Bendersky-Adamchik constants, hyperfactorials, and Ramanujan summation of series

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Stirling's constant

The story starts with the famous Stirling approximation for the factorial:

$$n! \sim \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n} \quad \text{as} \quad n \to \infty$$

which dates back from the middle of the 18th century. In fact, Stirling never explicitly stated this formula. The first appearance of this result occured in a letter from Euler to Goldbach dated June 1744.

The constant $\sqrt{2\pi}$ is known as the Stirling constant.

Stirling's asymptotic series

A much more accurate version of Stirling's formula was given a century later by Laplace:

$$\begin{aligned}\Gamma(x+1) &\sim \sqrt{2\pi}\, x^{x+\frac{1}{2}} e^{-x} \exp\left(\frac{B_2}{1\cdot 2x} + \frac{B_4}{3\cdot 4x^3} + \frac{B_6}{5\cdot 6x^5} + \cdots\right) \\ &= \sqrt{2\pi}\, x^{x+\frac{1}{2}} e^{-x} \left(1 + \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51840x^3} - \cdots\right)\end{aligned}$$

as $x \to +\infty$, where Γ is Euler's gamma function, and B_n are the Bernoulli numbers. This classical expansion is one of the oldest appearance of an asymptotic series. This series is improperly called Stirling's series even though Stirling never wrote it!

Glaisher-Kinkelin constant

A similar (but less well-known) approximation also applies to the hyperfactorial function:

$$\prod_{\nu=1}^{n} \nu^{\nu} = 1^1 2^2 \cdots n^n \sim A\, n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}}\, e^{-\frac{n^2}{4}} \quad \text{as} \quad n \to \infty\,.$$

The constant

$$A = \lim_{n\to\infty} \frac{\prod_{\nu=1}^{n} \nu^{\nu}}{n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}}\, e^{-\frac{n^2}{4}}} = 1.282427\ldots$$

is called Glaisher's constant or the Glaisher-Kinkelin constant after James Glaisher (1848-1928) and Hermann Kinkelin (1832-1913). Glaisher introduced this constant for the first time in 1877 in his research on the gamma function.

Glaisher-Kinkelin constant

The asymptotic expansion of the hyperfactorial is

$$1^1 2^2 \cdots n^n \sim A\, n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}}\, e^{-\frac{n^2}{4}} \exp\left(-\frac{B_4}{2\cdot 3\cdot 4n^2}-\frac{B_6}{4\cdot 5\cdot 6n^4}-\cdots\right)$$

$$= A\, n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}}\, e^{-\frac{n^2}{4}} \left(1+\frac{1}{720n^2}-\frac{1433}{7257600n^4}+\cdots\right)$$

as $n \to \infty$. All the coefficients in this asymptotic series can be computed by a recursive formula (Chen and Lin, 2013).

Bendersky's gamma function

In 1933, Bendersky studies the product $\prod_{\nu=1}^{n} \nu^{\nu^k}$ for $k = 0, 1, 2, \ldots$ which reduces to the classical factorial when $k = 0$, and to the classical hyperfactorial when $k = 1$. The reference for Bendersky's work is

L. Bendersky, Sur la function gamma généralisée (On the generalized gamma function), *Acta Mathematica* **61** (1933).

This masterpiece seems to be his one and only published article!

Bendersky's gamma function

For his purpose, Bendersky introduces a natural generalization Γ_k of the Γ-function, whose fundamental properties are $\Gamma_k(1) = 1$ and

$$\Gamma_k(x+1) = x^{x^k}\Gamma_k(x) \quad \text{for} \quad x > 0.$$

In particular,

$$\Gamma_k(n+1) = \prod_{\nu=1}^{n} \nu^{\nu^k} \quad \text{for} \quad k = 0, 1, 2, \ldots$$

Notably, $\Gamma_0 = \Gamma$, and Γ_1 is the Kinkelin hyperfactorial K-function.

Bendersky-Adamchik constants

For any integer $k \geq 0$, Bendersky shows the existence of a constant A_k and two polynomials P_k and Q_k of degree $k+1$ such that

$$\Gamma_k(x+1) \sim A_k \, x^{P_k(x)} e^{-Q_k(x)} \quad \text{as} \quad x \to +\infty \, .$$

The numbers A_k (for $k = 0, 1, 2, \ldots$) are called the *Bendersky-Adamchik constants* or *generalized Glaisher-Kinkelin constants*. In particular, the constant A_0 is nothing else than the Stirling constant, and the constant A_1 is the Glaisher-Kinkelin constant.

