Continuité de certaines fonctionnelles du mouvement brownien fractionnaire en fonction du paramètre de Hurst

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Nice, 24 novembre 2015

Plan

Introduction

Regularity of the fractional Brownian field

Continuity in H of the law of the hitting times of fBm

Fractional Brownian motion (fBm)

Fractional Brownian motion of Hurst parameter $H \in (0,1)$ can either be defined as:

1. the centred Gaussian process with covariance:

$$R_H(t,s) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}) \ , \ \ s,t \in \mathbb{R} \ .$$

2. a H-self-similar with stationary increments Gaussian process.

Discovered by [Kolmogorov 40] (in relation with turbulent fluid dynamics). For $H \neq 1/2$, an integral representation was given by [Mandelbrot & Van Ness 68]:

$$B_t^H = c_H \int_{\mathbb{R}} \left((t-s)_+^{H-1/2} - (-s)_+^{H-1/2} \right) \mathrm{d}\mathbb{W}_s$$

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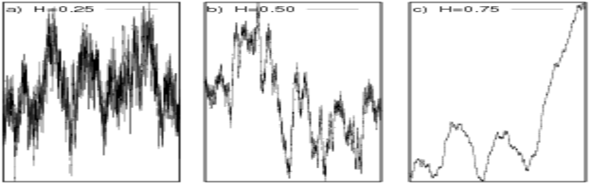
Few properties of the fBm

- Hölder continuity of the sample paths:



- For $H \neq 1/2$, fBm is *neither* a Markov process, nor a semimartingale.
- For $H > 1/2$, the noise is long-range dependent:

$$\sum_{k=0}^{\infty} \mathbb{E}\left(B_1^H \left(B_{1+k}^H - B_k^H\right)\right) = \infty$$

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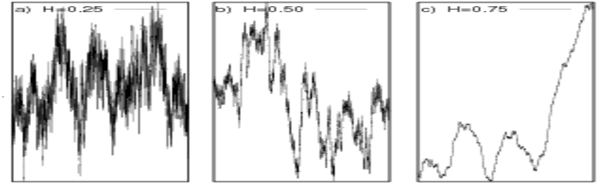

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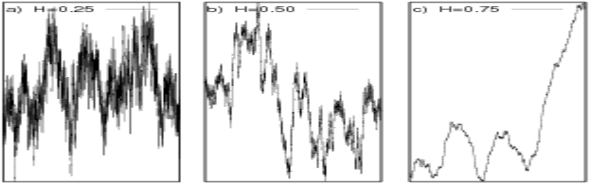

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Sensitivity in the Hurst parameter

Several authors have studied this problem for:

- the convergence in law of the Russo-Valois symmetric integral:

$$\int u_s^H \; dB_s^H \underset{H \to H_0}{\Longrightarrow} \int u_s^{H_0} \; dB_s^{H_0}, \quad \text{[Jolis \& Viles 10]};$$

- idem for the multiple Wiener-Ito integrals [Jolis & Viles 07];
- the local time of fBm, and Gaussian *random fields* [Wu & Xiao 09].

"*These kinds of results justify the use of $B^{\hat{H}}$ as a model in applied situations where the true value of the Hurst parameter is unknown and $\hat{H}$ is some estimation of it.*" [Jolis & Viles 10]

Remark

We can see at least two problems arising:

- *the first one mentioned in [Wu & Xiao 09]: these problems are more difficult for random fields;*
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On a bounded open domain $U \subset \mathbb{R}^N$ *with smooth boundary, let* $h \in (0,1)^N$ *and consider equation:*

$$\Delta u = \dot{\mathbb{W}}^{(h)} \quad \text{on } U ,$$

with condition $u = 0$ *on* ∂U.

u *depends on* h. $\Rightarrow$ *How does the law of* u *evolve subject to a small perturbation* $h + \delta h$*?*

Question (2)

The law of the hitting times of the fractional Brownian motion:

$$\tau_H = \inf\{t \geq 0 : \ B_t^H = 1\}$$

is unknown (and expected to be hard to obtain...).
What is the deviation from the law of the HT for Brownian motion:

$$\left|\mathbb{E}\left(e^{-\lambda\tau_H}\right) - \mathbb{E}\left(e^{-\lambda\tau_{1/2}}\right)\right|, \ \lambda \in \mathbb{R}_+ \ ?$$

