n^{(j)} = \sum_{k=1}^{n} \frac{1}{k^j} \ (j = 1, 2, \ldots, m),$$

which gives for instance Euler's famous identity:

$$\xi_1(2) = \sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\zeta(3).$$

4.1. The function β_1

The Euler series transformation (Proposition 3 above) provides an alternative expression for β_1.

Proposition 6. *For all $s \in \mathbb{C}$ with $\Re(s) \geq 1$, one has*

$$\beta_1(s) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{O_n^{(s)}}{n},$$

hence, for each integer $m \geq 1$,

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{O_n^{(m)}}{n} = \sum_{n=1}^{\infty} \frac{2^{n-1}}{\binom{2n}{n} n^2} P_{m-1}\big(O_n^{(1)}, \ldots, O_n^{(m-1)}\big). \tag{13}$$

Proof. By (4), one has for all $|z| < \frac{1}{2}$,

$$\sum_{n=1}^{+\infty} \frac{D(a)(n)}{n} z^n = -\sum_{n=1}^{+\infty} \frac{1}{n} S(a)(n) \left(\frac{z}{z-1}\right)^n.$$

If the series $\sum_{n=1}^{+\infty} \frac{D(a)(n)}{n} \frac{1}{2^n}$ is convergent, then, by the classical Abel lemma, we get

$$\sum_{n=1}^{+\infty} \frac{D(a)(n)}{n} \frac{1}{2^n} = \sum_{n=1}^{+\infty} \frac{1}{n} S(a)(n)(-1)^{n-1}.$$

By formula (5), the series $\sum_{n=1}^{\infty} \frac{1}{2^n n} D(\frac{1}{(2x-1)^s})(n)$ is convergent and

$$\beta_1(s) = \sum_{n=1}^{\infty} \frac{1}{2^n n} D\left(\frac{1}{(2x-1)^s}\right)(n) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n} S\left(\frac{1}{(2x-1)^s}\right)(n).$$

Then, using formula (1) for $D(\frac{1}{(2x-1)^m})(n)$, one obtains (13). □

Example 6.

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{O_n}{n} = \sum_{n=1}^{\infty} \frac{2^{n-1}}{\binom{2n}{n}} \frac{1}{n^2} = \frac{\pi^2}{16},$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{O_n^{(2)}}{n} = \sum_{n=1}^{\infty} \frac{2^{n-1}}{\binom{2n}{n}} \frac{O_n}{n^2} = \frac{7}{4}\zeta(3) - \frac{\pi}{2} G$$

$$\left([7, (2.36) \text{ and } (2.37)] \text{ with } u = 2 \text{ and } \theta = \frac{\pi}{2}\right), \tag{14}$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{O_n^{(3)}}{n} = \sum_{n=1}^{\infty} \frac{2^n}{\binom{2n}{n}} \frac{(O_n)^2}{(2n)^2} + \sum_{n=1}^{\infty} \frac{2^n}{\binom{2n}{n}} \frac{O_n^{(2)}}{(2n)^2} = \frac{\pi^4}{64} - G^2$$

$$\left([7, (2.38), (2.39), (2.40) \text{ and } (C.4)] \text{ with } u = 2 \text{ and } \theta = \frac{\pi}{2}\right). \tag{15}$$

Remark 4. In complete analogy with (13), one also has the following relation

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{H_n^{(m)}}{n} = \sum_{n=1}^{\infty} \frac{1}{2^n n^2} P_{m-1}\left(H_n^{(1)}, \ldots, H_n^{(m-1)}\right) \tag{16}$$

which is equivalent to that given by Choi and Srivastava [4, p. 66, formula (4.29)].

