New identities involving Cauchy numbers, harmonic numbers and zeta values

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Abstract In this article, we present a class of identities linking together Cauchy numbers, the special values of the Riemann zeta function and its derivative, and a generalization of the Roman harmonic numbers, which represents a significant refinement and improvement of our earlier work on the subject.

Keywords Cauchy numbers; Roman harmonic numbers; binomial identities; series with zeta values; Ramanujan summation of series.

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1 Introduction

Several years ago, we introduced a method based on the Ramanujan summation of series which enabled us to generate a number of interesting identities linking together Cauchy numbers (also known as Bernoulli numbers of the second kind), harmonic numbers and values of the Riemann zeta function at positive integers [4]. Thanks to new formulas recently proved in our last paper [6], we can refine and improve these earlier results by providing new closed form evaluations of the same kind (see Proposition 1 and Corollary 1). In order to do this, we consider a natural generalization of the Roman harmonic numbers that were first introduced in [5] (see Definition 2), and use a general transformation formula that links the Cauchy numbers to the Ramanujan summation of series [3, Theorem 18]. A noteworthy fact is the presence in most of our formulas of certain alternating series with zeta

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values (see Definition 3) that recently appeared in different contexts [1, 6, 7]. In the aim to help the reader to find his way among our various formulas, a summary of the most noteworthy identities, ranked in ascending order of complexity, is given in the penultimate section of the article.

2 Preliminaries : reminder of the main definitions and results

We first recall various definitions and results that appeared in our previous work, refering to the indicated references for the proof of these results.

2.1 Harmonic numbers

Definition 1. The generalized harmonic numbers $H_n^{(r)}$ are defined for non-negative integers n and r by

$$H_0^{(r)} = 0 \quad \text{and} \quad H_n^{(r)} = \sum_{j=1}^{n} \frac{1}{j^r} \quad \text{for } n \geq 1\,. \tag{1}$$

For $r = 1$, they reduce to classical harmonic numbers $H_n = H_n^{(1)}$. The sums

$$\mathcal{S}_{r,p} = \sum_{n=1}^{\infty} \frac{H_n^{(r)}}{n^p}$$

for positive integers $p \geq 2$ are called linear Euler sums. We recall Euler's celebrated formula for $\mathcal{S}_{1,p}$ [11, Eq. (3.6)]:

$$2\mathcal{S}_{1,p} = (p+2)\zeta(p+1) - \sum_{j=1}^{p-2} \zeta(p-j)\zeta(j+1) \qquad (p \geq 2)\,.$$

Definition 2 ([5])**.** The generalized Roman harmonic numbers $H_{n,k}^{(r)}$ are defined for non-negative integers n, r and k with $n \geq 1$, $r \geq 1$ and $k \geq 0$ by

$$H_{n,0}^{(r)} = \frac{1}{n^{r-1}} \quad \text{and} \quad H_{n,k}^{(r)} = \sum_{n \geq j_1 \geq \cdots \geq j_k \geq 1} \frac{1}{j_1\, j_2 \cdots j_k^r} \quad \text{for } k \geq 1\,. \tag{2}$$

In particular, for $k = 1$, they reduce to classical generalized harmonic numbers

$$H_{n,1}^{(r)} = H_n^{(r)}\,,$$

and for $k=2$, we have the expression

$$H_{n,2}^{(r)} = \sum_{j=1}^{n} \frac{H_j^{(r)}}{j} \,.$$

For $r=1$, the harmonic numbers $H_{n,k}^{(1)}$ will be noted $H_{n,k}$ in the remainder of the article.

Remark 1. The harmonic numbers $H_{n,k}$ are nothing else than the ordinary Roman harmonic numbers introduced three decades ago by Roman, Loeb and Rota (see [10] and the references to the original papers given inside). It is well-known (see e.g. [4, Eq. (18)], [10, Eq. (29)]) that the Roman harmonic number $H_{n,k} = H_{n,k}^{(1)}$ can be expressed as a polynomial in the generalized harmonic numbers $H_n^{(1)}, \cdots, H_n^{(k)}$. In particular, we have the expressions

$$H_{n,2} = \sum_{j=1}^{n} \frac{H_j}{j} = \frac{1}{2}(H_n)^2 + \frac{1}{2} H_n^{(2)} \,, \tag{3}$$

and

$$H_{n,3} = \frac{1}{6}(H_n)^2 + \frac{1}{2} H_n H_n^{(2)} + \frac{1}{3} H_n^{(2)} \tag{4}$$

that will be useful later on.

