Parameter estimation for SDEs related to stationary Gaussian processes

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Plan

Brownian case

Drift Estimation of FOU

- Ergodic case
- Non-ergodic case

Drift Estimation of OU driven FOU

- Continuous case
- Discrete case

Parameter estimation for SDEs related to stationary Gaussian processes

- Power variation
- Applications to Ornstein-Uhlenbeck processes

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Ornstein-Uhlenbeck processes

The Ornstein-Uhlenbeck process X_t driven by a certain type of noise Z_t is described by the following linear stochastic differential equation

$$X_t = X_0 - \theta \int_0^t X_s ds + Z_t.$$

Suppose we don't know the parameter θ, and we have (discrete or continuous) observations of $(X_t, 0 \leqslant t \leqslant T)$ up to the time instant T.
Then an important problem is to estimate the parameter θ based on the observations $(X_t, 0 \leqslant t \leqslant T)$ as $T \longrightarrow \infty$.

Ornstein-Uhlenbeck processes

If Z_t is the standard Brownian motion, this problem has been extensively studied (for example Kutoyants, Yu. A. Springer, (2004) and Liptser, R.S. and Shiryaev, A.N. Springer, 2001) and the references therein. The most popular approaches :

- the maximum likelihood estimators,
- the least squares estimators,
- the method of moments.

Other type of processes have also been studied. For example, when Z_t is an α-stable process, least squares approaches are proposed in Hu, Y. and Long COSA (2007) and SPA (2009).

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Drift Estimation of Fractional Ornstein-Uhlenbeck Process (FOU)

We consider the Ornstein-Uhlenbeck process $X = \{X_t, t \geqslant 0\}$ given by the following linear stochastic differential equation

$$X_0 = 0; \quad dX_t = -\theta X_t dt + dB_t, \quad t \geqslant 0,$$

where B is a fractional Brownian motion of Hurst index $H \in (0,1)$ and $\theta \in (-\infty, \infty)$ is an unknown parameter. The fractional Brownian motion $\{B_t, t \geqslant 0\}$ with Hurst parameter $H \in (0,1)$, is defined as a centered Gaussian process starting from zero with covariance

$$\mathbb{E}(B_t B_s) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

An interesting problem is to estimate the parameter θ when one observes the whole trajectory of X.

Maximum likelihood estimator for FOU

In **[Kleptsyna and Le Breton (2002)]**, the maximum likelihood estimator $\bar{\theta}_T$ for the parameter θ is obtained and has the following expression

$$\bar{\theta}_T = -\left\{\int_0^T Q^2(s)dw_s^H\right\}^{-1}\int_0^T Q(s)dZ_s\,,$$

where

$$k_H(t,s) = \kappa_H^{-1}s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H}\,, \quad w_t^H = \lambda_H^{-1}t^{2-2H}\,;$$
$$Q(t) = \frac{d}{dw_t^H}\int_0^t k_H(t,s)X_s ds\,; \quad Z_t = \theta\int_0^T Q(s)\ dw_s^H + M_t^H\,;$$
$$M_t^H = \int_0^t k_H(t,s)dB_s \text{ is a Gaussian martingale with } < M_t^H >= w_t^H.$$

It is proved that $\lim_{T\to\infty}\bar{\theta}_T = \theta$ almost surely.

Drift Estimation of FOU : Ergodic case

[Hu-Nualart 2010]

In the case $\theta > 0$ (corresponding to the ergodic case), Hu and Nualart (2010) studied the parameter estimation for θ by using the least squares estimator (LSE) defined as

$$\widehat{\theta}_t = -\frac{\int_0^t X_s \, \delta X_s}{\int_0^t X_s^2 ds}, \quad t \geqslant 0.$$

This LSE is obtained by the least squares technique, that is, $\widehat{\theta}_t$ (formally) minimizes

$$\theta \longmapsto \int_0^t \left| \dot{X}_s + \theta X_s \right|^2 ds.$$

Consistency

There is a major difference with respect to the standard Brownian motion case, because the process X being no longer a semimartingale, one cannot utilize the Itô integral to integrate with respect to it. However, one can choose, instead, the Skorohod integral which is connected to Young integral.
By using ergodic property, they proved the strong consistence of $\widehat{\theta}_t$.

Assume $H \in [\frac{1}{2}, 1)$. Then, as $t \to \infty$

$$\widehat{\theta}_t \longrightarrow \theta$$

almost surely.