Proposition 7. *Let*

$$\widetilde{H}_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}.$$

If s is such that $\Re(s) > 1$, then

$$\sum_{n=1}^{\infty} \frac{\widetilde{H_n}}{(2n-1)^s} = \left(1 - 2^{-s}\right)\zeta(s)\ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(2n-1)^s} - \beta_1(s). \tag{17}$$

Thus, for each integer $m > 1$,

$$\sum_{n=1}^{\infty} \frac{\widetilde{H}_n}{(2n-1)^m} = \left(1 - 2^{-m}\right)\zeta(m)\ln 2 + (-1)^m \ln 2 + 2\sum_{k=1}^{m}(-1)^{m-k}\beta(k) - \beta_1(m). \tag{18}$$

Proof. The first relation is a direct consequence of the following elementary result:

If the series $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$, $\sum_{n=1}^{\infty} a_n b_n$ and $\sum_{n=1}^{\infty} b_n \sum_{k=1}^{n} a_k$ are convergent, then the series $\sum_{n\geq 1} a_n \sum_{k=1}^{n} b_k$ is convergent and we have

$$\sum_{n=1}^{\infty} a_n \sum_{k=1}^{n} b_k = \sum_{n=1}^{\infty} a_n \sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} a_n b_n - \sum_{n=1}^{\infty} b_n \sum_{k=1}^{n} a_k.$$

Applied to $a_n = \frac{1}{(2n-1)^s}$ and $b_n = \frac{(-1)^{n-1}}{n}$, this relation gives (17). The second relation is a consequence of the first one by the following observation:

If s is an integer, $s = m$, then we have

$$\sum_{k=1}^{m} \frac{(-1)^k}{(2n-1)^k} = \frac{(-1)^m (2n-1)^{-m} - 1}{2n},$$

hence

$$\frac{(-1)^{n-1}}{n(2n-1)^m} = (-1)^m \frac{(-1)^{n-1}}{n} + 2\sum_{k=1}^{m}(-1)^{m-k}\frac{(-1)^{n-1}}{(2n-1)^k}$$

which gives formula (18). □

Example 7. Formula (18) gives respectively for $m = 2$ and $m = 3$ the following identities:

$$\sum_{n=1}^{\infty} \frac{\widetilde{H}_n}{(2n-1)^2} = \frac{\pi^2}{8}\ln 2 + \ln 2 - \frac{\pi}{2} + 2G - \frac{7}{4}\zeta(3) + \frac{\pi G}{2}, \tag{19}$$

$$\sum_{n=1}^{\infty} \frac{\widetilde{H}_n}{(2n-1)^3} = \frac{7}{8}\zeta(3)\ln 2 - \ln 2 + \frac{\pi}{2} - 2G + \frac{\pi^3}{16} + G^2 - \frac{\pi^4}{64}. \tag{20}$$

5. The values $\alpha_k(1)$ and $\beta_k(1)$

The special values of α_k and β_k at $s = 1$ are evaluated in terms of (generalized) log-sine functions (cf. [7,9]).

Proposition 8. *For each integer $k \geq 1$ and $\alpha \in \mathbb{R}$ such that $0 \leq \alpha \leq 1$, let $\mathrm{L}_k(\alpha)$ be the log-sine-type integral*[1]*:*

$$\mathrm{L}_k(\alpha) = \int_0^{\alpha\pi} u \ln^{k-1}\left(2\sin\frac{u}{2}\right) du = \pi^2 \int_0^{\alpha} x \ln^{k-1}\left(2\sin\frac{\pi x}{2}\right) dx.$$

Then, one has

$$2\alpha_k(1) = \sum_{n=1}^{\infty} \frac{2^{2n}}{\binom{2n}{n}} \frac{1}{n^{k+1}} = 2^{k-1} \sum_{i=1}^{k} (-1)^{i-1} \frac{(\ln 2)^{k-i}}{(i-1)!(k-i)!} \mathrm{L}_i(1), \tag{21}$$

$$2\beta_k(1) = \sum_{n=1}^{\infty} \frac{2^{n}}{\binom{2n}{n}} \frac{1}{n^{k+1}} = \sum_{i=1}^{k} (-1)^{i-1} \frac{2^{i-1}(\ln 2)^{k-i}}{(i-1)!(k-i)!} \mathrm{L}_i\left(\frac{1}{2}\right). \tag{22}$$