In addition, by Abel's lemma, we have the reciprocity formula (cf. [8, Eq. (2.2)])

$$\sum_{j=1}^{n} \frac{H_j^{(r)}}{j} + \sum_{j=1}^{n} \frac{H_j}{j^r} = H_n H_n^{(r)} + H_n^{(r+1)} \qquad (r \geq 1)$$

which leads to the useful relation

$$H_{n,2}^{(r)} = H_n H_n^{(r)} + H_n^{(r+1)} - \sum_{j=1}^{n} \frac{H_j}{j^r} \qquad (r \geq 1) \,. \tag{5}$$

The generalized Roman harmonic numbers $H_{n,k}^{(r)}$ also verify the following binomial identity [5, Eq. (4.7)]:

$$H_{n,k}^{(r)} = \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} \frac{H_{j,r-1}}{j^k} \qquad (k \geq 1) \,. \tag{6}$$

By inverse binomial transform[1], formula (6) also admits a reciprocal:

$$\frac{H_{n,r-1}}{n^k} = \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} H_{j,k}^{(r)} \qquad (k \geq 1) \,. \tag{7}$$

1. If $b(n) = \frac{1}{n} \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} j\, a(j)$, then $a(n) = \frac{1}{n} \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} j\, b(j)$ (cf. [5, Definition 5 and Corollary 1]).

Example 1. For $r = 1$, formula (6) reduces to

$$H_{n,k} = \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} \frac{1}{j^k}$$

which is a classical property of Roman harmonic numbers [10, Eq. (20)], and for $r = 2$, it may be written

$$H_{n,k}^{(2)} = \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} \frac{H_j}{j^k}.$$

In particular, we have

$$H_n^{(2)} = \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} \frac{H_j}{j}$$

and

$$\sum_{j=1}^{n} \frac{H_j^{(2)}}{j} = \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} \frac{H_j}{j^2}.$$

2.2 Alternating series with zeta values

Definition 3 ([1, 6, 7])**.** The numbers τ_p are defined for all positive integers p by the series representation

$$\tau_p = \sum_{k=1}^{\infty} (-1)^{k+p} \frac{\zeta(k+p)}{k}. \tag{8}$$

For $p \geq 2$, they verify the following identity [1, Proposition 7], [6, Lemma 1]:

$$\sum_{n=1}^{\infty} \frac{\ln(n+1)}{n^p} = -\zeta'(p) - (-1)^p \tau_p. \tag{9}$$

For $p = 1$, we have [1, Theorem 2 (a)]

$$\tau_1 = \int_0^1 \frac{\psi(x+1) + \gamma}{x} \, dx,$$

where ψ is the digamma function and $\gamma = -\psi(1)$ is Euler's constant.

2.3 Cauchy numbers

Definition 4 ([2, 4, 9])**.** The classical Cauchy numbers c_n are defined for $n \geq 1$ by

$$c_n = \int_0^1 x(x-1)\cdots(x-n+1)\,dx\,.$$

The Cauchy numbers alternate in sign. Using the notations introduced in [4], we consider the sequence $\{\lambda_n\}_n$ of non-alternating Cauchy numbers defined by

$$\lambda_n = (-1)^{n-1}c_n \qquad (n \geq 1)\,.$$

The first terms of the sequence are the following:

$$\lambda_1 = \frac{1}{2}\,,\ \lambda_2 = \frac{1}{6}\,,\ \lambda_3 = \frac{1}{4}\,,\ \lambda_4 = \frac{19}{30}\,,\ \lambda_5 = \frac{9}{4}\,,\ \lambda_6 = \frac{863}{84}\,,\ \text{etc.}$$