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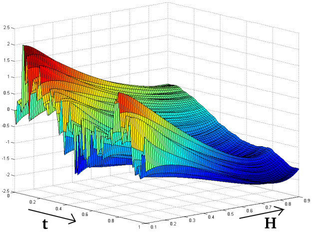

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$$\mathbb{E}(X_s^H X_t^H) = \frac{1}{2}\left(\|s\|^{2H} + \|t\|^{2H} - \|s-t\|^{2H}\right)$$

- Fractional Brownian sheet ([Kamont '96]): for $H = (H_1, \ldots, H_N) \in (0,1)^N$, $s,t \in \mathbb{R}_+^N$,

$$\mathbb{E}(W_s^H W_t^H) = \frac{1}{2^N} \prod_{k=1}^{N} \left(|s_k|^{2H_k} + |t_k|^{2H_k} - |t_k - s_k|^{2H_k}\right)$$

- The multiparameter fBm [Herbin & Merzbach '06]: $H \in (0, 1/2]$, $s,t \in \mathbb{R}_+^N$,

$$\mathbb{E}(B_s^H B_t^H) = \frac{1}{2}\left(\lambda([0,s])^{2H} + \lambda([0,t])^{2H} - \lambda([0,s] \triangle [0,t])^{2H}\right) .$$

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- The multiparameter fBm [Herbin & Merzbach '06].

Multiparameter extensions of fBm

All the aforementionned processes have a covariance of the form:

$$(s,t) \in \mathbb{R}^N \mapsto \frac{1}{2}\left(\mu(U_t)^{2H} + \mu(U_s)^{2H} - \mu(U_s \bigtriangleup U_t)^{2H}\right)$$

for some measure μ and parametric family $t \in \mathbb{R}^N \mapsto U_t \in \mathcal{B}(\mathbb{R}^N)$.

Example (Brownian sheet)

$$k_{1/2}^{(N)} = \frac{1}{2}\left(\lambda([0,t]) + \lambda([0,s]) - \lambda([0,t] \triangle [0,s])\right)$$
$$= \lambda\left([0,t] \cap [0,s]\right)$$

Example (Lévy fractional Brownian motion (Centsov's construction))

S_N the unit sphere of $\mathbb{R}^N$, $S = S_N \times (0,\infty)$,
μ the product measure of the uniform measure on S_N with the Lebesgue measure on $(0,\infty)$.

$$\mathbb{E}\left(X_t^H X_s^H\right) = \frac{1}{2}\left(\mu(U_t)^{2H} + \mu(U_s)^{2H} - \mu(U_s \triangle U_t)^{2H}\right) \ ,$$

where for any $t \in \mathbb{R}^N$,

$$U_t = \{(s,r) \in S : \ r < \langle s,t \rangle\} \ .$$

The L^2-fractional Brownian motion

Let $(T, \mathcal{T}, m)$ be a measurable space, $f, g \in L^2(T, m)$, $h \in (0, 1/2]$:

$$\begin{aligned} k_h : (f, g) \mapsto & \frac{1}{2} \left(m(f^2)^{2h} + m(g^2)^{2h} - m((f-g)^2)^{2h} \right) \\ & = \frac{1}{2} \left(\|f\|_{L^2(T,m)}^{4h} + \|g\|_{L^2(T,m)}^{4h} - \|f-g\|_{L^2(T,m)}^{4h} \right) \end{aligned}$$

is positive definite.

$\Rightarrow \boldsymbol{B}^h$ the centred Gaussian process with covariance k_h.

Includes the SIfBm, the fractional Brownian sheet, the Lévy fBm, and many other Gaussian processes.

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Abstract Wiener spaces

Definition ([Gross '67])

Given a separable Hilbert space $\mathcal{H}$, a Banach space E and a measure μ on E, we say that $(\mathcal{H}, E, \mu)$ is an *abstract Wiener space* if:

- $\mathcal{H}$ is densely and continuously embedded into E (which we denote by $\mathcal{H} \hookrightarrow E$); denote by $S : E^* \to \mathcal{H}$ the canonical injection:

$$E^* \overset{S}{\hookrightarrow} \mathcal{H}^* \equiv \mathcal{H} \overset{S^*}{\hookrightarrow} E \ ;$$