Proof. The proof is similar to that given in [13, §4]. Let

$$J_k(x) = \frac{1}{2^k} \sum_{n=1}^{\infty} \frac{(2x)^{2n}}{\binom{2n}{n} n^{k+1}}.$$

Then, one has for $k \geq 1$,

$$J_k(x) = \int_0^x \frac{J_{k-1}(u)}{u} du.$$

By a classical identity due to Euler (cf. [13,14]), one also has

$$J_1(x) = (\arcsin x)^2,$$

hence

$$J_0(x) = \frac{2x \arcsin x}{\sqrt{1-x^2}}.$$

It is easily verified that

[1] With the notation of [9], $\mathrm{L}_k(\alpha)$ is $-\mathrm{Ls}_{k+1}^{(1)}(\alpha\pi)$.

$$2\alpha_k(1) = 2^k J_k(1) = 2^k \int_0^1 \frac{J_{k-1}(x)}{x} dx,$$

and

$$2\beta_k(1) = 2^k J_k\left(\frac{\sqrt{2}}{2}\right) = 2^k \int_0^{\frac{\sqrt{2}}{2}} \frac{J_{k-1}(x)}{x} dx.$$

By $(k-1)$ integrations by parts and the change of variable $x = \sin\frac{u}{2}$, we get

$$J_k(1) = \frac{(-1)^{k-1}}{(k-1)!} \int_0^1 \ln^{k-1}(x) \frac{J_0(x)}{x} dx = \frac{(-1)^{k-1}}{2(k-1)!} \int_0^\pi u \ln^{k-1}\left(\sin\frac{u}{2}\right) du,$$

and

$$J_k\left(\frac{\sqrt{2}}{2}\right) = \frac{(-1)^{k-1}}{(k-1)!} \int_0^{\frac{\sqrt{2}}{2}} \ln^{k-1}(\sqrt{2}x) \frac{J_0(x)}{x} dx = \frac{(-1)^{k-1}}{2(k-1)!} \int_0^{\frac{1}{2}\pi} u \ln^{k-1}\left(\sqrt{2}\sin\frac{u}{2}\right) du.$$

It remains to use the binomial expansions of

$$\ln^{k-1}\left(\sin\frac{u}{2}\right) = \left[\ln\left(\frac{1}{2}\right) + \ln\left(2\sin\frac{u}{2}\right)\right]^{k-1},$$

and

$$\ln^{k-1}\left(\sqrt{2}\sin\frac{u}{2}\right) = \left[\ln\left(\frac{1}{\sqrt{2}}\right) + \ln\left(2\sin\frac{u}{2}\right)\right]^{k-1}$$

to obtain formulas (21) and (22). □

Proposition 9. *For all α such that $0 \le \alpha \le 1$, we have*

$$\text{a)} \quad \mathrm{L}_1(\alpha) = \frac{\pi^2}{2}\alpha^2,$$

$$\text{b)} \quad \mathrm{L}_2(\alpha) = \zeta(3) - \sum_{n=1}^{\infty} \frac{\cos(\pi n\alpha)}{n^3} - \alpha\pi \sum_{n=1}^{\infty} \frac{\sin(\pi n\alpha)}{n^2},$$

$$\text{c)} \quad \mathrm{L}_3(\alpha) = \frac{\pi^4}{16}\left(\alpha^4 - \frac{8}{3}\alpha^3 + 2\alpha^2\right) + 2\sum_{n=1}^{\infty} \frac{H_n}{(n+1)^3} \cos\big(\pi(n+1)\alpha\big) + 2\pi\alpha \sum_{n=1}^{\infty} \frac{H_n}{(n+1)^2} \sin\big(\pi(n+1)\alpha\big) - \frac{1}{2}\zeta(4). \tag{23}$$

Proof. The assertion a) is trivially verified and b) is a classical identity (cf. [9, formula (7.53)]). It remains to prove c). We use the following expansion:

$$\operatorname{Log}^2(1-z) = 2\sum_{n=1}^{\infty} H_n \frac{z^{n+1}}{n+1}$$

to get

$$\operatorname{Log}^2\bigl(1-e^{-i\pi x}\bigr) = 2\sum_{n=1}^{\infty} H_n \frac{e^{-i\pi x(n+1)}}{n+1}.$$