3 Series with Cauchy numbers and Roman harmonic numbers

We first recall the transformation formula [3, Theorem 18] that links the Cauchy numbers to the Ramanujan summation of series: if a is a function analytic in the half-plane $P = \{\mathrm{Re}(z) > 0\}$ such that there exists a constant $C > 0$ with

$$|a(z)| < C\,2^{|z|} \quad \text{for all } z \in P\ ,$$

then

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} \sum_{j=1}^{n} (-1)^{j-1}\binom{n}{j}\,j\,a(j) = \sum_{n\geq 1}^{\mathcal{R}} a(n) \tag{10}$$

where $\sum_{n\geq 1}^{\mathcal{R}}$ denotes the $\mathcal{R}$-sum of the series i.e. the sum of the series in the sense of Ramanujan's summation method [3, 4]. In particular, using formula (6) and its reciprocal (7), we deduce from (10) the following reciprocal identities:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} H_{n,k}^{(r)} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_{n,r-1}}{n^{k+1}} \qquad (k \geq 0,\ r \geq 1). \tag{11}$$

and

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n^{k+1}} H_{n,r-1} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_{n,k}^{(r)}}{n} \qquad (k \geq 0,\ r \geq 1). \tag{12}$$

We can now state our main result.

Proposition 1. *For any integer $p \geq 2$, we have the following identities:*

a)

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} H_{n,p-1} = \zeta(p) - \frac{1}{p-1}\,. \tag{13}$$

b)

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n^2} H_{n,p-1} = \gamma\zeta(p) + \zeta(p+1) - \mathcal{S}_{1,p} - \zeta'(p) - (-1)^p\tau_p - \sigma_p \tag{14}$$

with

$$\sigma_p = \frac{1+(-1)^p}{p} + \sum_{j=1}^{p-2}(-1)^j\zeta(p-j)\left\{\frac{(j-1)!(p-1-j)!}{(p-1)!} - \frac{1}{j}\right\}.$$

c)

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} H^{(2)}_{n,p-1} = \mathcal{S}_{1,p} - \sum_{j=1}^{p-2}\frac{(-1)^{p-j}}{j}\,\zeta(p-j) + (-1)^p\zeta'(p) + \tau_p\,. \tag{15}$$

Proof. Applying (12) with $r=p$ gives for $k=0$

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} H_{n,p-1} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H^{(p)}_{n,0}}{n} = \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n^p} = \zeta(p) - \frac{1}{p-1}$$

(cf. [3, Eq. (1.22)]), and for $k=1$

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n^2} H_{n,p-1} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H^{(p)}_{n}}{n}\,.$$

Hence formula (14) results from [6, Eq. (11)]. Specializing(11) with $k=p-1$ gives for $r=2$

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} H^{(2)}_{n,p-1} = \sum_{n\geq 1}^{\mathcal{R}} \frac{H_n}{n^p}\,.$$

Hence formula (15) results from [6, Eq. (7)]. □

Summing (14) and (15), we obtain the following corollary:

Corollary 1. *For any even integer $p\geq 2$, we have the relation*

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n^2}\left\{H_{n,p-1} + nH^{(2)}_{n,p-1}\right\} = \gamma\zeta(p) + \zeta(p+1) - \sum_{j=0}^{p-2}(-1)^j a_{j,p}\,\zeta(p-j) - \frac{2}{p} \tag{16}$$

with

$$a_{0,p} = 0\,, \quad \textit{and} \quad a_{j,p} = \frac{(j-1)!(p-1-j)!}{(p-1)!} \quad \textit{for } j\geq 1\,.$$

Remark 2. a) Since $H_{n,0} = 1$, formulae (13) and (14) also extend to the case $p = 1$ for which they reduce to the identities

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} = \gamma$$

(cf. [2, 4, 9]), and

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n^2} = \frac{1}{2}\gamma^2 - \frac{1}{2}\zeta(2) + \gamma_1 + \tau_1$$

where γ_1 is the first Stieltjes constant (cf. [6, Eq. (14)]).

b) Taking into account the identity (9), formula (14) can be seen as a refinement of [4, Eq. (27)].

Example 2. For the first values of p, formulas (13), (14) and (15) translate respectively into the following identities:

1) For $p = 2$,

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} H_n = \zeta(2) - 1\,, \tag{17}$$

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n^2} H_n = \gamma\zeta(2) - \zeta(3) - \zeta'(2) - \tau_2 - 1\,, \tag{18}$$

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} H_n^{(2)} = 2\zeta(3) + \zeta'(2) + \tau_2\,. \tag{19}$$

2) For $p = 3$,

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} \sum_{j=1}^{n} \frac{H_j}{j} = \zeta(3) - \frac{1}{2}\,, \tag{20}$$