A general formula which allows to evaluate explicitly these polynomials P_k and Q_k is given by:

$$P_k(x) = \frac{x^{k+1}}{k+1} + \frac{x^k}{2} + \sum_{r=1}^{[\frac{k+1}{2}]} \frac{B_{2r}}{(2r)!} \left(\prod_{j=1}^{2r-1} (k-j+1) \right) x^{k+1-2r} ,$$

and

$$Q_k(x) = \frac{x^{k+1}}{(k+1)^2} - \sum_{r=1}^{[\frac{k+1}{2}]+\frac{(-1)^k-1}{2}} \frac{B_{2r}}{(2r)!} \left\{ \prod_{j=1}^{2r-1} (k-j+1) \sum_{j=1}^{2r-1} \frac{1}{k-j+1} \right\} x^{k+1-2r} .$$

Bendersky-Adamchik constants

In particular, for the first values of k, this somewhat cumbersome (but efficient) formula gives the following polynomials:

$$P_0(x) = x + \frac{1}{2} \quad \text{and} \quad Q_0(x) = x\,,$$
$$P_1(x) = \frac{x^2}{2} + \frac{x}{2} + \frac{1}{12} \quad \text{and} \quad Q_1(x) = \frac{x^2}{4}\,,$$
$$P_3(x) = \frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{6} \quad \text{and} \quad Q_1(x) = \frac{x^3}{9} - \frac{x}{12}\,,$$

etc.

Bendersky himself successfully calculated P_k and Q_k for $k \leq 4$.

Bendersky-Adamchik constants

Example: In the case $k=2$ (hyper-hyperfactorial function), we have

$$1^1 2^4 \cdots n^{n^2} \sim A_2\, n^{\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}} e^{-\frac{n^3}{9}+\frac{n}{12}} \quad \text{as} \quad n \to \infty\,,$$

with $A_2 = 1.030916\ldots$ More precisely,

$$1^1 2^4 \cdots n^{n^2} \sim A_2\, n^{\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}} e^{-\frac{n^3}{9}+\frac{n}{12}} \left(1 - \frac{1}{360n} + \frac{1}{259200n^2} + \cdots\right)$$

All the coefficients of this asymptotic series can be computed recursively (Wang, 2017).

Adamchik's formula

In 1998, Adamchik rediscovers the constants A_k and gives a nice expression of the constants $\ln A_k$ in terms of the derivatives of the Riemann zeta function. Let us recall that

$$\ln A_k = \lim_{n\to\infty} \left\{ \sum_{\nu=1}^{n} \nu^k \ln \nu - P_k(n) \ln n + Q_k(n) \right\}$$

This expression (called Adamchik's formula) is the following:

$$\ln A_k = \frac{H_k B_{k+1}}{k+1} - \zeta'(-k)\,,$$

where H_k is the kth harmonic number (with the usual convention $H_0 = 0$).

Adamchik's formula

Thanks to the Adamchik formula, we can derive from the functional equation of zeta the following identities:

$$\ln A_{2k-1} = \frac{B_{2k}}{2k}\left(\gamma + \ln 2\pi - \frac{\zeta'(2k)}{\zeta(2k)}\right) \quad \text{for} \quad k \geq 1\,,$$

where γ is Euler's constant, and

$$\ln A_{2k} = \frac{B_{2k}}{4} \cdot \frac{\zeta(2k+1)}{\zeta(2k)} \quad \text{for} \quad k \geq 1\,.$$

In particular,

$$\begin{aligned}
\ln A_0 &= \ln\sqrt{2\pi} = -\zeta'(0) \\
\ln A_1 &= \frac{1}{12} - \zeta'(-1) = \frac{\gamma + \ln 2\pi}{12} - \frac{\zeta'(2)}{12\,\zeta(2)} \\
\ln A_2 &= -\zeta'(-2) = \frac{\zeta(3)}{24\,\zeta(2)}\,.
\end{aligned}$$

Link with the Ramanujan summation

The Bendersky-Adamchik constants have an interesting interpretation in terms of the Ramanujan summation of divergent series that motivates my interest in these constants.

If $\sum_{n\geq 1}^{\mathcal{R}} a_n$ denotes the $\mathcal{R}$-sum of the series $\sum_{n\geq 1} a_n$ (i.e. the sum of the series in the sense of Ramanujan's summation method), then we have

$$\sum_{n\geq 1}^{\mathcal{R}} n^k \ln n = \ln A_k - \frac{H_k B_{k+1}}{k+1} - \frac{1}{(k+1)^2} \quad \text{for} \quad k = 0, 1, 2, \ldots$$

An equivalent expression for this sum is

$$\sum_{n\geq 1}^{\mathcal{R}} n^k \ln n = \int_0^1 \ln \Gamma_k(x+1)\, dx\,,$$

where Γ_k is Bendersky's gamma function.