Since

$$\begin{aligned}\operatorname{Log}^2\bigl(1-e^{-i\pi x}\bigr) &= \operatorname{Log}^2\bigl(e^{-i\pi x/2}\bigl(e^{i\pi x/2}-e^{-i\pi x/2}\bigr)\bigr)\\ &= \left(-i\pi x/2 + i\pi/2 + \ln\left(2\sin\frac{1}{2}x\pi\right)\right)^2\\ &= -\frac{\pi^2}{4}(x-1)^2 + \ln^2\left(2\sin\frac{\pi x}{2}x\right) + i\Im\bigl(\operatorname{Log}^2\bigl(1-e^{-i\pi x}\bigr)\bigr),\end{aligned}$$

one has

$$\ln^2\left(2\sin\frac{\pi x}{2}\right) = \frac{\pi^2}{4}(x-1)^2 + \Re\bigl(\operatorname{Log}^2\bigl(1-e^{-i\pi x}\bigr)\bigr),$$

hence

$$\ln^2\left(2\sin\frac{\pi x}{2}\right) = \frac{\pi^2}{4}(x-1)^2 + 2\sum_{n=1}^{\infty} H_n \frac{\cos(\pi(n+1)x)}{n+1}.$$

Integrating, this gives

$$\begin{aligned}\int_0^{\alpha} x\ln^2\left(2\sin\frac{\pi x}{2}\right)dx &= \frac{\pi^2}{4}\int_0^{\alpha} x(x-1)^2\,dx\\ &\quad + 2\sum_{n=1}^{\infty}\frac{H_n}{n+1}\int_0^{\alpha} x\cos\bigl(\pi(n+1)x\bigr)\,dx. \qquad (24)\end{aligned}$$

The permutation of $\sum$ and $\int$ in (24) is justified by the following Lemma 2 and the dominated convergence theorem. The integrals in the right-hand side of (24) are easily computed by

$$\int_0^{\alpha} x(x-1)^2\,dx = \frac{\alpha^4}{4} - \frac{2\alpha^3}{3} + \frac{\alpha^2}{2},$$

and

$$\int_0^\alpha x\cos\bigl(\pi(n+1)x\bigr)dx = \frac{\cos(\pi(n+1)\alpha)}{\pi^2(n+1)^2} + \frac{\alpha}{\pi(n+1)}\sin\bigl(\pi(n+1)\alpha\bigr) - \frac{1}{\pi^2(n+1)^2}.$$

Thus, we deduce from (24) the following expression for $\mathrm{L}_3(\alpha)$:

$$\mathrm{L}_3(\alpha) = \frac{\pi^4}{4}\left(\frac{\alpha^4}{4} - \frac{2\alpha^3}{3} + \frac{\alpha^2}{2}\right) + 2\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^3}\cos\bigl(\pi(n+1)\alpha\bigr) + 2\pi\alpha\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^2}\sin\bigl(\pi(n+1)\alpha\bigr) - 2\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^3}.$$

Moreover, one has

$$\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^3} = \sum_{n=1}^{\infty}\frac{H_n}{n^3} - \zeta(4) = \frac{5}{4}\zeta(4) - \zeta(4) = \frac{1}{4}\zeta(4),$$

and this gives (23). □

Lemma 2. *The partial sums*

$$\sum_{n=1}^{k} H_n\frac{x\cos(\pi(n+1)x)}{n+1}$$

are uniformly bounded for $x \in]0,1[$.