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n^2} \sum_{j=1}^{n} \frac{H_j}{j} = \gamma\zeta(3) - \frac{1}{4}\zeta(4) - \frac{1}{2}\zeta(2) - \zeta'(3) + \tau_3\,, \tag{21}$$

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} \sum_{j=1}^{n} \frac{H_j^{(2)}}{j} = \frac{5}{4}\zeta(4) - \zeta(2) - \zeta'(3) + \tau_3\,. \tag{22}$$

Remark 3. a) Formula (17) is a well-known series representation for $\zeta(2)$ which seems to appear for the first time in [9] (see also [2] and [4]).

b) By means of (3), formula (20) can be rewritten under the following equivalent form:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n} \left\{ (H_n)^2 + H_n^{(2)} \right\} = 2\zeta(3) - 1 \tag{23}$$

which coincides with [2, Eq. (9)]. Subtracting (19) from (23) enables to deduce easily another interesting identity:

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n} (H_n)^2 = -\zeta'(2) - \tau_2 - 1\,. \tag{24}$$

c) Subtracting (18) from (24) and replacing $H_n - \frac{1}{n}$ by H_{n-1} inside this expression leads to the surprisingly simple relation

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n} H_n\, H_{n-1} = \zeta(3) - \gamma\zeta(2)\,. \tag{25}$$

d) By means of [6, Eq. (15)] we can also prove the following dual formula of (22):

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n} \sum_{j=1}^{n} \frac{H_j}{j^2} = \frac{7}{4}\zeta(4) + \zeta(2) + 2\zeta'(3) - 2\tau_3 - 1\,. \tag{26}$$

Summing (22) and (26) and using the relation (5) with $r = 2$ leads to the following formula

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n} \left\{ H_n^{(3)} + H_n H_n^{(2)} \right\} = 3\zeta(4) + \zeta'(3) - \tau_3 - 1\,. \tag{27}$$

Furthermore, by applying (13) with $p = 4$ and using the polynomial expression (4), we have

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n} \left\{ (H_n)^3 + 3H_n H_n^{(2)} + 2H_n^{(3)} \right\} = 6\zeta(4) - 2\,. \tag{28}$$

Thus, a simple comparison of (27) and (28) gives yet another interesting formula

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\, n} \left\{ (H_n)^3 + H_n H_n^{(2)} \right\} = -2\zeta'(3) + 2\tau_3\,. \tag{29}$$

4 Summary of main formulas

The most noteworthy identities are listed below in increasing order of complexity.

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} H_n = \frac{\pi^2}{6} - 1 \tag{A}$$

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} \left\{ (H_n)^2 + H_n^{(2)} \right\} = 2\zeta(3) - 1 \tag{B}$$

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} H_n \, H_{n-1} = \zeta(3) - \gamma \frac{\pi^2}{6} \tag{C}$$

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n^2} \left\{ H_n + n H_n^{(2)} \right\} = \gamma \frac{\pi^2}{6} + \zeta(3) - 1 \tag{D}$$

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} (H_n)^2 = -\zeta'(2) - \tau_2 - 1 \tag{E}$$

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} H_n^{(2)} = 2\zeta(3) + \zeta'(2) + \tau_2 \tag{F}$$

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n^2} H_n = \gamma \frac{\pi^2}{6} - \zeta(3) - \zeta'(2) - \tau_2 - 1 \tag{G}$$

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} \left\{ (H_n)^3 + H_n H_n^{(2)} \right\} = -2\zeta'(3) + 2\tau_3 \tag{H}$$

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n} \left\{ H_n^{(3)} + H_n H_n^{(2)} \right\} = \frac{\pi^4}{30} + \zeta'(3) - \tau_3 - 1 \tag{I}$$

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n!\,n^2} \left\{ (H_n)^2 + H_n^{(2)} \right\} = 2\gamma\zeta(3) - \frac{\pi^4}{180} - \frac{\pi^2}{6} - 2\zeta'(3) + 2\tau_3 \tag{J}$$

5 Conclusion

Theoretically, the reciprocal identities (11) and (12) should allow us to compute a large number of infinite sums involving Cauchy numbers and (generalized) Roman harmonic numbers. Unfortunately, finding an explicit evaluation of the $\mathcal{R}$-sums in the right-hand side of these identities remains a difficult task in most of cases. This highlights both the interest and the practical limits of our method.

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