Alternating series involving multizeta values

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MICOPAM Conference, Evry

August 2019

Main reference

The main reference for this talk is my recent article:

A note on some alternating series involving zeta and multiple zeta values
Journal of Mathematical Analysis and Applications, **475** (2019),
1831–1841.

Available on my website: math.unice.fr/$\sim$coppo/

Euler's constant

The famous Euler-Mascheroni constant

$$\gamma := \lim_{n\to\infty} \left\{ \sum_{j=1}^{n} \frac{1}{j} - \ln n \right\} = 0.5772156649\ldots$$

was first introduced by Euler and has been computed with high accuracy since the middle of the 18th century.

One of the main reasons of the importance of this constant lies in its close relation with the Riemann zeta function.

Stieltjes constants

The Riemann zeta-function

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s}, \qquad \Re(s) > 1$$

is a meromorphic function in the entire complex plane with one simple pole at $s = 1$. The Laurent expansion of the Riemann zeta-function at $s = 1$ is given by

$$\zeta(s) = \frac{1}{s-1} + \gamma + \sum_{k=1}^{\infty} \frac{(-1)^k (s-1)^k}{k!} \gamma_k$$

The numbers γ_k are called Stieltjes constants.

Stieltjes constants

A classical expression of the Stieltjes constants is the following:

$$\gamma_k = \lim_{n\to\infty} \left\{ \sum_{j=1}^{n} \frac{\ln^k j}{j} - \frac{\ln^{k+1} n}{k+1} \right\}$$

In the specific case $k = 1$, we have

$$\gamma_1 = \lim_{n\to\infty} \left\{ \sum_{j=1}^{n} \frac{\ln j}{j} - \frac{1}{2} \ln^2 n \right\} = -0.07281584548\ldots$$

The constant γ_1 will reappear at the end of this presentation.

Euler's and Mascheroni's series for γ

A conditionally convergent series representation of γ is the following:

$$\gamma = \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n}$$

(Euler, 1735)

An absolutely convergent series representation of γ is the following:

$$\gamma = \sum_{n=1}^{\infty} \frac{|b_n|}{n} = \frac{1}{2} + \frac{1}{24} + \frac{1}{72} + \frac{19}{2880} + \frac{3}{800} + \frac{863}{362\,880} + \cdots$$

(Mascheroni, 1790)

Bernoulli numbers of the second kind

The numbers b_n are the Bernoulli numbers of the second kind (also called Gregory coefficients). The numbers $c_n = n!b_n$ are the Cauchy numbers. The sequence $\{b_n\}_n$ can be computed recursively by $b_0 = 1$ and

$$\sum_{k=0}^{n} \frac{(-1)^k b_k}{n-k+1} = 0 \quad \text{for } n \geq 1\,,$$

or explicitly by

$$b_n = \frac{1}{n!}\int_0^1 x\,(x-1)\,(x-2)\cdots(x-n+1)\,dx, \qquad n = 1, 2, 3, \ldots$$

Bernoulli numbers of the second kind

The Bernoulli numbers of the second kind alternate in sign. The first one are

$$b_1 = \frac{1}{2}\,, b_2 = \frac{-1}{12}\,, b_3 = \frac{1}{24}\,, b_4 = \frac{-19}{720}\,, b_5 = \frac{3}{160}\,, b_6 = \frac{-863}{60\,480}\,, \ldots$$

Asymptotically, they behave as

$$|b_n| \sim \frac{1}{n\,(\ln n)^2}\,, \qquad n \to +\infty$$

Bernoulli numbers of the second kind

A remark on the Mascheroni's series: if the absolute values are removed in the series, it may be shown that the sum

$$\sum_{n=1}^{\infty} \frac{b_n}{n} = \mathrm{li}(2) - \gamma = 0.46794811521\dots$$

where $\mathrm{li}(x)$ is the logarithmic integral function.

The sequence $\{\nu_k\}$

For each integer k with $k \geqslant -1$, we now consider the shifted series

$$\nu_k := \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n+k} \,.$$

They form a sequence $\{\nu_k\}_k$ of conditionally convergent series parametrized by k. In particular, we have

$$\nu_0 = \gamma \,.$$

Suryanarayana's formula

A quite simple formula for ν_1 is

$$\nu_1 = \frac{\gamma}{2} - \frac{1}{2}\ln(2\pi) + 1$$

(Suryanarayana, 1974)

A short proof of this formula is given by Singh and Verma (Yokohama Mathematical Journal, 31 (1983)).

The case $k \geq 1$

For any integer $k \geq 1$, it can be shown that a general formula for ν_k is

$$\nu_k = \frac{\gamma}{k+1} - \frac{1}{2}\ln(2\pi) + \sum_{j=1}^{k-1}(-1)^j \binom{k}{j} \zeta'(-j) + C_k$$

where C_k is a positive rational number whose explicit expression is given by

$$C_k = \frac{1}{k} + \sum_{j=1}^{k-1} \binom{k}{j} \frac{B_{j+1}\, H_j}{j+1}$$

In this expression, $\{H_n\}_n$ are the harmonic numbers,

$$H_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

and $\{B_n\}_n$ are the Bernoulli numbers.

Bernoulli numbers

The Bernoulli numbers are defined by their generative function

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \qquad (|x| < 2\pi)$$

They can be computed recursively by $B_0 = 1$ and

$$\sum_{k=0}^{n} \frac{B_k}{k!(n-k+1)!} = 0$$

The first one are

$B_1 = \frac{-1}{2},\ B_2 = \frac{1}{6},\ B_3 = 0,\ B_4 = \frac{-1}{30},\ B_5 = 0,\ B_6 = \frac{1}{42}$, etc.