Proof. Let $S_n(x) = x\sum_{j=1}^{n}\cos(\pi(j+1)x)$. A summation by parts gives

$$\sum_{n=1}^{k}\frac{H_n}{n+1}x\cos\bigl(\pi(n+1)x\bigr) = \sum_{n=1}^{k}S_n(x)\left(\frac{H_n}{n+1} - \frac{H_{n+1}}{n+2}\right) + \frac{H_k}{k+1}S_k(x),$$

and one has

$$\bigl|S_n(x)\bigr| = \left|x\sum_{j=1}^{n}\cos\bigl(\pi(j+1)x\bigr)\right| \le \frac{2x}{\sin(\pi x/2)}.$$

It follows that, for all $x \in]0,1[$,

$$\left|\sum_{n=1}^{k} xH_n\frac{\cos(\pi(n+1)x)}{n+1}\right| \le \frac{2x}{\sin(\pi x/2)}\left(\frac{H_1}{2} - \frac{H_{k+1}}{k+2} + \frac{H_k}{k+1}\right) \le \frac{Cx}{\sin(\pi x/2)} \le C'. \qquad \square$$

Example 8. Formulae (21) and (22) give for $k = 2$ the following identities:

$$\sum_{n=1}^{\infty} \frac{2^{2n}}{\binom{2n}{n}} \frac{1}{n^3} = \pi^2 \ln 2 - \frac{7}{2}\zeta(3),$$

$$\sum_{n=1}^{\infty} \frac{2^{n}}{\binom{2n}{n}} \frac{1}{n^3} = \frac{\pi^2}{8} \ln 2 + \pi G - \frac{35}{16}\zeta(3),$$

which were known of Ramanujan (cf. [2, p. 269]).

6. New formulae for Ramanujan's constant $G(1)$

In Chapter 9 of his notebooks (cf. [2, p. 255, Entry 11]), Ramanujan introduced two generating functions[2]:

$$F(x) := \sum_{n=1}^{\infty} \frac{O_n x^{2n}}{(2n)^2} \quad \text{and} \quad G(x) := \sum_{n=1}^{\infty} \frac{O_n x^{2n}}{(2n)^3},$$

then, he writes the following functional relation:

$$G(x) + G\left(\frac{1-x}{1+x}\right) = F(x)\log(x) + F\left(\frac{1-x}{1+x}\right)\log\left(\frac{1-x}{1+x}\right) - \frac{1}{16}\log^2(x)\log^2\left(\frac{1-x}{1+x}\right) + C, \tag{ii}$$

with

$$C = \frac{\pi}{4}\sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+1)^3} - \frac{\pi}{3\sqrt{3}}\sum_{n=0}^{\infty} \frac{1}{(2n+1)^3}.$$

Unfortunately, this beautiful formula for C given by Ramanujan turns out to be erroneous since, letting x tend to 1 in (ii), one sees easily that the constant C must be equal to

$$G(1) = \sum_{n=1}^{\infty} \frac{O_n}{(2n)^3} \quad \text{(cf. [2, p. 257], or [11] for more details).}$$

However, the calculation of $\alpha_3(1)$ and $\alpha_2(2)$ provides two interesting formulae for the constant $G(1)$.

[2] These functions are respectively denoted by ϕ and ψ in the original manuscript: cf. [10, p. 108].

Proposition 10. *Let* $G(1)=\sum_{n=1}^{\infty}\frac{O_n}{(2n)^3}$ *be the Ramanujan constant. One has*

$$G(1)=\frac{7}{8}\zeta(3)\ln 2-\frac{\pi^4}{384}-\frac{1}{8}\pi^2(\ln 2)^2+2\sum_{n=1}^{\infty}\frac{2^{2n}}{\binom{2n}{n}}\frac{1}{(2n)^4}. \tag{25}$$

Proof. One has

$$\begin{aligned}
G(1)=\sum_{n=1}^{\infty}\frac{O_n}{(2n)^3}&=\sum_{n=1}^{\infty}\frac{H_{2n}-\frac{1}{2}H_n}{(2n)^3}\\
&=\sum_{n=1}^{\infty}\frac{H_{2n}}{(2n)^3}-\sum_{n=1}^{\infty}\frac{1}{2}\frac{H_n}{(2n)^3}\\
&=\sum_{n=1}^{\infty}\frac{1+(-1)^n}{2}\frac{H_n}{n^3}-\sum_{n=1}^{\infty}\frac{1}{2^4}\frac{H_n}{n^3}\\
&=\frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n}{n^3}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^nH_n}{n^3}-\frac{1}{2^4}\sum_{n=1}^{\infty}\frac{H_n}{n^3}\\
&=\frac{35}{64}\zeta(4)-\frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^3}.
\end{aligned}$$