Asymptotic distribution

The LSE $\widehat{\theta}_t$ of θ is asymptotically normal if $H \in [\frac{1}{2}, \frac{3}{4})$:
Assume $H \in [\frac{1}{2}, \frac{3}{4})$. Then, as $t \to \infty$

$$\sqrt{t}(\widehat{\theta}_t - \theta) \xrightarrow{\text{law}} \mathcal{N}(0, \theta\sigma_H).$$

[Nualart-Ortiz-Latorre]
Let $\{F_n, n \geqslant 1\}$ be a sequence of random variables in the *pth* Wiener chaos, $p \geqslant 2$, such that $\lim_{n\to\infty} E(F_n^2) = 1$. Then the following conditions are equivalent:

- F_n converges in law to $N(0, 1)$ as n tends to infinity.
- $\|DF_n\|_{\mathcal{H}}^2$ converges in L^2 to a constant as n tends to infinity.

Non-ergodic case

[Belfadli-Es-Ouknine 2011]: Non-ergodic case corresponding to $\theta < 0$ in

$$X_0 = 0; \quad dX_t = -\theta X_t dt + dB_t, \quad t \geqslant 0,$$

. More precisely, we estimate θ by the LSE $\widehat{\theta}_t = -\frac{\int_0^t X_s \ \delta X_s}{\int_0^t X_s^2 ds}$,

where in this case, the integral $\int_0^t X_s dX_s$ is interpreted as a Young integral.

Asymptotic behavior

Assume $H \in (\frac{1}{2}, 1)$. Then, as $t \longrightarrow \infty$,

$$\widehat{\theta}_t \longrightarrow \theta \text{ almost surely as } t \longrightarrow \infty.$$

and

$$e^{-\theta t}\left(\widehat{\theta}_t - \theta\right) \xrightarrow{\texttt{law}} -2\theta\mathcal{C}(1),$$

with $\mathcal{C}(1)$ the standard Cauchy distribution with the probability density function $\frac{1}{\pi(1+x^2)}$; $x \in \mathbb{R}$.

Non-ergodic case

[El Machkouri-Es-Ouknine 2015]: Non-ergodic case corresponding to $\theta < 0$ in

$$X_0 = 0; \quad dX_t = -\theta X_t dt + dG_t, \quad t \geqslant 0,$$

Using the formula

$$f_t g_t = f_0 g_0 + \int_0^t g_u df_u + \int_0^t f_u dg_u$$

we can rewrite $\widehat{\theta}_t$ as follows,

$$\widehat{\theta}_t = -\frac{\int_0^t X_s \, dX_s}{\int_0^t X_s^2 ds} == \frac{X_t^2}{2\int_0^t X_s^2 ds}, \quad t \geqslant 0.$$

Asymptotic behavior

(A1) The process G has Hölder continuous paths of strictly positive order.

(A2) For every $t \geqslant 0$, $E\left(G_t^2\right) \leqslant ct^{2\gamma}$ for some positive constants c and γ.

Theorem

Assume that (A1) and (A2) hold. Then

$$\widehat{\theta}_t \to \theta \text{ almost surely as } t \to \infty.$$

Asymptotic behavior

(A3) The limiting variance of $e^{-\theta t} \int_0^t e^{\theta s} dG_s$ exists as $t \to \infty$ i.e., there exists a constant $\sigma_G > 0$ such that

$$\lim_{t\to\infty} E\left[\left(e^{-\theta t}\int_0^t e^{\theta s} dG_s\right)^2\right] \longrightarrow \sigma_G^2.$$

(A4) For all fixed $s \geqslant 0$

$$\lim_{t\to\infty} E\left(G_s e^{-\theta t}\int_0^t e^{\theta r} dG_r\right) = 0.$$

Asymptotic behavior

Theorem

Assume that (A1), (A2), (A3) and (A4) hold. Then, as $t \to \infty$,

$$e^{\theta t}\left(\widehat{\theta}_t - \theta\right) \xrightarrow{\texttt{law}} \frac{2\sigma_G}{\sqrt{E\left(Z_\infty^2\right)}}\mathcal{C}(1),$$

with $\mathcal{C}(1)$ is the standard Cauchy distribution with the probability density function $\frac{1}{\pi(1+x^2)}$; $x \in \mathbb{R}$.