- and

$$\hat{\mu}(\xi) = \int_E e^{i\langle \xi, x\rangle} \ \mu(\mathrm{d}x) = e^{-\frac{1}{2}\|S\xi\|^2_{\mathcal{H}}} \ , \ \forall \xi \in E^*.$$

Proposition (see e.g. [Stroock '10])

Let $\mathcal{H}$ and $\mathcal{H}_\mu$ be two separable Hilbert spaces, $\mathcal{H}_\mu$ being endowed with a Wiener space structure $(\mathcal{H}_\mu, E, \mu)$. $\mathcal{H}$ can also be endowed with such a structure by isometry, i.e. if $u : \mathcal{H}_\mu \to \mathcal{H}$ is a linear isometry, $(\mathcal{H}, \tilde{u}(E), \tilde{u}_*\mu)$ is a Wiener space.

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Wiener space of the fBm

- For any $h \in (0,1)$, the Wiener space of the fBm is given by [Decreusefond & Üstünel '99]:

$$\left(\underbrace{\mathcal{I}_{0+}^{h+1/2}(L^2[0,1])}_{\left(\mathcal{H}_h,\ (\cdot,\cdot)_{\mathcal{H}_h}\right)}, C_0([0,1]), \mathcal{W}_h \right) .$$

- $\boldsymbol{B}$ will be the white noise on $C_0([0,1])$ of control measure $\mathcal{W}_{1/2}$:

$$\forall U, V \in \mathcal{B}\left(C_0([0,1])\right), \quad \mathbb{E}\left(\boldsymbol{B}(U)\boldsymbol{B}(V)\right) = \mathcal{W}_{1/2}(U \cap V) .$$

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Proposition ([R. '15])

For any $h \in (0,1)$*, there exists a linear map* $\mathcal{K}_h : \mathcal{H}_h \to C_0([0,1])^*$ *such that:*

$$R_h(s,t) = \int_{C_0([0,1])} \langle \mathcal{K}_h R_h(\cdot,s), x\rangle \, \langle \mathcal{K}_h R_h(\cdot,t), x\rangle \, d\mathcal{W}(x) \ .$$

As a consequence, the process:

$$B_{h,t} = \int_{C_0([0,1])} \langle \mathcal{K}_h R_h(\cdot,t), x\rangle \, \mathrm{d}\boldsymbol{B}_x \ ,$$

is a fractional Brownian field on $(0,1)\times[0,1]$.

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- Denote $\mathcal{H}(k_h)$ the Reproducing kernel Hilbert space (RKHS) of the kernel $k_h : L^2(T,m) \times L^2(T,m) \to \mathbb{R}$;
- For any $h \in (0, 1/2]$, let u_h be a linear isometry between $\mathcal{H}_h$ and $\mathcal{H}(k_h)$:

$$(\mathcal{H}(k_h), E_h, \mu_h) := (\mathcal{H}(k_h), \tilde{u}_h(C_0[0,1]), (\tilde{u}_h)_* \mathcal{W}_h) \ .$$

  This defines a family of AWS $(\mathcal{H}(k_h), E_h, \mu_h)$, $h \in (0, 1/2]$;
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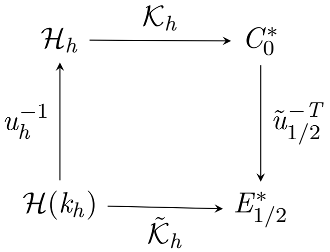

$\boldsymbol{B}$ is now a white noise on $E_{1/2}$ with control measure $(\tilde{u}_{1/2})_* \mathcal{W}$.

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With the previous notations,

$$k_h(f,g) = \int_{E_{1/2}} \langle \tilde{\mathcal{K}}_h k_h(f,\cdot), x\rangle \ \langle \tilde{\mathcal{K}}_h k_h(g,\cdot), x\rangle \ \mu_{1/2}(dx)$$

and

$$\boldsymbol{B}_{h,f} = \int_{E_{1/2}} \langle \tilde{\mathcal{K}}_h k_h(f,\cdot), x\rangle \ \mathrm{d}\boldsymbol{B}_x$$

is a fractional Brownian field on $(0,1/2] \times L^2(T,m)$*, i.e. for any fixed* h, $\{\boldsymbol{B}_{h,f}, f \in L^2\}$ *has covariance* k_h.

