Stochastic regularization effects of semi-martingales on random functions

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Stochastic regularization

Itô-Wentzell-Tanaka trick

STOCHASTIC REGULARIZATION IN A NUTSHELL

The following slides are based on the lecture notes of Franco Flandoli (2015) and on his St. Flour lecture Notes "Random Perturbation of PDEs and Fluid Dynamic Models" (2010).

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The map u is smooth and solves the Heat equation:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u, \quad u(0,\cdot) = \varphi(\cdot),$$

and

$$u(t,x) = \int_{\mathbb{R}^d} P_t^{\text{heat}}(x-y)\varphi(y)dy.$$

A SECOND EXAMPLE

- Consider the following ODE:

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  for some $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$.

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- Take for instance $d = 1$ and $b(t, x) := b(x) := 2\mathsf{sgn}(x)\sqrt{|x|}$ and $x_0 := 0$, then every function of the form

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What is then a good solution?

A SECOND EXAMPLE

- Add some noise:

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- Selection of solutions: Assume that for any σ there exists a unique solution, then let $\mathbb{P}_\sigma$ denotes its law. Then prove that $(\mathbb{P}_\sigma)_{\sigma>0}$ is tight and converges in law (as σ tends to 0) to some measure supported on the set of solutions to the ODE.

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  - For instance, Bafico and Baldi (81′) proved that for $b(x) = 2\mathsf{sgn}(x)\sqrt{|x|}$ and $x_0 = 0$ it converges to:

    $$\frac{1}{2}\delta_{+t^2} + \frac{1}{2}\delta_{-t^2}.$$

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- How does it work?

A SECOND EXAMPLE

- Recall that:

$$X_t = x_0 + \int_0^t b(s, X_s)ds + \sigma B_t$$

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- Example: use the celebrated Itô-Tanaka formula for $b = \delta_a$ and for B:

$$\int_0^t \delta_a(B_s) ds = |B_t - a| - |a| - \int_0^t \mathsf{sgn}(B_s - a) dB_s.$$

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- Idea: to express $\int_0^t b(s, X_s)ds$ by means of more regular objects

THE ITÔ-TANAKA TRICK

- Apply Itô's formula with a smooth mapping U:

$$U(t, X_t) = U(T, X_T) - \int_t^T \left(\frac{\partial U}{\partial t} + b \cdot \nabla U + \frac{1}{2}\sigma^2 \Delta U \right)(s, X_s) ds$$
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- So if U is solution to the Fokker-Planck (Backward) PDE

$$\frac{\partial U}{\partial t} + b \cdot \nabla U + \frac{\sigma^2}{2}\Delta U = -b, \quad U(T,x) = 0,$$

then

Théorème (Itô-Tanaka Trick)

$$\int_0^T f(s,X_s)ds = -U(0,X_0) - \int_0^T \nabla U(s,X_s)dB_s, \ \mathbb{P} - a.s..$$

and so

$$X_t = x_0 + U(0,x_0) - U(t,X_t) + \sigma \int_0^t (\nabla U(s,X_s) + Id.)dB_s.$$

APPLICATIONS OF THE ITÔ-TANAKA TRICK TO SPDES

- The Itô-Tanaka Trick can be used to obtain new results in linear transport equations by introducing a stochastic perturbation (see *Flandoli, Gubinelli, Priola; 10'; Invent. Math.*).

APPLICATIONS OF THE ITÔ-TANAKA TRICK TO SPDES

- The Itô-Tanaka Trick can be used to obtain new results in linear transport equations by introducing a stochastic perturbation (see *Flandoli, Gubinelli, Priola; 10'; Invent. Math.*).
- Limitation to other problems: (Flandoli *et al.*)

  "The generalization to nonlinear transport equations, where b depends on u itself, would be a major next step for applications to fluid dynamics but it turns out to be a difficult problem. Specifically there are already some difficulties in dealing with a vector field b which depends itself on the random perturbation W. There is no obvious extension of the Itô-Tanaka trick to integrals of the form $\int_0^T f(\omega, s, X_s^x(\omega))ds$ *with random* f*."*

Stochastic regularization

Itô-Wentzell-Tanaka trick

GENERALIZATIONS TO RANDOM MAPPINGS

The problem pointed out previously is to provide an expression for:

$$\int_0^T f(s, \omega, X_s) ds,$$

where f is now random (previously we had $f = b$ where b was deterministic) in a predictable way.

- If we reproduce the ideas before we need to consider the Fokker-Planck SPDE:

$$U(t, x) = -\int_t^T \left(\frac{1}{2}\Delta + b(s, \omega, x) \cdot \nabla \right) U(s, x) ds - \int_t^T f(s, \omega, x) ds.$$

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- But: in that case $U(t,x)$ is not adapted (even if the data b, f are adapted) so you can not use classical Itô calculus and the previous approach fails.