Applications

Fractional Brownian motion:

Fractional Brownian motion with Hurst parameter $H \in (0,1)$:

$$E\left(B_t^H B_s^H\right) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

By assuming that $H > \frac{1}{2}$, Belfadli et al. studied the LSE $\widehat{\theta}_t$. We extend their result to the case $H \in (0,1)$. Moreover, we offer an elementary proof.

Applications

Sub-fractional Brownian motion:
Subfractional Brownian motion with parameter $H \in (0,1)$:

$$E\left(S_t^H S_s^H\right) = t^{2H} + s^{2H} - \frac{1}{2}\left((t+s)^{2H} + |t-s|^{2H}\right).$$

Bifractional Brownian motion:
Bifractional Brownian motion with parameters $(H,K) \in (0,1)^2$:

$$E(B_s^{H,K} B_t^{H,K}) = \frac{1}{2^K}\left(\left(t^{2H} + s^{2H}\right)^K - |t-s|^{2HK}\right).$$

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OUFOU with continuous observation

[El Onsy-Es-Viens 2014]:

Suppose now that $X = \{X_t, t \geqslant 0\}$ is an Ornstein-Uhlenbeck process driven by fractional Ornstein-Uhlenbeck process $V = \{V_t, t \geqslant 0\}$ given by the following linear stochastic differential equations

$$\begin{cases} X_0 = 0; \quad dX_t = -\theta X_t dt + dV_t, \quad t \geqslant 0 \\ V_0 = 0; \quad dV_t = -\rho V_t dt + dB_t^H, \quad t \geqslant 0, \end{cases}$$

where $B^H = \left\{B_t^H, t \geqslant 0\right\}$ is a fBm of Hurst index $H \in (\frac{1}{2}, 1)$, whereas $\theta > 0$ and $\rho > 0$ are considered as unknown parameters such that $\theta \neq \rho$.

OUFOU with continuous observation

Consider the following estimators :

$$\widehat{\theta}_T = -\frac{\int_0^T X_t \delta X_t}{\int_0^T X_t^2 dt}$$

and

$$\widehat{\rho}_T = -\frac{\int_0^T \widehat{V}_t \delta \widehat{V}_t}{\int_0^T \widehat{V}_t^2 dt}.$$

where $\widehat{V}_t = X_t + \widehat{\theta}_T \int_0^t X_t dt$ for every $t \leqslant T$.

OUFOU with continuous observation

Theorem

If $H \in (\frac{1}{2}, \frac{3}{4})$, we have

$$\widehat{\theta}_T = -\frac{\int_0^T X_t \delta X_t}{\int_0^T X_t^2 dt} \longrightarrow \theta^* = \theta + \rho \quad a.s.$$

If $H \in (\frac{1}{2}, \frac{3}{4})$

$$\widehat{\rho}_T = -\frac{\int_0^T \widehat{V}_t \delta \widehat{V}_t}{\int_0^T \widehat{V}_t^2 dt} \longrightarrow \rho^* = \frac{\theta\rho(\theta+\rho)\eta^{\Sigma}}{\eta^X + (\theta+\rho)^2\eta^{\Sigma}} \quad a.s.,$$

and

$$\sqrt{T}\left(\widehat{\theta}_T - \theta^*, \widehat{\rho}_T - \rho^*\right) \xrightarrow{law} \mathcal{N}(0, \Sigma^T \Gamma \Sigma)$$

OUFOU with discrete observation

From a practical point of view, in parametric inference, it is more realistic and interesting to consider asymptotic estimation for fractional diffusion processes based on discrete observations. Assume that the process X is observed equidistantly in time with the step size Δ_n: $t_i = i\Delta_n, i = 0, \ldots, n$, and $T_n = n\Delta_n$ denotes the length of the 'observation window'. We will construct two estimators $\check{\theta}_n$ and $\check{\rho}_n$ of θ and ρ respectively based on the sampling data $X_{t_i}, i = 0, \ldots, n$.