In particular, we have the following identities:

$$\sum_{n\geq 1}^{\mathcal{R}} \ln n = \int_0^1 \ln\Gamma(x+1)\,dx = \ln\sqrt{2\pi} - 1\,,$$

and

$$\sum_{n\geq 1}^{\mathcal{R}} n\ln n = \int_0^1 \ln K(x+1)\,dx = \ln A - \frac{1}{3}\,.$$

Furthermore, we mention another (independant) notable result:

$$\sum_{n\geq 1}^{\mathcal{R}} n^{-1}\ln n = \gamma_1\,,$$

where $\gamma_1 = \lim_{n\to\infty}\left\{\sum_{\nu=1}^{n}\frac{\ln\nu}{\nu} - \frac{1}{2}\ln^2 n\right\}$ is the first Stieltjes constant.

Cauchy numbers

We now present some new convergent series towards $\ln A_k$ which can be deduced from Adamchik's formula and from an expression of $\zeta'(-k)$ given in a more recent study (see Coppo and Young, 2016). As these series involve the Cauchy numbers, we make a few preliminary remarks.

The non-alternating Cauchy numbers, denoted by λ_n, are positive rational numbers which can be defined recursively by means of the relation

$$\sum_{k=1}^{n-1} \frac{\lambda_k}{k!\,(n-k)} = \frac{1}{n} \quad \text{for} \quad n \geq 2\,.$$

The first ones are the following:

$$\lambda_1 = \frac{1}{2}\,,\ \lambda_2 = \frac{1}{6}\,,\ \lambda_3 = \frac{1}{4}\,,\ \lambda_4 = \frac{19}{30}\,,\ \lambda_5 = \frac{9}{4}\,,\ \lambda_6 = \frac{863}{84}\,,\ \text{etc.}$$

Cauchy numbers

The numbers λ_n are closely linked to the Bernoulli numbers of the second kind b_n defined by their generating function

$$\frac{x}{\ln(1+x)} = \sum_{n=0}^{\infty} b_n x^n, \quad |x| < 1\,,$$

through the relation

$$\lambda_n = n!\,|b_n| \quad \text{for} \quad n \geq 1\,.$$

In particular, we can easily deduce from this relation the integral expression

$$\lambda_n = \int_0^1 x(1-x)\cdots(n-1-x)\,dx \quad \text{for} \quad n \geq 2\,.$$

New convergent series for $\ln A_k$

The convergent representation:

$$\ln A_0 = \ln\sqrt{2\pi} = \sum_{n=2}^{\infty} \frac{\lambda_n}{n!(n-1)} + \frac{1}{2}\gamma + \frac{1}{2}\,,$$

is a fairly known result. In addition, we have the following new identities:

$$\ln A_1 = \sum_{n=3}^{\infty} \frac{\lambda_n}{n!(n-2)} + \frac{1}{12}\gamma + \frac{1}{8}\,,$$

$$\ln A_2 = \sum_{n=4}^{\infty} \frac{\lambda_n\,(n-1)}{n!\,(n-2)(n-3)} - \frac{1}{24}\,,$$

$$\ln A_3 = \sum_{n=5}^{\infty} \frac{\lambda_n\, n(n-1)}{n!\,(n-2)(n-3)(n-4)} - \frac{1}{120}\gamma - \frac{29}{240}\,,$$

$$\ln A_4 = \sum_{n=6}^{\infty} \frac{\lambda_n\,(n-1)^2(n+4)}{n!\,(n-2)(n-3)(n-4)(n-5)} - \frac{113}{480}\,.$$

New convergent series for $\ln A_k$

A general expression for these sums is given by the following formula:

$$\ln A_{2k} = \sum_{n=2k+2}^{\infty} \frac{\lambda_n}{n!} \left\{ \sum_{r=1}^{2k} \frac{(-1)^r r!\, S(2k,r)}{n-1-r} \right\} + C_{2k} \quad \text{for} \quad k \geq 1\,,$$

and

$$\ln A_{2k-1} = \sum_{n=2k+1}^{\infty} \frac{\lambda_n}{n!} \left\{ \sum_{r=1}^{2k-1} \frac{(-1)^{r-1} r!\, S(2k-1,r)}{n-1-r} \right\} + \frac{B_{2k}}{2k}\,(H_{2k}+\gamma) + C_{2k-1} \quad \text{for} \quad k \geq 1\,,$$

with $S(k,r)$ the Stirling numbers of the second kind, $C_1 = 0$, and

$$C_k = (-1)^k \sum_{r=1}^{k-1} (-1)^r r!\, S(k,r) \sum_{j=r+2}^{k+1} \frac{\lambda_j}{j!\,(j-1-r)} \quad \text{for} \quad k \geq 2\,.$$

Main references for this talk

B. Candelpergher, *Ramanujan Summation of Divergent Series*, Lecture Notes in Math. 2185, Springer, 2017.

M-A. Coppo and P.T. Young, On shifted Mascheroni series and hyperharmonic numbers, *J. Number Theory*, **169** (2016).

M-A. Coppo, Generalized Glaisher-Kinkelin constants and Ramanujan summation of series, to appear in *Research in Number Theory*

All available on my website: https://math.univ-cotedazur.fr/~coppo/