An equivalent expression for ν_k in the case $k \geq 1$ is the following:

$$\nu_k = \frac{\gamma}{k+1} + \frac{1}{k} - \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \ln(A_j)$$

where A_k are the generalized Glaisher-Kinkelin constants defined by

$$\ln(A_k) = \frac{B_{k+1}\, H_k}{k+1} - \zeta'(-k) \quad \text{for } k \geq 0$$

In particular, $A_0 = \sqrt{2\pi}$ is the Stirling constant and $A_1 = A$ is the the Glaisher-Kinkelin constant

$$A = e^{\frac{1}{12} - \zeta'(-1)} = 1.282427129\ldots$$

The case $k = -1$

The case $k = -1$ is a special case of particular interest. By definition, we have

$$\nu_{-1} := \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\zeta(k+1)}{k} = 1.2577468869\ldots$$

This constant admits the following expressions:

- $$\nu_{-1} = \sum_{n=1}^{\infty} \frac{1}{n} \ln\left(1 + \frac{1}{n}\right)$$
- $$\nu_{-1} = -\sum_{n=2}^{\infty} \zeta'(n)$$
- $$\nu_{-1} = \int_{-\infty}^{+\infty} \frac{\zeta(\frac{3}{2} + ix)}{(1+2ix)\cosh(\pi x)}\, dx$$

This last formula is due to Blagouchine and can be easily proved by the residue theorem.

Higher Gregory coefficients

We consider now the Gregory coefficients of higher order defined by

$$G_n^{(k)} := \frac{1}{n!}\sum_{j=1}^{n}\frac{s(n,j)}{j+k} \qquad (k \geq 0, n \geq 1)$$

In this formula, $s(n,j)$ denotes the Stirling numbers of the first kind.
The Gregory coefficients of higher order are representable by the integral

$$G_n^{(k)} = \frac{(-1)^{n+1}}{n!}\int_0^1 x^k(1-x)(2-x)\cdots(n-1-x)\,dx$$

In the specific case $k=1$, we recover the ordinary Gregory coefficients since

$$G_n^{(1)} = b_n\,.$$

Higher Gregory coefficients

As for the b_n, the Gregory coefficients $G_n^{(k)}$ alternate in sign.

$$G_n^{(k)} = (-1)^{n+1}|G_n^{(k)}|$$

It can be shown shown that

$$\nu_{k-1} = \sum_{n=1}^{\infty} \frac{|G_n^{(k)}|}{n} \qquad (k \geq 0)$$

In the specific case $k = 1$, we recover the Mascheroni series for γ.

Multizeta values

For integers $p \geq 0$ and $k \geq -1$, we consider now the more general alternating series

$$\nu_{k,p} := \sum_{n=2}^{\infty} \frac{(-1)^n}{n+k} \zeta(n, \underbrace{1, \ldots, 1}_{p})$$

where

$$\zeta(s_1, s_2, \cdots, s_k) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}}$$

In the specific case $p = 0$, we have

$$\nu_{k,0} = \nu_k \,.$$

Multizeta values

It can be shown that for all integers $p \geq 0$ and $k \geq -1$, we have

$$\nu_{k,p} = \sum_{n=1}^{\infty} \frac{|G_n^{(k+1)}|}{n^{p+1}}$$

In the specific case $k = 0$, we get

$$\nu_{0,p} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n, \underbrace{1, \ldots, 1}_{p}) = \sum_{n=1}^{\infty} \frac{|b_n|}{n^{p+1}}$$

The case $p = 1$

In particular, for $p = 1$,

$$\nu_{0,1} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n,1) = \sum_{n=1}^{\infty} \frac{|b_n|}{n^2} = 0.5290529699\ldots$$

We have the following nice relation

$$\nu_{0,1} = \nu_{-1} + \gamma_1 + \frac{1}{2}\gamma^2 - \frac{1}{2}\zeta(2)$$

where γ_1 is the first Stieltjes constant.

Link with the harmonic zeta function

The Apostol-Vu harmonic zeta function ζ_H defined for $\Re(s) > 1$ by

$$\zeta_H(s) := \sum_{n=1}^{\infty} \frac{H_n}{n^s}$$

is an analytic function in the half-plane $\Re(s) > 1$ which can be extended meromorphically in the whole complex plane with a double pole at $s = 1$ and an infinity of simple poles at the integers $0, -1, -3, -5, -7, -9, \ldots$.

Link with the harmonic zeta function

The special values of the harmonic zeta function at negative even integers are $\zeta_H(-2k) = -B_{2k}/4k + B_{2k}/2$. The special values at positive integers are given by

$$2\zeta_H(n) = (n+2)\zeta(n+1) - \sum_{r=1}^{n-2} \zeta(r+1)\zeta(n-r) \qquad (n \geq 2).$$

This last formula was first obtained by Euler in a famous article dated 1775 and several times rediscovered afterwards.

Link with the harmonic zeta function

It can be shown that

$$\sum_{n=2}^{\infty}(-1)^n\frac{\zeta_H(n)}{n} = \gamma_1 + \frac{1}{2}\gamma^2 + \frac{1}{2}\zeta(2) = 0.916240149\ldots$$

where γ_1 is the first Stieltjes constant.

Furthermore, in a neighborhood of $s = 1$, we have the expansion

$$\zeta_H(s) = \frac{1}{(s-1)^2} + \frac{\gamma}{(s-1)} + \frac{1}{2}\gamma^2 + \frac{1}{2}\zeta(2) + O(s-1)\,.$$