OUFOU with discrete observation

Let us consider the two following auxiliary estimators $\widetilde{\theta}_n$ and $\widetilde{\rho}_n$ of θ^* and ρ^* respectively. $\widehat{\theta}_T = -\frac{\int_0^T X_t \delta X_t}{\int_0^T X_t^2 dt}$ becomes

$$\widetilde{\theta}_n = \frac{\frac{1}{T_n}\int_0^{T_n} X_t \delta X_t}{Q_n(X)}$$

and $\widehat{\rho}_T = -\frac{\int_0^T \widehat{V}_t \delta \widehat{V}_t}{\int_0^T \widehat{V}_t^2 dt}$ becomes

$$\widetilde{\rho}_n(\Sigma) = \frac{\frac{1}{T_n}\int_0^{T_n} \widehat{V}_t^n \delta \widehat{V}_t^n}{Q_n(X) + (\widetilde{\theta}_n)^2 Q_n(\Sigma)}$$

where

$$Q_n(Z) = \frac{1}{n}\sum_{i=1}^{n}(Z_{t_{i-1}})^2$$

and

$$\widehat{V}_t^n = X_t + \widehat{\theta}_n \Sigma_t, \quad 0 \leqslant t \leqslant T_n$$

OUFOU with discrete observation

We have

$$\widetilde{\theta}_n = \frac{\frac{1}{T_n}\int_0^{T_n} X_t \delta X_t}{Q_n(X)} := \frac{J_n^{\theta}(X)}{Q_n(X)}$$

and

$$\widetilde{\rho}_n(\Sigma) = \frac{\frac{1}{T_n}\int_0^{T_n} \widehat{V}_t^n \delta \widehat{V}_t^n}{Q_n(X) + (\widetilde{\theta}_n)^2 Q_n(\Sigma)} := \frac{J_n^{\rho}(\widehat{V})}{Q_n(X) + (\widetilde{\theta}_n)^2 Q_n(\Sigma)}$$

where

$$J_n^{\theta}(X) = (\rho + \theta)\eta^X + o(\frac{1}{\sqrt{T_n}}).$$

where $o(\frac{1}{\sqrt{T_n}})$ denotes a random variable such that $\sqrt{T_n} o(\frac{1}{\sqrt{T_n}})$ converges to zero almost surely .
and

$$J_n^{\rho}(\widehat{V}) = -\rho\theta(\rho + \theta)\eta^{\Sigma} + o(\frac{1}{\sqrt{T_n}}).$$

OUFOU with discrete observation

$$\begin{cases} \check{\theta}_n + \check{\rho}_n = \frac{H\Gamma(2H)[(\check{\rho}_n)^{2-2H} - (\check{\theta}_n)^{2-2H}]}{(\check{\rho}_n - \check{\theta}_n)Q_n(X)} \\ \frac{(\check{\theta}_n)^2 - (\check{\rho}_n)^2}{\left[(\check{\theta}_n)^{2-2H} - (\check{\rho}_n)^{2-2H} + \left(\check{\rho}_n + \check{\theta}_n\right)^2\left((\check{\rho}_n)^{-2H} - (\check{\theta}_n)^{-2H}\right)\right]} = \frac{H\Gamma(2H)}{Q_n(X) + (\check{\theta}_n + \check{\rho}_n)^2 Q_n(\Sigma)} \end{cases}$$

We find that the definition of $\left(\check{\theta}_n, \check{\rho}_n\right)$ is equivalent to the following:

$$F\left(\check{\theta}_n, \check{\rho}_n\right) = (Q_n(X), Q_n(\Sigma))$$

where F is a positive function of the variables (x, y) in $(0, +\infty)^2$ defined by: for every $(x, y) \in (0, +\infty)^2$

$$F(x, y) = H\Gamma(2H) \times \begin{cases} \frac{1}{y^2 - x^2}\left(y^{2-2H} - x^{2-2H}, x^{-2H} - y^{-2H}\right) & \text{if } x \neq y \\ \left((1 - H)x^{-2H}, Hx^{-2H-2}\right) & \text{if } x = y. \end{cases}$$

OUFOU with discrete observation

We have

$$J_F(x,y) = 2\times$$

$$\begin{pmatrix} \frac{(1-H)x^{1-2H}\left(x^2-y^2\right)-x\left(x^{2-2H}-y^{2-2H}\right)}{\left(x^2-y^2\right)^2} & \frac{(1-H)y^{1-2H}\left(y^2-x^2\right)-y\left(y^{2-2H}-x^{2-2H}\right)}{\left(x^2-y^2\right)^2} \\ \frac{Hx^{-2H-1}\left(x^2-y^2\right)+x\left(x^{-2H}-y^{-2H}\right)}{\left(x^2-y^2\right)^2} & \frac{Hy^{-2H-1}\left(y^2-x^2\right)+y\left(y^{-2H}-x^{-2H}\right)}{\left(x^2-y^2\right)^2} \end{pmatrix}$$

The determinant of $J_F(x,y)$ is non-zero on $(0,+\infty)^2$. Hence

$$\left(\widehat{\theta}_n, \widehat{\rho}_n\right) = G\left(Q_n(X), Q_n(\Sigma)\right).$$

where G is the inverse function of F.