$\boldsymbol{B}$ is now a white noise on $E_{1/2}$ with control measure $(\tilde{u}_{1/2})_*\mathcal{W}$.

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Regularity of the increments

Theorem ([R. '15])

There exists a fractional Brownian field $\boldsymbol{B}$ indexed over $(0,1/2] \times L^2(T,m)$ whose covariance satisfies:

$\forall \eta \in (0,1/4)$ and any compact subset D of $L^2(T,m)$, $\exists C_{\eta,D} \equiv C > 0$ such that for any $f, f' \in D$, and any $h, h' \in [\eta, 1/2-\eta]$,

$$\mathbb{E}\left(\boldsymbol{B}_{h,f} - \boldsymbol{B}_{h',f'}\right)^2 \leq C\left\{(h-h')^{2-\epsilon} + m\left((f-f')^2\right)^{2(h\wedge h')}\right\} .$$

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Application to solutions of an SPDE

On a bounded open domain $U \subset [0,1]^N$ with smooth boundary:

$$\Delta u = \dot{\mathbb{W}}^h \quad \text{on } U, \tag{$\mathcal{L}_h$}$$

and with the condition that $u = 0$ on ∂U.

Proposition ([R. '15])

Let u_h and $u_{h'}$ be the mild solutions to $(\mathcal{L}_h)$ and $(\mathcal{L}_{h'})$ respectively. Then, for all $\varphi \in C_c^\infty(U)$,

$$\mathbb{E}\left(\langle u_h, \varphi\rangle - \langle u_{h'}, \varphi\rangle\right)^2 \leq M_\eta \|\varphi\|_{\mathcal{H}}^2 \, (h-h')^2 L(h-h')^2 \ .$$

Plan

Introduction

Regularity of the fractional Brownian field

Continuity in H of the law of the hitting times of fBm

Example: the Leaky Integrate-and-Fire model

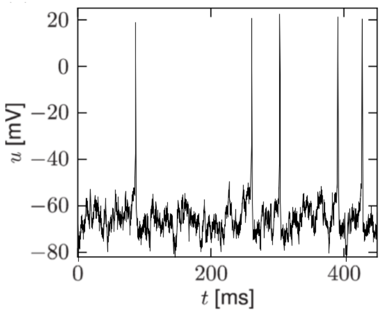

Figure : Membrane potential of a neuron

Let $\theta_1 = \inf\{t > 0 : \; V_t = 1\}$ and

$$\begin{cases} \mathrm{d}V_t = (I_t - \lambda V_t)\,\mathrm{d}t + \sigma \mathrm{d}B_t \\ V_{\theta_k^+} = 0 \\ \theta_{k+1} = \inf\{t > \theta_k : \; V_t = 1\}\;. \end{cases}$$

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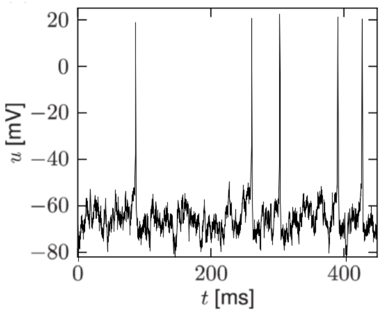

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Hitting times of the fractional Brownian motion - a few properties

Joint work with Denis Talay.

$$\tau_H = \inf\{t \geq 0 : \; B_t^H = 1\}$$

- Laplace transform: [Decreusefond and Nualart 08], for $H \geq 1/2$,

$$\mathbb{E}\left(e^{-\lambda \tau_H^{2H}}\right) \leq e^{-\sqrt{2\lambda}} \; , \; \forall \lambda \geq 0 \; ,$$

  and for $H \leq 1/2$,

$$\mathbb{E}\left(e^{-\lambda \tau_H^{2H}}\right) \geq e^{-\sqrt{2\lambda}} \; , \; \forall \lambda \geq 0 \; .$$

- Asymptotic behaviour: [Molchan 99]

  $\log \mathbb{P}(\tau_H > t) = -(1-H)\log(t)\,(1 + o(1))$, lorsque $t \to \infty$.