Moreover, by (23) with $\alpha=1$, one also has

$$\begin{aligned}
\mathrm{L}_3(1)&=\frac{25}{8}\zeta(4)-2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{H_n}{n^3}\\
&=\frac{\pi^4}{96}+4G(1).
\end{aligned}\tag{26}$$

Then, applying formula (21) with $k=3$, it results from (26) that

$$2\alpha_3(1)=\sum_{n=1}^{\infty}\frac{2^{2n}}{\binom{2n}{n}}\frac{1}{n^4}=\pi^2(\ln 2)^2-7\zeta(3)\ln 2+\frac{\pi^4}{48}+8G(1) \tag{27}$$

and this relation is equivalent to (25). □

Remark 5. We have seen before (cf. Example 4, formula (9)) that

$$\alpha_1(2)=\sum_{n=1}^{\infty}\frac{2^{2n-1}}{\binom{2n}{n}}\frac{O_n}{n^2}=\frac{7}{2}\zeta(3),$$

and one also knows (cf. [2, p. 259]) that

$$4F(1)=\sum_{n=1}^{\infty}\frac{O_n}{n^2}=\frac{7}{4}\zeta(3).$$

Thus

$$F(1) = \sum_{n=1}^{\infty} \frac{O_n}{(2n)^2} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{2^{2n}}{\binom{2n}{n}} \frac{O_n}{(2n)^2}. \tag{28}$$

The calculation of $\alpha_2(2)$ provides a nice expression of the Ramanujan constant $G(1)$ similar to (28).

Proposition 11. *Let $G(1) = \sum_{n=1}^{\infty} \frac{O_n}{(2n)^3}$ be the Ramanujan constant. One has the following formula:*

$$G(1) = \frac{7}{8}\zeta(3)\ln 2 - \frac{\pi^4}{256} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{2^{2n}}{\binom{2n}{n}} \frac{O_n}{(2n)^3}. \tag{29}$$

Proof. Applying [7, (2.44) and (2.45)] with $u = 4$ and $\theta = \pi$, one obtains, after calculations, an expression of $\alpha_2(2)$ involving $\mathrm{L}_3(1)$ which may be simplified using formula (26). Finally, we get the following relation:

$$\alpha_2(2) = 7\zeta(3)\ln 2 - \frac{\pi^4}{32} - 8G(1) \tag{30}$$

and this relation is equivalent to (29). □

Remark 6. Since $\frac{7}{8}\zeta(3) = \sum_{n\geq 0} \frac{1}{(2n+1)^3}$ and $\frac{\pi^3}{32} = \sum_{n\geq 0} \frac{(-1)^n}{(2n+1)^3}$, formula (29) may be rewritten as

$$G(1) = 2\ln(2) \sum_{n\geq 0} \frac{1}{(4n+1)^3} - \big(\pi/8 + \ln(2)\big) \sum_{n\geq 0} \frac{(-1)^n}{(2n+1)^3} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{2^{2n}}{\binom{2n}{n}} \frac{O_n}{(2n)^3} \tag{31}$$

which gets closer to the erroneous formula given by Ramanujan for C.

7. Conclusion

With the aim of defining a natural framework for the study of the special values of zeta functions of Arakawa–Kaneko-type, we were led to consider polylogarithmic series in the generic form:

$$F_k(a, z) = \sum_{n=1}^{\infty} D(a)(n) \frac{z^n}{n^k} \quad \text{with } k \in \mathbb{N},\ z \in \mathbb{C}, \quad \text{and}$$

$$D(a)(n+1) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} a(j+1).$$

We studied the important cases $a(x) = x^{-s}$ and $a(x) = (2x-1)^{-s}$ for $z = 1$ and $z = 1/2$. Though limited in practice to small values of k and s, our approach provided plenty of nice formulae with interest both in number theory and in physics.

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