OUFOU with discrete observation

Theorem

Assume $H \in (\frac{1}{2}, 1)$ (even if $\Delta_n = 1$).

$$(\check{\theta}_n, \check{\rho}_n) \longrightarrow (\theta, \rho)$$

almost surely as $n \to \infty$.

Theorem

Suppose that $H \in (\frac{1}{2}, \frac{3}{4})$ and $\Delta_n \to 0$. Then

$$\sqrt{T_n}\left(\check{\theta}_n - \theta, \check{\rho}_n - \rho\right) \xrightarrow{law} \mathcal{N}(0, M^T \Sigma^T \Gamma \Sigma M)$$

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Method of moments approach [Es-Viens 2015]

Consider, for every $q \in \mathbb{N}^*$ even, the following power variation

$$P_{q,n}(Z) := \frac{1}{n} \sum_{i=0}^{n-1} (Z_i)^q.$$

Define $\delta_Z(q) := E\left[(Z_0)^q\right] = \frac{q!}{(\frac{q}{2})! 2^{q/2}} \left[E\left(Z_0^2\right)\right]^{q/2}$.

Theorem (almost sure convergence)

Suppose that Z is ergodic. Then, as $n \to \infty$

$$P_{q,n}(Z) \longrightarrow \delta_Z(q) \ a.s..$$

Let $c_{q,2k} = \frac{1}{(2k)!} \int_{-\infty}^{\infty} \frac{e^{-x^2/2}}{\sqrt{2\pi}} x^q H_{2k}(x)\, dx$ be the coefficients of the monomial x^q expanded in the basis of Hermite polynomials:

$$x^q = \sum_{k=0}^{q/2} c_{q,2k} H_{2k}(x).$$

Power variation

Then we can write,

$$
\begin{aligned}
V_{P_{q,n}}(Z) &= \sqrt{n}\left(P_{q,n}(Z) - E\left[(Z_0)^q\right]\right) \\
&= \frac{[r_Z(0)]^{q/2}}{\sqrt{n}} \sum_{i=0}^{n-1} \left(E\left[\left(\frac{Z_i}{\sqrt{r_Z(0)}}\right)^q\right] - E\left[\left(\frac{Z_0}{\sqrt{r_Z(0)}}\right)^q\right]\right) \\
&= \frac{[r_Z(0)]^{q/2}}{\sqrt{n}} \sum_{i=0}^{n-1} \sum_{k=1}^{q/2} c_{q,2k}(Z) H_{2k}(Y_i) \\
&= \sum_{k=1}^{q/2} c_{q,2k}(Z) \frac{[r_Z(0)]^{q/2}}{\sqrt{n}} \sum_{i=0}^{n-1} I_{2k}\left(\varepsilon_i^{\otimes 2k}\right) \\
&= \sum_{k=1}^{q/2} I_{2k}\left(c_{q,2k}(Z) \frac{[r_Z(0)]^{q/2}}{\sqrt{n}} \sum_{i=0}^{n-1} \varepsilon_i^{\otimes 2k} \right)
\end{aligned}
$$

where $r_Z(k) := E(Z_0 Z_k)$ and $Y_i = Y(\varepsilon_i) = \frac{Z_i}{\sqrt{r_Z(0)}}$.

Power variation

Furthermore,

$$
\begin{aligned}
& E\left[V^2_{P_{q,n}}(Z)\right] \\
= \quad & [r_Z(0)]^q \sum_{k=1}^{q/2} c^2_{q,2k}(Z) \frac{(2k)!}{n} \sum_{i,j=0}^{n-1} |r_Y(i-j)|^{2k} \\
= \quad & [r_Z(0)]^q \sum_{k=1}^{q/2} c^2_{q,2k}(Z)(2k)! \left(1 + \frac{2}{n} \sum_{j=1}^{n-1} (n-1-j)|r_Y(j)|^{2k}\right) \\
= \quad & [r_Z(0)]^q \sum_{k=1}^{q/2} c^2_{q,2k}(Z)(2k)! \left(1 + 2\sum_{j=1}^{n-1} |r_Y(j)|^{2k} - \frac{2}{n} \sum_{j=1}^{n-1} j|r_Y(j)|^{2k}\right)
\end{aligned}
$$