Hitting times of the fBm

- The previous bounds are not sharp, in fact ([R. & Talay '15]):
  - Molchan's estimate and a Tauberian theorem yields:

$$H \in (1/3, 1) \Rightarrow \mathbb{E}\left(\exp(-\lambda \tau_H^{2H})\right) \sim 1 - C\ \lambda^{\frac{(1-H)}{2H}},\ \lambda \to 0;$$

  - from $\mathbb{P}(\tau_H^{2H} \leq \epsilon) \underset{\epsilon \to 0}{\sim} \Psi(\epsilon^{-1/2})$, where Ψ is the Gaussian tail distribution function, and De Bruijn's Tauberian theorem:

$$-\log \mathbb{E}\left(\exp(-\lambda \tau_H^{2H})\right) \sim \sqrt{2\lambda}\ , \qquad \lambda \to \infty\ .$$

Similarly,

$$\mathbb{E}\left(\exp(-\lambda \tau_H)\right) \sim 1 - C\ \lambda^{1-H} \quad \text{as } \lambda \to 0\ ,$$

$$-\log \mathbb{E}\left(\exp(-\lambda \tau_H)\right) \sim \left(1 + \frac{1}{2H}\right)\ H^{\frac{1}{(2H+1)}}\ \lambda^{\frac{2H}{2H+1}} \quad \text{as } \lambda \to \infty\ .$$

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Main results

Theorem ([R. & Talay '15])

There exists a constant $\alpha > 0$ *such that for any* $\epsilon > 0$ *small, there is* $c_\epsilon > 0$ *satisfying: for any* $\lambda \geq 0$ *and any* $H \in [1/2, 1)$,

$$\left| \mathbb{E}\left(e^{-\lambda \tau_H}\right) - \mathbb{E}\left(e^{-\lambda \tau_{1/2}}\right) \right| \leq c_\epsilon (H - 1/2)^{1/3-\epsilon} \; e^{-\alpha\sqrt{\lambda}} \; .$$

Consider the fractional SDE, for $H \geq 1/2$:

$$\mathrm{d}X_t^H = b(X_t^H)\ \mathrm{d}t + \sigma(X_t^H)\ \mathrm{d}B_t^H \qquad (\mathcal{E}_H)$$

(Young integral for $H > 1/2$, Stratonovich for $H = 1/2$).

Assumption $(\mathcal{H})$

1. $b, \sigma \in C_b^1(\mathbb{R})$;
2. $\exists \sigma_{min} > 0$ *such that* $\inf_{x\in\mathbb{R}} |\sigma(x)| \geq \sigma_{min}$.

Theorem ([R. & Talay '15])

Let X^H (resp. $X^{1/2}$) be solution to $(\mathcal{E}_H)$ (resp. $(\mathcal{E}_{1/2})$), with b and σ satisfying $(\mathcal{H})$. There exist constants $\alpha, \lambda_0 > 0$ such that for any $\epsilon > 0$ small, there is $c_\epsilon > 0$ satisfying: for any $\lambda \geq \lambda_0$ and any $H \in [1/2, 1)$,

$$\left| \mathbb{E}\left(e^{-\lambda \tau_H^X}\right) - \mathbb{E}\left(e^{-\lambda \tau_{1/2}^X}\right) \right| \leq c_\epsilon (H - 1/2)^{1/3-\epsilon}\ e^{-\alpha\sqrt{\lambda}}\ .$$

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Divergence operator of fBm

Let $T > 0$ be a time horizon and $\mathcal{H}_H$ denote the Cameron-Martin of the fBm.
For $H > 1/2$,

$$K_H(\theta, \sigma) := c_H \ (H - 1/2) \ \sigma^{1/2-H} \int_\sigma^\theta u^{H-1/2}(u-\sigma)^{H-3/2} \ \mathrm{d}u.$$

Set $\tilde{c}_H = (H - 1/2)c_H$ and, for $H > 1/2$,

$$\begin{aligned} K_H^* u(s) &= \int_s^T \frac{\partial K_H}{\partial \theta}(\theta, s) \ u(\theta) \ \mathrm{d}\theta \\ &= \tilde{c}_H \int_s^T \left(\frac{\theta}{s}\right)^{H-1/2} (\theta - s)^{H-3/2} \ u(\theta) \ \mathrm{d}\theta \ . \end{aligned}$$

Let δ_H (resp. δ) be the Skorokhod integral on $\mathcal{H}_H$ (resp. $\mathcal{H}_{1/2}$).
For any u such that $K_H^* u \in \operatorname{dom} \delta$,

$$\delta_H(u) = \delta(K_H^* u) \ .$$

Sketch of proof (HT of fBm)