GENERALIZATIONS TO RANDOM MAPPINGS

- Idea: make it adapted, and consider rather the following Fokker-Planck BSPDE:

$$U^a(t,x) = -\int_t^T \mathcal{L}_s U^a(s,x)ds - \int_t^T f(s,\omega,x)ds - \int_t^T Z(s,x)dB_s,$$

with $\mathcal{L}_s := \frac{1}{2}\Delta + b(s,\omega,x)\cdot\nabla$.

If solvable, U^a and Z are two predictable processes.

ITÔ-WENTZELL-TANAKA TRICK

Théorème (Duboscq, R.)

Assume that U^a and Z exist and are regular enough, then

$$\int_0^T f(s,\omega,X_s)ds = -U^a(0,X_0) - \int_0^T \left(\nabla U^a(s,X_s) + Z(s,X_s)\right) dB_s - \int_0^T \nabla Z(s,X_s)ds,\ \mathbb{P}-a.s..$$

Now we need to study the BSPDE and the regularity of (U^a, Z).

To this end, we make use of the Malliavin calculus.

SOME ELEMENTS OF MALLIAVIN CALUCLUS

We consider $\mathcal{S}$ the set of *simple* random fields F:

$$F : \Omega \times \mathbb{R}^d \to \mathbb{R}$$

$$F(\omega, x) := \varphi\left(B_{t_1}(\omega), \cdots, B_{t_n}(\omega), x\right), \ \varphi \in \mathcal{C}^\infty(\mathbb{R}^{n+d}), \ n \geq 1, \ t_i \in [0, T].$$

For any such F we set:

$$DF : \Omega \times \mathbb{R}^d \to L^p([0, T], dt)$$

defined as

$$D_t F := \sum_{i=1}^{n} \partial_i \varphi\left(B_{t_1}, \cdots, B_{t_n}, x\right) \mathbf{1}_{t \leq t_i}, \quad t \in [0, T].$$

$\mathbb{D}^{1,m,p}$:= closure of $\mathcal{S}$ with respect to the Malliavin-Sobolev semi-norm:

$$\|F\|^p_{\mathbb{D}^{1,m,p}} := \mathbb{E}\left[\|F\|^p_{W^{m,p}(\mathbb{R}^d)}\right] + \int_0^T \mathbb{E}\left[\|D_\theta F\|^p_{W^{m,p}(\mathbb{R}^d)}\right] d\theta.$$

ANALYSIS OF THE BSPDE

Théorème (Duboscq, R.)

Let $p, q \geq 2$. Assume that b, f are adapted and belong to $L^q([0,T];\mathbb{D}^{1,0,p})$ (+additional properties on Db, Df). There exists a unique strong (predictable) solution to the Fokker-Planck BSPDE

$$\int_0^T \mathbb{E}[\|U^a(t,\cdot)\|_{W^{2,p}(\mathbb{R}^d}]^{q/p} + \mathbb{E}[\|Z(t,\cdot)\|_{W^{2,p}(\mathbb{R}^d}]^{q/p} dt < +\infty$$

Futhermore, we have the following representation of U^a

$$U^a(t,x) = \mathbb{E}\left[-\int_t^T P^X_{t,r} f(r,x)dr \middle| \mathcal{F}_t\right]. \tag{1}$$

In addition, for a.e. (t,x), $U^a(t,x)$ is Malliavin differentiable ($\int_0^T \|U^a(t,\cdot)\|^q_{\mathbb{D}^{1,2,p}} dt < +\infty$), and for a.e. $x \in \mathbb{R}^d$, a version of the process $(Z(t,x))_{t\in[0,T]}$ is given by

$$Z(t,x) = D_t U^a(t,x) = \mathbb{E}\left[-\int_t^T D_t P^X_{t,r} f(r,x)dr \middle| \mathcal{F}_t\right]. \tag{2}$$

...

ANALYSIS OF THE BSPDE

Théorème (Duboscq, R.)

... Finally, U^a admits the following mild (a.k.a. Duhamel's formula) representation

$$U^a(t,x) = -\int_t^T P^X_{t,r} f(r,x)dr - \int_t^T P^X_{t,r} Z(r,x)dB_r, \tag{3}$$

where $P^X\phi$ is the unique solution to:

$$P^X_{s,t}\phi(x) = \phi(x) - \int_s^t \mathcal{L}_r P^X_{r,t}\phi(x)dr, \quad 0 \le s \le t.$$

ANALYSIS OF THE BSPDE

Remarques

- We are not working in L^2
- We provide an explicit representation which is a counterpart of the one for linear BSDEs (no reversibility of the semigroup)
- Malliavin differentiability in $L^p - L^q$ spaces is not completely trivial...there are catches
- Duhamel's formula in that context is new