Power variation

Lemma

Let $(Z_k)_{k \geqslant 0}$ be a stationary Gaussian sequence with $E(Z_0^2) < \infty$, and let $\lambda > 0$ and $q \in \mathbb{N}^*$ even. Consider the sequence

$$R_{P,q}(\lambda, Z_k) := \left(Z_k - e^{-\lambda k} Z_0\right)^q - (Z_k)^q .$$

Then for every $p \geqslant 1$ there exits a constant $c(\lambda, q)$ depending on λ, q and $E(Z_0^2)$ such that

$$\left\| \frac{1}{n} \sum_{k=0}^{n-1} R_{P,q}(\lambda, Z_k) \right\|_{L^p(\Omega)} \leqslant \frac{c(\lambda, q)}{n}.$$

Moreover for every $\varepsilon > 0$, $\lim_{n\to\infty} \frac{1}{n^\varepsilon} \sum_{k=0}^{n-1} R_{P,q}(\lambda, Z_k) \longrightarrow 0$. a.s.

Power variation

Theorem

Let $\left(V_{P_{q,n}}(Z)\right)_{n\geqslant 0}$ *and* $\left(R_{P,q}(\lambda, Z_k)\right)_{n\geqslant 0}$ *be the sequences defined in the above.*
1) Then there exist C *depending on* λ*,* q *and* $r_Z(0)$ *such that*

$$d_W\left(\sqrt{\frac{n}{E\left[V^2_{P_{q,n}}(Z)\right]}}\left(P_{q,n}\left(Z_k - e^{-\lambda k}Z_0\right) - \delta_Z(q)\right), N\right)$$

$$\leqslant \quad \frac{C}{E\left[V^2_{P_{q,n}}(Z)\right]}\left(\sqrt{\frac{E\left[V^2_{P_{q,n}}(Y)\right]}{n}}\right.$$

$$\left. + \sqrt{E\left[V^2_{P_{2,n}}(Y)\right]\sqrt{\kappa_4(V_{P_{2,n}}(Y)) + \kappa_4(V_{P_{2,n}}(Y))}}\right).$$

where $Y = Z/\sqrt{r_Z(0)}$ *and* $\kappa_4(Y) = E[Y^4] - 3E[Y^2]^2$.

Power variation

On the other hand if $\sum_{k\in\mathbb{Z}} |r_Y(k)|^2 < \infty$, we can write

$$d_W\left(\sqrt{\frac{n}{v_{P_q}(Z)}}\left(P_{q,n}\left(Z_k - e^{-\lambda k}Z_0\right) - \eta_Z(q)\right), N\right)$$

$$\leqslant \quad \frac{C}{v_{P_q}(Z)}\left(\sqrt{\frac{v_{P_q}(Z)}{n}} + \sqrt{E\left[V^2_{P_{2,n}}(Y)\right]\sqrt{\kappa_4(V_{P_{2,n}}(Y)) + \kappa_4(V_{P_{2,n}}(Y}\right.$$

$$+ \left|v_q(Z) - E\left[V^2_{P_{q,n}}(Y)\right]\right|\Bigg).$$

where $v_{P_q}(Z) := \lim_{n\to\infty} E\left[V^2_{P_{q,n}}(Z)\right]$.

2) Assume that $r_Y(k) \sim c|k|^{-\alpha}$ with $0 < \alpha < \frac{1}{2}$. Then

$$\frac{n^\alpha}{\sqrt{v_q(Z)}}\left(P_{q,n}\left(Z_k - e^{-\lambda k}Z_0\right) - \eta_Z(q)\right) \xrightarrow{law} \frac{c_{q,2}}{\sqrt{D}}F_\infty.$$

Fractional Ornstein-Uhlenbeck process

$X = \{X_t, t \geqslant 0\}$ is an Ornstein-Uhlenbeck process driven by a fractional Brownian motion $B^H = \{B_t^H, t \geqslant 0\}$ of Hurst index $H \in (0,1)$. That is, X is the solution of

$$X_0 = 0; \quad dX_t = -\theta X_t dt + dB_t^H, \quad t \geqslant 0,$$

where$\theta > 0$ is considered as unknown parameter.
The solution X has the following explicit expression:

$$X_t = \int_0^t e^{-\theta(t-s)} dB_s^H.$$

We can also write

$$X_t = Z_t^\theta - e^{-\theta t} Z_0^\theta$$

where

$$Z_t^\theta = \int_{-\infty}^t e^{-\theta(t-s)} dB_s^H.$$

Moreover, Y^θ is an ergodic stationary Gaussian process.