By Ito's formula for fractional Brownian motion:

$$\mathbb{E}\left(e^{-\lambda\tau_H}\right) - \mathbb{E}\left(e^{-\lambda\tau_{1/2}}\right) = \mathbb{E}\left[\lambda\int_0^{\tau_H}(2Hs^{2H-1}-1)u_\lambda(B_s^H)\ e^{-\lambda s}\ \mathrm{d}s\right] + \mathbb{E}\left[\sqrt{\lambda}\,\delta_H\left(\mathbf{1}_{[0,t]}u_\lambda(B_\cdot^H)\ e^{-\lambda\cdot}\right)\Big|_{t=\tau_H}\right]$$

where $u_\lambda(x) = \mathbb{E}^x\left(e^{-\lambda\tau_{1/2}}\right)$.

1. The first integral is easy;
2. for the last one, *no Doob's stopping theorem*;
3. hence, control its supremum using Garsia-Rodemich-Rumsey's lemma and control on the moments:

$$\mathbb{E}\left\|D_\cdot(\bar{X}_s(\cdot)-\bar{X}_t(\cdot))\right\|^2_{L^2[0,T]^2} \lesssim (H-1/2)^2\ (t-s)^-\ e^{-2\alpha\sqrt{\lambda}},$$

where $\bar{X}_z(v) = \{K_H^* - \mathrm{Id}\}\left(\mathbf{1}_{[n,z)}\ u_\lambda(B_\cdot^H)\phi_\eta(B_\cdot^H)\ e^{-\lambda\cdot}\right)(v)$.

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Sketch of proof (HT of fractional SDE)

More terms in $\mathbb{E}\left(e^{-\lambda\tau_H^X}\right) - \mathbb{E}\left(e^{-\lambda\tau_{1/2}^X}\right)$.

Additional ingredients:

1. A bound on $\mathbb{E}\left(\sup_{t\in[0,T]} |X_t^H - X_t^{1/2}|\right)$;
2. A good (Gaussian-type) estimate of the density $p_t^{X^H}(x)$:
   $\forall t \in (0,\infty), \forall x \in \mathbb{R}$

$$p_t^{X^H}(x) \le \frac{C}{\sqrt{2\pi t^{2H}}} \exp\left(-\frac{(x-x_0)^2}{2t^{2H}}\right) e^{Ct^{2-2H}},$$

where C depends only on b, σ (not H, t).

Perspectives

- In the first part, treat more general SPDEs than $\Delta u = \dot{\mathbb{W}}^h$ (e.g. multiplicative noise);
- In the second part, remove the condition $\lambda \geq \lambda_0$, possibly by adding a kind of ergodicity condition on b, σ;
- Treat $H \leq 1/2$ (rough paths), ...

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Measuring the Hurst parameter

For a time series $\{\theta_k\}_{k\geq 0}$, the R/S statistics is given by:

$$RS(n) = \frac{\max_{0\leq k\leq n}(\theta_k - k\theta_n/n) - \min_{0\leq k\leq n}(\theta_k - k\theta_n/n)}{\sqrt{\frac{1}{n}\sum_{k=1}^{n}(\theta_k - \theta_{k-1} - \theta_n/n)^2}}\ ,\ n \geq 1.$$

Example

Replacing θ_k with B_k^H,

$$n^{-H} RS(n) \Rightarrow \sup_{t\leq 1}(B_t^H - tB_1^H) - \inf_{t\leq 1}(B_t^H - tB_1^H)\ .$$

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Reproducing Kernel Hilbert Space

Definition

Let (T, m) be a complete separable metric space and R a continuous covariance function on $T \times T$. There exists a unique Hilbert space $H(R)$ such that:

1. $H(R)$ is a space of functions from $T \to \mathbb{R}$, and for all $t \in T$, $R(\cdot, t) \in H(R)$;
2. the scalar product is given by: $\forall t \in T, \forall f \in H(R)$,

$$(f, R(\cdot, t))_{H(R)} = f(t).$$

This is a *separable* Hilbert space. It satisfies $H(R) = \overline{\mathrm{Span}\{R(\cdot, t), t \in T\}}^{\|\cdot\|_{H(R)}}$.