Fractional Ornstein-Uhlenbeck process

Lemma

Let $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$*,* $m, m' > 0$*. Then,*

$$E\left[\left(Z_0^\theta\right)^2\right] = H\Gamma(2H)\theta^{-2H}$$

and for large $|t|$

$$E\left[Z_0^\theta Z_t^\theta\right] \sim \frac{H(2H-1)}{\theta^2}|t|^{2H-2}.$$

Construction of the estimators

Fix $q \geqslant 2$ and assume that q is even.
We can write

$$P_{q,n}(X) \quad = \quad P_{q,n}(Z^{\theta}) + \frac{1}{n}\sum_{k=0}^{n-1} R_{P,q}(\theta, Z_k^{\theta})$$

Since Z^{θ} is ergodic we conclude that, almost surely,

$$\begin{aligned}
\lim_{n\longrightarrow\infty} P_{q,n}(X) &= \lim_{n\longrightarrow\infty} P_{q,n}(Z^{\theta}) \\
&= \gamma_{Z^{\theta}}(q) \\
&= \frac{q!}{(\frac{q}{2})!2^{q/2}}\left(H\Gamma(2H)\theta^{-2H} - 1\right)^{q/2} \\
&:= \mu_q(\theta).
\end{aligned}$$

Hence we obtain the following estimator for θ

$$\widehat{\theta}_{q,n} = \mu_q^{-1}\left[P_{q,n}(X)\right].$$

Consistency

As consequence, we have the following strong consistence of $\widehat{\theta}_{q,n}$.

Theorem

Let $H \in (0,1)$. Then, as $n \longrightarrow \infty$

$$\widehat{\theta}_{q,n} \longrightarrow \theta$$

almost surely.

Asymptotic distribution

Theorem

Denote $N \sim \mathcal{N}(0,1)$. If $H \in (0, \frac{3}{4}]$, then there exists C depending on q, H and θ such that

$$d_W\left(\sqrt{\frac{n}{E\left[V_{q,n}^2(Z^\theta)\right]}}\left(\mu_q\left(\widehat{\theta}_{q,n}\right) - \mu_q(\theta)\right), N\right)$$

$$\leqslant \quad C \begin{cases} n^{-\frac{1}{4}}, & \text{if } 0 < H < \dfrac{5}{8} \\ n^{-\frac{1}{4}} \log^{\frac{3}{4}}(n), & \text{if } H = \dfrac{5}{8} \\ n^{2H-\frac{3}{2}}, & \text{if } \dfrac{5}{8} < H < \dfrac{3}{4} \\ \log^{-\frac{1}{4}}(n), & \text{if } H = \dfrac{3}{4}. \end{cases}$$

Asymptotic distribution

If $H = \frac{3}{4}$

$$\sqrt{\frac{n}{\log(n)}}\left(\mu_q\left(\widehat{\theta}_{q,n}\right) - \mu_q(\theta)\right) \xrightarrow{law} \mathcal{N}\left(0, 4b_{q,2}^2(Z^\theta)\right)$$

where in this case $E\left[V_{q,n}^2(Z^\theta)\right] \sim 4b_{q,2}^2(Z^\theta)\log(n)$. In the case when $H \in (\frac{3}{4}, 1)$, we have

$$\frac{1}{n^{2H-\frac{3}{2}}}\left(\mu_q\left(\widehat{\theta}_{q,n}\right) - \mu_q(\theta)\right) \xrightarrow{law} \frac{b_{q,2}(Z^\theta)}{\sqrt{D}}F_\infty$$

where F_∞ is defined above.

Asymptotic distribution

Theorem

If $H \in (0, \frac{3}{4}]$, then

$$\sqrt{\frac{n}{E\left[V_{q,n}^2(Z^\theta)\right]}}\left(\widehat{\theta}_{q,n} - \theta\right) \xrightarrow{law} \mathcal{N}\left(0, \left(\mu_q'(\theta)\right)^{-2}\right).$$

If $H \in (\frac{3}{4}, 1)$, then

$$\frac{1}{n^{2H-\frac{3}{2}}}\left(\widehat{\theta}_{q,n} - \theta\right) \xrightarrow{law} \frac{b_{q,2}}{\mu_q'(\theta)\sqrt{D}} F_\infty$$

Because,

$$\sqrt{n}\left(\mu_q(\widehat{\theta}_{q,n}) - \mu_q(\theta)\right) = \mu_q'(\xi_{q,n})\sqrt{n}\left(\widehat{\theta}_{q,n} - \theta\right)$$

where $\xi_{q,n}$ is a random variable between θ and $\widehat{\theta}_{q,n}$.

Fractional Ornstein-Uhlenbeck process with the second kind

Let $U = \{U_t, t \geqslant 0\}$ be a fOU process with the second kind defined as

$$U_0 = 0, \text{ and } dU_t = -\alpha U_t dt + dY_t^{(1)}, \quad t \geqslant 0,$$

where $Y_t^{(1)} = \int_0^t e^{-s} dB_{a_s}$ with $a_s = He^{\frac{s}{H}}$ and $B = \{B_t, t \geqslant 0\}$ is a fBm.

U_t admits an explicit solution

$$\begin{aligned} U_t &= e^{-\alpha t} \int_0^t e^{\alpha s} dY_s^{(1)} = e^{-\alpha t} \int_0^t e^{(\alpha-1)s} dB_{a_s} \\ &= H^{(1-\alpha)H} e^{-\alpha t} \int_{a_0}^{a_t} r^{(\alpha-1)H} dB_r. \end{aligned}$$

Hence $U_t = U_t^\alpha + R(\alpha, U^\alpha)$

where

$U_t^\alpha = e^{-\alpha t} \int_{-\infty}^t e^{(\alpha-1)s} dB_{a_s} = H^{(1-\alpha)H} e^{-\alpha t} \int_0^{a_t} r^{(\alpha-1)H} dB_r.$

Fractional Ornstein-Uhlenbeck process with the second kind

Lemma

Let $H \in (\frac{1}{2}, 1)$. Then,

$$E\left[(U_0^\alpha)^2\right] = \frac{(2H-1)H^{2H}}{\alpha}\beta(1-H+\alpha H, 2H-1).$$

and for large $|t|$

$$r_{U^\alpha}(t) = E\left[U_0^\alpha U_t^\alpha\right] = O\left(e^{-\min\{\alpha, \frac{1-H}{H}\}t}\right).$$

Fractional Ornstein-Uhlenbeck process with the second kind

Since U^α is ergodic we conclude that almost surely

$$\lim_{n\longrightarrow\infty} Q_{q,n}(U) = \lim_{n\longrightarrow\infty} Q_{q,n}(U^\alpha) = \frac{q!}{(\frac{q}{2})!2^{q/2}}\left(E\left[(U_0^\alpha)^2\right]-1\right)^{q/2}$$

$$= \frac{q!}{(\frac{q}{2})!2^{q/2}}\left(\frac{(2H-1)H^{2H}}{\alpha}\beta(1-H+\alpha H, 2H-1)-1\right)^{q/2}$$

$$:= \nu_q(\alpha).$$

Hence we obtain the following estimator for α

$$\widehat{\alpha}_{q,n} = \nu_q^{-1}\left[Q_{q,n}(U)\right].$$

By construction, we have the strong consistence of $\widehat{\alpha}_{q,n}$.

Theorem

Let $H \in \left(\frac{1}{2}, 1\right)$. Then, as $n \longrightarrow \infty$

$$\widehat{\alpha}_{q,n} \longrightarrow \alpha, \text{ almost surely.}$$

Fractional Ornstein-Uhlenbeck process with the second kind

Theorem

Let $H \in (\frac{1}{2}, 1)$. Then

$$\sqrt{\frac{n}{E\left[V_{q,n}^2(U^\alpha)\right]}}\left(\widehat{\alpha}_{q,n} - \alpha\right) \xrightarrow{law} \mathcal{N}\left(0, \left(\nu_q'(\alpha)\right)^{-2}\right).$$

(generalization of the work of Azmoodeh and Viitasaari (2014))

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THANK YOU FOR YOUR ATTENTION